Integrais impróprias (16/12/2019 - 18/12/2019) Vitor Oliveira Ponte
Livro: Cálculo A.
Tópico 6-15, página 290.
Questões: 4-7, e 9-14.

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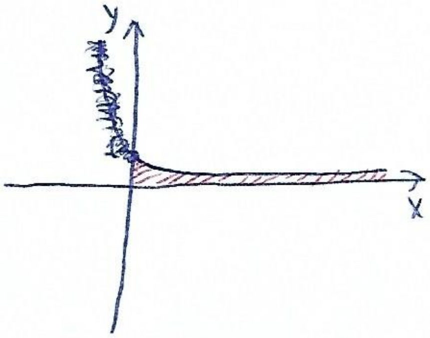
(4) Encontrar a área sob a curva $y = e^{-x}$, $x \geq 0$.



$a = 0$ e $b = +\infty$

$$A = \int_0^{+\infty} e^{-x}\,dx = \lim_{b\to+\infty} \int_0^b e^{-x}\,dx \qquad \begin{matrix} u = -x \\ du = -dx \\ -du = dx \end{matrix} \qquad \lim_{b\to+\infty} -\int_0^b e^u\,du$$

$$= \lim_{b\to+\infty} -e^u \Big|_0^b = \lim_{b\to+\infty} -e^{-x}\Big|_0^b = \lim_{b\to+\infty}\left[\underbrace{-e^{-\infty}}_{\to 0} + e^{-0}\right] = \boxed{1\text{ u.a.}}$$

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(5) Investigar a integral imprópria $\int_7^{+\infty} \frac{1}{(x-5)^2}\,dx$.

$$I = \lim_{b\to+\infty} \int_7^b \frac{1}{(x-5)^2}\,dx \qquad \begin{matrix} u = x-5 \\ du = dx \end{matrix} \qquad \lim_{b\to+\infty} \int_7^b \frac{1}{u^2}\,du = \lim_{b\to+\infty} \int_7^b u^{-2}\,du = \lim_{b\to+\infty} \frac{u^{-1}}{-1}\Big|_7^b$$

$$= \lim_{b\to+\infty} -\frac{1}{u}\Big|_7^b = \lim_{b\to+\infty} -\frac{1}{x-5}\Big|_7^b = \lim_{b\to+\infty}\left[\underbrace{-\frac{1}{+\infty-5}}_{\to 0} + \frac{1}{7-5}\right] = \frac{1}{2} \text{ (Convergente)}$$

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(6) Mostrar que $\int_1^{+\infty} \frac{dx}{\sqrt{x}}$ é divergente.

$$I = \lim_{b\to+\infty} \int_1^b \frac{dx}{x^{\frac{1}{2}}} = \lim_{b\to+\infty} \int_1^b x^{-\frac{1}{2}}\,dx = \lim_{b\to+\infty} \frac{x^{\frac{1}{2}}}{1/2}\Big|_1^b = \lim_{b\to+\infty} 2\sqrt{x}\Big|_1^b = \lim_{b\to+\infty}\left[\underbrace{2\sqrt{+\infty}}_{\to+\infty} - 2\sqrt{1}\right]$$

$$= \boxed{+\infty} \text{ (divergente)}$$

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(7) Verificar se a integral $\int_{-\infty}^0 e^{5x}$ converge. Em caso positivo, determinar seu valor

$$I = \lim_{a\to-\infty} \int_a^0 e^{5x}\,dx = \qquad \begin{matrix} u = 5x \\ du = 5dx \\ \frac{du}{5} = dx \end{matrix} \qquad \lim_{a\to-\infty} \int_a^0 e^u \frac{du}{5} = \lim_{a\to-\infty} \frac{1}{5}\int_a^0 e^u\,du$$

$$= \lim_{a\to-\infty} \frac{1}{5} e^{5x}\Big|_a^0 = \lim_{a\to-\infty} \frac{1}{5}\left[e^{5\cdot 0} - \underbrace{e^{5\cdot-\infty}}_{\to 0}\right] = \boxed{\frac{1}{5}} \text{ (converge)}$$

9) Encontrar a área sob o gráfico da curva $y = (x+1)^{-3/2}$, $x \geq 15$

$$\lim_{b \to +\infty} \int_{15}^{b} (x+1)^{-3/2}\,dx = \qquad u = x+1 \quad du = dx \qquad \lim_{b\to+\infty} \int_{15}^{b} u^{-3/2}\,du = \lim_{b \to +\infty} \frac{u^{-1/2}}{-1/2} \Big|_{15}^{b}$$

$$= \lim_{b\to+\infty} -\frac{2}{\sqrt{u}}\Big|_{15}^{b} = \lim_{b\to+\infty} -\frac{2}{\sqrt{x+1}}\Big|_{15}^{b} = \lim_{b\to+\infty}\left[-\frac{2}{\sqrt{\infty+1}} + \frac{2}{\sqrt{15+1}}\right]$$

$$= \frac{2}{4} = \boxed{\frac{1}{2}\,u.a.}$$

10) Encontrar a área sob o gráfico de $y = \frac{1}{(x+1)^2}$ para $x \geq 1$.

$$\lim_{b\to+\infty} \int_{1}^{b} \frac{1}{(x+1)^2}\,dx = \qquad u = x+1 \quad du = dx \qquad = \lim_{b\to+\infty}\int_{1}^{b} \frac{du}{u^2} = \lim_{b\to+\infty}\int_{1}^{b} u^{-2}\,du = \lim_{b\to+\infty} \frac{u^{-1}}{-1}\Big|_{1}^{b}$$

$$= \lim_{b\to+\infty} -\frac{1}{u}\Big|_{1}^{b} = \lim_{b\to+\infty} -\frac{1}{x+1}\Big|_{1}^{b} = \lim_{b\to+\infty}\left[-\frac{1}{\infty+1} + \frac{1}{1+1}\right] = \lim_{b\to+\infty}\left[-\frac{1}{\infty} + \frac{1}{2}\right] = \boxed{\frac{1}{2}\,u.a.}$$

11) Engenheiros da Petrobrás estimaram que um poço de petróleo pode produzir óleo a uma taxa de: $P(t) = 80e^{-0,04t} - 80e^{-0,1t}$ milhares de barris por mês, onde t representa o tempo medido em meses, a partir do momento em que foi feita a estimativa. Determinar o potencial de produção de óleo desse poço a partir dessa data.

$$\lim_{b\to+\infty} \int_{0}^{b} [80e^{-0,04t} - 80e^{-0,1t}]\,dt = \lim_{b\to+\infty} \int_{0}^{b} 80e^{-0,04t}\,dt - \int_{0}^{b} 80e^{-0,1t}\,dt$$

$u = -0,04t$ $\quad du = -0,04\,dt$ $\quad \frac{du}{-0,04} = dt$

$v = -0,1t$ $\quad dv = -0,1\,dt$ $\quad \frac{dv}{-0,1} = dt$

$$\lim_{b\to+\infty} \frac{80}{-0,04}\int_{0}^{b} e^{u}\,du - \frac{80}{-0,1}\int_{0}^{b} e^{v}\,dv = \lim_{b\to+\infty} -2000\,e^{u}\Big|_{0}^{b} + 800\,e^{v}\Big|_{0}^{b}$$

$$= \lim_{b\to+\infty} -2000e^{-0,04t}\Big|_{0}^{b} + 800e^{-0,1t}\Big|_{0}^{b} = \lim_{b\to+\infty} -2000[e^{-0,04\infty} - e^{-0,04\cdot 0}] + 800[e^{-0,1\cdot\infty} - e^{-0,1\cdot 0}]$$

$$= -2000[-1] + 800[-1] = \boxed{1200 \text{ milhares de barris.}}$$

(12) Investigar as integrais impróprias seguintes.

a) $\int_{-\infty}^{0} e^x dx = \lim_{a \to -\infty} \int_a^0 e^x dx = \lim_{a\to-\infty} e^x \Big/_a^0 = \lim_{a\to-\infty} [\overset{1}{e^0} - \overset{0}{e^{-\infty}}] = 1$ (Convergente)

b) $\int_{-\infty}^{0} x e^{-x^2} dx = \lim_{a\to-\infty} \int_a^0 e^{-x^2} x dx = \begin{matrix} u = -x^2 \\ du = -2x\,dx \end{matrix} \to -\frac{du}{2} = x dx = \lim_{a\to-\infty} \int_a^0 e^u - \frac{du}{2}$

$= \lim_{a\to-\infty} -\frac{1}{2}\int_a^0 e^u du = \lim_{a\to-\infty} -\frac{1}{2} e^u \Big/_a^0 = \lim_{a\to-\infty} -\frac{1}{2} e^{-x^2} \Big/_a^0 = \lim_{a\to-\infty} -\frac{1}{2}[e^{-0^2} - e^{+\infty^2}]$

$= \lim_{a\to-\infty} -\frac{1}{2}[\overset{1}{e^0} - \overset{\infty}{e^\infty}] = 1 - \infty = -\infty$ (Divergente)

c) $\int_1^{+\infty} \ln|x| dx = \lim_{b\to+\infty} \int_1^b \ln|x| dx =$ $\quad \int u\,dv = u.v - \int v.du \quad \begin{matrix} u = \ln|x| \\ du = \frac{1}{x} \end{matrix} \quad \boxed{\begin{matrix} v = x \\ dv = dx \end{matrix}}$

$\int \ln|x| dx = \ln|x|.x - \int x.\frac{1}{x} dx = \ln|x|.x - \int dx = \ln|x|.x - x = x(\ln|x| - 1)$

$\lim_{b\to+\infty} x(\ln|x|-1)\Big/_1^b = \lim_{b\to+\infty} [\infty(\ln|\infty|-1) - 1(\ln|1|-1)] = \lim_{b\to+\infty}[\infty\ln|\infty| - \infty - \ln|1| + 1]$

$= [\overset{0}{\cancel{\infty-\infty}}] - \overset{0}{\cancel{\ln|1|}} + 1 = 1$ (Convergente)

d) $\int_{-\infty}^{+\infty} \frac{dx}{9+x^2} = \int_{-\infty}^0 \frac{dx}{9+x^2} + \int_0^{+\infty} \frac{dx}{9+x^2} = \lim_{a\to-\infty}\int_a^0 \frac{dx}{9+x^2} + \lim_{b\to+\infty}\int_0^b \frac{dx}{9+x^2}$ $\quad 9+x^2 = (3-x)^2$

$\lim_{a\to-\infty}\int_a^0 \frac{dx}{(3-x)^2} + \lim_{b\to+\infty}\int_0^b \frac{dx}{(3-x)^2}$ $\quad \begin{matrix} u = 3-x \\ du = -dx \\ -du = dx \end{matrix} \quad \lim_{a\to-\infty}\int_a^0 -\frac{du}{u^2} + \lim_{b\to+\infty}\int_0^b -\frac{du}{u^2}$

$= \lim_{a\to-\infty} -\int_a^0 u^{-2} du - \lim_{b\to+\infty}\int_0^b u^{-2} du = \lim_{a\to-\infty} \frac{1}{u}\Big/_a^0 + \lim_{b\to+\infty} \frac{1}{u}\Big/_0^b = \lim_{a\to-\infty} \frac{1}{9+x^2}\Big/_a^0 + \lim_{b\to+\infty} \frac{1}{9+x^2}\Big/_0^b$

$\lim_{a\to-\infty}\left[\frac{1}{9} - \overset{0}{\cancel{\frac{1}{\infty}}}\right] + \lim_{b\to+\infty}\left[\overset{0}{\cancel{\frac{1}{\infty}}} - \frac{1}{9}\right] = \frac{1}{9} - \frac{1}{9} = 0$ (Convergente)

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e) $\int_e^{+\infty} \frac{dx}{x(\ln x)^2} = \lim_{b\to+\infty}\int_e^b \frac{dx}{x(\ln x)^2}$ $\quad u=\ln x$, $du=\frac{dx}{x}$ $\quad \lim_{b\to+\infty}\int_e^b \frac{du}{u^2}$

$\downarrow$ 0 → limite de domínio

$$\lim_{s\to0^-}\int_e^s u^{-2}\,du + \lim_{r\to0^+}\int_r^{+\infty} u^{-2}\,du = \lim_{s\to0^-} -\frac{1}{u}\Big|_e^s + \lim_{r\to0^+} -\frac{1}{u}\Big|_r^{+\infty}$$

$$= \lim_{s\to0^-} -\frac{1}{\ln x}\Big|_e^s + \lim_{r\to0^+} -\frac{1}{\ln x}\Big|_r^{+\infty} = \lim_{s\to0^-} -\left[\cancelto{\infty}{\frac{1}{\ln s}} - \cancelto{1}{\frac{1}{\ln e}}\right] + \lim_{r\to0^+} -\left[\cancelto{0}{\frac{1}{\infty}} - \cancelto{\infty}{\frac{1}{\ln r}}\right]$$

$= -\cancel{\infty} + 1 + \cancel{\infty} = 1$ (Convergente)

f) $\int_0^{+\infty}\frac{4\,dx}{x+1} = 4\lim_{b\to+\infty}\int_0^b \frac{dx}{x+1}$ $\quad u=x+1$, $du=dx$ $\quad 4\lim_{b\to+\infty}\int_0^b \frac{du}{u} = 4\lim_{b\to+\infty} \ln|u|\Big|_0^b$

$= 4\lim_{b\to+\infty} \ln|x+1|\Big|_0^b = 4\lim_{b\to+\infty}\left[\cancelto{\infty}{\ln|\infty|} - \cancelto{0}{\ln|1|}\right] = \infty$ (Divergente)

g) $\int_0^{+\infty} r.e^{-rx}\,dx,\ r>0 = r\int_0 e^{-rx}\,dx$ $\quad u=-rx$, $du=-r\,dx$, $\frac{du}{-r}=dx$ $\quad = \frac{\cancel{r}}{-\cancel{r}}\int_0 e^u\,du$

$= -\lim_{b\to+\infty}\int_0^b e^u\,du = -\lim_{b\to+\infty} e^u\Big|_0^b = -\lim_{b\to+\infty} e^{-rx}\Big|_0^b = -\lim_{b\to+\infty}\left[\cancelto{0}{e^{-\infty}} - \cancelto{1}{e^{0}}\right]$

$= 1$ (Convergente

h) $\int_{-\infty}^{+\infty}\frac{4x^3}{(x^4+3)^2}\,dx = 4\int_{-\infty}^{+\infty}\frac{x^3}{(x^4+3)^2}\,dx$ $\quad u=x^4+3$, $du=4x^3\,dx$, $\int\frac{du}{4}=x^3\,dx$ $\quad \int_{-\infty}^{+\infty}\frac{du}{u^2} = -\frac{1}{u}\Big|_{-\infty}^{+\infty}$

$$= \lim_{a\to-\infty} -\frac{1}{u}\Big|_a^0 + \lim_{b\to+\infty} -\frac{1}{u}\Big|_0^b = \lim_{a\to-\infty} -\left[\frac{1}{x^4+3}\right]_a^0 + \lim_{b\to+\infty} -\left[\frac{1}{x^4+3}\right]_0^b$$

$$= \lim_{a\to-\infty} -\left[\frac{1}{3} - \cancelto{0}{\frac{1}{\infty}}\right] + \lim_{b\to+\infty} -\left[\cancelto{0}{\frac{1}{\infty}} - \frac{1}{3}\right] = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$ (Convergente)

(13) Determinar a área sob a curva $Y = \frac{1}{\sqrt{4-x}}$, no intervalo $[0, 4)$.

$$A = \int_0^4 \frac{1}{\sqrt{4-x}}\,dx = \lim_{s\to4^-}\int_0^s \frac{1\,dx}{\sqrt{4-x}} \qquad \begin{array}{l} u = 4-x \\ du = -dx \\ -du = dx \end{array} \qquad \lim_{s\to4^-} -\int_0^s \frac{du}{u^{1/2}} \to \int \frac{du}{u^{1/2}} = \frac{2u^{3/2}}{3}$$

$$= \lim_{s\to4^-} -\frac{2\sqrt{u^3}}{3}\Big|_0^s = \lim_{s\to4^-} -\frac{2\sqrt{(4-x)^3}}{3}\Big|_0^s = \lim_{s\to4^-} -\frac{2}{3}\left[\overset{2}{\cancel{\sqrt{(4-s)^3}}} - \overset{8}{\cancel{\sqrt{4^3}}}\right]$$

$$= -\frac{2}{3}\cdot -6 = \frac{12}{3} = \boxed{4\,u.a.}$$

~~Investigar as integrais impróprias~~

(14) a) $\int_0^1 \frac{dx}{\sqrt{1-x}}$ $\quad \lim_{s\to1^-}\int_0^s \frac{dx}{(1-x)^{1/2}}$ $\quad \begin{array}{l} u = 1-x \\ du = -dx \\ -du = dx \end{array}$ $\quad -\lim_{s\to1^-}\int_0^s \frac{du}{u^{1/2}}$

$$\lim_{s\to1^-} -\frac{2}{3}\left[u^{3/2}\right]_0^s = \lim_{s\to1^-} -\frac{2}{3}\left[\sqrt{(1-x)^3}\right]_0^s = \lim_{s\to1^-} \ominus \frac{2}{3}\left[\overset{0}{\cancel{\sqrt{(1-s)^3}}} \ominus 1\right] = \boxed{\frac{2}{3}} \text{ (Convergente)}$$

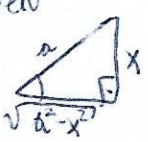
b) $\int_{-1}^1 \frac{dx}{x^2}$ $\quad \frac{dx}{0^2} = \frac{dx}{0}$ O é o limite de domínio $[-1,0)$ e $(0,1]$ $\quad \lim_{s\to0^-}\int_{-1}^s \frac{dx}{x^2} + \lim_{n\to0^+}\int_n^1 \frac{dx}{x^2}$

$$\lim_{s\to0^-} -\frac{1}{x}\Big|_{-1}^s + \lim_{n\to0^+} -\frac{1}{x}\Big|_n^1 = \lim_{s\to0^-} -\left[\overset{+\infty}{\cancel{\frac{1}{s}}} + 1\right] + \lim_{n\to0^+} -\left[1 - \overset{0}{\cancel{\frac{1}{n}}}\right] = -\infty - 1 - 1 = \boxed{-\infty}$$

(Divergente)

c) $\int_0^3 \frac{dx}{\sqrt{9-x^2}} = \lim_{s\to3^-}\int_0^s \frac{dx}{(9-x^2)^{1/2}}$ $\quad$ 1º caso $\quad x = 3\,\text{sen}\,\theta \quad \sqrt{a^2-x^2} = 3\cos\theta$

$dx = 3\cos\theta\,d\theta$



$\text{sen}\,\theta = \frac{CO}{H} = \frac{x}{a} =$

$\text{sen}\,\theta = \frac{x}{a} = \frac{x}{3}$

$\theta = \text{arcsen}\left(\frac{x}{3}\right)$

$$\int \frac{3\cos\theta\,d\theta}{3\cos\theta} = \int d\theta = \theta = \text{arcsen}\left(\frac{x}{3}\right) + C$$

$$\lim_{s\to3^-} \text{arcsen}\left(\frac{x}{3}\right)\Big|_0^s = \lim_{s\to3^-}\left[\overset{\frac{\pi}{4}}{\cancel{\text{arcsen}\left(\frac{s}{3}\right)}} - \overset{0}{\cancel{\text{arcsen}(0)}}\right] = \boxed{\frac{\pi}{4} \text{ (Convergente)}}$$

d) $\displaystyle\int_0^5 \frac{x\,dx}{\sqrt{25-x^2}}$ $\qquad \displaystyle\lim_{s\to 5^-} \int_0^s \frac{x\,dx}{\sqrt{25-x^2}}$

$u = 25 - x^2$

$du = -2x\,dx$

$-\dfrac{du}{2} = x\,dx$

$$-\frac{1}{2}\int \frac{du}{u^{1/2}} = -\frac{1}{2}\cdot\frac{2}{3}\cdot u^{3/2} = -\frac{1}{3}\sqrt{u^3} \Rightarrow \lim_{s\to 5^-} -\frac{1}{3}\sqrt{u^3}\Big|_0^s$$

$$= \lim_{s\to 5^-} -\frac{1}{3}\left[\sqrt{(25-x^2)^3}\right]\Big|_0^s = \lim_{s\to 5^-} -\frac{1}{3}\left[\overset{0}{\sqrt{(25-5^2)^3}} - \overset{125}{\sqrt{(25)^3}}\right] = +\frac{15}{3} = 5 \text{ (Convergente)}$$

e) $\displaystyle\int_{-2}^{2} \frac{x\,dx}{1-x}$ $\qquad \dfrac{x\,dx}{1-1} = \dfrac{x\,dx}{0}$ ↳ Rolmite de domínio

$$\lim_{s\to 1^-}\int_{-2}^{s} \frac{x\,dx}{1-x} + \lim_{r\to 1^+}\int_{r}^{2}\frac{x\,dx}{1-x}$$

$$\begin{array}{r|l} x & 1-x \\ \hline -x+1 & -1 \\ \hline 1 & \end{array}$$

$$\int \left[-1 + \frac{1}{1-x}\right]dx = \int -dx + \int \frac{dx}{1-x} \qquad \begin{array}{l} u = 1-x \\ du = -dx \\ -du = dx \end{array}$$

$$-\int dx - \int \frac{du}{u} = -x - \ln|u| = -x - \ln|1-x| + c$$

$$\lim_{s\to 1^-}\left[-x-\ln|1-x|\right]\Big|_{-2}^{s} + \lim_{r\to 1^+}\left[-x-\ln|1-x|\right]\Big|_{r}^{2} =$$

$$= \lim_{s\to 1^-}\left[-\underset{0}{s} - \underset{-\infty}{\ln|1-s|} - (+2-\ln|1+2|)\right] + \lim_{r\to 1^+}\left[-2-\underset{\nexists}{\ln|1-2|} - (-r-\underset{\nexists}{\ln|1-r|})\right]$$

$\nexists$ (Divergente)

f) $\displaystyle\int_0^{+\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$ $\qquad u = -\sqrt{x} = -x^{\frac{1}{2}}$, $\quad du = -\frac{1}{2}x^{-\frac{1}{2}}\,dx \Rightarrow -du\,2 = \frac{1}{\sqrt{x}}dx$

$$-2\int_0^{+\infty} e^u\,du = \lim_{b\to+\infty} -2\int_0^b e^u\,du = \lim_{b\to+\infty} -2\,e^u\Big|_0^b$$

$$= \lim_{b\to+\infty} -2\,e^{-\sqrt{x}}\Big|_0^b = \lim_{b\to+\infty} -2\left[\overset{0}{e^{-\sqrt{b}}} - \overset{1}{e^0}\right] = \boxed{2 \text{ (Convergente)}}$$

g) $\displaystyle\int_1^{+\infty} \frac{dx}{(x-1)^3}$ $\qquad u = x-1$, $\quad du = dx$

$$\int \frac{du}{u^3} = \int u^{-3}\,du = \frac{u^{-2}}{-2} = -\frac{1}{2u^2} \Rightarrow \lim_{b\to+\infty} -\frac{1}{2}\cdot\frac{1}{u^2}\Big|_1^b$$

$$\lim_{b\to+\infty} -\frac{1}{2}\cdot\frac{1}{(x-1)^2}\Big|_1^b = \lim_{b\to+\infty} -\frac{1}{2}\left[\underset{0}{\frac{1}{(b-1)^2}} - \underset{\nexists}{\frac{1}{0}}\right] = \nexists \quad \boxed{\text{(Divergente)}}$$