Integração por substituição trigonométrica (09/10/2019 - 14/10/2019) Vitor Oliveira Rupp

Livro: Cálculo A.

Tópico 7.4, página 311.

Questões: 45, 46, 54-58, 60, 61 e 63.

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Nos exercícios 45 a ~~67~~ 63 calcular a integral indefinida:

(45) $\int \frac{dx}{x^2\sqrt{x^2-5}}$

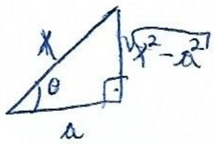
$\sqrt{x^2-5} = a\,tg\theta = \sqrt{5}\,tg\theta$

$x = a\sec\theta = \sqrt{5}\sec\theta$

$dx = \sqrt{5}\sec\theta\, tg\theta\, d\theta$

$\int \frac{\cancel{\sqrt{5}}\sec\theta\,\cancel{tg\theta}\,d\theta}{(\sqrt{5}\sec\theta)^2\cancel{\sqrt{5}}\,\cancel{tg\theta}} = \int \frac{\sec\theta\, d\theta}{5\cdot\sec^2\theta}$

$= \int \frac{d\theta}{5\sec\theta} = \frac{1}{5}\int \frac{1}{\sec\theta}\,d\theta = \frac{1}{5}\cdot \operatorname{sen}\theta + c$



$\operatorname{sen}\theta = \frac{CO}{H} = \boxed{\frac{\sqrt{x^2-a^2}}{x}} = \frac{\sqrt{5}\,tg\theta}{x}$

$= \boxed{\frac{1}{5}\cdot\frac{\sqrt{x^2-5}}{x} + c}$

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(46) $\int \frac{dt}{\sqrt{9-16t^2}} = \int \frac{dt}{\sqrt{3^2-(4t)^2}}$

$u = 4t$

$du = 4dt \rightarrow \frac{du}{4} = dt$

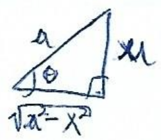
$= \int \frac{du/4}{\sqrt{3^2-u^2}} = \frac{1}{4}\int \frac{du}{\sqrt{3^2-u^2}}$

$u = a\operatorname{sen}\theta = 3\operatorname{sen}\theta$

$du = 3\cos\theta\, d\theta$

$\sqrt{3^2-u^2} = a\cos\theta = 3\cos\theta$

$= \frac{1}{4}\int \frac{3\cos\theta\, d\theta}{3\cos\theta} = \frac{1}{4}\int d\theta = \frac{1}{4}\theta + c$



$u = 3\operatorname{sen}\theta \rightarrow \theta = \frac{u}{3\operatorname{sen}} \rightarrow \theta = \operatorname{arcsen}\left(\frac{u}{3}\right) = \operatorname{arcsen}\left(\frac{4t}{3}\right)$

$= \boxed{\frac{1}{4}\cdot \operatorname{arcsen}\left(\frac{4t}{3}\right) + c}$

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(54) $\int \frac{e^x}{\sqrt{e^{2x}+1}}\,dx$

$u = e^x$

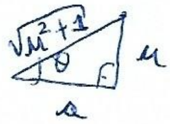
$du = e^x dx$

$\int \frac{du}{\sqrt{u^2+1}} = \operatorname{arctg}(u) + c = \boxed{\operatorname{arctg}(e^x) + c}$

↳ Sub Tri → $u = a\,tg\theta = tg\theta$ $\quad \sqrt{u^2+1} = \sec\theta$

$du = \sec^2\theta\, d\theta$

$= \int \frac{\sec^2\theta\, d\theta}{\sec\theta} = \int \sec\theta\, d\theta = \ln|\sec\theta + tg\theta| + c$



$\sec\theta = \frac{H}{CA} = \frac{\sqrt{e^x+1}}{1}$

$tg\theta = \frac{CO}{CA} = \frac{e^x}{1}$

$= \ln|\sqrt{e^x+1} + e^x| + c$

$= \ln|\sqrt{e^x} + e^x + 1| + c = \operatorname{arctg}(e^x) + c$

(55) $\displaystyle\int \frac{x^2}{\sqrt{2-x^2}}\,dx$ $x = a\,\text{sen}\,\theta = \sqrt{2}\,\text{sen}\,\theta$ $dx = \sqrt{2}\cos\theta\,d\theta$ $\sqrt{2-x^2} = a\cos\theta = \sqrt{2}\cos\theta$

$$= \int \frac{(\sqrt{2}\,\text{sen}\,\theta)^2}{\cancel{\sqrt{2}\cos\theta}} \cdot \cancel{\sqrt{2}\cos\theta}\,d\theta$$

$$= \int 2\,\text{sen}^2\theta\,d\theta = 2\int \text{sen}^2\theta\,d\theta = 2\int \frac{1-\cos(2\theta)}{2}\,d\theta = \int d\theta - \int \cos(2\theta)\,d\theta = \theta - \int \cos(u)\,\frac{du}{2}$$

$u = 2\theta$

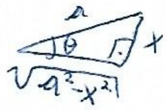
$du = 2\,d\theta \rightarrow \frac{du}{2} = d\theta$

$$= \theta - \frac{1}{2}\text{sen}(u) + c = \theta - \frac{1}{2}\text{sen}(2\theta) + c$$

$x = \sqrt{2}\,\text{sen}\,\theta$

$\theta = \frac{x}{\sqrt{2}\,\text{sen}} \Rightarrow \text{arcsen}\left(\frac{x}{\sqrt{2}}\right)$

$$= \text{arcsen}\left(\frac{x}{\sqrt{2}}\right) - \frac{1}{2}\cdot\frac{2x}{\sqrt{2}} + c$$



$\text{sen}\,2\theta = \frac{2\,CO}{H} = 2\frac{x}{a}$

$2\frac{x}{\sqrt{2}}$

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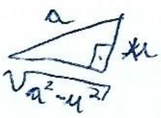
(56) $\displaystyle\int \frac{e^x}{\sqrt{4-e^{2x}}}\,dx$ $u = e^x$ $du = e^x\,dx$ $= \displaystyle\int \frac{du}{\sqrt{4-u^2}}$ $\sqrt{4-u^2} = 2\cos\theta$ $u = 2\,\text{sen}\,\theta$ $du = 2\cos\theta\,d\theta$

$$= \int \frac{2\cos\theta\,d\theta}{2\cos\theta} = \int d\theta = \theta + c$$



$u = 2\,\text{sen}\,\theta \rightarrow \theta = \frac{u}{2\,\text{sen}} = \text{arcsen}\left(\frac{u}{2}\right) = \text{arcsen}\left(\frac{e^x}{2}\right)$

$$\text{arcsen}\left(\frac{e^x}{2}\right) + c$$

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(57) $\displaystyle\int \frac{x+1}{\sqrt{x^2-1}}\,dx$ $\sqrt{x^2-1} = \text{tg}\,\theta$ $x = \sec\theta$ $dx = \sec\theta\,\text{tg}\,\theta\,d\theta$

$$= \int \left(\frac{\sec\theta+1}{\cancel{\text{tg}\,\theta}}\right)\cdot \sec\theta\,\cancel{\text{tg}\,\theta}\,d\theta = \int (\sec\theta+1)\cdot\sec\theta\,d\theta$$

$$= \int [\sec^2\theta + \sec\theta]\,d\theta$$

$$= \int \sec^2\theta\,d\theta + \int \sec\theta\,d\theta = \text{tg}\,\theta + \ln|\sec\theta + \text{tg}\,\theta| + c$$

$$= \sqrt{x^2-1} + \ln|x + \sqrt{x^2-1}| + c$$

$\text{tg}\,\theta = \frac{CO}{CA} = \frac{\sqrt{x^2-1}}{1}$

$\sec\theta = \frac{H}{CA} = \frac{x}{a} = x$

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(58) $\displaystyle\int \frac{\sqrt{1+x^2}}{x^3}\,dx$ $x = \text{tg}\,\theta$ $dx = \sec^2\theta\,d\theta$ $\sqrt{1+x^2} = \sec\theta$

$$\int \frac{\sec\theta}{\text{tg}^3\theta}\sec^2\theta\,d\theta = \int \frac{\sec^3\theta}{\text{tg}^3\theta}\,d\theta$$

$$= \int \sec^3\theta\,\text{cotg}^3\theta\,d\theta = \int \frac{1}{\cancel{\cos^3\theta}}\cdot\frac{\cancel{\cos^3\theta}}{\text{sen}^3\theta}\,d\theta = \int \frac{1}{\text{sen}^3\theta}\,d\theta = \int \text{cosec}^3\theta\,d\theta = \int \text{cosec}\,\theta\,\text{cosec}^2\theta\,d\theta$$

$u = \text{cosec}\,\theta \longrightarrow du = -\text{cosec}\,\theta\cdot\text{cotg}\,\theta\,d\theta$

$dv = \text{cosec}^2\theta\,d\theta \rightarrow v = -\text{cotg}\,\theta$

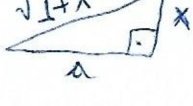
$\int \text{cosec}\,\theta\cdot\text{cosec}^2\theta\,d\theta = \text{cosec}\,\theta\cdot -\text{cotg}\,\theta - \int -\text{cotg}\,\theta\cdot -\text{cosec}\,\theta\cdot\text{cotg}\,\theta\,d\theta \rightarrow -\text{cosec}\,\theta\cdot\text{cotg}\,\theta - \int \text{cotg}^2\theta\,\text{cosec}\,\theta\,d\theta$

$\int \text{cosec}\,\theta\cdot\text{cosec}^2\theta\,d\theta = -\text{cosec}\,\theta\,\text{cotg}\,\theta - \int \text{cosec}\,\theta[\text{cosec}^2\theta - 1]\,d\theta \rightarrow -\text{cosec}\,\theta\cdot\text{cotg}\,\theta - \int[\text{cosec}^3\theta - \text{cosec}\,\theta]\,d\theta$

$\int \text{cosec}^3\theta\,d\theta = -\text{cosec}\,\theta\cdot\text{cotg}\,\theta - \int \text{cosec}^3\theta\,d\theta + \int \text{cosec}\,\theta\,d\theta$

$2\int \text{cosec}^3\theta\,d\theta = -\text{cosec}\,\theta\cdot\text{cotg}\,\theta + \ln|\text{cosec}\,\theta - \text{cotg}\,\theta| + c$

$\int \text{cosec}^3\theta\,d\theta = \dfrac{-\text{cosec}\,\theta\cdot\text{cotg}\,\theta + \ln|\text{cosec}\,\theta - \text{cotg}\,\theta| + c}{2}$



$\text{cosec}\,\theta = \frac{H}{CO} = \frac{\sqrt{1+x^2}}{x}$

$\text{cotg}\,\theta = \frac{CA}{CO} = \frac{1}{x}$

$$\frac{-\frac{\sqrt{1+x^2}}{x}\cdot\frac{1}{x}+\ln\left|\frac{\sqrt{1+x^2}}{x}-\frac{1}{x}\right|+c}{2} = \boxed{\frac{1}{2}\cdot\frac{-\sqrt{1+x^2}}{x^2}+\ln\left|\frac{\sqrt{1+x^2}}{x}-\frac{1}{x}\right|+c}$$

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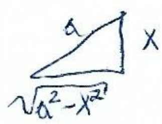
(60) $\displaystyle\int\frac{(x+1)}{\sqrt{4-x^2}}dx = \int\frac{x+1}{\sqrt{2^2-x^2}}dx$ $x=2\,sen\theta$, $dx=2\cos\theta\, d\theta$ $\sqrt{4-x^2}=2\cos\theta$ $\displaystyle=\int\frac{2\,sen\theta+1}{\cancel{2\cos\theta}}\cancel{2\cos\theta}\, d\theta = \int[2\,sen\theta+1]\,d\theta$

$\displaystyle=\int 2\,sen\theta\, d\theta+\int d\theta = -2\cos\theta+\theta+c$



$\cos\theta = \frac{CA}{H}=\frac{\sqrt{a^2-x^2}}{2}$

$x=2\,sen\theta \to \theta=\frac{x}{2\,sen}$

$\to\theta = arcsen\left(\frac{x}{2}\right)$

$$= -\frac{\cancel{2}\sqrt{4-x^2}}{\cancel{2}}+arcsen\left(\frac{x}{2}\right)+c = \boxed{-\sqrt{4-x^2}+arcsen\left(\frac{x}{2}\right)+c}$$

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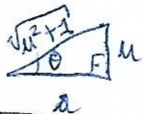
(61) $\displaystyle\int\frac{6x+5}{\sqrt{9x^2+1}}dx = \int\frac{6x+5}{\sqrt{(3x)^2+1}}dx$ $u=3x$, $du=3dx\to\frac{du}{3}=dx$ $\displaystyle\int\frac{2u+5}{\sqrt{u^2+1}}\cdot\frac{du}{3}=\frac{1}{3}\int\frac{2u+5}{\sqrt{u^2+1}}du$

$u=tg\theta$ $\sqrt{u^2+1}=sec\theta$ $\displaystyle\frac{1}{3}\int\frac{2tg\theta+5}{\cancel{sec\theta}}\cdot sec^2\theta\, d\theta = \frac{1}{3}\int[2tg\theta+5]\,sec\theta\, d\theta$

$du=sec^2\theta\, d\theta$

$\displaystyle=\frac{1}{3}\int[2tg\, sec\theta+5sec\theta]\,d\theta = \frac{1}{3}\int 2\,tg\theta\, sec\theta+\frac{1}{3}\int 5\,sec\theta\, d\theta = \frac{2}{3}\int tg\theta\, sec\theta\, d\theta+\frac{5}{3}\int sec\theta\, d\theta$

$\displaystyle=\frac{2}{3}sec\theta+\frac{5}{3}\ln|sec\theta+tg\theta|+c$



$sec\theta=\frac{H}{CA}=\frac{\sqrt{u^2+1}}{a}=\sqrt{9x^2+1}$

$tg\theta=\frac{CO}{CA}=\frac{u}{a}=3x$

$$=\boxed{\frac{2}{3}\sqrt{9x^2+1}+\frac{5}{3}\ln\left|\sqrt{9x^2+1}+3x\right|+c}$$

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(63) $\displaystyle\int\sqrt{4-x^2}\,dx=\int\sqrt{2^2-x^2}\,dx$ $x=a\,sen\theta=2\,sen\theta$, $dx=2\cos\theta\, d\theta$ $\sqrt{2^2-x^2}=2\cos\theta$

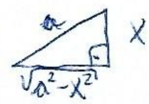
$\displaystyle\int 2\cos\theta\cdot2\cos\theta\, d\theta=\int 4\cos^2\theta\, d\theta = 4\int\cos^2\theta\, d\theta = 4\int\frac{1+\cos(2\theta)}{2}d\theta = 4\int\frac{1}{2}d\theta+4\int\frac{\cos(2\theta)}{2}d\theta$

$u=2\theta$

$du=2d\theta\to\frac{du}{2}=d\theta$

$\displaystyle=2\int d\theta+4\int\frac{\cos(u)}{2}\cdot\frac{du}{2} = 2\theta+sen(2\theta)+c$

$sen\theta=\frac{CO}{H}=\frac{x}{a}$



$x=2\,sen\theta$

$\theta=\frac{x}{2\,sen}$

$\theta=arcsen\left(\frac{x}{2}\right)$

$$=\boxed{2arcsen\left(\frac{x}{2}\right)+\frac{x}{2}+c}$$