Example

TBox:

$$\texttt{MixedTeam} \equiv \texttt{Team} \sqcap \exists \texttt{hasMember.Male} \sqcap \exists \texttt{hasMember.Female}$$

$$\texttt{Male} \equiv \neg \texttt{Female}$$

ABox:

$$\texttt{MixedTeam}(\texttt{FC})$$

$$(\forall \texttt{hasMember.Male})(\texttt{FC})$$

The above knowledge base is unsatisfiable. How can we prove it using tableaus?

Example (cont'd)

After unfolding the definition of `MixedTeam`, we have:

$$(\texttt{Team} \sqcap \exists \texttt{hasMember}.\texttt{Male} \sqcap \exists \texttt{hasMember}.\texttt{Female})(\texttt{FC})$$

$$(\forall \texttt{hasMember}.\texttt{Male})(\texttt{FC})$$

After unfolding the definition of `Male`, we have:

$$(\texttt{Team} \sqcap \exists \texttt{hasMember}.\neg \texttt{Female} \sqcap \exists \texttt{hasMember}.\texttt{Female})(\texttt{FC})$$

$$(\forall \texttt{hasMember}.\neg \texttt{Female})(\texttt{FC})$$

Example (cont'd)

The closed tableau showing unsatisfiability is as follows:

1	$(\mathtt{Team} \sqcap \exists \mathtt{hasMember}.\neg \mathtt{Female} \sqcap \exists \mathtt{hasMember}.\mathtt{Female})(\mathtt{FC})$	given
2	$(\forall \mathtt{hasMember}.\neg \mathtt{Female})(\mathtt{FC})$	given
3	$(\mathtt{Team} \sqcap \exists \mathtt{hasMember}.\neg \mathtt{Female})(\mathtt{FC})$	1, $\sqcap$-rule
4	$(\exists \mathtt{hasMember}.\mathtt{Female})(\mathtt{FC})$	1, $\sqcap$-rule
5	$\mathtt{hasMember}(\mathtt{FC}, \mathtt{a})$	4, $\exists$-rule
6	$\mathtt{Female}(\mathtt{a})$	4, $\exists$-rule
7	$\neg \mathtt{Female}(\mathtt{a})$	2, 5, $\forall$-rule
8	**Clash**	6, 7

General TBoxes

General TBoxes include concept definitions and concept inclusions.

Example:

$$\texttt{Woman} \equiv \texttt{Person} \sqcap \texttt{Female}$$

$$\texttt{Person} \sqsubseteq \exists \texttt{hasParent}.\texttt{Person}$$

Tableau Techniques for General TBoxes

To construct a tableau proof for general TBoxes, we proceed as follows:

1. Given a of TBox $\mathcal{T}$, we construct a set of concepts $\widehat{\mathcal{T}}$ as follows:
   - Each concept definition is equivalently re-written as two concept inclusions.
   - Each concept inclusion $C \sqsubseteq D$ is rewritten as $\neg C \sqcup D$.
2. We compute the negation normal form $nnf(\widehat{\mathcal{T}})$ of $\widehat{\mathcal{T}}$ as the set of the negation normal forms of its members.

Example

$$\mathcal{T} = \{\ \texttt{Woman} \equiv \texttt{Person} \sqcap \texttt{Female},\ \texttt{Person} \sqsubseteq \exists \texttt{hasParent.Person}\ \}$$

$\mathcal{T}$ can be equivalently rewritten as follows:

$$\widehat{\mathcal{T}} = \{\ \texttt{Woman} \sqsubseteq \texttt{Person} \sqcap \texttt{Female},\ \texttt{Person} \sqcap \texttt{Female} \sqsubseteq \texttt{Woman},$$

$$\texttt{Person} \sqsubseteq \exists \texttt{hasParent.Person}\ \}$$

Then

$$\widehat{\mathcal{T}} = \{\ \neg\texttt{Woman} \sqcup (\texttt{Person} \sqcap \texttt{Female}),\ \neg(\texttt{Person} \sqcap \texttt{Female}) \sqcup \texttt{Woman},$$

$$\neg\texttt{Person} \sqcup \exists \texttt{hasParent.Person}\ \}$$

Example (cont'd)

Then

$$\texttt{nnf}(\widehat{\mathcal{T}}) = \{\ \neg\texttt{Woman} \sqcup (\texttt{Person} \sqcap \texttt{Female}),\ \neg\texttt{Person} \sqcup \neg\texttt{Female} \sqcup \texttt{Woman},$$

$$\neg\texttt{Person} \sqcup \exists\texttt{hasParent}.\texttt{Person}\ \}$$

Rationale

The rationale behind the construction of $\widehat{\mathcal{T}}$ is the following.

Given any Tbox $\mathcal{T}$ such that $\widehat{\mathcal{T}} = \{C_1, \ldots, C_n\}$, it is easy to see that $\mathcal{T}$ is equivalent to

$$\top \sqsubseteq C_1 \sqcap \cdots \sqcap C_n.$$

How do we prove this?

We have to prove that for every interpretation $\mathcal{I}$:

$$\mathcal{I} \models \mathcal{T} \text{ iff } \mathcal{I} \models \top \sqsubseteq C_1 \sqcap \cdots \sqcap C_n.$$

Rationale (cont'd)

Try first proving a simple version:

For any concepts C and D and every interpretation $\mathcal{I}$:

$$\mathcal{I} \models C \sqsubseteq D \text{ iff } \mathcal{I} \models \top \sqsubseteq \neg C \sqcup D.$$

The $\sqsubseteq$-rule

We now introduce a new inference rule.

If a is an individual that appears in $\mathcal{A}$ and C is a concept in $\widehat{\mathcal{T}}$
then

$$\mathcal{A} := \mathcal{A} \cup \{C(a)\}.$$

Example

Let us assume we are given the following knowledge base:

$$\texttt{Woman} \equiv \texttt{Person} \sqcap \texttt{Female}$$

$$\texttt{Person}(\texttt{ANN}),\ \texttt{Female}(\texttt{ANN}),\ \neg\texttt{Woman}(\texttt{ANN})$$

Can we use tableau to prove that it is unsatisfiable?

Example (cont'd)

If $\mathcal{T}$ is the above Tbox, then

$$\mathtt{nnf}(\widehat{\mathcal{T}}) = \{\, \neg\mathtt{Woman} \sqcup (\mathtt{Person} \sqcap \mathtt{Female}),\ \neg\mathtt{Person} \sqcup \neg\mathtt{Female} \sqcup \mathtt{Woman}\, \}$$

Example (cont'd)

If we use abbreviations $\mathtt{A}, \mathtt{F}, \mathtt{P}, \mathtt{W}$ to save space, the complete proof is as follows:

1	$\mathtt{P}(\mathtt{A})$	given						
2	$\mathtt{F}(\mathtt{A})$	given						
3	$\neg\mathtt{W}(\mathtt{A})$	given						
4	$(\neg\mathtt{P} \sqcup \neg\mathtt{F} \sqcup \mathtt{W})(\mathtt{A})$	by $\sqsubseteq$ for $\mathtt{A}$						
5	$(\neg\mathtt{P})(\mathtt{A})$		6	$(\neg\mathtt{F})(\mathtt{A}) \sqcup \mathtt{W}(\mathtt{A})$	by 4, $\sqcup$			
	Clash	by 1,5	7	$(\neg\mathtt{F})(\mathtt{A})$		8	$\mathtt{W}(\mathtt{A})$	by 6, $\sqcup$
				Clash	by 2,7		**Clash**	by 3,8