Volume de um sólido de revolução (25/11/2019 - 27/11/2019)

Livro: Cálculo A.

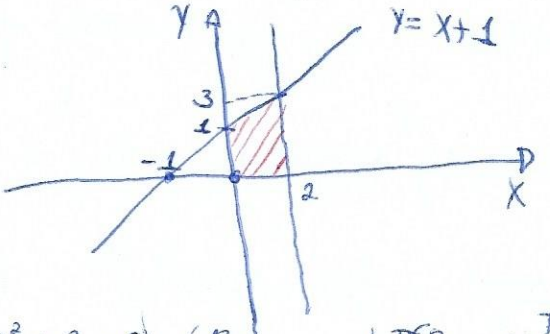
Tópico 8.7, página 359.

Questões: 1-9, 11-15 e 19.

Nos exercícios 1 a 5, determinar o volume do sólido de revolução gerado pela rotação, em torno do eixo dos X, da região R delimitada pelos gráficos das equações dadas.

① $y = x + 1$, $x = 0$, $x = 2$ e $y = 0$

$a = 0$ e $b = 2$



$$V = \pi\int_0^2 (x+1)^2\,dx = \pi\int_0^2 (x^2+2x+1)\,dx$$

$$= \pi\left(\frac{x^3}{3} + x^2 + x\right)\Big|_0^2 = \pi\left(\frac{2^3}{3} + 2^2 + 2 - 0\right) = \pi\left(\frac{8}{3} + 4 + 2\right) = \pi\left(\frac{8}{3} + 6\right) = \pi\left(\frac{8}{3} + \frac{18}{3}\right) = \boxed{\frac{26\pi}{3}\text{u.v.}}$$

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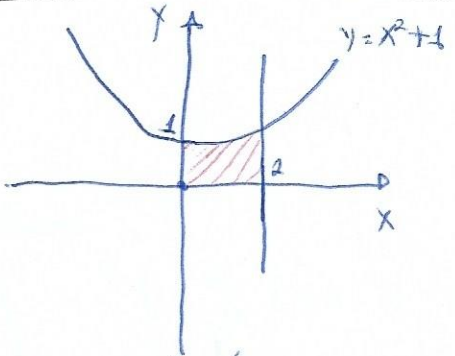
② $y = x^2 + 1$, $x = 0$, $x = 2$ e $y = 0$

$a = 0$ e $b = 2$.



$$V = \pi\int_0^2 (x^2+1)^2\,dx = \pi\int_0^2 (x^4 + 2x^2 + 1)\,dx$$

$$= \pi\left(\frac{x^5}{5} + \frac{2x^3}{3} + x\right)\Big|_0^2 = \pi\left(\frac{2^5}{5} + \frac{2}{3}\cdot 2^3 + 2 - 0\right) = \pi\left(\frac{32}{5} + \frac{16}{3} + 2\right) = \pi\left(\frac{96 + 80 + 30}{15}\right) = \boxed{\frac{206\pi}{15}\text{u.v.}}$$

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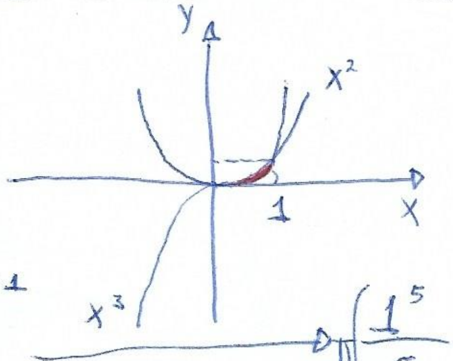
③ $y = x^2$ e $y = x^3$

$a = 0$ e $b = 1$



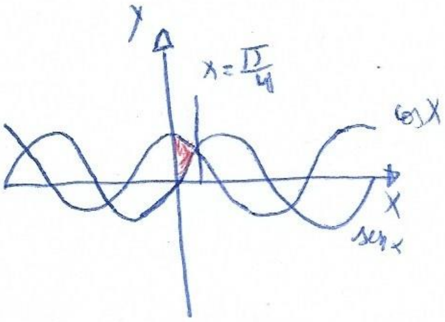
$$V = \pi\int_0^1 \left([x^2]^2 - [x^3]^2\right)dx$$

$$= \pi\int_0^1 (x^4 - x^6)\,dx = \pi\left(\frac{x^5}{5} - \frac{x^7}{7}\right)\Big|_0^1 \longrightarrow \pi\left(\frac{1^5}{5} - \frac{1^7}{7} - 0\right) = \pi\left(\frac{1}{5} - \frac{1}{7}\right) =$$

$$= \pi\left(\frac{7-5}{35}\right) = \boxed{\frac{2\pi}{35}\text{u.v.}}$$

(4) $y = \cos x,\ y = \operatorname{sen} x,\ x = 0$ e $x = \frac{\pi}{4}$

$a = 0$ e $b = \frac{\pi}{4}$



$$V = \pi \int_0^{\frac{\pi}{4}} \left([\cos x]^2 - [\operatorname{sen} x]^2 \right) dx$$

$$= \pi \int_0^{\frac{\pi}{4}} (\cos^2 x - \operatorname{sen}^2 x)\, dx = \pi \int_0^{\frac{\pi}{4}} \left(\frac{1}{2}(1 + \cos 2x) - \frac{1}{2}(1 - \cos 2x) \right) dx$$

$$= \pi \int_0^{\frac{\pi}{4}} \frac{1 + \cos 2x}{2} dx - \pi \int_0^{\frac{\pi}{4}} \frac{1 - \cos 2x}{2} dx = \pi \int_0^{\frac{\pi}{4}} \frac{1}{2} + \pi \int_0^{\frac{\pi}{4}} \cos 2x - \pi \int_0^{\frac{\pi}{4}} \frac{1}{2} + \pi \int_0^{\frac{\pi}{4}} \cos 2x$$

$$= \pi \left(\cancel{\frac{1}{2}x} + \frac{\operatorname{sen} 2x}{2} - \cancel{\frac{1}{2}x} + \frac{\operatorname{sen} 2x}{2} \right) \Big|_0^{\frac{\pi}{4}} = \pi\, \cancel{2} \left(\frac{\operatorname{sen} 2x}{\cancel{2}} \right) \Big|_0^{\frac{\pi}{4}} = \pi \operatorname{sen} 2x \Big|_0^{\frac{\pi}{4}} =$$

$$= \pi \operatorname{sen}\left(2 \frac{\pi}{4}\right) - \pi \operatorname{sen} 0 = \pi \operatorname{sen}\left(\frac{\pi}{2}\right) - \pi \operatorname{sen} 0 = \pi - 0 = \boxed{\pi\ u.v.}$$

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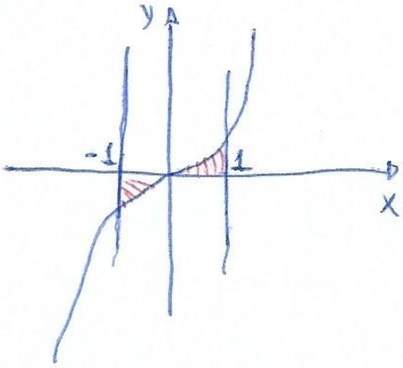
(5) $y = x^3,\ x = -1,\ x = 1$ e $y = 0$

$a = -1$ e $b = 1$



$$V = \pi \int_{-1}^{1} [x^3]^2 = \pi \int_{-1}^{1} x^6 = \pi \left(\frac{x^7}{7} \right) \Big|_{-1}^{1}$$

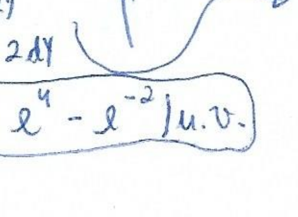
$$= \frac{(+1)^7}{7} - \frac{(-1)^7}{7} = \frac{1}{7} + \frac{1}{7} = \boxed{\frac{2}{7}\ u.v.}$$

Nos exercícios de 6 a 10, determine o volume do sólido de revolução gerado pela rotação, em torno do eixo dos y, da região limitada pelos gráficos das equações dadas.

(6) $y = \ln x,\ y = -1,\ y = 2$ e $x = 0$

$a = -1$ e $b = 2$

$y = \ln x$
$e^y = x$



$$V = \pi \int_{-1}^{2} [e^y]^2\, dy = \pi \int_{-1}^{2} e^{2y}\, dy$$

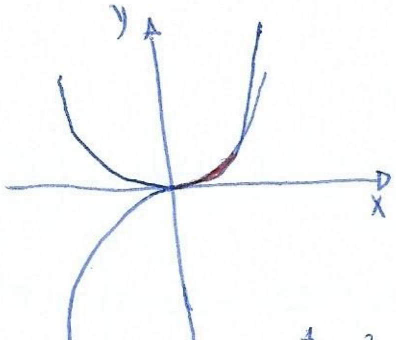
$u = 2y$
$du = 2dy$

$$2\pi \int_{-1}^{2} e^u\, du = 2\pi (e^u) \Big|_1^2$$

$$= 2\pi\, e^{2y} \Big|_1^2 = 2\pi e^4 - 2\pi e^{-2} = \boxed{2\pi(e^4 - e^{-2})\ u.v.}$$

7) $y = x^3$ e $y = x^2$

$a = 0$ e $b = 1$

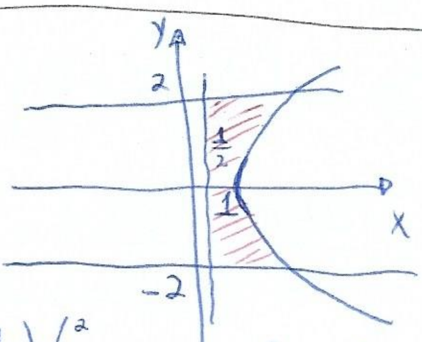


$\sqrt[3]{y} = x$ e $\sqrt{y} = x$

$$V = \pi \int_0^1 \left([\sqrt[3]{y}]^2 - [\sqrt{y}]^2 \right) dy = \pi \int_0^1 \left([\sqrt[3]{y^2}] - y \right) dy = \pi \int_0^1 \left(y^{\frac{2}{3}} - y \right) dy = \pi \left(\frac{3y^{\frac{5}{3}}}{5} - \frac{y^2}{2} \right) \Big|_0^1$$

$$= \pi \left(\frac{3 \cdot \sqrt[3]{1^5}}{5} - \frac{1^2}{2} - 0 \right) = \pi \left(\frac{3}{5} - \frac{1}{2} \right) = \pi \, \frac{6-5}{10} = \boxed{\frac{\pi}{10} \text{ u.v.}}$$

8) $x = y^2 + 1$, $x = \frac{1}{2}$, $y = -2$ e $y = 2$

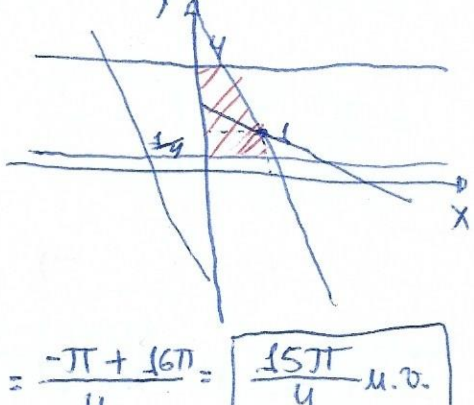


$$V = \pi \int_{-2}^{2} \left[y^2 + 1 - \frac{1}{2} \right]^2 dy = \pi \int_{-2}^{2} \left[y^2 + \frac{1}{2} \right]^2 dy$$

$$= \pi \int_{-2}^{2} \left(y^4 + y^2 + \frac{1}{4} \right) dy = \pi \left(\frac{y^5}{5} + \frac{y^3}{3} + \frac{y}{4} \right) \Big|_{-2}^{2} = \pi \left(\frac{2^5}{5} + \frac{2^3}{3} + \frac{2}{4} - \left(\frac{(-2)^5}{5} + \frac{(-2)^3}{3} + \frac{(-2)}{4} \right) \right)$$

$$= \pi \left(\frac{32}{5} + \frac{8}{3} + \frac{1}{2} + \frac{32}{5} + \frac{8}{3} + \frac{1}{2} \right) = \pi \left(\frac{64}{5} + \frac{16}{3} + 1 \right) = \pi \left(\frac{192 + 80 + 15}{15} \right) \boxed{= \frac{\pi 287}{15} \text{ u.v.}}$$

9) $y = \frac{1}{x}$, $x = 0$, $y = \frac{1}{4}$ e $y = 4$



$y = \frac{1}{x} \rightarrow x = \frac{1}{y}$

$$V = \pi \int_{\frac{1}{4}}^{4} [1/y]^2 \, dy = \pi \int_{\frac{1}{4}}^{4} \left(\frac{1}{y^2} \right) dy = \pi \int_{\frac{1}{4}}^{4} y^{-2} \, dy$$

$$= \pi \frac{y^{-1}}{-1} \Big|_{\frac{1}{4}}^{4} = -\frac{\pi}{y} \Big|_{1/4}^{4} = -\frac{\pi}{4} - \left(-\frac{\pi}{1/4} \right) = -\frac{\pi}{4} + 4\pi = \frac{-\pi + 16\pi}{4} = \boxed{\frac{15\pi}{4} \text{ u.v.}}$$

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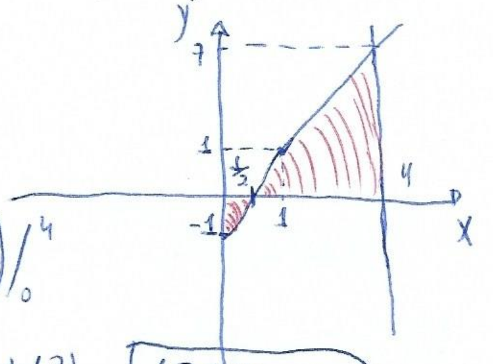
Nos exercícios 11 a 16, determinar o volume do sólido de revolução gerado pela rotação das regiões indicadas, ao redor dos eixos dados.

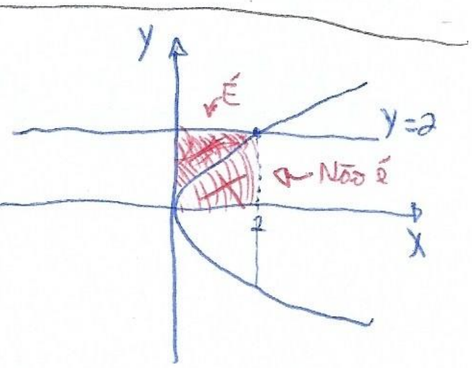
(11) $y = 2x - 1$, $y = 0$, $x = 0$, $x = 4$; ao redor do eixo dos X



$$V = \pi \int_0^4 [2x-1]^2\, dx = \pi \int_0^4 (4x^2 - 4x + 1)\, dx = \pi \left(\frac{4x^3}{3} - 2x^2 + x \right) \Big|_0^4$$

$$= \pi \left(\frac{4 \cdot 4^3}{3} - 2 \cdot 4^2 + 4 \right) = \pi \left(\frac{256}{3} - 32 + 4 \right) = \pi \left(\frac{256 - 96 + 12}{3} \right) = \boxed{\frac{172\pi}{3} \text{ u.v.}}$$

(12) $y^2 = 2x$, $x = 0$, $y = 0$ e $y = 2$; ao redor do eixo dos Y.

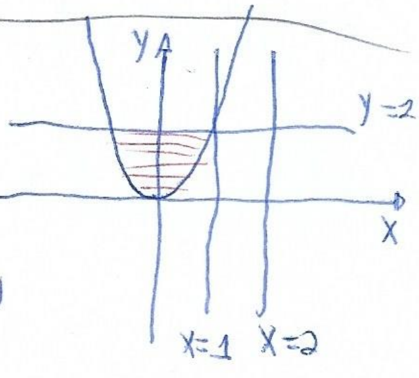


$$y^2 = 2x \rightarrow \frac{y^2}{2} = x$$

$$V = \pi \int_0^2 \left[\frac{y^2}{2} \right]^2 dy = \pi \int_0^2 \frac{y^4}{4}\, dy = \frac{\pi}{4} \int_0^2 y^4\, dy = \frac{\pi}{4} \cdot \frac{y^5}{5} \Big|_0^2$$

$$= V = \frac{\pi}{4} \cdot \frac{2^5}{5} = \frac{16\pi}{20} \overset{\div 2}{\underset{\div 2}{=}} \boxed{\frac{8\pi}{10} \text{ u.v.}}$$

(13) $y = 2x^2$, $x = 1$, $x = 2$ e $y = 2$; ao redor do eixo $y = 2$



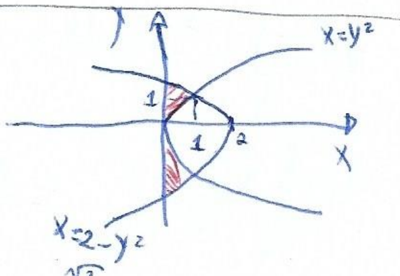
X	-3	-2	-1	0	1	2	3
Y	18	8	2	0	2	8	18

a = 0 e b = 2

$$y = 2x^2 \rightarrow \frac{y}{2} = x^2 \rightarrow \sqrt{\frac{y}{2}} = x \qquad V = \pi \int_0^2 \left[\cancel{\sqrt{\frac{y}{2}}} \right]^{\cancel{2}} dy = \pi \int_0^2 \frac{y}{2}\, dy$$

$$= \frac{\pi}{2} \cdot \frac{y^2}{2} \Big|_0^2 = \frac{\pi}{2} \cdot \frac{2^2}{2} = \frac{\pi}{\cancel{2}} \cdot \cancel{2} = \boxed{\pi \text{ u.v.}}$$

(14) $x = y^2$ e $\underbrace{x = 2 - y^2}$; ao redor do eixo dos Y



Y	-3	-2	-1	0	1	2	3
X	-7	-2	1	2	1	-2	-7

$x = 2 - y^2$

$0 = 2 - y^2 \rightarrow -2 = -y^2$

$y^2 = 2 \rightarrow y = \sqrt{2} = a$ e b

$$V = \pi \int_{-\sqrt{2}}^{\sqrt{2}} \left([y^2]^2 - [2 - y^2]^2 \right) dy = \pi \int_{-\sqrt{2}}^{\sqrt{2}} (\cancel{y^4} - \cancel{y^4} + 4y^2 - 4)\, dy = \pi \int_{-\sqrt{2}}^{\sqrt{2}} (4y^2 - 4)\, dy$$

$$= \pi\left(\frac{4y^3}{3} - 4y\right)\Big|_{-\sqrt{2}}^{\sqrt{2}} = \pi\left(\frac{4\sqrt{2}^3}{3} - 4\sqrt{2} - \left(\frac{4(-\sqrt{2})^3}{3} - 4\sqrt{2}\right)\right)$$

$$= \pi\left(\frac{4.\sqrt{2^3}}{3} - 4\sqrt{2} - \left(-\frac{4\sqrt{2^3}}{3} - 4\sqrt{2}\right)\right) = \pi\left(\frac{4.\sqrt{8}}{3} - \cancel{4\sqrt{2}} + \frac{4\sqrt{8}}{3} + \cancel{4\sqrt{2}}\right)$$

$$= \boxed{\pi\,\frac{8\sqrt{8}}{3}\ u.v.}$$

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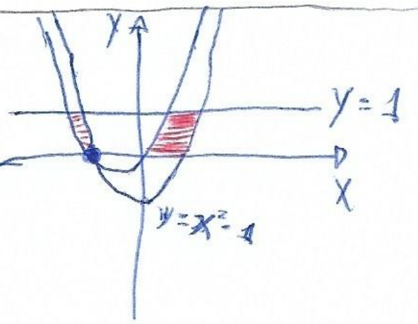
(15) $y = x + x^2$, $y = x^2 - 1$ e $x = 0$; ao redor do eixo $y = 1$



$y = x + x^2$

x	-3	-2	-1	0	1	2	3
y	6	2	0	0	2	6	12

$y = x^2 - 1$

x	-3	-2	-1	0	1	2	3
y	8	3	0	-1	0	3	8

$y = x + x^2 \rightarrow \sqrt{y} = x + x \rightarrow \sqrt{y} = 2x \rightarrow \frac{\sqrt{y}}{2} = x$

$y = x^2 - 1 \rightarrow y + 1 = x^2 \rightarrow \sqrt{y+1} = x$

$$V = \pi\int_0^1 \left(\left[\sqrt{y+1}\right]^2 - \left[\frac{\sqrt{y}}{2}\right]^2\right)dy = \pi\int_0^1 \left(y + 1 - \frac{y}{\sqrt{2}}\right)dy = \pi\left(\frac{y^2}{2} + y - \frac{y^2}{2\sqrt{2}}\right)\Big|_0^1$$

$$= \pi\left(\frac{1}{2} + 1 - \frac{1}{2\sqrt{2}}\right) = \pi\left(\frac{\sqrt{2} + 2\sqrt{2} - 1}{2\sqrt{2}}\right) = \pi\left(\frac{3\sqrt{2} - 1}{2\sqrt{2}}\right) = \pi\left(\sqrt{2} - \frac{1}{2\sqrt{2}}\right) =$$

$$= \boxed{\pi\sqrt{2} - \frac{\pi}{2\sqrt{2}}\ u.v.}$$

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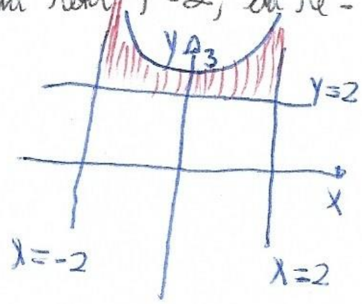
(19) Calcular o volume do sólido gerado pela rotação, em torno da reta $y = 2$, da região limitada por $y = 3 + x^2$, $x = -2$, $x = 2$ e $y = 2$.



x	-3	-2	-1	0	1	2	3
y	12	7	4	3	4	7	12

$a = -2$
$b = 2$

$$V = \pi\int_{-2}^{2}\left(\left[3 + x^2 - 2\right]^2\right)dx = \pi\int_{-2}^{2}(x^2 - 1)^2\,dx = \pi\int_{-2}^{2}(x^4 - 2x^2 + 1)\,dx$$

$$\pi\left(\frac{x^5}{5}-\frac{2x^3}{3}+x\right)\Big/_{-2}^{2}=\pi\left(\frac{2^5}{5}-\frac{2.2^3}{3}+2-\left(\frac{(-2)^5}{5}-\frac{2.(-2)^3}{3}-2\right)\right)$$

$$=\pi\left(\frac{32}{5}-\frac{16}{3}+2+\frac{32}{5}-\frac{16}{3}+2\right)=\left(\frac{64}{5}-\frac{32}{3}+4\right)\pi=\pi\left(\frac{192-160+60}{15}\right)$$

$$=\pi\left(\frac{92}{15}\right)=\boxed{\frac{92\pi}{15}\,u.v.}$$