Componentes de um vetor e mudança de base

Livro: Boldrini

Tópico 4.8 página 129.

Questões: 29, 30, 33 e 35.

Vetor Oliveira Ropke

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(29) Sejam $\beta=\{(1,0),(0,1)\}$, $\beta_1=\{(-1,1),(1,1)\}$, $\beta_2=\{(\sqrt{3},1),(\sqrt{3},-1)\}$ e $\beta_3=\{(2,0),(0,2)\}$ bases ordenadas de $\mathbb{R}^2$.

a) Ache as matrizes de mudança de base:

i) $[I]_{\beta}^{\beta_1}$ $\beta=\{(1,0)(0,1)\}$
$\beta_1=\{(-1,1),(1,1)\}$

$(-1,1)=a(1,0)+b(0,1)$
$(-1,1)=(a,0)+(0,b)$
$(-1,1)=(a,b)$

$\begin{cases} a=-1 \\ b=1 \end{cases}$

$(1,1)=c(1,0)+d(0,1)$
$(1,1)=(c,0)+(0,d)$
$(1,1)=(c,d)$

$\begin{cases} c=1 \\ d=1 \end{cases}$

$$[I]_{\beta}^{\beta_1}=\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

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ii) $[I]_{\beta_1}^{\beta}$ $\beta=\{(1,0),(0,1)\}$
$\beta_1=\{(-1,1),(1,1)\}$

$(1,0)=a(-1,1)+b(1,1)$
$(1,0)=(-a,a)+(b,b)$
$(1,0)=(-a+b,a+b)$

$\begin{cases} -a+b=1 \rightarrow b=1+a \\ a+b=0 \rightarrow a=-b \end{cases}$

$a=-(1+a)$
$a=-1-a$
$a+a=-1$
$2a=-1$
$\boxed{a=\frac{-1}{2}}$

$b=1+\left(-\frac{1}{2}\right) \rightarrow b=1-\frac{1}{2}$

$\boxed{b=\frac{1}{2}}$

$(0,1)=c(-1,1)+d(1,1)$
$(0,1)=(-c,c)+(d,d)$
$(0,1)=(-c+d,c+d)$

$\begin{cases} -c+d=0 \rightarrow d=c=\frac{1}{2} \\ c+d=1 \end{cases}$

0.5 0.5

$\frac{1}{2}$ $\frac{1}{2}$

$$[I]_{\beta_1}^{\beta}=\begin{bmatrix} \frac{-1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

iii) $[I]^{\beta}_{B_2}$ $\beta = \{(1,0),(0,1)\}$
$B_2 = \{(\sqrt{3}, 1), (\sqrt{3}, -1)\}$

$(1,0) = a(\sqrt{3},1) + b(\sqrt{3},-1)$
$(1,0) = (\sqrt{3}a, a) + (\sqrt{3}b, -b)$
$(1,0) = (\sqrt{3}a + \sqrt{3}b, a-b)$

$\frac{1}{2}$ $\frac{1}{2}$
0.5 0.5

$$\begin{cases} \sqrt{3}a + \sqrt{3}b = 1 \\ a - b = 0 \rightarrow \boxed{a = b} \end{cases}$$

$(0,1) = c(\sqrt{3},1) + d(\sqrt{3},-1)$
$(0,1) = (\sqrt{3}c, c) + (\sqrt{3}d, -d)$
$(0,1) = (\sqrt{3}c + \sqrt{3}d, c-d)$

$$\begin{cases} \sqrt{3}c + \sqrt{3}d = 0 \rightarrow \sqrt{3}c = -\sqrt{3}d \rightarrow \sqrt{3}d+1 = -\sqrt{3}d \\ c - d = 1 \rightarrow c = d+1 \end{cases}$$

$$[I]^{\beta}_{B_2} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ d+1 & c-1 \end{bmatrix}$$

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iv) $[I]^{\beta}_{B_3}$ $\beta = \{(1,0),(0,1)\}$
$B_3 = \{(2,0),(0,2)\}$

$(1,0) = a(2,0) + b(0,2)$
$(1,0) = (2a,0) + (0,2b)$
$(1,0) = (2a, 2b)$

$$\begin{cases} 2a = 1 \rightarrow \boxed{a = \frac{1}{2}} \\ 2b = 0 \rightarrow \boxed{b = 0} \end{cases}$$

$(0,1) = c(2,0) + d(0,2)$
$(0,1) = (2c,0) + (0,2d)$
$(0,1) = (2c, 2d)$

$$\begin{cases} 2c = 0 \rightarrow \boxed{c = 0} \\ 2d = 1 \rightarrow \boxed{d = \frac{1}{2}} \end{cases}$$

$$[I]^{\beta}_{B_3} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

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b) Quais são as coordenadas do vetor $v = (3,-2)$ em relação à base:

i) β $\beta = \{(1,0),(0,1)\}$

$$\beta = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad [v]_{\beta'} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \quad [v]_{\beta} = \beta \cdot [v]_{\beta'} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1.3 & 0.(-2) \\ 0.3 & 1.(-2) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3+0 \\ 0+-2 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 \\ -2 \end{bmatrix} \quad [v]_{\beta} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

ii) β_1 $\quad \beta_1 = \{(-1,1),(1,1)$

$(3,-2) = a(-1,1) + b(1,1)$

$(3,-2) = (-a,a) + (b,b)$

$(3,-2) = (-a+b,\ a+b)$

$\begin{cases} -a+b=3 \to b = 3+a \\ a+b=-2 \to a+3+a=-2 \to 2a=-2-3 \to 2a=-5 \end{cases}$

$\boxed{a = \frac{-5}{2}}$

$b = 3+a$

$b = 3 - \frac{5}{2} \to b = \frac{6-5}{2} \to \boxed{b = \frac{1}{2}}$

$[v]_{\beta_1} = \begin{bmatrix} \frac{-5}{2} \\ \frac{1}{2} \end{bmatrix}$

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iii) β_2 $\quad \beta_2 = \{(\sqrt{3},1),(\sqrt{3},-1)\}$

$(3,-2) = a(\sqrt{3},1) + b(\sqrt{3},-1)$

$(3,-2) = (\sqrt{3}a, a) + (\sqrt{3}b, -b)$

$(3,-2) = (\sqrt{3}a + \sqrt{3}b,\ a-b)$

$\begin{cases} \sqrt{3}a + \sqrt{3}b = 3 \longrightarrow \sqrt{3}\,b-2+\sqrt{3}b = 3 \\ a - b = -2 \to a = b-2 \end{cases}$

$[v]_{\beta_2} = \begin{pmatrix} b-2 \\ a+2 \end{pmatrix}$

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iv) β_3 $\quad \beta_3 = \{(2,0),(0,2)\}$

$(3,-2) = a(2,0) + b(0,2)$

$(3,-2) = (2a,0) + (0,b2)$

$(3,-2) = (2a,\ 2b)$

$\begin{cases} 2a = 3 \to \boxed{a = \frac{3}{2}} \\ 2b = -2 \to \boxed{b = -1} \end{cases}$

$[v]_{\beta_3} = \begin{bmatrix} \frac{3}{2} \\ -1 \end{bmatrix}$

c) As coordenadas de um vetor v em relação à base B_1 são dadas por $[v]_{B_1} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$

Quais são as coordenadas de v em relação à base:

i) B $B\{(1,0),(0,1)\}$

I) Primeiro, deve-se achar esse vetor:

$$(x,y) = 4(-1,1) + 0(1,1)$$
$$(x,y) = (-4,4)$$

$$\begin{cases} x = -4 \\ y = 4 \end{cases}$$

$$[v] = \begin{bmatrix} -4 \\ 4 \end{bmatrix}$$

II) Voltando para a questão

$$(-4,4) = a(1,0) + b(0,1)$$
$$(-4,4) = (a,0) + (0,b)$$
$$(-4,4) = (a,b)$$

$$\begin{cases} a = -4 \\ \boxed{b = 4} \end{cases}$$

$$[v]_B = \begin{bmatrix} -4 \\ 4 \end{bmatrix}$$

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ii) B_2 $B_2 = \{(\sqrt{3}, 1), (\sqrt{3}, -1)\}$

$$(-4,4) = a(\sqrt{3}, 1) + b(\sqrt{3}, -1)$$
$$(-4,4) = (\sqrt{3}a, a) + (\sqrt{3}b, -b)$$
$$(-4,4) = (\sqrt{3}a + \sqrt{3}b, a-b)$$

$$\begin{cases} \sqrt{3}a + \sqrt{3}b = -4 \rightarrow \sqrt{3}a + \sqrt{3}a - 4 = -4 \rightarrow 2\sqrt{3}a = 0 \\ a - b = 4 \rightarrow a = b + 4 \end{cases}$$

$$\rightarrow \sqrt{3}a = 0$$

$$\boxed{a = 0}$$

$$b = a - 4 \rightarrow b = -4$$

$$[v]_{B_2} = \begin{bmatrix} 0 \\ -4 \end{bmatrix}$$

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iii) B_3 $B_3 = \{(2,0),(0,2)\}$

$$(-4,4) = a(2,0) + b(0,2)$$
$$(-4,4) = (2a,0) + (0,2b)$$
$$(-4,4) = (2a,2b)$$

$$\begin{cases} 2a = -4 \rightarrow a = \frac{-4}{2} = -2 \\ 2b = 4 \rightarrow b = \frac{4}{2} = 2 \end{cases}$$

$$[v]_{B_3} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

(30) Se $[I]_{\alpha}^{\alpha'} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$ ache

a) $[v]_{\alpha}$ onde $[v]_{\alpha'} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$

$$[v]_{\alpha} = [I]_{\alpha}^{\alpha'} \cdot [v]_{\alpha'}$$

$$[v]_{\alpha} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1\cdot(-1) + 1\cdot 2 + 0\cdot 3 \\ 0\cdot(-1) + (-1)\cdot 2 + 1\cdot 3 \\ 1\cdot(-1) + 0\cdot 2 + (-1)\cdot 3 \end{bmatrix} = \begin{bmatrix} -1+2+0 \\ 0+(-2)+3 \\ -1+0-3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix}$$

$$\boxed{[v]_{\alpha} = \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix}}$$

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b) $[v]_{\alpha'}$ onde $[v]_{\alpha} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$ $\quad \left([I]_{\alpha}^{\alpha'}\right)^{-1} = [I]_{\alpha'}^{\alpha}$

$$[v]_{\alpha'} = [I]_{\alpha'}^{\alpha} \cdot [v]_{\alpha} \qquad [I]_{\alpha'}^{\alpha} = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 \end{array}\right] L_3 \to L_3 + (-1)L_1$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \end{array}\right] L_2 \to L_2(-1) \to \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & +1 & -1 & 0 & -1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} L_1 \to L_1 + (-1)L_2 \\ \\ L_3 \to L_3 + L_2 \end{array} \to$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & -2 & -1 & -1 & 1 \end{array}\right] L_3 \to \frac{L_3}{-2} \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1/2 \end{array}\right] \begin{array}{l} L_1 \to L_1 + (-1)L_3 \\ L_2 \to L_2 + L_3 \end{array} \to$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/2 & 1/2 \\ 0 & 1 & 0 & 1/2 & -1/2 & -1/2 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1/2 \end{array}\right] \qquad [I]_{\alpha'}^{\alpha} = \begin{bmatrix} 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 \end{bmatrix}$$

$$[v]_{\alpha'} = \begin{bmatrix} 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1/2\cdot(-1) + 1/2\cdot 2 + 1/2\cdot 3 \\ 1/2\cdot(-1) + (-1/2)\cdot 2 + (-1/2)\cdot 3 \\ 1/2\cdot(-1) + 1/2\cdot 2 + (-1/2)\cdot 3 \end{bmatrix}$$

$$\cancel{-\frac{1}{2}} + \frac{2}{2} + \cancel{\frac{3}{2}}$$

$$\begin{pmatrix} -\frac{1}{2}+\frac{2}{2}+\frac{3}{2} \\ -\frac{1}{2}-\frac{2}{2}-\frac{3}{2} \\ -\frac{1}{2}+\frac{2}{2}-\frac{3}{2} \end{pmatrix} = \begin{pmatrix} \frac{4}{2} \\ -\frac{6}{2} \\ -\frac{2}{2} \end{pmatrix} = \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}$$

$$\boxed{[v]_{\alpha'} = \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}}$$

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(33) Seja V o espaço vetorial de matrizes 2×2 triangulares superiores. Sejam

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \text{ e } B_1 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\}$$

duas bases de V. Ache $[I]^{B_1}_{B}$.

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$$

$a = 1$
$b = 0$
$c = 0$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = d\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + e\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + f\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} d & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & e \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & f \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} d & e \\ 0 & f \end{bmatrix}$$

$d = 1$
$e = 1$
$f = 0$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = g\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + h\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + i\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} g & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & h \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & i \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} g & h \\ 0 & i \end{bmatrix}$$

$g = 1$

$h = 1$

$i = 1$

$$[I]_{\beta}^{\beta_1} = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

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(35) Se α é base de um espaço vetorial, qual é a matriz de mudança de base $[I]_{\alpha}^{\alpha}$?

Tomando como exemplo $\alpha = \{(1,2), (3,4)\}$, temos

$(1,2) = a(1,2) + b(3,4)$

$(1,2) = (a, 2a) + (3b, 4b)$

$(1,2) = (a+3b, 2a+4b)$

$\begin{cases} a+3b=1 \\ 2a+4b=2 \end{cases}$ $\to a = 1-3b \to a = 1-3\cdot 0$ $\boxed{a=1}$

$2(1-3b)+4b=2$

$\cancel{2}-6b+4b=\cancel{2}$

$-2b=0$

$\boxed{b=0}$

$(3,4) = c(1,2) + d(3,4)$

$(3,4) = (c, 2c) + (3d, 4d)$

$(3,4) = (c+3d, 2c+4d)$

$\begin{cases} c+3d=3 \\ 2c+4d=4 \end{cases}$ $\to c = 3-3d \to c = 3-3\cdot 1$ $\boxed{c=0}$

$2(3-3d)+4d=4$

$6-6d+4d=4$

$-2d=-2$

$\boxed{d=1}$

$$[I]_{\alpha}^{\alpha} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Dá igual à matriz identidade