Integração por partes (23/09/2013) Vitor Oliveira Popke

Livro: Cálculo A.

Tópico 6.6, página 255.

Questões: 1-8, 20, 21, 23, 25, 29, 33 e 36.

Página 255

Resolver as seguintes integrais usando a técnica de integração por partes.

$\int u \cdot dv = u \cdot v - \int v \cdot du$

(1) $\int x \, \text{sen}\, 5x \, dx$

$\int \text{sen}(5x)\, dx$

$a = 5x$

$da = 5\, dx \to \frac{da}{5} = dx$

$\int \text{sen}(a) \frac{da}{5} \to \frac{1}{5}\int \text{sen}(a)\, da$

$\to \frac{1}{5} \cdot -\cos(a) + c \to \boxed{\frac{-\cos(5x)}{5}} +$

$u = x \to du = dx$

$dv = \text{sen}(5x)dx \to v = \frac{-\cos(5x)}{5}$

$\int x\,\text{sen}(5x)\, dx = x \cdot \frac{-\cos(5x)}{5} - \int \frac{-\cos(5x)}{5}\, dx$

$\int \frac{-\cos(5x)}{5} dx$

$b = 5x$

$db = 5\,dx \to \frac{db}{5} = dx$

$\int \frac{-\cos(b)}{5} \cdot \frac{db}{5} = \frac{1}{25}\int -\cos(b)\, db$

$\to \frac{1}{25} \cdot -\text{sen}(5x) + c \to \boxed{\frac{-\text{sen}(5x) + c}{25}}$

$\int x\,\text{sen}(5x)\, dx = -\frac{x\cos(5x)}{5} + \frac{\text{sen}(5x)}{25} + c$

$\int x\,\text{sen}(5x)\, dx = \boxed{\frac{-5x\cos(5x) + \text{sen}(5x)}{25} + c}$

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(2) $\int \ln(1-x)\, dx$

$a = 1-x$

$da = -dx \to -da = dx$

$\int \ln(a) \cdot -da = -\int \ln(a)\, da \leftarrow \int u \cdot dv = u \cdot v - \int v \cdot du$

$u = \ln(a) \to du = \frac{1}{a} da$

$dv = +da \to v = +a$

$-\int \ln(a)\, da = \ln(a) a - \int a \cdot \frac{1}{a} da$

$= \ln(a) a - \int \frac{a}{a} da \to \ln(a) a - \int da \to \ln(a) a - a$

$-\int \ln(a)\, da = \ln(a) a - a$

$\int \ln(a)\, da = -\ln(a) a + a$

$= -(\ln(1-x))(1-x) + 1 - x$

$= \boxed{(\ln(1-x) - 1)(x-1) + c}$

③ $\int t e^{4t} dt$

$a = 4t$

$da = 4dt \to \frac{da}{4} = dt \qquad \int t e^{a} \frac{da}{4} = \frac{1}{4}\int t e^{a} da$

$u = t \to du = dt = \frac{da}{4}$

$dv = e^{a} da \to v = e^{a}$

$\frac{1}{4}\int t e^{a} \cancel{dt} = t e^{a} - \int e^{a} dt$

$= te^{a} - \int e^{a}\frac{da}{4} \to te^{a} - \frac{1}{4}\int e^{a} da \to te^{a} - \frac{1}{4} e^{a} + c$

$\frac{1}{4}\int te^{a} da = te^{a} - \frac{e^{a}}{4} + c \Rightarrow \int te^{a} da = \frac{te^{a}}{4} - \frac{e^{a}}{16} + c$

$\Rightarrow \int te^{a} da = \frac{4te^{a} - e^{a}}{16} + c = \frac{4te^{4t} - e^{4t}}{16} + c = \boxed{\frac{(4t-1)e^{4t}}{16} + c}$

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④ $\int (x+1)\cos(2x)\, dx$

$u = x+1 \qquad du = dx$

$dv = \cos(2x)\, dx \qquad v = \frac{sen(2x)}{2}$

$a = 2x$

$da = 2dx \to \frac{da}{2} = dx$

$\int \cos(a)\frac{da}{2} \to \frac{1}{2}\int\cos(a)\, da$

$\to \frac{sen(a)}{2} + c$

$\boxed{\frac{sen(2x)}{2} + c}$

$\int (x+1)\cos(2x)\,dx = (x+1).\frac{sen(2x)}{2} - \int \frac{sen(2x)}{2}\, dx$

$= \frac{(x+1)sen(2x)}{2} - \frac{1}{2}\int sen(2x)\,dx$

$= \frac{(x+1)sen(2x)}{2} - \frac{1}{2}\int sen(a)\frac{da}{2} \to \frac{(x+1)sen(2x)}{2} - \frac{1}{4} - \cos(a)$

$\to \frac{(x+1)sen(2x)}{2} + \frac{\cos(2x)}{4} = \frac{2((x+1)sen(2x)) + \cos(2x)}{4} = \boxed{\frac{(2x+2)2sen(2x) + \cos(2x)}{4} + c}$

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⑤ $\int x \ln|3x|\, dx$

$u = \ln|3x| \to du = \frac{dx}{x}$

$dv = x\,dx \to v = \frac{x^2}{2}$

$\int \ln|3x|.x\,dx = \ln|3x|.\frac{x^2}{2} - \int\frac{x^2}{2}.\frac{dx}{x}$

$= \frac{\ln|3x|x^2}{2} - \int\frac{x}{2}dx = \frac{\ln|3x|x^2}{2} - \frac{1}{2}\int x\,dx$

$= \frac{\ln|3x|x^2}{2} - \frac{1}{2}.\frac{x^2}{2} = \frac{\ln|3x|x^2}{2} - \frac{x^2}{4} = \frac{2\ln|3x|x^2 - x^2}{4} = \boxed{\frac{x^2(2\ln|3x|-1)}{4} + c}$

⑥ $\int \cos^3 x\,dx = \int \cos^2 x \cos x\,dx$ $\quad (\cos^2 x)' = 2\cos x \cdot \cos x' = 2\cos x.(-\operatorname{sen} x)$
$= -2\cos x . \operatorname{sen} x$

$u = \cos^2 x \quad du = -2\cos x . \operatorname{sen} x\,dx$
$dv = \cos x\,dx \rightarrow v = \operatorname{sen} x$

$\int \cos^2 x . \cos x\,dx = \cos^2 x . \operatorname{sen} x - \int \operatorname{sen} x.(-2\cos x . \operatorname{sen} x\,dx)$
$= \cos^2 x . \operatorname{sen} x + 2\int \operatorname{sen} x(\cos x . \operatorname{sen} x\,dx) = \cos^2 x . \operatorname{sen} x + 2\int \underbrace{\cos x . \operatorname{sen}^2 x\,dx}$

$w = \operatorname{sen}^2 x \quad dw = 2\operatorname{sen} x . \cos x\,dx$
$dy = \cos x\,dx \quad y = \operatorname{sen} x$

$\int \cos x . \operatorname{sen}^2 x\,dx = \operatorname{sen}^2 x . \operatorname{sen} x - \int \operatorname{sen} x\, 2\operatorname{sen} x . \cos x\,dx$
$\int \cos x . \operatorname{sen}^2 x\,dx = \operatorname{sen}^3 x - 2\int \operatorname{sen}^2 x . \cos x\,dx$
$\int \cos x . \operatorname{sen}^2 x\,dx + 2\int \cos x . \operatorname{sen}^2 x\,dx = \operatorname{sen}^3 x \rightarrow 3\int \cos x . \operatorname{sen}^2 x\,dx = \operatorname{sen}^3 x \rightarrow$

$\rightarrow \int \cos x . \operatorname{sen}^2 x\,dx = \dfrac{\operatorname{sen}^3 x}{3}$

$\rightarrow \boxed{\cos^2 x . \operatorname{sen} x + 2\dfrac{\operatorname{sen}^3 x}{3} + C}$

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⑦ $\int e^x \cos\dfrac{x}{2}\,dx$ $\quad u = \cos\dfrac{x}{2} \quad du = \quad (\cos(a))' = \cos(a).a' = -\operatorname{sen}(a).\dfrac{x}{2}' = \dfrac{1}{2}.-\operatorname{sen}.x'$
$dv = e^x\,dx \rightarrow v = e^x \qquad du = -\dfrac{\operatorname{sen}\left(\frac{x}{2}\right)}{2}\,dx$

$\int \cos\dfrac{x}{2} e^x\,dx = \cos\left(\dfrac{x}{2}\right).e^x - \int e^x . \dfrac{-\operatorname{sen}\left(\frac{x}{2}\right)}{2}\,dx = \cos\left(\dfrac{x}{2}\right)e^x + \dfrac{1}{2}\int \underbrace{e^x . +\operatorname{sen}\left(\dfrac{x}{2}\right)\,dx}$

$w = \operatorname{sen}\left(\dfrac{x}{2}\right) \rightarrow dw = \dfrac{\cos\left(\frac{x}{2}\right)}{2}$
$dy = e^x\,dx \rightarrow y = e^x$

$\int e^x . \operatorname{sen}\left(\dfrac{x}{2}\right)dx = \operatorname{sen}\left(\dfrac{x}{2}\right)e^x - \int e^x \dfrac{\cos\left(\frac{x}{2}\right)}{2} = \operatorname{sen}\left(\dfrac{x}{2}\right)e^x - \dfrac{1}{2}\int e^x \cos\left(\dfrac{x}{2}\right)dx$

$\rightarrow \int \cos\left(\dfrac{x}{2}\right)e^x\,dx = \cos\left(\dfrac{x}{2}\right).e^x + \dfrac{1}{2}\int\left(\operatorname{sen}\left(\dfrac{x}{2}\right)e^x - \dfrac{1}{2}\int e^x \cos\left(\dfrac{x}{2}\right)dx\right)dx$
$\int \cos\left(\dfrac{x}{2}\right)e^x\,dx + \dfrac{1}{2}\int \cos\left(\dfrac{x}{2}\right)e^x\,dx = \cos\left(\dfrac{x}{2}\right)e^x + \dfrac{1}{2}\iint \operatorname{sen}\left(\dfrac{x}{2}\right)e^x\,dx$
$\dfrac{3}{2}\int \cos\left(\dfrac{x}{2}\right)e^x\,dx = \cos\left(\dfrac{x}{2}\right)e^x + \dfrac{1}{2}\iint \operatorname{sen}\left(\dfrac{x}{2}\right)e^x\,dx$

$$\int \cos\left(\frac{x}{2}\right)e^x\,dx = \frac{2}{3}\cdot \cos\left(\frac{x}{2}\right)e^x + \frac{1}{2}\int\!\!\int \operatorname{sen}\frac{x}{2}\, e^x\,dx$$

$$= \frac{2\cos\left(\frac{x}{2}\right)e^x}{3} + \frac{2}{6}\int\!\!\int \operatorname{sen}\left(\frac{x}{2}\right)e^x\,dx = \frac{2\cos\left(\frac{x}{2}\right)e^x}{3} + \frac{2\operatorname{sen}\left(\frac{x}{2}\right)e^x}{6}$$

$$= \frac{4\left(\cos\left(\frac{x}{2}\right)e^x\right) + 2\operatorname{sen}\left(\frac{x}{2}\right)e^x}{6} = \frac{2\left(2\left(\cos\left(\frac{x}{2}\right)e^x\right) + \operatorname{sen}\left(\frac{x}{2}\right)e^x\right)}{6} = \boxed{\frac{2\left(\cos\left(\frac{x}{2}\right)e^x\right) + \operatorname{sen}\left(\frac{x}{2}\right)e^x}{3} + c}$$

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8) $\int \sqrt{x}\,\ln x\,dx = \int x^{\frac{1}{2}}\ln x\,dx$

$$u = \ln x \quad du = \frac{dx}{x}$$
$$dv = x^{\frac{1}{2}}dx \quad v = \frac{2x^{\frac{3}{2}}}{3}$$

$$\int x^{\frac{1}{2}}\ln x\,dx = \ln|x|\cdot\frac{2x^{3/2}}{3} - \int \frac{2x^{\frac{3}{2}}}{3}\cdot\frac{dx}{x} = \ln|x|\cdot\frac{2x^{3/2}}{3} - \int\frac{2x^{\frac{1}{2}}\,dx}{3} =$$

$$= \ln|x|\cdot\frac{2x^{3/2}}{3} - \frac{2}{3}\int x^{\frac{1}{2}}\,dx = \ln|x|\cdot\frac{2x^{3/2}}{3} - \frac{2}{3}\cdot\frac{2x^{3/2}}{3} = \frac{\ln|x|\cdot 2x^{3/2}}{3} - \frac{4x^{3/2}}{9}$$

$$\frac{6\ln|x|\cdot x^{3/2} - 4x^{3/2}}{9} = \frac{2(3\ln|x|\cdot x^{3/2} - 2x^{3/2})}{9} = \boxed{\frac{2(3\ln|x|\cdot\sqrt{x^3} - 2\sqrt{x^3})}{9} + c}$$

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20) $\int x^2\ln x\,dx$

$$u = \ln x \to du = \frac{dx}{x}$$
$$dv = x^2dx \to v = \frac{x^3}{3}$$

$$\int x^2\ln x\,dx = \frac{\ln|x|\cdot x^3}{3} - \int\frac{x^3}{3}\cdot\frac{dx}{x} = \frac{\ln|x|\cdot x^3}{3} - \frac{1}{3}\int x^2\,dx = \frac{\ln|x|x^3}{3} - \frac{1}{3}\cdot\frac{x^3}{3}$$

$$= \frac{\ln|x|x^3}{3} - \frac{x^3}{9} = \boxed{\frac{3\ln|x|x^3 - x^3}{9} + c} = \boxed{\frac{\ln|x|x^3 - x^3}{3} + c}$$

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21) $\int x^2e^x\,dx$

$$u = x^2 \quad du = 2x\,dx$$
$$dv = e^xdx \quad v = e^x$$

$$\int x^2e^x\,dx = x^2e^x - \int e^x 2x\,dx = x^2e^x - 2\int e^x x\,dx$$

$$w = x \quad dw = dx$$
$$dy = e^xdx \quad y = e^x$$

$$\int e^x x\,dx = xe^x - \int e^x\,dx = xe^x - e^x \longrightarrow x^2e^x - 2(xe^x - e^x) = \boxed{x^2e^x - 2xe^x + 2e^x + c}$$

ou

$$\boxed{e^x(x^2 - 2x + 2) + c}$$

(23) $\int (x-1)\sec^2 x\,dx$

$u = x-1 \quad du = dx$
$dv = \sec^2 x\,dx \quad v = \operatorname{tg} x$

$= (x-1).\operatorname{tg}x - \int \operatorname{tg}x\,dx = (x-1).\operatorname{tg}x - \int \frac{\operatorname{sen}x}{\cos x}dx = (x-1).\operatorname{tg}x - \int \operatorname{sen}x\,dx.\frac{1}{\cos x}$

$= (x-1)\operatorname{tg}x - \underbrace{\int \operatorname{sen}x.\sec x\,dx}$

$w = \sec x \quad dw = \rightarrow (\sec x)' = \left(\frac{1}{\cos x}\right)' = -\frac{\cos x'}{\cos^2 x} = -\frac{-\operatorname{sen}x}{\cos^2 x} = \frac{\operatorname{sen}x}{\cos^2 x} = \frac{\operatorname{sen}x}{\cos x}.\frac{1}{\cos x}$

$dy = \operatorname{sen}x\,dx \quad y = -\cos x$

$dw = \operatorname{tg}x.\sec x\,dx$

$\sec x\,{-\cos x} - \int -\cos x.\operatorname{tg}x.\sec x\,dx = -\sec x.\cos x + \int \frac{\cos x}{\cos x}\operatorname{tg}x\,dx = -\sec x.\cos x + \int \operatorname{tg}x\,dx$

$= -\sec x.\cos x - \ln(|\cos x|) + c = -\frac{1}{\cos}.\cos x - \ln|\cos x| + c = \boxed{-\ln|\cos x| + c}$

$\rightarrow \boxed{(x-1)\operatorname{tg}x + \ln|\cos x| + c}$

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(25) $\int x^n \ln x\,dx,\ n \in \mathbb{N}$

$u = \ln x \quad du = \frac{dx}{x}$
$dv = x^n dx \rightarrow v = \frac{x^{n+1}}{n+1}$

$= \frac{\ln|x|\,x^{n+1}}{n+1} - \int \frac{x^{n+1}}{n+1}.\frac{dx}{x} = \frac{\ln|x|\,x^{n+1}}{n+1} - \frac{1}{n+1}\int x^n dx = \frac{\ln|x|\,x^{n+1}}{n+1} - \frac{1}{n+1}.\frac{x^{n+1}}{n+1}$

$= \frac{\ln|x|\,x^{n+1}}{n+1} - \frac{x^{n+1}}{n^2+2n+1} + c = \boxed{\frac{(n+1)(\ln|x|\,x^{n+1}) - x^{n+1}}{n^2+2n+1} + c}$

Página 256

(29) $\int x^5 e^{x^2}dx$

$u = x^5 \quad du = 5x^4 dx$
$dv = e^{x^2}dx \quad v = \frac{e^{x^2}}{2x}$

$\int e^{x^2}dx \quad a = x^2$
$da = 2x\,dx \rightarrow \frac{da}{2} = x\,dx$

$\int e^a \frac{da}{2x} = \frac{1}{2x}\int e^a\,da \rightarrow \frac{da}{2x} = dx$

$\frac{x^5.e^{x^2}}{2x} - \int \frac{e^{x^2}}{2x}5x^4 dx = \frac{x^5.e^{x^2}}{2x} - \frac{5}{2}\underbrace{\int e^{x^2}x^3 dx}$

$u' = x^3 \quad du' = 3x^2 dx$
$dv' = e^{x^2}dx \quad v' = \frac{e^{x^2}}{2x}$

$\rightarrow \frac{x^3.e^{x^2}}{2x} - \int \frac{e^{x^2}}{2x}3x^2 dx = \frac{x^2.e^{x^2}}{2} - \frac{3}{2}\int e^{x^2}x\,dx$

$u'' = X \quad du'' = dX$
$dv'' = e^{X^2} \quad v'' = \frac{e^{X^2}}{2X}$

$$\frac{X.e^{X^2}}{2X} - \int \frac{e^{X^2}}{2X}\,dX = \frac{e^{X^2}}{2} - \frac{1}{2}\int \frac{e^{X^2}}{X}\,dX = \frac{e^{X^2}}{2} - \frac{1}{2}\int e^{X^2}\ln|X|$$

$u''' = \ln|X| \quad du''' = \frac{dX}{X}$
$dv''' = e^{X^2} \quad v''' = \frac{e^{X^2}}{2X}$

$$\frac{\ln|X|e^{X^2}}{2X} - \int \frac{e^{X^2}}{2X}\frac{dX}{X} =$$

$$\rightarrow \int e^{X^2}\ln|X| = \frac{\ln|X|e^{X^2}}{2X}$$

$$\rightarrow \frac{e^{X^2}}{2} - \frac{1}{2}\cdot\frac{\ln|X|e^{X^2}}{2X} = \frac{e^{X^2}}{2} - \frac{\ln|X|e^{X^2}}{4X} = \frac{2Xe^{X^2} - \ln|X|e^{X^2}}{4X}$$

$$\rightarrow \frac{X^2e^{X^2}}{2} - \frac{3}{2}\,\frac{2Xe^{X^2} - \ln|X|e^{X^2}}{4X} = \frac{X^2e^{X^2}}{2} - \frac{6Xe^{X^2} + 3\ln|X|e^{X^2}}{8X} = \frac{4X^3e^{X^2} - 6Xe^{X^2} + 3\ln|X|e^{X^2}}{8X}$$

$$\rightarrow \frac{X^5e^{X^2}}{2X} - \frac{5}{2}\cdot\frac{4X^3e^{X^2} - 6Xe^{X^2} + 3\ln|X|e^{X^2}}{8X} = \frac{X^5e^{X^2}}{2X} - \frac{20X^3e^{X^2} + 30Xe^{X^2} - 15\ln|X|e^{X^2}}{8X}$$

$$= \frac{4X^5e^{X^2} - 20X^3e^{X^2} + 30Xe^{X^2} - 15\ln|X|e^{X^2}}{8X} = \frac{X^4e^{X^2} - 5X^2e^{X^2} + 8e^{X^2} - 4\ln|X|e^{X^2}}{2}$$

$$\boxed{= \frac{(X^4 - 5X^2 + 8 - 4\ln|X|)e^{X^2}}{2} + C}$$

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(33) $\int \cos(\ln x)\,dx \qquad a = \ln|X| \quad da = \frac{dx}{X} \rightarrow X\,da = dX$

$\int \cos(a)\,X\,da \qquad u = X \rightarrow du = da$
$dv = \cos(a)\,da \rightarrow v = \mathrm{sen}(a)$

$= X.\mathrm{sen}(a) - \int \mathrm{sen}(a)\,da$

$= X.\mathrm{sen}(\ln|X|) - \int \frac{\mathrm{sen}(\ln|X|)\,dX}{X} \qquad a' = \ln|X| \quad da' = \frac{dX}{X}$

$= X.\mathrm{sen}(\ln|X|) - \int \mathrm{sen}(a')\,da' = \boxed{X.\mathrm{sen}(\ln|X|) + \cos(\ln|X|) + C}$

36) $\int \frac{1}{x^3} e^{\frac{1}{x}} \, dx$ $\qquad a = \frac{1}{x} \quad da = -\frac{1}{x^2} dx \rightarrow -da = \frac{1}{x^2} dx$

$$\int \frac{e^a}{x} \, -da = -\int e^a \cdot \frac{1}{x} \, da = -\int e^a \cdot a \, da$$

$u = a \quad du = da$

$dv = e^a \quad v = e^a$

$$= a \cdot e^a - \int e^a \, da = a \cdot e^a - e^a = \frac{e^{\frac{1}{x}}}{x} - e^{\frac{1}{x}} \rightarrow -\frac{e^{\frac{1}{x}}}{x} + e^{\frac{1}{x}} = \frac{-e^{\frac{1}{x}} + x e^{\frac{1}{x}}}{x}$$

$$= \boxed{\frac{(-1+x)e^{\frac{1}{x}}}{x} + c}$$