Integração de funções racionais por meio de frações parciais (16/10/2019 - 23-10/2019)

Livro: Cálculo A.

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Tópico 7.6, página 325.

Questões: 1-5, 8, 10, 17, 19 e 20.

(1) $\int \frac{2x^3}{x^2+x}dx = 2\int \frac{x^3}{x^2+x}dx = 2\int \frac{x^2}{x+1}dx$

$$\begin{array}{r|l} x^2 & x+1 \\ -x^2-1 & x \\ \hline -1 & \end{array}$$

$p(x) = q(x) \cdot s(x) + r(x)$

$\frac{p(x)}{q(x)} = \frac{q(x) \cdot s(x) + r(x)}{q(x)} = s(x) + \frac{r(x)}{q(x)}$

$2\int\left(x + \frac{-1}{x+1}\right)dx = 2\int x\,dx + 2\int \frac{-1}{x+1}dx = 2\int x\,dx - 2\int \frac{dx}{x+1}$ $\quad u = x+1$, $du = dx$ $\quad = 2\int x\,dx - 2\int \frac{du}{u}$

$= \cancel{2}\frac{x^2}{\cancel{2}} - 2\ln|u| + c = \boxed{x^2 - 2\ln|x+1| + c}$

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(2) $\int \frac{2x+1}{2x^2+3x-2}dx = \int \frac{x+1}{x^2+3x-2}dx$ $\quad \frac{Cx+D}{x^2+3x-2}$ $\quad (x+1) = (Cx+D)$ $\quad D, C = 1$

$= \int \frac{x}{x^2+3x-2}dx$ $\quad u = x^2+3x-2$

$du = 2x+3\,dx \rightarrow \frac{du}{5} = x\,dx$

$= \int \frac{du}{5u} = \frac{1}{5}\int \frac{du}{u} = \frac{1}{5}\ln|u| + c = \boxed{\frac{1}{5} \cdot \ln|x^2+3x-2| + c}$

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(3) $\int \frac{x-1}{x^3+x^2-4x-4}dx = \int \frac{x-1}{x(x^2+x) - 4(x+1)}dx = \int \frac{x-1}{x(x^2+x)(-4x-4)}dx = \int \frac{x-1}{(x-2)(x+1)(x+2)}dx$

$\frac{x-1}{(x-2)(x+1)(x+2)} = \frac{A}{x-2} + \frac{A_1}{x+1} + \frac{A_2}{x+2} = \frac{(x+1)(x+2)A + (x-2)(x+2)A_1 + (x-2)(x+1)A_2}{(x-2)(x+1)(x+2)}$

$x - 1 = (x+1)(x+2)A + (x-2)(x+2)A_1 + (x-2)(x+1)A_2$

P/ $x = 2$

$2-1 = (2+1)(2+2)A + 0 + 0$

$1 = 3 \cdot 4A$

$1 = 12A$

$\boxed{A = \frac{1}{12}}$

P/ $x = -1$

$-1-1 = 0 + (-1-2)(-1+2)A_1 + 0$

$-2 = -3A_1$

$\boxed{A_1 = \frac{2}{3}}$

P/ $x = -2$

$-2-1 = 0+0+(-2-2)(-2+1)A_2$

$-3 = -4 \cdot -1\,A_2$

$-3 = 4A_2$

$\boxed{A_2 = -\frac{3}{4}}$

$\frac{16}{16} - \frac{21}{16} = \frac{5}{16}$

$\frac{A}{x-2} + \frac{A_1}{x+1} + \frac{A_2}{x+2} = \frac{1/12}{x-2} + \frac{2/3}{x+1} + \frac{-3/4}{x+2} \rightarrow \frac{1}{12}\int \underbrace{\frac{dx}{x-2}}_{\substack{u=x-2 \\ du=dx}} + \frac{2}{3}\int \underbrace{\frac{dx}{x+1}}_{\substack{u=x+1 \\ du=dx}} - \frac{3}{4}\int \underbrace{\frac{dx}{x+2}}_{\substack{u=x+2 \\ du=dx}}$

$$\boxed{\frac{1}{12}\cdot \ln|x-2| + \frac{2}{3}\ln|x+1| - \frac{3}{4}\ln|x+2| + c}$$

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④ $$\int \frac{3x^2}{2x^3-x^2-2x+1}\,dx = \frac{3}{2}\int \frac{x^2}{x^3-\frac{x^2}{2}-x+\frac{1}{2}}\,dx$$

$$x^3-x-\frac{x^2}{2}+\frac{1}{2} = x(x^2-1)-\frac{1}{2}(x^2-1) = (x-1)(x+1)(2x-1)$$

$$3\int \frac{x^2}{(x-1)(x+1)(2x-1)}\,dx \Rightarrow \frac{x^2}{(x-1)(x+1)(2x-1)} = \frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{2x-1}$$

$$\frac{(x+1)(2x-1)A+(x-1)(2x-1)B+(x-1)(x+1)C}{(x-1)(x+1)(2x-1)} \Rightarrow (x+1)(2x-1)A+(x-1)(2x-1)B+(x-1)(x+1)C = x^2$$

P/ $x=1$

$(1+1)(2-1)A = 1^2$

$2A=1$

$\boxed{A=\frac{1}{2}}$

P/ $x=-1$

$-2.(-2-1)B=(-1)^2$

$-2.(-3)B=1$

$6B=1$

$\boxed{B=\frac{1}{6}}$

P/ $x=\frac{1}{2}$

$\left(\frac{1}{2}-1\right)\left(\frac{1}{2}+1\right)C = \left(\frac{1}{2}\right)^2$

$-\frac{1}{2}\cdot\frac{3}{2}C = \frac{1}{4}$

$-\frac{3}{4}C = \frac{1}{4} \Rightarrow$

$C = \frac{1}{4}\cdot -\frac{4}{3} = -\frac{4}{12} = \boxed{-\frac{1}{3}}$

$$\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{2x-1} = \frac{1/2}{x-1}+\frac{1/6}{x+1}+\frac{-1/3}{2x-1}$$

$$\Rightarrow \frac{1}{2}\int\frac{dx}{x-1}+\frac{1}{6}\int\frac{dx}{x+1}-\frac{1}{3}\int\frac{dx}{2x-1}$$

$u=x-1$, $du=dx$ | $u=x+1$, $du=dx$ | $u=2x-1$, $du=2dx$, $\frac{du}{2}=dx$

$$= \boxed{\frac{1}{2}\ln|x-1| + \frac{1}{6}\ln|x+1| - \frac{1}{3}\cdot\frac{\ln|2x-1|}{2} + c}$$

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⑤ $$\int \frac{x^2+5x+4}{x^2-2x+1}\,dx$$

$$\begin{array}{l|l} x^2+5x+4 & x^2-2x+1 \\ \hline -x^2+2x-1 & 1 \\ \hline 7x+3 & \end{array}$$

$$\int dx + \int \frac{7x+3}{x^2-2x+1}\,dx$$

$2+25$, 25

$$x^2-2x+1 = (x-1)(x-1) = (x-1)^2$$

$$\frac{A}{(x-1)^2}+\frac{B}{x-1} = \frac{(x-1)A+(x-1)^2B}{(x-1)^2} = \frac{7x+3}{(x-1)^2}$$

$$(x-1)A+(x-1)^2B = 7x+3$$

P/ $x=0$

$-A+B=3$

$B=3+A$

P/ $x=-1$

$-2A+4B=-7+3$

$-2A+4B=-4 \quad \div 2$

$-A+2B=-2$

$A-2B=2 \Rightarrow A=2+2B$

$B=3+2+2B$

$B-2B=3+2$

$-B=5 \Rightarrow \boxed{B=-5}$

$A=2+2.(-5) \Rightarrow A=2-10$

$\boxed{A=-8}$

$$\frac{-8}{(x-1)^2}+\frac{-5}{x-1} \Rightarrow -8\int\underbrace{\frac{dx}{(x-1)^2}}_{\substack{u=x-1\\du=dx}}-5\int\underbrace{\frac{dx}{x-1}}_{\substack{u=x-1\\du=dx}} = -8\int\frac{du}{u^2}-5\int\frac{du}{u} = -8\int u^{-2}-5\ln|u|+c$$

$$= -8\frac{u^{-1}}{-1}-5\ln|x-1|+c = \boxed{8\frac{1}{(x-1)}-5\ln|x-1|+c}$$

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(8) $$\int\frac{dx}{x^3-4x^2} = \int\frac{dx}{x^2(x-4)} \qquad \frac{1}{x^2(x-4)} = \frac{A}{x-4}+\frac{B}{x^2}+\frac{C}{x} = \frac{Ax^2+B(x-4)+Cx(x-4)}{x^2(x-4)}$$

P/x = 4
$$1 = A4^2$$
$$\boxed{A=\frac{1}{16}}$$

P/x = 0
$$1 = B(-4)$$
$$\boxed{B=-\frac{1}{4}}$$

P/x = 1
$$1=\frac{1}{16}+-\frac{1}{4}(1-4)+C(1-4) \rightarrow 1=\frac{1}{16}+\frac{3}{4}+(-3)C$$
$$\rightarrow \frac{1}{16}=\frac{1}{16}+\frac{12}{16}-\frac{48}{16}C \rightarrow \frac{1}{16}=-\frac{35C}{16} \rightarrow \boxed{C=-\frac{1}{35}}$$

$$\rightarrow \int\frac{1/16}{x-4}dx+\int\frac{-1/4}{x^2}dx+\int\frac{-1/35}{x}dx = \frac{1}{16}\int\underbrace{\frac{dx}{x-4}}_{\substack{u=x-4\\du=dx}}-\frac{1}{4}\int\frac{dx}{x^2}-\frac{1}{35}\int\frac{dx}{x}$$

$$=\frac{1}{16}\int\frac{du}{u}-\frac{1}{4}\int x^{-2}dx-\frac{1}{35}\int\frac{dx}{x} = \boxed{\frac{1}{16}\ln|x-4|+\frac{1}{4x}-\ln|x|+c}$$

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(10) $$\int\frac{5dx}{x^3+4x} = 5\int\frac{dx}{x^3+4x} = 5\int\frac{dx}{x^2(x+4)} \qquad \frac{1}{x^2(x+4)} = \frac{A}{x+4}+\frac{B}{x^2}+\frac{C}{x}$$

$$\frac{Ax^2+B(x+4)+C(x+4)x}{(x+4)x^2} = \frac{1}{x^2(x+4)}$$

P/x = -4
$$A16 = 1$$
$$\boxed{A=\frac{1}{16}}$$

P/x = 0
$$B4 = 1$$
$$\boxed{B=\frac{1}{4}}$$

P/x = 1
$$\frac{1}{16}+\frac{1}{4}\cdot5+C5=1 \rightarrow \frac{1}{16}+\frac{5}{4}+C5=1$$
$$\rightarrow \frac{1}{16}+\frac{20}{16}+C5=1 \rightarrow \frac{21}{16}+C5=1 \rightarrow C5=1-\frac{21}{16}$$
$$\rightarrow C5 = -\frac{5}{16} \rightarrow C=\frac{-\frac{5}{16}}{5} \rightarrow C=-\frac{5}{16}\cdot5 = \boxed{-\frac{25}{16}}$$

$$\Rightarrow 5\int\frac{1/16}{x+4}dx+5\int\frac{1/4}{x^2}dx+5\int\frac{-25/16}{x}dx = \frac{5}{16}\int\underbrace{\frac{dx}{x+4}}_{\substack{u=x+4\\du=dx}}+\frac{5}{4}\int\frac{dx}{x^2}-\frac{125}{16}\int\frac{dx}{x}$$

$$\boxed{\frac{5}{16}\cdot \ln|x+4| + \frac{5}{4}\cdot\frac{x^{-1}}{-1} - \frac{125}{16}\ln|x| + C}$$

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(17) $\displaystyle\int \frac{dx}{x^3+9x} = \int \frac{dx}{x^2(x+9)}$ $\qquad \displaystyle\frac{1}{x^2(x+9)} = \frac{A}{x+9} + \frac{B}{x^2} + \frac{C}{x} = \frac{Ax^2 + B(x+9) + C(x+9)x}{x^2(x+9)}$

$Ax^2 + B(x+9) + C(x+9)x = 1$

P/x = −9	P/x = 0	P/x = 1
$A\,81 = 1$	$B\,9 = 1$	$\frac{1}{81} + \frac{1}{9}\cdot 10 + C\cdot 10 = 1 \rightarrow \frac{1}{81} + \frac{10}{9} + C\cdot 10 = 1$
$\boxed{A = \frac{1}{81}}$	$B = \frac{1}{9}$	$\rightarrow \frac{1}{81} + \frac{90}{81} + C\cdot 10 = 1 \rightarrow \frac{91}{81} + C\cdot 10 = 1 \rightarrow C\cdot 10 = 1 - \frac{91}{81}$
		$\rightarrow C\cdot 10 = -\frac{10}{81} \rightarrow \boxed{C = -\frac{100}{81}}$

$$\Rightarrow \int \frac{1/81\,dx}{x+9} + \int \frac{1/9\,dx}{x^2} + \int \frac{-100/81\,dx}{x} = \frac{1}{81}\int \underbrace{\frac{dx}{x+9}}_{\substack{u = x+9 \\ du = dx}} + \frac{1}{9}\int\frac{dx}{x^2} - \frac{100}{81}\int\frac{dx}{x}$$

$$= \boxed{\frac{1}{81}\ln|x+9| + \frac{1}{9}\cdot\frac{1}{x} - \frac{100}{81}\ln|x| + C}$$

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(19) $\displaystyle\int \frac{x^3+x^2+2x+1}{x^3-1}\,dx$

$$\begin{array}{l|l} x^3 + x^2 + 2x + 1 & x^3 - 1 \\ \hline -x^3 \qquad\qquad\quad +1 & 1 \\ \hline x^2 + 2x + 2 & \end{array}$$

$$= \int dx + \boxed{\int \frac{x^2+2x+2}{x^3-1}\,dx} \longrightarrow \frac{x^2+2x+2}{x^3-1} = \frac{x^2+2x+2}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

$$= \frac{\boxed{(x^2+x+1)A + (x-1)(Bx+C)}}{(x-1)(x^2+x+1)} \longrightarrow (x^2+x+1)A + (x-1)(Bx+C) = x^2+2x+2$$

P/x = 1	P/x = 0	
$(1+1+1)A = 1+2+2$	$A + (-1)C = 2$	$\frac{5}{3} = 2 + C \rightarrow \frac{5}{3} - 2 = C$
$3A = 5$	$A - C = 2$	$\boxed{C = -\frac{1}{3}}$
$\boxed{A = \frac{5}{3}}$	$\boxed{A = 2 + C}$	

P/x = -1

$(1-1+1)\frac{5}{3} + (-2)\left(B(-1) + \left(\frac{-1}{3}\right)\right) = 1-2+2$

$\frac{5}{3} + 2B + \frac{2}{3} = 1 \rightarrow 2B = 1 - \frac{2}{3} - \frac{5}{3} \rightarrow 2B = \frac{-4}{3} \rightarrow \boxed{B = -\frac{8}{3}}$

$\frac{A}{x-1} + \frac{Bx+C}{x^2+x+1} \Rightarrow \int \frac{5/3}{x-1}dx + \int \frac{(-8/3)x - 1/3}{x^2+x+1}dx = \frac{5}{3}\int \underbrace{\frac{dx}{x-1}}_{\substack{u = x-1 \\ du = dx}} + \frac{1}{3}\int \underbrace{\frac{2x+1}{x^2+x+1}}_{\substack{u = x^2+x+1 \\ du = 2x+1\,dx}} dx$

$\frac{5}{3}\int \frac{du}{u} + \int \frac{du}{u} = \frac{5}{3}\ln|x-1| + \frac{1}{3}\ln|x^2+x+1| + c$

$\boxed{x + \frac{5}{3}\ln|x-1| + \ln|x^2+x+1| + c}$

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(20) $\int \frac{x^3\,dx}{(x^2+2)^2}$

$$\begin{array}{r|l} x^3 & x^2+2 \\ \hline -x^3-2x & x \\ -2x & \end{array}$$

$\int x\,dx + \int \frac{-2x}{(x^2+2)^2}dx = \int x\,dx - \int \frac{2x}{(x^2+2)^2}dx$

$\frac{2x}{(x^2+2)^2} = \frac{A}{(x^2+2)^2} + \frac{B}{x^2+2} = \frac{A + (x^2+2)B}{(x^2+2)^2} = A + (x^2+2)B = 2x$

P/x = 0

$A + 2B = 0$

$\boxed{A = -2B}$

$\boxed{A = -4}$

P/x = 1

$A + 3B = 2$

$3B = 2 - A$

$3B = 2 + 2B$

$3B - 2B = 2$

$\boxed{B = 2}$

$\frac{A}{(x^2+2)^2} + \frac{B}{x^2+2} = \frac{-4}{(x^2+2)^2} + \frac{2}{x^2+2} \Rightarrow \int \frac{-4\,dx}{(x^2+2)^2} + \frac{2\,dx}{x^2+2}$ $u = x^2+2$, $du = 2x\,dx$

$\int \frac{-2du}{u^2} + \int \frac{du}{u} = -2\int u^{-2}du + \ln|u| = \frac{2}{u} + \ln|u| + c = \frac{2}{x^2+2} + \ln|x^2+2| + c$

$\boxed{\frac{x^2}{2} - \frac{2}{x^2+2} + \ln|x^2+2| + c}$