Coordenadas polares: Comprimento de arco (20/01/2020)

Livro: Cálculo A

Tópico 8.11, página 379

Questões: 33-38.

$$L = \int_{\theta_0}^{\theta_1} \sqrt{f'(\theta)^2 + f(\theta)^2}\, d\theta$$

Página 380

Nos exercícios 33 a 37, encontrar o comprimento de arco da curva dada.

(33) $r = e^{\theta}$, entre $\theta = 0$ e $\theta = \pi/3$

$f(\theta) = e^{\theta}$	$f'(\theta) = e^{\theta}$	$\theta_0 = 0$	$\theta_1 = \pi/3$

$$L = \int_0^{\pi/3} \sqrt{(e^{\theta})^2 + (e^{\theta})^2}\, d\theta = \int_0^{\pi/3} \sqrt{2(e^{\theta})^2}\, d\theta = \int_0^{\pi/3} \sqrt{2}\,\cancel{\sqrt{(e^{\theta})^2}}\, d\theta = \int_0^{\pi/3} \sqrt{2}\, e^{\theta}\, d\theta = \sqrt{2}\int_0^{\pi/3} e^{\theta}\, d\theta$$

$$= \sqrt{2}\, e^{\theta}\Big|_0^{\pi/3} = \sqrt{2}\left[e^{\pi/3} - \cancel{e^{0}}^{\,1}\right] = \boxed{\sqrt{2}\,(e^{\pi/3} - 1)\, u.c.}$$

(34) $r = 1 + \cos\theta$

$f(\theta) = 1 + \cos\theta$	$f'(\theta) = -\operatorname{sen}\theta$	$\theta_0 = 0$	$\theta_1 = \pi$

Simetria

$$L = 2\int_0^{\pi} \sqrt{f'(\theta)^2 + f(\theta)^2}\, d\theta$$

$$L = 2\int_0^{\pi} \sqrt{(-\operatorname{sen}\theta)^2 + (1+\cos\theta)^2}\, d\theta = 2\int_0^{\pi} \sqrt{\operatorname{sen}^2\theta + 1 + 2\cos\theta + \cos^2\theta}\, d\theta$$

$$= 2\int_0^{\pi} \sqrt{1 + 2\cos\theta + \underbrace{\operatorname{sen}^2\theta + \cos^2\theta}_{1}}\, d\theta = 2\int_0^{\pi} \sqrt{1 + 2\cos\theta + 1}\, d\theta = 2\int_0^{\pi} \sqrt{2 + 2\cos\theta}\, d\theta$$

$1 + \cos\theta = 2\cos^2\left(\frac{\theta}{2}\right)$

$$2\int_0^{\pi} \sqrt{2 \cdot 2\cos^2\left(\tfrac{\theta}{2}\right)}\, d\theta = 2\int_0^{\pi} \sqrt{4\cos^2\left(\tfrac{\theta}{2}\right)}\, d\theta = 2\int_0^{\pi} \sqrt{4}\sqrt{\cos^2\left(\tfrac{\theta}{2}\right)}\, d\theta$$

$$= 2\int_0^{\pi} 2\cos\left(\tfrac{\theta}{2}\right) = 4\int_0^{\pi} \cos\left(\tfrac{\theta}{2}\right)$$

$u = \frac{\theta}{2}$ $\quad du = \frac{d\theta}{2} \to 2du = d\theta$ $\quad \theta_0 = u_0 = \frac{0}{2} = 0$ $\quad \theta_1 = u_1 = \frac{\pi}{2} = \frac{\pi}{2}$

$$= 4\int_0^{\pi/2} \cos(u)\, 2du = 8\int_0^{\pi/2} \cos(u)\, du = 8\operatorname{sen}(u)\Big|_0^{\pi/2} = 8\left[\cancel{\operatorname{sen}\left(\tfrac{\pi}{2}\right)}^{\,1} - \cancel{\operatorname{sen}(0)}^{\,0}\right] = \boxed{8\, u.c.}$$

(35) $r = 2a\operatorname{sen}\theta$

$f(\theta) = 2a\operatorname{sen}\theta$	$f'(\theta) = 2a\cos\theta$	$\theta_0 = 0$	$\theta_1 = \pi$

$$L = \int_0^{\pi} \sqrt{(2a\operatorname{sen}\theta)^2 + (2a\cos\theta)^2}\, d\theta = \int_0^{\pi} \sqrt{(2a)^2(\operatorname{sen}^2\theta) + (2a)^2(\cos^2\theta)}\, d\theta$$

$$= \int_0^{\pi} \sqrt{(2a)^2(\underbrace{\operatorname{sen}^2\theta + \cos^2\theta}_{1})}\, d\theta = \int_0^{\pi} \sqrt{(2a)^2}\, d\theta = \int_0^{\pi} 2a\, d\theta = 2a\int_0^{\pi} d\theta = 2a\theta\Big|_0^{\pi} = 2a[\pi - 0]$$

$$= \boxed{2a\pi\, u.c.}$$

(36) $r = 3\theta^2$, de $\theta = 0$ até $\theta = 2\pi/3$

$\boxed{f(\theta) = 3\theta^2}$ $\boxed{f'(\theta) = 3.2\theta = 6\theta}$ $\boxed{\theta_0 = 0}$ $\boxed{\theta_1 = 2\pi/3}$

$$L = \int_0^{\frac{2\pi}{3}} \sqrt{(6\theta)^2 + (3\theta^2)^2}\, d\theta = \int_0^{\frac{2\pi}{3}} \sqrt{36\theta^2 + (3\theta^2)^2}\, d\theta = \int_0^{\frac{2\pi}{3}} \sqrt{3\theta^2(12 + 1^2)}\, d\theta = \int_0^{\frac{2\pi}{3}} \sqrt{3\theta^2\, 13}\, d\theta$$

$$= \int_0^{\frac{2\pi}{3}} \sqrt{39\theta^2}\, d\theta = \int_0^{\frac{2\pi}{3}} \sqrt{39}\,\theta\, d\theta = \sqrt{39}\int_0^{\frac{2\pi}{3}} \theta\, d\theta = \sqrt{39}\,\frac{\theta^2}{2}\Big|_0^{\frac{2\pi}{3}} = \frac{\sqrt{39}}{2}\left[\left(\frac{2\pi}{3}\right)^2 - \cancel{\frac{0^2}{2}}^{\,0}\right]$$

$$= \frac{\sqrt{39}}{2}\left[\frac{4\pi^2}{9}\right] = \frac{4\pi^2\sqrt{39}}{18} = \boxed{\frac{2\pi^2\sqrt{39}}{9}\ u.c.}$$

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(37) $r = e^{2\theta}$, de $\theta = 0$ até $\theta = \frac{3\pi}{2}$.

$\boxed{f(\theta) = e^{2\theta}}$ $\boxed{f'(\theta) = 2e^{2\theta}}$ $\boxed{\theta_0 = 0}$ $\boxed{\theta_1 = \frac{3\pi}{2}}$

$$L = \int_0^{\frac{3\pi}{2}} \sqrt{e^{2\theta} + 2e^{2\theta}}\, d\theta = \int_0^{\frac{3\pi}{2}} \sqrt{e^{2\theta}(1+2)}\, d\theta = \int_0^{\frac{3\pi}{2}} \sqrt{3e^{2\theta}}\, d\theta = \int_0^{\frac{3\pi}{2}} \sqrt{3}\sqrt{e^{2\theta}}\, d\theta =$$

$$= \sqrt{3}\int_0^{\frac{3\pi}{2}} \sqrt{e^{2\theta}}\, d\theta = \sqrt{3}\int_0^{\frac{3\pi}{2}} e^{\theta}\, d\theta = \sqrt{3}\, e^{\theta}\Big|_0^{\frac{3\pi}{2}} = \sqrt{3}\left[e^{\frac{3\pi}{2}} - \cancel{e^0}^{\,1}\right]$$

$$= \boxed{\sqrt{3}\left[e^{\frac{3\pi}{2}} - 1\right] u.c.}$$

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(38) Achar o comprimento da cardioide $r = 10(1 - \cos\theta)$.

$f(\theta) = 10 - 10\cos\theta$ $\boxed{f'(\theta) = 10\,\text{sen}\,\theta}$ $\boxed{\theta_0 = 0}$ $\boxed{\theta_1 = \pi}$ $\boxed{\text{Simetria: } L = 2\int_{\theta_0}^{\theta_1} \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta}$

$$L = 2\int_0^{\pi} \sqrt{(10 - 10\cos\theta)^2 + (10\,\text{sen}\,\theta)^2}\, d\theta = 2\int_0^{\pi} \sqrt{100 - 200\cos\theta + 100\cos^2\theta + 100\,\text{sen}^2\theta}\, d\theta$$

$$= 2\int_0^{\pi} \sqrt{100 - 200\cos\theta + 100\underbrace{(\cos^2\theta + \text{sen}^2\theta)}_{1}}\, d\theta = 2\int_0^{\pi} \sqrt{100 - 200\cos\theta + 100}\, d\theta = 2\int_0^{\pi} \sqrt{200 - 200\cos\theta}\, d\theta$$

$$= 2\int_0^{\pi} \sqrt{200(1 - \cos\theta)}\, d\theta = 2\int_0^{\pi} \sqrt{200\left(2\,\text{sen}^2\left(\frac{\theta}{2}\right)\right)}\, d\theta = 2\int_0^{\pi} \sqrt{400\,\text{sen}^2\left(\frac{\theta}{2}\right)}\, d\theta = 2\int_0^{\pi} \sqrt{400}\sqrt{\text{sen}^2\left(\frac{\theta}{2}\right)}\, d\theta$$

$$= 2\int_0^{\pi} 20\,\text{sen}\left(\frac{\theta}{2}\right) d\theta = 40\int_0^{\pi} \text{sen}\left(\frac{\theta}{2}\right) d\theta$$

$u = \frac{\theta}{2}$

$du = \frac{d\theta}{2} \rightarrow 2du = d\theta$

$$80\int_0^{\pi} \text{sen}(u)\, du = 80\left(-\cos\left(\frac{\theta}{2}\right)\right)\Big|_0^{\pi}$$

$$= -80\left[\cancel{\cos\left(\frac{\pi}{2}\right)}^{\,0} - \cancel{\cos(0)}^{\,1}\right] = -80(-1)$$

$\boxed{80\,u.c.}$