Coordenadas polares: área de figuras planas (22/01/2020)

Livro: Cálculo A.

Tópico 8.11, página 379

Questões: 47-56, 58 e 59.

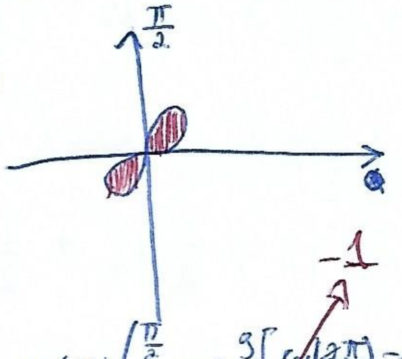
$$A=\frac{1}{2}\int_{\theta_0}^{\theta_1}[f(\theta)]^2\,d\theta$$

Página 381

Nos exercícios 47 a 56, calcular a área limitada pela curva dada.

(47) $r^2 = 9\,sen\,2\theta \rightarrow r=\sqrt{9\,sen\,2\theta} \rightarrow \cancel{3\sqrt{sen\,2\theta}} = f(\theta)$

$\theta_0=0$ e $\theta_1=\frac{\pi}{2}$ $\quad f(\theta)=\sqrt{9\,sen\,2\theta}$



$$A=\frac{1}{2}\int_0^{\pi/2}\left[\sqrt{9\,sen2\theta}\right]^2 d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} 9\,sen2\theta\, d\theta = \frac{9}{2}\int_0^{\frac{\pi}{2}} sen2\theta\, d\theta$$

$u=2\theta$, $du=2\,d\theta$, $\frac{du}{2}=d\theta$

$$\frac{9}{2}\int_0^{\frac{\pi}{2}} sen(u)\frac{du}{2} = \frac{9}{4}\int_0^{\frac{\pi}{2}} sen(u)\,du = \frac{9}{4}\,-\cos(2\theta)\Big|_0^{\frac{\pi}{2}} = -\frac{9}{4}\Big[\underbrace{\cos\left(\tfrac{2\pi}{2}\right)}_{-1}-\underbrace{\cos(0)}_{1}\Big]$$

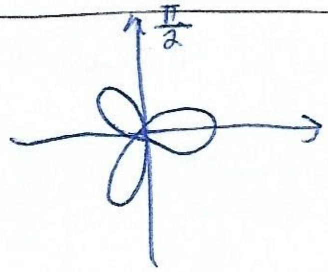
$$= -\frac{9}{4}(-1-1) = \frac{18}{4} = \frac{9}{2}$$

(Simetria) O gráfico se repete duas vezes. $\frac{9}{2}\cdot 2 = \boxed{9\ u.a.}$

(48) $r=\cos 3\theta$ $\quad f(\theta)=\cos 3\theta$

$\theta=0$ $\quad \theta=\pi$

n é ímpar e igual a 3.



$$A=\frac{1}{2}\int_0^{\pi}[\cos 3\theta]^2 d\theta = \frac{1}{2}\int_0^{\pi} \cos^2 3\theta\, d\theta$$

$u=3\theta$, $\frac{du}{3}=d\theta$

$$= \frac{1}{2}\cdot\frac{1}{3}\int_0^{\pi}\cos^2 u\,du = \frac{1}{3}\left[\int_0^{\pi}\frac{1+\cos 2u}{2}\right] = \frac{1}{3}\left[\int_0^{\pi}\frac{1}{2}du + \int_0^{\pi}\frac{\cos 2u}{2}du\right]$$

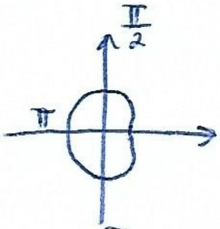
$v=2u$, $\frac{dv}{2}=du$

$$\frac{1}{3}\left[\frac{1}{2}u\Big|_0^{\pi} + \frac{sen(v)}{2}\Big|_0^{\pi}\right] = \frac{1}{3}\left[\frac{3\theta}{2}\Big|_0^{\pi} + \frac{sen(6\theta)}{2}\Big|_0^{\pi}\right]$$

$$= \frac{1}{3}\left[\frac{3\pi}{2} - \cancel{\frac{3\cdot 0}{2}} + \cancel{\frac{sen(6\pi)}{2}} - \cancel{\frac{sen(6\cdot 0)}{2}}\right] = \frac{3\pi}{6} = \boxed{\frac{\pi}{2}\ u.a.}$$

(49) $r = 2 - \cos\theta \qquad f(\theta) = 2 - \cos\theta$

$\theta_0 = 0 \qquad \theta_1 = \pi$



$$A = \frac{1}{2}\int_0^{\pi} [2-\cos\theta]^2 d\theta = \frac{1}{2}\int_0^{\pi} [4 - 4\cos\theta + \cos^2\theta] d\theta = \frac{1}{2}\left[\int_0^{\pi} 4\, d\theta - \int_0^{\pi} 4\cos\theta\, d\theta + \int_0^{\pi} \cos^2\theta\, d\theta\right]$$

$$= \frac{1}{2}\left[4\theta\Big|_0^{\pi} - 4\,\mathrm{sen}\,\theta\Big|_0^{\pi} + \int_0^{\pi} \frac{1+\cos(2\theta)}{2}\, d\theta\right] = \frac{1}{2}\left[4\pi - \cancel{4(\mathrm{sen}\,\pi - \mathrm{sen}\,0)}^{\,0} + \int_0^{\pi} \frac{1}{2}\, d\theta + \int_0^{\pi} \frac{\cos(2\theta)}{2}\, d\theta\right]$$

$$= \frac{1}{2}\left[4\pi + \frac{1}{2}\theta\Big|_0^{\pi} + \frac{1}{2}\int_0^{\pi} \cos(2\theta)\, d\theta\right] \qquad \begin{matrix} u = 2\theta \\ \frac{du}{2} = d\theta \end{matrix} \qquad = \frac{1}{2}\left[4\pi + \frac{1}{2}\pi + \frac{1}{2}\int_0^{\pi} \cos(u)\, \frac{du}{2}\right]$$

$$= \frac{1}{2}\left[4\pi + \frac{\pi}{2} + \frac{1}{4}\int_0^{\pi} \cos(u)\, du\right] = \frac{1}{2}\left[\frac{9\pi}{2} + \frac{1}{4}\,\mathrm{sen}(2\theta)\Big|_0^{\pi}\right] = \frac{1}{2}\left[\frac{9\pi}{2} + \cancel{\frac{1}{4}(\mathrm{sen}(2\pi) - \mathrm{sen}(0))}^{\,0}\right]$$

$$= \frac{1}{2} \cdot \frac{9\pi}{2} = \boxed{\frac{9\pi}{4}\ u.a.}$$

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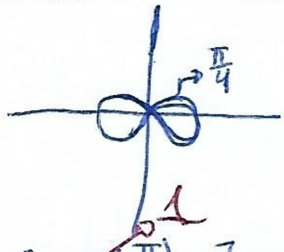
(50) $r^2 = 16\cos 2\theta \rightarrow \sqrt{16\cos 2\theta} \rightarrow f(\theta)$

$\theta_0 = 0 \qquad \theta_1 = \frac{\pi}{4}$



$$A = \frac{1}{2}\int_0^{\frac{\pi}{4}} \left[\cancel{\sqrt{16\cos 2\theta}}\right]^{\cancel{2}} d\theta = \frac{1}{2}\int_0^{\frac{\pi}{4}} 16\cos 2\theta\, d\theta$$

$$= \frac{16}{2}\int_0^{\frac{\pi}{4}} \cos 2\theta\, d\theta = 8\int_0^{\frac{\pi}{4}} \cos 2\theta\, d\theta = \frac{8}{2}\,\mathrm{sen}\,2\theta\Big|_0^{\frac{\pi}{4}} = 4\left[\cancel{\mathrm{sen}\left(\frac{\pi}{2}\right)}^{\,1} - 0\right] = 4$$

São 4 partes então

$4 * 4 = \boxed{16\ u.a.}$

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(51) $r = 3\,\mathrm{sen}\,2\theta \rightarrow f(\theta)$

$\theta_0 = 0 \qquad \theta_1 = \pi$

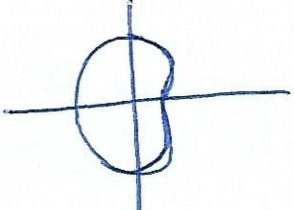
$$A = \frac{1}{2}\int_0^{\pi} [3\,\mathrm{sen}\,2\theta]^2 d\theta = \frac{1}{2}\int_0^{\pi} 9\,\mathrm{sen}^2 2\theta\, d\theta$$

$$= \frac{9}{2}\int_0^{\pi} \mathrm{sen}^2 2\theta\, d\theta \qquad \begin{matrix} u = 2\theta \\ \frac{du}{2} = d\theta \end{matrix} \qquad \frac{9}{2}\int_0^{\pi} \mathrm{sen}^2 u\, \frac{du}{2} = \frac{9}{4}\int_0^{\pi} \mathrm{sen}^2 u\, du = \frac{9}{4}\int_0^{\pi} \frac{1-\cos(2u)}{2}\, du$$

$$= \frac{9}{4}\left[\int_0^{\pi} \frac{1}{2}\, du - \frac{1}{2}\int_0^{\pi} \cos(2u)\, du\right] = \frac{9}{4}\left[\frac{4\theta}{2}\Big|_0^{\pi} - \frac{1}{4}\,\mathrm{sen}(4\theta)\Big|_0^{\pi}\right] = \frac{9}{4}\left[2\pi - \cancel{\frac{1}{4}\,\mathrm{sen}(4\pi)}^{\,0}\right]$$

$$= \frac{18\pi}{4} = \boxed{\frac{9\pi}{2}\ u.a.}$$

(52) $r = 3 - 2\cos\theta \rightarrow f(\theta)$



$\theta_0 = 0 \quad \theta_1 = \pi$

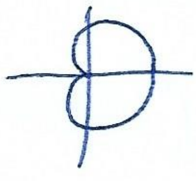
$$A = \frac{1}{2}\int_0^{\pi}[3-2\cos\theta]^2\,d\theta = \frac{1}{2}\int_0^{\pi}[9 - 12\cos\theta + 4\cos^2\theta]\,d\theta$$

$$= \frac{1}{2}\left[\int_0^{\pi} 9\,d\theta - \int_0^{\pi} 12\cos\theta\,d\theta + \int_0^{\pi} 4\cos^2\theta\,d\theta\right] = \frac{1}{2}\left[9\theta\Big|_0^{\pi} - \cancelto{0}{12\left(\operatorname{sen}\theta\Big|_0^{\pi}\right)} + 4\int_0^{\pi}\frac{1+\cos(2\theta)}{2}\,d\theta\right]$$

$$= \frac{1}{2}\left[9\pi + 4\left[\int_0^{\pi}\frac{1}{2}\,d\theta + \int_0^{\pi}\frac{\cos(2\theta)}{2}\,d\theta\right]\right] = \frac{1}{2}\left(9\pi + 4\left[\frac{\theta}{2}\Big|_0^{\pi} + \cancelto{0}{\frac{1}{4}\operatorname{sen}(2\theta)\Big|_0^{\pi}}\right]\right)$$

$$= \frac{1}{2}\left(9\pi + 4\left[\frac{\pi}{2} + 0\right]\right) = \frac{1}{2}(9\pi + 2\pi) = \frac{1}{2}(11\pi) = \boxed{\frac{11\pi}{2}} \times 2 = \boxed{11\pi\, u.a}$$

(53) $r = 4(1+\cos\theta) \rightarrow 4 + 4\cos\theta \rightarrow f(\theta)$



$\theta_0 = 0 \quad \theta_1 = \pi$

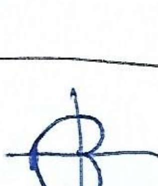
$$A = \frac{1}{2}\int_0^{\pi}[4+4\cos\theta]^2\,d\theta = \frac{1}{2}\int_0^{\pi}[16 + 32\cos\theta + 16\cos^2\theta]\,d\theta$$

$$= \frac{1}{2}\left[\int_0^{\pi}16\,d\theta + \int_0^{\pi}32\cos\theta\,d\theta + \int_0^{\pi}16\cos^2\theta\,d\theta\right] = \frac{1}{2}\left[16\theta\Big|_0^{\pi} + \cancelto{0}{32\operatorname{sen}\theta\Big|_0^{\pi}} + 16\left(\int_0^{\pi}\frac{1+\cos(2\theta)}{2}\,d\theta\right)\right]$$

$$= \frac{1}{2}\left[16\pi + 16\left(\int_0^{\pi}\frac{1}{2}\,d\theta + \int_0^{\pi}\frac{\cos(2\theta)}{2}\right)\right] = \frac{1}{2}\left[16\pi + 16\left(\frac{\theta}{2}\Big|_0^{\pi} + \cancelto{0}{\frac{1}{4}\operatorname{sen}(2\theta)\Big|_0^{\pi}}\right)\right]$$

$$= \frac{1}{2}\left[16\pi + 16\left(\frac{\pi}{2}\right)\right] = \frac{1}{2}[16\pi + 8\pi] = \frac{24\pi}{2} = 12\pi \underbrace{\times 2}_{\text{Simetria}} = \boxed{24\pi\, u.a.}$$

(54) $r = 4(1-\cos\theta) \rightarrow 4 - 4\cos\theta \rightarrow f(\theta)$

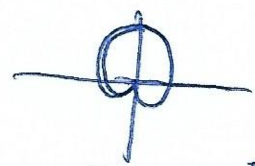


$\theta_0 = 0 \quad \theta_1 = \pi$

$$A = \frac{1}{2}\int_0^{\pi}[4-4\cos\theta]^2\,d\theta = \frac{1}{2}\int_0^{\pi}[16 - 32\cos\theta + 16\cos^2\theta]\,d\theta$$

$$= \frac{1}{2}\left[\int_0^{\pi}16\,d\theta - \cancelto{0}{\int_0^{\pi}32\cos\theta\,d\theta} + \int_0^{\pi}16\cos^2\theta\,d\theta\right] = \frac{1}{2}\left[16\theta + 16\left(\frac{\pi}{2}\right)\right] = \frac{1}{2}[16\theta + 8\pi] = \frac{1}{2}[16\pi + 8\pi] = \frac{24\pi}{2}$$

$$= 12\pi \underbrace{\times 2}_{\text{Simetria}} = \boxed{24\pi\, u.a}$$



(55) $r = 4(1+\operatorname{sen}\theta) \to 4+4\operatorname{sen}\theta \to f(\theta)$

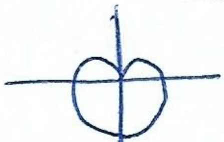
$\theta_1 = 0 \quad \theta_2 = \pi/2$

$$A = \frac{1}{2}\int_0^{\frac{\pi}{2}} [4+4\operatorname{sen}\theta]^2 d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} [16+32\operatorname{sen}\theta + 16\operatorname{sen}^2\theta]\, d\theta = \frac{1}{2}\left[16\theta\Big|_0^{\frac{\pi}{2}} + 32(-\cos(\theta))\Big|_0^{\frac{\pi}{2}} + 16\left(\frac{1}{2}\theta - \int_0 \frac{\cos(2\theta)}{2}\right)\right.$$

$$= \frac{1}{2}\left[8\pi - 32\left(\cos\left(\frac{\pi}{2}\right) - \cos(0)\right) + 16\left(\frac{\theta}{2}\Big|_0^{\pi} - \frac{1}{4}\underbrace{\operatorname{sen}(2\theta)\Big|_0^{\pi}}_{\to 0}\right)\right] =$$

$$= \frac{1}{2}\left[8\pi - 1 + 16\left(\frac{\pi}{2}+0\right)\right] = \frac{1}{2}[8\pi - 1 + 8\pi] = \frac{16\pi - 1}{2} = \left(8\pi - \frac{1}{2}\right) \times \frac{2}{1} = \boxed{16\pi - 1\,u.a.}$$

simetria



(56) $r = 4(1-\operatorname{sen}\theta) \to 4 - 4\operatorname{sen}\theta \to f(\theta)$

$\theta_1 = 0$ e $\theta_2 = \pi$

$$A = \frac{1}{2}\int_0^{\pi} [4-4\operatorname{sen}\theta]^2 d\theta = \frac{1}{2}\int_0^{\pi} [16 - 32\operatorname{sen}\theta + 16\operatorname{sen}^2\theta]\, d\theta = \frac{1}{2}\left[\int_0^{\pi} 16 d\theta - \int_0^{\pi} 32\operatorname{sen}\theta d\theta + \int_0^{\pi} 16\operatorname{sen}^2\theta\, d\theta\right]$$

$$= \frac{1}{2}\left[16\theta\Big|_0^{\pi} - 32(-\cos(\theta))\Big|_0^{\pi} + 16\int_0^{\pi} \frac{1-\cos(2\theta)}{2} d\theta\right]$$

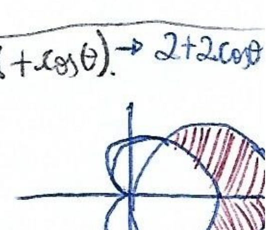
$$= \frac{1}{2}\left[16\pi + 32\underbrace{(\cos(\pi) - \cos(0))}_{\to -2} + 16\left(\int_0^{\pi} \frac{1}{2} d\theta - \int_0^{\pi} \frac{\cos(2\theta)}{2} d\theta\right)\right]$$

$$= \frac{1}{2}\left[16\pi - 64 + 16\left(\frac{\theta}{2}\Big|_0^{\pi} - \frac{1}{4}\cos(2\theta)\Big|_0^{\pi}\right)\right] = \frac{1}{2}\left[16\pi - 64 + 16\left(\frac{\pi}{2} - \frac{1}{4}\underbrace{(\cos(2\pi) - \cos(0))}_{\to 0}\right)\right]$$

$$= \frac{1}{2}\left[16\pi - 64 + 16\left(\frac{\pi}{2} - 0\right)\right] = \frac{1}{2}[16\pi - 64 + 8\pi] = \frac{1}{2}[24\pi - 64] = (12\pi - 32) \times \frac{2}{1}$$

$\boxed{24\pi - 64\,u.a.}$ simetria

(58) Encontrar a área interior ao círculo $r = 6\cos\theta$ e exterior a $r = 2(1+\cos\theta) \to 2+2\cos\theta$



$6\cos\theta = 2+2\cos\theta \to 4\cos\theta = 2 \to \cos\theta = \frac{1}{2} \to \theta = \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$

$\arccos\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

$$A = \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} [6\cos\theta]^2 d\theta - \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} [2+2\cos\theta]^2 d\theta = \frac{1}{2}\left[\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} 36\cos^2\theta\, d\theta - \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} [4+8\cos\theta + 4\cos^2\theta]\, d\theta\right]$$

$$= \frac{1}{2}\left[36\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{1+\cos(2\theta)}{2} d\theta - \left[4\theta\Big|_{\frac{\pi}{3}}^{\frac{2\pi}{3}} + 8\underbrace{\left[\operatorname{sen}\left(\frac{2\pi}{3}\right) - \operatorname{sen}\left(\frac{\pi}{3}\right)\right]}_{\to 0} + 4\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{1+\cos(2\theta)}{2}\right]\right]$$

$$= \frac{1}{2}\left[36\left(\frac{\theta}{2}\Big|_{\frac{\pi}{3}}^{\frac{2\pi}{3}} + \frac{1}{4}\,sen(2\theta)\Big|_{\frac{\pi}{3}}^{\frac{2\pi}{3}}\right) + 4\left(\frac{\theta}{2}\Big|_{\frac{\pi}{3}}^{\frac{2\pi}{3}} + \frac{1}{4}\,sen(2\theta)\Big|_{\frac{\pi}{3}}^{\frac{2\pi}{3}}\right)\right]$$

$$= \frac{1}{2}\left[36\left(\frac{2\pi}{3}\cdot 2 - \frac{\pi}{3}\cdot 2 + \frac{1}{4}\left(sen\left(2\cdot\frac{2\pi}{3}\right) - sen\left(2\cdot\frac{\pi}{3}\right)\right)\right) + 4\left(\frac{2\pi}{3}\cdot 2 - \frac{\pi}{3}\cdot 2 + \frac{1}{4}\left(sen\left(2\cdot\frac{2\pi}{3}\right) - sen\left(2\cdot\frac{\pi}{3}\right)\right)\right)\right]$$

$$= \frac{1}{2}\left[36\left(\frac{4\pi}{3} - \frac{2\pi}{3} + \frac{1}{4}\left(-\sqrt{3}\right)\right) + 4\left(\frac{4\pi}{3} - \frac{2\pi}{3} + \frac{1}{4}\left(-\sqrt{3}\right)\right)\right]$$

$$= \frac{1}{2}\left[36\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{4}\right) + 4\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{4}\right)\right] = \frac{1}{2}\left[\frac{72\pi}{3} - \frac{36\sqrt{3}}{4} + \frac{8\pi}{3} - \frac{4\sqrt{3}}{4}\right]$$

$$= \frac{1}{2}\left[24\pi - 9\sqrt{3} + \frac{8\pi}{3} - \sqrt{3}\right] = \frac{1}{2}\left[\frac{80\pi}{3} - 10\sqrt{3}\right] = \frac{80\pi}{6} - 5\sqrt{3} = \frac{40\pi}{3} - 5\sqrt{3}$$

$$= \frac{40\pi}{3} - \frac{15\sqrt{3}}{3} = \boxed{\frac{1}{3}\left(40\pi - 15\sqrt{3}\right)\ u.a.}$$

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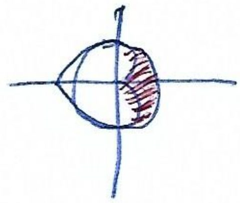
59 Encontrar a área interior ao círculo $r = 4$ e exterior à cardioide $r = 4(1-\cos\theta)$

$$4 = 4 - 4\cos\theta \longrightarrow \cos\theta = -\frac{0}{4}$$

$$0 = -4\cos\theta \qquad \theta = \frac{1}{2}\pi$$



$$A = 2\left\{\frac{1}{2}\left[\int_0^{\frac{\pi}{2}} (4)^2\, d\theta - \int_0^{\frac{\pi}{2}} (4-4\cos\theta)^2\, d\theta\right]\right\}$$

$$= 2\left\{\frac{1}{2}\left[4\frac{\pi}{2} - \int_0^{\frac{\pi}{2}} (16 - 32\cos\theta + 16\cos^2\theta)\, d\theta\right]\right\}$$

$$= 2\left\{\frac{1}{2}\left[2\pi - 16\frac{\pi}{2} + 32\left(\overset{1}{sen\left(\frac{\pi}{2}\right)} - sen(0)\right) - 16\int_0^{\frac{\pi}{2}} \frac{1+\cos(2\theta)}{2}\, d\theta\right]\right\}$$

$$= 2\left\{\frac{1}{2}\left[-\frac{12\pi}{2} + 32 - 16\left(\frac{\theta}{2}\Big|_0^{\frac{\pi}{2}} + \cancel{\frac{1}{4}\left(sen\left(2\frac{\pi}{2}\right) - sen(0)\right)}\right)\right]\right\}$$

$$= 2\left\{\frac{1}{2}\left[-6\pi + 32 - 16(\pi + 0)\right]\right\} = 2\left\{\frac{1}{2}\left[-6\pi + 32 - 16\pi\right]\right\} = 2\left\{\frac{1}{2}\left[32 - 22\pi\right]\right\}$$

$$= 2\left\{16 - 11\pi\right\} = \boxed{32 - 22\pi\ u.a.}$$