Analysis of Irreversible Phenomena via τ-Manifold: A Formal τ-Analysis Approach

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Notation and Conventions

- τ = The τ-space basis/irreversibility constant.
- z = A complex τ-independent variable in the definition of q.
- w = A complex τ-dependent variable multiplied by the basis τ in the definition of q.
- q = The variable responsible to carry the irreversible information; $q = z + \tau w$.
- S_τ = τ-sphere representing the hyperbolic manifold of irreversibilities.
- $\mathbb{T}^n$ = n-dimensional τ-space (n degrees of freedom of irreversibility).
- $\chi_m \left| e_m \right\rangle$ = The tensor representation for the τ-independent variable z (when the degree of irreversibility is greater than one).
- $\psi_l \left| e_l \right\rangle$ = The tensor representation of the τ-dependent variable w (when the degree of irreversibility is greater than one).
- $f_\tau(q)$ = An integrable; differentiable function whose domain is contained in the τ-sphere; $f_\tau(q) = \text{ᚢ} + \tau \text{ᚠ}$.
- $\text{ᚢ}(\chi_m, \psi_l)$ = A complex τ-independent subfunction of f_τ.
- $\text{ᚠ}(\chi_m, \psi_l)$ = A complex τ-dependent subfunction of f_τ.
- $\text{ɰ}_\sigma \left| e_\sigma \right\rangle$ = The tensor representation of the τ-independent subfunction ᚢ (when the degree of irreversibility is greater than one).
- $\text{ɧ}_j \left| e_j \right\rangle$ = The tensor representation of the τ-dependent subfunction ᚠ (when the degree of irreversibility is greater than one).
- ᚺ = An integrable; differentiable function whose domain is contained in the τ-sphere (most used to denote a τ-function in its hyperbolic form)
- $q^{\blacksquare}$ = The stable form of the τ-variable.

- $f_\tau(q)^{\blacksquare}$ = The stable form of the τ-function.
- $f_\tau(q^{\blacksquare})$ = The τ-function of the stable τ-variable.
- $\{\mathcal{G}\}$ = Generic space.
- κ = The attached variable of the generic space.
- $\{S\}$ = A homomorphic subspace of the generic space - the desired space.
- s = The attached variable of the desired space.
- $f(\kappa)$ = An integrable function in the generic space.
- $\widetilde{f}(s)$ = An integrable function in the desired space.
- $\hat{ᛘ}_s$ = The irreversibility operator from a desired space onto a generic space.
- $\hat{ᛘ}_s^{-1}$ = The irreversibility operator from a generic space onto a desired space.
- $\hat{ᛗ}$ = $\hat{ᛘ}_s \hat{ᛘ}_s^{-1}$ - if $\hat{ᛗ}$ equals the identity operator $\hat{\mathrm{I}}$, then the process is reversible.
- ю = The irreversibility eigenvalue; since $\hat{ᛗ} = \hat{\mathrm{I}}$, then ю $= 1$ - another indication of reversibility.
- ӁК = Jacobian Transition Matrix, defined in S_τ.
- Я = The radius of the hyperbolic τ-sphere.
- ξ = The scape angle of the τ-sphere.
- $\{\mathcal{C}_{\mathbb{T}\times\mathbb{C}}\}$ = A partition of S_τ defining the sub circumference which consists on a $\mathbb{T}\times\mathbb{C}$ plane.
- $\{\mathcal{C}_{\mathbb{T}\times\mathbb{R}}\}$ = A partition of S_τ defining the sub circumference which consists on a $\mathbb{T}\times\mathbb{R}$ plane.
- $\{\mathcal{C}_{\mathbb{C}\times\mathbb{R}}\}$ = A partition of S_τ defining the sub circumference which consists on a $\mathbb{C}\times\mathbb{R}$ plane (the Argand-Gauss plane).
- Π = A singularity in the τ-manifold.
- $\underbrace{f}(\kappa)$ = Distributional form of a function in the generic space.
- K = The Kernel of a transformation ᛘ.
- $\mathcal{G}^{\mathcal{X}}$ = Generic topological manifold of order $\mathcal{X}$.
- $\rho_c\,|e_c\rangle$ = The tensor representation for the desirable variable of index c $(\rho_c \in \{S\})$.

- $\lambda_c \, |e_d\rangle$ = The tensor representation for the generic variable of index d ($\lambda_d \in \{\mathcal{G}\}$).
- y_{dc} = Elements of the matrix form of the co-tensor κ_{dc}.
- κ_{dc} = The co-tensor position; $\kappa_{dc} \in \mathcal{G}^{\mathcal{X}}$.
- $\mathcal{H}_{cd}(\kappa_{dc})$ = A tensor field depending of the generic co-tensor κ_{dc}.
- $\hbar_{cd}$ = An element of the tensor field $\mathcal{H}_{cd}$.
- $\mathscr{D}\kappa_{dc}$ = The co-tensor differential
- ∇_{dc} = The co-tensor gradient operator.
- X = An arbitrary matrix/tensor in the generic space - $X \in \mathcal{G}^{\mathcal{X}}$.
- $\not{X}$ = Dense arbitrary matrix/tensor.
- $\not{X}$ = The dense form of the matrix/tensor X.
- $\Phi(X)$ = The dense functional, which carries a matrix X to its dense form $\not{X}$.
- $\mathcal{H}^{\blacksquare}_{cd}(\kappa_{dc})$ = The stable form of the tensor field $\mathcal{H}_{cd}$.
- $\mathrm{Int}(A)$ = The Internal subset $\mathrm{Int}(A) \subseteq A$ of an arbitrary set A.
- $\mathscr{T}$ = The topology of an arbitrary set, given by the conditions stated in the Section 7.0.
- $\nabla\mathcal{G}^{\mathcal{X}}$ = The boundary of the topological manifold $\mathcal{G}^{\mathcal{X}}$.
  - Although the usual notation for a boundary of a manifold is represented by the *Del* symbol - ∂ - I desired to emphasize the fact that $\mathcal{G}^{\mathcal{X}}$ possesses an order $\mathcal{X}$, and then used the symbol *Nabla* - ∇ - to indicate a multidimensional manifold.
- $\vartheta^{\mathcal{Z}}$ = A set defined as the axis of rotation of a planar cycle, with elements given by dimensions; $\mathcal{Z}$ determines the quantity of dimensions contained on the axis - $1 \leq \mathcal{Z} \leq \mathcal{X}$.

Abstract

Irreversibility is a fundamental topic in both pure and applied sciences, including mathematics, mathematical physics, and complex systems. Despite extensive efforts by leading researchers, traditional methods often rely on approximations or computationally intensive algorithms. Consequently, many irreversible phenomena — such as entropy evolution or complex economic fluctuations — remain only partially understood.

The introduction of the novel concepts in this paper contained concerns about encoding irreversibility as a dispersive phenomena in a hyperbolic topological manifold - regarding singularities created by the dynamics of the irreversible phenomena - breaking out complex entangled systems (via tensors) to a topological interpretation of the manifold of irreversibility, called τ-space.

1 Introduction

Irreversibility is a fundamental topic in both pure and applied sciences, including mathematics, mathematical physics, and complex systems. Despite extensive efforts by leading researchers, traditional methods often rely on approximations or computationally intensive algorithms. Consequently, many irreversible phenomena — such as entropy evolution or complex economic fluctuations — remain only partially understood. In this preprint I will introduce a novel formalism for treating irreversibility not merely as a unavoidable loss of information, but as a structured domain – a τ-manifold, systematically designed to contain and control information dissipation.

To formalize an irreversible process, we introduce a unit τ satisfying:

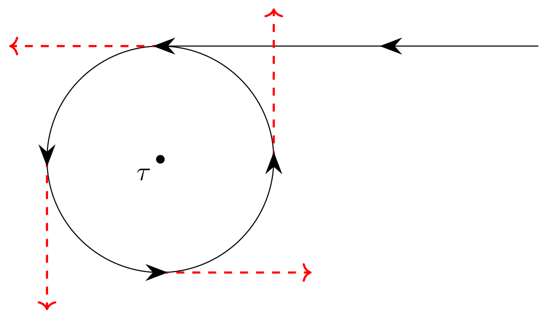
$$\sqrt{\tau} = -1$$

Squaring both sides yields $\tau = 1$, however, the original negative sign is lost in the squaring process, indicating an intrinsic loss of information. This property exemplifies the irreversible nature encoded in τ and motivates its use as a foundational unit within the τ-manifold.

2 The τ-Sphere

To follow the chronological sequence of the theory, for an advance on mathematical abstraction I shall introduce the idea of time-current. The time is one of the most logical and direct ideals to comprehend irreversibility, in the sight that, a change in some parameter in some point in a given time position, may affect future events chaotically.

The interpretation of time-current is directly connected with the definition of circulation - described as follows:

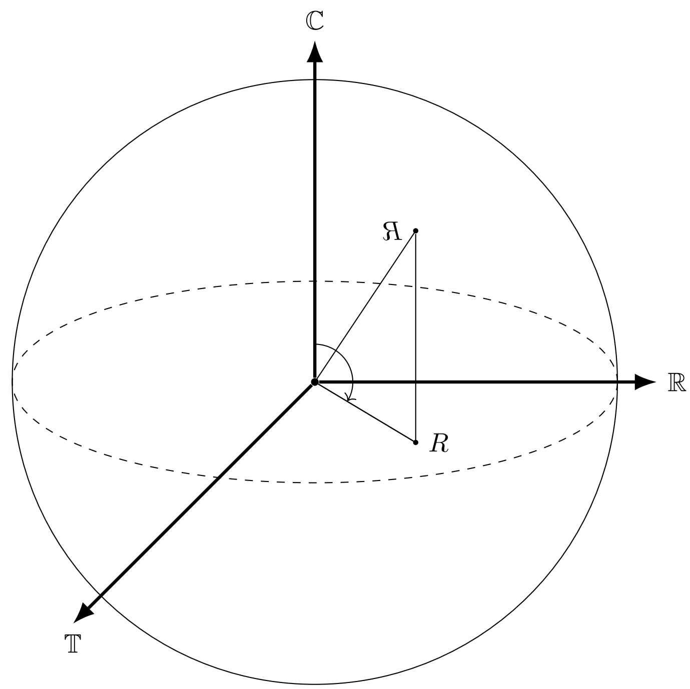


Once a perturbation at t_0 is done, the time current takes the path towards a point t; the path walked by the current passes along a circumference, which represents geometrically the τ-space. According to the degree of reversibility of the analyzed system, the current scapes by an angle ξ compared with the original route, and such an angle describes the degree of irreversibility of the system. By convention, because the degree of irreversibility is not periodic, the angle ξ is denoted to be hyperbolic, $-\infty < \xi < \infty$, such that, when $\xi = 0$, there is absence of deviation, and when $\xi \to \infty$ the complete irreversibility happens (in hyperbolic coordinates it's similar to affirm orthogonality).

Definition I: The τ-sphere - S_τ, is a remarkable artifice to interpret the τ-space geometry: It consists of poles representing the supremum of $sup(S_\tau)$:

$$\sup(S_\tau) = \begin{cases} \pm i, \textbf{latitudinally} \\ \pm 1, \textbf{longitudinally} \end{cases}$$

The previously seen circulation consists in a sub circumference of S_τ. Inside the sphere the pure τ domain is evident; the center of the sphere is the τ-unit, far from a distance $\Re$ of the vertexes (that changes according to the chosen direction).



Theorem I: The τ-unit τ is a basis for both S_τ and $\mathbb{C}$, but not for $\mathbb{R}$.

Proof. Let $\sqrt{\tau} = -1$, then:

1) $\tau^{\frac{1}{4}} = i$

2) $\tau^2 \neq 1$, otherwise, information dispersion occurs.

Therefore, rational powers of the τ-unit are represented as complex numbers, but the same is not true for real numbers $\square$

3 Mathematical Framework

This section will be entirely dedicated to analyzing what these simple properties of S_τ imply in a more systematic mathematical framework.
The current situation requires an introduction of operators; their function in τ-analysis is extremely important and provides a direct and clear interpretation and application using the concepts explored hitherto.

Let $\hat{\Upsilon} : \mathbb{C} \to \mathbb{T}$ be the called τ-inversion operator, which diverse meanings depending the analyzed situations – I shall present some of them in detail further.

For every operator, there must be a function in which the same can be applied:

Remark: We are going to use Dirac notation for inner products; bra's $\langle\ |$ and ket's $|\ \rangle$, due to many similarities our approach has with Quantum Mechanics.

Let $q = z + \tau w;\ z, w \in \mathbb{C}$, be a τ-variable – this is, a well-defined variable inside S_τ. Then, let $f_\tau(q)$ be a function $f_\tau : \mathbb{T} \to \mathbb{T}$. The following process will be, with caution, to adopt a similar approach to what is done in Complex Analysis and derive precisely the two kinds of τ-functions:

1. τ-trivial functions
2. τ-analytic functions

For a well behaved τ-function $f_\tau : \mathbb{T} \to \mathbb{T}$ in which is integrable; differentiable by each one of its components (z, w); continuous under the period of integration we define a τ-integral of f_τ as follows:

$$\oint_{S_\tau} f_\tau(q)dq$$

Where f_τ is identified as a field, and S_τ as a path in which the field acts.

3.1 Definition of τ-trivial functions

Lemma: Given $f_\tau(q) = \text{ᚢ}(z, w) + \tau\text{ᚠ}(z, w)$:

$$\oint_{S_\tau} f_\tau(q)dq = \oiint_{\mathbb{T}} \langle\, \text{ᚢ}(z, w) + \tau\, \text{ᚠ}(z, w)|dz + \tau dw\rangle$$

• By the Generalized Stokes Theorem, we get: *(See Section 7 for a detailed derivation, regarding dense generic manifolds)*

$$\oiint_{\mathbb{T}} \langle \text{ᚢ}(z, w) + \tau\text{ᚠ}(z, w)\, |dz + \tau dw\rangle$$

$=$

$$\oiint_{\mathbb{T}\mapsto\mathbb{C}^2} (\nabla \times f_\tau)dzdw = \oiint_{\mathbb{T}\mapsto\mathbb{C}^2} \left(\frac{\partial\text{ᚠ}(z, w)}{\partial z} - \frac{\partial\text{ᚢ}(z, w)}{\partial w}\right) dzdw$$

• τ-trivial functions are those in which $\dfrac{\partial\text{ᚠ}(z, w)}{\partial z} = \dfrac{\partial\text{ᚢ}(z, w)}{\partial w}$, such that the τ-integral is zero.

τ-trivial functions led to an instantaneous understanding about the rotation the field f_τ enforces – the rotation is zero, therefore the angle ξ has an absence of deviation, then, τ-trivial functions represent the called irreversible static functions.

3.2 Definition of τ-analytic functions

Let $q = z + \tau w$; $f_\tau(q) = f_\tau(q(z,w))$, then:

$$\oint_{S_\tau} f_\tau(q)dq = \oint_{S_\tau} f_\tau(q(z,w))dq$$

Analyzing a specific case – when $w = 0 \Rightarrow q = z \ \in \mathbb{C}$ – we are led to the following corollary:

<u>**Corollary I**</u>: The case $w = 0$ implies $q = z$, then $f_\tau(q\,(z,w)) = f_\tau\,(q\,(z,0)) = f_\tau(q\,(z))$, however, $q = z$, therefore, $f_\tau(q\,(z,0)) \mapsto f(z)$ – implying the absence of the influence of S_τ. Such an absence implies a remarkable simplification to the standard τ-integral:

$$\oint_{S_\tau} f_\tau(q)dq \mapsto \oint_{\Gamma\in\mathbb{C}} f(z)dz$$

• Therefore, such an integral is on the complex domain – which is clearly reversible. This concludes that the complex space $\mathbb{C}$ is a proper subset of the τ-space T: Given $q \ \in \mathbb{T}$; $q = z + \tau w; w = 0$ implies $q = z$; z is a variable in the complex space.
Thus, $z \in \mathbb{C}$; $z = q(z, w = 0) \in \mathbb{T} \ \Rightarrow \mathbb{C} \ \subset \mathbb{T} \Rightarrow \mathbb{C}$ is a proper subset of $\mathbb{T}$.

Now, the context requires the introduction of the **reversibility operator** ($\hat{\text{ᛘ}}$): Such an operator takes a function in the complex manifold $f(z) : \mathbb{C} \to \mathbb{C}$ and maps the latter in the τ-manifold - $\hat{\text{ᛘ}} : \mathbb{C} \to \mathbb{T}$. Symmetrically, $\hat{\text{ᛘ}}^{-1}$ is defined as being a mapping of T onto C – $\hat{\text{ᛘ}}^{-1} : \mathbb{T} \to \mathbb{C}$.

The junction of $\hat{\text{ᛘ}}$ and $\hat{\text{ᛘ}}^{-1}$ forms the operator $\hat{\text{ᛗ}} = \hat{\text{ᛘ}}\hat{\text{ᛘ}}^{-1}$. Applying the new definitions to the latter analyzed case – $w = 0$: $\hat{\text{ᛘ}}\hat{\text{ᛘ}}^{-1} |f_\tau(q(z,0))\rangle = \hat{\text{ᛗ}}\,|f_\tau(q(z,0))\rangle = I\,|f_\tau(q(z,0)))\rangle\,; I$ is called the identity eigenvalue. In such a case the irreversible component of the τ-variable, w, is zero, so then, $f_\tau(q\,(z,0)) \mapsto f(z)$, and behaves statically with respect to dissipation of information.

For generalize the operator approach for an arbitrary well-behaved endofunction f_τ is crucial the addition of the variable w – which carries the irreversibility. Then, the influence of the reversibility operator under a potentially irreversible τ-function requires a scale of measurement of the dissipation – an eigenvalue labeled ю, such that:

$$\hat{\text{ᛗ}}\,|f_\tau(q(z,w))\rangle = \text{ю}\,|f_\tau(q(z,w))\rangle$$

Such a relation is well known in linear algebra – the eigenvalue relation. τ-functions that respect the eigenvalue relation (those which are eigenfunctions of ю) are said to be τ-analytic.

The reversibility operators can be recognized as homomorphisms betwixt $\mathbb{C}$ and $\mathbb{T}$; and this relation, assuming that $\hat{\Upsilon}$ consists in a mapping, is explored in *Theorem II*:

Theorem II (Structural Reversibility Condition): Let $\hat{\Upsilon} : \mathbb{C}^1 \to \mathbb{T}^1$ a mapping from $\mathbb{C}^1$ onto $\mathbb{T}^1$ of a bijective function $f_\tau(q); q \in \mathbb{T}$ - then $\hat{\Upsilon}$ is an anti-homomorphism from $\mathbb{C}$ to $\mathbb{T}$ and $\hat{\Upsilon}^{-1}$ is a homomorphism from $\mathbb{T}$ to $\mathbb{C}$, and, since the eigenvalue ю$= 1$, then $\mathbb{C}^1 \cong \mathbb{T}^1$

Remark: The notation $A \cong\ B$ for two sets denotes "A is isomorphic to B".

Proof. Let $f_\tau(q)$: $\mathbb{C}^1 \to \mathbb{C}^1; g_\tau$: $\mathbb{C}^1 \to \mathbb{C}^1$

$\Rightarrow\ \ \hat{\Upsilon}^{-1}|f_\tau(q(z,w)) = f(z); \hat{\Upsilon}^{-1}|g_\tau(q(z,w))\rangle = g(z) \Rightarrow \hat{Ϻ}\left|\begin{matrix} f_\tau \\ g_\tau \end{matrix}\right\rangle = ю\left|\begin{matrix} f_\tau \\ g_\tau \end{matrix}\right\rangle \Rightarrow =$I, otherwise:

$= \hat{\Upsilon}^{-1}$:$\mathrm{T}^1 \to \mathbb{C}^1 \to \mathbb{T}^{\blacksquare 1}$, i.e., $\hat{Ϻ} : \mathbb{T}^1 \to \mathbb{T}^{\blacksquare 1}$, unless ю $= 1$ is the unique case in which $\mathbb{T}^1 \cong \mathbb{T}^{\blacksquare 1}$, which proves the statement two.

- $\mathbb{T}^{\blacksquare 1}$ *is called the stable subset of* $\mathbb{T}^1$*, and such a relation shall be explained in the next section.* Since $\hat{\Upsilon}$ returns to a different set of the which $\hat{\Upsilon}^{-1}$ has departed; and as *Corollary I* affirms that $\mathbb{C}$ is a proper subset of $\mathbb{T}$, then:

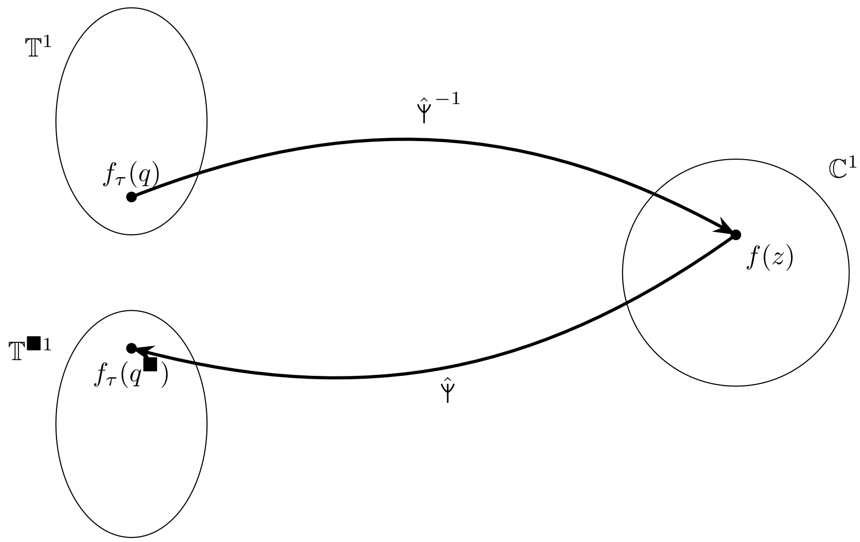
1. $\hat{\Upsilon}^{-1} : \mathbb{T}^1 \twoheadrightarrow \mathbb{C}^1$

2. $\hat{\Upsilon} : \mathbb{C}^1 \twoheadrightarrow \mathbb{T}^{\blacksquare 1}$

Which proves statement one, and completes the proof □

- Below is an schematic representation of a bijective function $f_\tau(q)$ that satisfies *Theorem II*:

$\hat{\Upsilon}^{-1}$



Remark: $f_\tau(q)$ is an endofunction in $\mathbb{T}^1$, while $f_\tau(q^{\blacksquare})$ is an endofunction in $\mathbb{T}^{\blacksquare 1}$.

Corollary II: For a well-behaved endofunction $f_\tau(q(z,w))$, its stable form, $f_\tau^{\blacksquare}(q(z,w))$, under a τ-integration is equivalent to the complex integration of $f_\tau^{\blacksquare}(q(z,0)) = f^{\blacksquare}(z)$:

$$\underset{S_\tau}{\oint\!\!\!\!\!\oplus} \ \hat{ᛗ}|f_\tau(q(z,w))\rangle \, dq = \underset{S_\tau}{\oint\!\!\!\!\!\oplus} \ f_\tau^{\blacksquare}(q(z,w))dq$$

By *Corollary I*:

$$\underset{S_\tau}{\oint\!\!\!\!\!\oplus} \ f_\tau^{\blacksquare}(q(z))dq \mapsto \oint_{\Gamma\in\mathbb{C}} f^{\blacksquare}(z)dz.$$

4 Degrees of Freedom of Irreversibility Multiple τ-Dimensions

A diversity of observable phenomena – most of them, have a complex system of dependencies of variables – and therefore a more entangled chain of irreversible factors; that when blended, transforms a predictable system into pseudo-chaotic or even chaotic systems.

The target of this section is to develop a higher comprehension of pseudo-chaotic systems, and how the latter can be fragmented into isolated and simpler systems to be analyzed separately.

Pseudo-chaotic systems appear chaotic only because **repeated anti-homomorphisms** distort a reversible structure.

4.1 Anti-homomorphism and the codification of irreversibility

Definition II: Let $\hat{\text{ᛉ}}$; $\hat{\text{ᛉ}}^{-1}$ represent an anti-homomorphism and homomorphism, respectively (as proved in *Theorem II*); $\hat{\text{ᛉ}} : \mathbb{C}^1 \twoheadrightarrow \mathbb{T}^1; \hat{\text{ᛉ}}^{-1} : \mathbb{T}^1 \twoheadrightarrow \mathbb{C}^1$; $\hat{\text{ᛗ}} = \hat{\text{ᛉ}}\hat{\text{ᛉ}}^{-1}$ and a τ-function $f_\tau(q) = \text{ᚢ}(z, w) + \tau\text{ᚠ}(z, w)$. To analyze the nth degree of freedom of irreversibility of $f_\tau(q)$ there is a necessity of extend the τ-space from $\mathbb{T}^1$ to $\mathbb{T}^n$. Then, each component of f_τ takes the following form:

$$\text{ᚢ} = \sum_{\sigma=1}^{n} \text{ɰ}_\sigma \left| e_\sigma \right\rangle$$

$$\text{ᚠ} = \sum_{j=1}^{n} \text{ɧ}_j \left| e_j \right\rangle$$

; the previously defined eigenvalue relation takes the pattern:

$$\begin{pmatrix} \hat{\text{ᛗ}}_1 & 0 & \cdots & 0 & 0 \\ 0 & \hat{\text{ᛗ}}_2 & \cdots & 0 & 0 \\ 0 & 0 & \hat{\text{ᛗ}}_3 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \hat{\text{ᛗ}}_n \end{pmatrix} \left| \begin{matrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_n \end{matrix} \right\rangle = \mathrm{IO} \left| \begin{matrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_n \end{matrix} \right\rangle$$

The corresponding definition leads us to understand why *Theorem II* limits the degree of freedom of $\mathbb{T}^n$ to $n = 1$. Concerning to $n > 1$ degrees of liberty, it is mandatory to analyze each eigenvalue component by its own properties regarding the space they are embedded, given respective automorphic operators $\hat{\text{ᛗ}}_n$ in $\mathbb{T}^n$.

4.2 τ-holomorphic functions and the Dispersion Derivative

The cartesian representation of a τ-function with n degrees of freedom is not trivial – in the sight that $3n$ dimensions would be necessary to graph the latter. For this subtopic of *Section 3*, we will follow strictly the derivation made by Cauchy of the conditions for a holomorphic function, however, extending it to τ-functions.

Aiming to keep the clearness and the geometric interpretation of τ-functions, the approach to be used will have much in common with the complex analysis:

• Let:

$$q^{\blacksquare} \in \mathbb{T}^n; \quad q^{\blacksquare} = \sum\nolimits_{m=1}^{n} \chi_m^{\blacksquare} \left|e_m\right\rangle + \sum_{l=1}^{n} \psi_l^{\blacksquare} \left|e_l\right\rangle$$

$$\Rightarrow \text{ᚠ}(\chi_m^{\blacksquare}, \psi_l^{\blacksquare}) = \sum_{\sigma=1}^{n} \text{ɰ}_\sigma(\chi_m^{\blacksquare}, \psi_l^{\blacksquare}) \left|e_\sigma\right\rangle$$

$$\Rightarrow \text{Ո}(\chi_m^{\blacksquare}, \psi_l^{\blacksquare}) = \sum_{j=1}^{n} \mathfrak{h}_j(\chi_m^{\blacksquare}, \psi_l^{\blacksquare}) \left|e_j\right\rangle$$

$$\partial \text{Ո} = \frac{\partial}{\partial \chi_m^{\blacksquare}} \sum_{j=1}^{n} \mathfrak{h}_j \left|e_j\right\rangle d\chi_m^{\blacksquare} + \frac{\partial}{\partial \psi_l^{\blacksquare}} \sum_{j=1}^{n} \mathfrak{h}_j \left|e_j\right\rangle d\psi_l^{\blacksquare}$$

$$= \sum_{j=1}^{n} \frac{\partial \mathfrak{h}_j}{\partial \chi_m^{\blacksquare}} \left|e_j\right\rangle d\chi_m^{\blacksquare} + \sum_{j=1}^{n} \frac{\partial \mathfrak{h}_j}{\partial \psi_l^{\blacksquare}} \left|e_j\right\rangle d\psi_l^{\blacksquare}$$

$$; \partial \text{ᚠ} = \frac{\partial}{\partial \chi_m^{\blacksquare}} \sum_{\sigma=1}^{n} \text{ɰ}_\sigma \left|e_\sigma\right\rangle d\chi_m^{\blacksquare} + \frac{\partial}{\partial \psi_l^{\blacksquare}} \sum_{\sigma=1}^{n} \text{ɰ}_\sigma \left|e_\sigma\right\rangle d\psi_l^{\blacksquare}$$

$$= \sum_{\sigma=1}^{n} \frac{\partial \text{ɰ}_\sigma}{\partial \chi_m^{\blacksquare}} \left|e_\sigma\right\rangle d\chi_m^{\blacksquare} + \sum_{\sigma=1}^{n} \frac{\partial \text{ɰ}_\sigma}{\partial \psi_l^{\blacksquare}} \left|e_\sigma\right\rangle d\psi_l^{\blacksquare}$$

$$\Longrightarrow \partial \text{Ո}_j = \begin{bmatrix} \dfrac{\partial \mathfrak{h}_j}{\partial \chi_m^{\blacksquare}} & \dfrac{\partial \mathfrak{h}_j}{\partial \psi_l^{\blacksquare}} \end{bmatrix} \times \begin{bmatrix} d\chi_m^{\blacksquare} \\ d\psi_l^{\blacksquare} \end{bmatrix}$$

$$\Longrightarrow \partial \text{ᚠ}_\sigma = \begin{bmatrix} \dfrac{\partial \text{ɰ}_\sigma}{\partial \chi_m^{\blacksquare}} & \dfrac{\partial \text{ɰ}_\sigma}{\partial \psi_l^{\blacksquare}} \end{bmatrix} \times \begin{bmatrix} d\chi_m^{\blacksquare} \\ d\psi_l^{\blacksquare} \end{bmatrix}$$

$$\therefore \begin{bmatrix} \partial \text{Ո}_j \\ \partial \text{ᚠ}_\sigma \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \mathfrak{h}_j}{\partial \chi_m^{\blacksquare}} & \dfrac{\partial \mathfrak{h}_j}{\partial \psi_l^{\blacksquare}} \\ \dfrac{\partial \text{ɰ}_\sigma}{\partial \chi_m^{\blacksquare}} & \dfrac{\partial \text{ɰ}_\sigma}{\partial \psi_l^{\blacksquare}} \end{bmatrix} \times \begin{bmatrix} d\chi_m^{\blacksquare} \\ d\psi_l^{\blacksquare} \end{bmatrix} = \text{Ӂ} \times \begin{bmatrix} d\chi_m^{\blacksquare} \\ d\psi_l^{\blacksquare} \end{bmatrix}$$

• Ӂ denotes the matrix of the derivatives of the components of $\partial \text{Ո}_j ; \partial \text{ᚠ}_\sigma$ – called the Jacobian Transition Matrix.

The conditions for derivable functions in the τ-space is provided by multiplying the τ-variable q_{ml} by its stable differential form $dq_{ml}^{\blacksquare}$;(the indexes $m; l$ indicate the mth complex component and the lth pure τ-component):

Let $q_{ml} = \chi_m \left|e_m\right\rangle + \psi_l \left|e_l\right\rangle ; dq_{ml}^{\blacksquare} = d\chi_m^{\blacksquare} \left|e_m\right\rangle + d\psi_l^{\blacksquare} \left|e_l\right\rangle$

$$\Rightarrow q \; dq^{\blacksquare} = (\chi_m \left|e_m\right\rangle + \psi_l \left|e_l\right\rangle)(d\chi_m^{\blacksquare} \left|e_m\right\rangle + d\psi_l^{\blacksquare} \left|e_l\right\rangle)$$

$$= \chi_m \left|e_m\right\rangle d\chi_m^{\blacksquare} \left|e_m\right\rangle + \chi_m \left|e_m\right\rangle d\psi_l^{\blacksquare} \left|e_l\right\rangle + \psi_l \left|e_l\right\rangle d\chi_m^{\blacksquare} \left|e_m\right\rangle + \psi_l \left|e_l\right\rangle d\psi_l^{\blacksquare} \left|e_l\right\rangle$$

$$= \psi_l \left|e_l\right\rangle d\psi_l^{\blacksquare} \left|e_l\right\rangle + \chi_m \left|e_m\right\rangle d\chi_m^{\blacksquare} \left|e_m\right\rangle^2 + \left(\chi_m d\psi_l^{\blacksquare} \left|e_l\right\rangle + \psi_l d\chi_m^{\blacksquare} \left|e_l\right\rangle\right) \left|e_m\right\rangle$$

• **<u>Remark</u>**: $\left|e_m\right\rangle^2 \equiv \tau^2$, which is remained as τ^2, but treated as an outside element of the τ-vector, as demonstrates the organization of the elements above.

$$\Longrightarrow q \ dq^{\blacksquare} = \begin{bmatrix} \psi_l & \tau^2 \chi_m \\ \chi_m & \psi_l \end{bmatrix} \times \begin{bmatrix} d\chi_m^{\blacksquare} \\ d\psi_l^{\blacksquare} \end{bmatrix}$$

$$\therefore \begin{cases} \begin{bmatrix} \partial \text{ᑎ}_j \\ \partial \text{ᚠ}_\sigma \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \mathfrak{h}_j}{\partial \chi_m^{\blacksquare}} & \dfrac{\partial \mathfrak{h}_j}{\partial \psi_l^{\blacksquare}} \\ \dfrac{\partial \text{ɰ}_\sigma}{\partial \chi_m^{\blacksquare}} & \dfrac{\partial \text{ɰ}_\sigma}{\partial \psi_l^{\blacksquare}} \end{bmatrix} \times \begin{bmatrix} d\chi_m^{\blacksquare} \\ d\psi_l^{\blacksquare} \end{bmatrix} \\ q \ dq^{\blacksquare} = \begin{bmatrix} \psi_l & \tau^2 \chi_m \\ \chi_m & \psi_l \end{bmatrix} \times \begin{bmatrix} d\chi_m^{\blacksquare} \\ d\psi_l^{\blacksquare} \end{bmatrix} \end{cases}$$

<u>Definition III</u>: $f_\tau(q)$ is holomorphic **iff** $q \ dq^{\blacksquare} \equiv \begin{bmatrix} d\chi_m^{\blacksquare} \\ d\psi_l^{\blacksquare} \end{bmatrix}$

That's clear, because the matrix of the derivatives of f_τ must be X̆K, then:

$$q \ dq^{\blacksquare} \equiv \begin{bmatrix} d\chi_m^{\blacksquare} \\ d\psi_l^{\blacksquare} \end{bmatrix} \Longleftrightarrow$$

$$\begin{bmatrix} \psi_l & \tau^2 \chi_m \\ \chi_m & \psi_l \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \mathfrak{h}_j}{\partial \chi_m^{\blacksquare}} & \dfrac{\partial \mathfrak{h}_j}{\partial \psi_l^{\blacksquare}} \\ \dfrac{\partial \text{ɰ}_\sigma}{\partial \chi_m^{\blacksquare}} & \dfrac{\partial \text{ɰ}_\sigma}{\partial \psi_l^{\blacksquare}} \end{bmatrix}$$

$$\Rightarrow \psi_l = \frac{\partial \mathfrak{h}_j}{\partial \chi_m^{\blacksquare}}$$

$$\Rightarrow \tau^2 \chi_m = \frac{\partial \mathfrak{h}_j}{\partial \psi_l^{\blacksquare}} \therefore \chi_m = \tau^2 \frac{\partial \mathfrak{h}_j}{\partial \psi_l^{\blacksquare}}$$

$$\Rightarrow \psi_l = \frac{\partial \text{ɰ}_\sigma}{\partial \psi_l^{\blacksquare}}$$

$$\Rightarrow \chi_m = \frac{\partial \text{ɰ}_\sigma}{\partial \chi_m^{\blacksquare}}$$

$$\Rightarrow \frac{\partial \mathfrak{h}_j}{\partial \chi_m^{\blacksquare}} = \frac{\partial \text{ɰ}_\sigma}{\partial \psi_l^{\blacksquare}}$$

$$\Rightarrow \tau^2 \frac{\partial \text{ɧ}_j}{\partial \psi_l^{\blacksquare}} = \frac{\partial \text{ɰ}_\sigma}{\partial \chi_m^{\blacksquare}}$$

- Here, one can easily assimilate the conditions for holomorphic τ-functions with the τ-trivial functions' condition, and in fact, both represent the same assumption - under different approaches - regarding a generalized form of the curl of a τ-endofunction f_τ in the domain $\mathbb{T}^n$.

- Therefore, conceptually, the holomorphy of a τ-function says about its conformity regarding a homomorphic mapping of $\mathbb{T}^n$ onto itself, i.e, **the local preservation of the scape-angle** ξ - that implies the local preservation of the singularity (*see Section 2 for definitions regarding the relation betwixt the scape-angle and singularities*);

→ The preservation of the singularity, by definition, implies the preservation of the eigenvalue in the application of $\hat{\text{ᛗ}}$, which means that the τ-eigenfunction is **angularly invariant** under transformations;

* *farther, such interpretation is going to be crucial for simplifying problems regarding tensor functionals, in Section 7.*

4.3 Application: Normalization of f_τ

• The τ-function is a remarkable artifice to decode dispersion of information – highlighted when the latter is in $\mathbb{T}^n$, then, the dispersion is broken-out in an nth dimension vector field. However, questions such as if any τ-function is allowed, and if which of the latter have a real meaning need to be retorted:

- In the following derivation, notations such as $\xi_\alpha; \xi_\beta$ will represent two different scape-angles from the τ-sphere; $\text{ᚺ}(q)$ used to denote a well-behaved function in the τ-sphere – which is τ-integrable; τ-analytic and endomorphic. $\hat{\text{ᛗ}}_\alpha; \hat{\text{ᛗ}}_\beta$ are going to be used to represent transformations of the τ-functions depending on $\alpha; \beta$, respectively; $\text{ю}_\alpha; \text{ю}_\beta$ the eigenvalues.

• Given two τ-functions $\text{ᚺ}_\alpha(q); \text{ᚺ}_\beta(q)$ in hyperbolic coordinates - $\text{ᚺ}(\text{Я}, \xi_\alpha); \text{ᚺ}(\text{Я}, \xi_\beta)$, their inner product is defined as follows:

$$\langle \text{ю}_\alpha \text{ᚺ}(\text{Я}, \xi_\alpha) | \text{ю}_\beta \text{ᚺ}(\text{Я}, \xi_\beta) \rangle = \text{ю}_\alpha \text{ю}_\beta \delta_{\alpha\beta}$$

; $\delta_{\alpha\beta}$ is the Kronecker Delta, defined as follows: $\delta_{\alpha\beta} = \begin{cases} 1, \alpha = \beta \\ 0, \alpha \neq \beta \end{cases}$

Therefore, the inner product is just different of zero when the angles are equal. Then:

$$\langle ю_\alpha \text{ᚺ}(\text{Я},\xi_\alpha)|ю_\beta \text{ᚺ}(\text{Я},\xi_\beta)\rangle = \oiint_{\mathbb{T}^2} \overline{(ю_\alpha \text{ᚺ}(\text{Я},\xi_\alpha)}(ю_\beta \text{ᚺ}(\text{Я},\xi_\beta))d\text{Я}d\xi = ю_\alpha ю_\beta \delta_{\alpha\beta}$$

$$\Rightarrow \oiint_{\mathbb{T}^2} \overline{(ю_\alpha \text{ᚺ}(\text{Я},\xi_\alpha)}(ю_\beta \text{ᚺ}(\text{Я},\xi_\beta))d\text{Я}d\xi =_{eigenvalue} \oiint_{\mathbb{T}^2} \overline{(\hat{\text{ᛗ}}_\alpha \text{ᚺ}(\text{Я},\xi_\alpha)}(\hat{\text{ᛗ}}_\beta \text{ᚺ}(\text{Я},\xi_\beta))d\text{Я}d\xi$$

<u>Theorem III</u>: If $\text{ᚺ}^{\blacksquare}_\alpha$ and $\text{ᚺ}^{\blacksquare}_\beta$ are the stable form of ᚺ_α and ᚺ_β, respectively, $\hat{\text{ᛘ}}^{-1}$ and $\hat{\text{ᛘ}}$ are homomorphisms and anti-homomorphisms, respectively ($\hat{\text{ᛘ}}^{-1}_\alpha; \hat{\text{ᛘ}}^{-1}_\beta : \mathbb{T}^n \twoheadrightarrow \mathbb{C}^n; \hat{\text{ᛘ}}_\alpha, \hat{\text{ᛘ}}_\beta : \mathbb{C}^n \twoheadrightarrow \mathbb{T}^n$) – (as proved in *Theorem II*), then the stable function of the product of ᚺ_α and ᚺ_β is also stable.

Proof. If $\hat{\text{ᛘ}}^{-1}$ and $\hat{\text{ᛘ}}$ are homomorphisms and anti-homomorphisms, respectively, then:

$$\hat{\text{ᛘ}}^{-1}|\varsigma\Phi\rangle = \hat{\text{ᛘ}}^{-1}|\varsigma\rangle\,\hat{\text{ᛘ}}^{-1}|\Phi\rangle$$
$$;\hat{\text{ᛘ}}|\varsigma\Phi\rangle = \hat{\text{ᛘ}}|\Phi\rangle\,\hat{\text{ᛘ}}|\varsigma\rangle$$

For any two τ-functions $\varsigma;\Phi \in \mathbb{T}^n$.

$$\Rightarrow \hat{\text{ᛘ}}\hat{\text{ᛘ}}^{-1}|\varsigma\Phi\rangle = \hat{\text{ᛘ}}\left|\hat{\text{ᛘ}}^{-1}|\varsigma\rangle\,\hat{\text{ᛘ}}^{-1}|\Phi\right. = \hat{\text{ᛘ}}\hat{\text{ᛘ}}^{-1}|\Phi\rangle\,\hat{\text{ᛘ}}\hat{\text{ᛘ}}^{-1}|\varsigma\rangle$$

$$\Rightarrow \hat{\text{ᛗ}}|\varsigma\Phi\rangle = \hat{\text{ᛗ}}|\Phi\rangle\,\hat{\text{ᛗ}}|\varsigma\rangle$$

- Applying $\text{ᚺ}_\alpha;\text{ᚺ}_\beta$:

$\hat{\text{ᛗ}}\left|\text{ᚺ}_\alpha\text{ᚺ}_\beta\right\rangle = \hat{\text{ᛗ}}\left|\text{ᚺ}_\beta\right\rangle\hat{\text{ᛗ}}\left|\text{ᚺ}_\alpha\right\rangle = \text{ᚺ}^{\blacksquare}_\alpha\text{ᚺ}^{\blacksquare}_\beta$, which is stable.

This completes our proof, and provides the necessary information to proceed with the main derivation □

- Applying *Theorem III* to the double integral:

$$\oiint_{\mathbb{T}^2} \overline{(\hat{\text{ᛗ}}_\alpha \text{ᚺ}(\text{Я},\xi_\alpha)}(\hat{\text{ᛗ}}_\beta \text{ᚺ}(\text{Я},\xi_\beta))d\text{Я}d\xi = \oiint_{\mathbb{T}^2} \overline{(\text{ᚺ}^{\blacksquare}(\text{Я},\xi_\beta)}(\text{ᚺ}^{\blacksquare}(\text{Я},\xi_\alpha))d\text{Я}d\xi$$

- By *Corollary I; Corollary II*:

$$\oiint_{\mathbb{T}^2} \overline{(\text{ᚺ}^{\blacksquare}(\text{Я},\xi_\beta)}(\text{ᚺ}^{\blacksquare}(\text{Я},\xi_\alpha))d\text{Я}d\xi = \oiint_{\Gamma\in\mathbb{C}^2} \overline{(\text{ᚺ}^{\blacksquare}(\text{Я},\xi_\beta)}(\text{ᚺ}^{\blacksquare}(\text{Я},\xi_\alpha))d\text{Я}d\xi = ю_\alpha ю_\beta \delta_{\alpha\beta}$$

- And for $\beta = \alpha$:

$$\oiint_{\Gamma\in\mathbb{C}^2} \overline{(\text{Ⱨ}^{\blacksquare}(\text{Я}, \xi_\beta)}(\text{Ⱨ}^{\blacksquare}(\text{Я}, \xi_\alpha))d\text{Я}d\xi = \oiint_{\Gamma\in\mathbb{C}^2} \mid \text{Ⱨ}^{\blacksquare}(\text{Я}, \xi_\alpha) \mid^2 d\text{Я}d\xi = \text{ю}_\alpha^2$$

- The index α can be removed without any loss of generality:

$$\therefore \oiint_{\Gamma\in\mathbb{C}^2} \mid \text{Ⱨ}^{\blacksquare}(\text{Я}, \xi) \mid^2 d\text{Я}d\xi = \text{ю}^2$$

5 The Foundations of the τ-Line Integral Algorithm

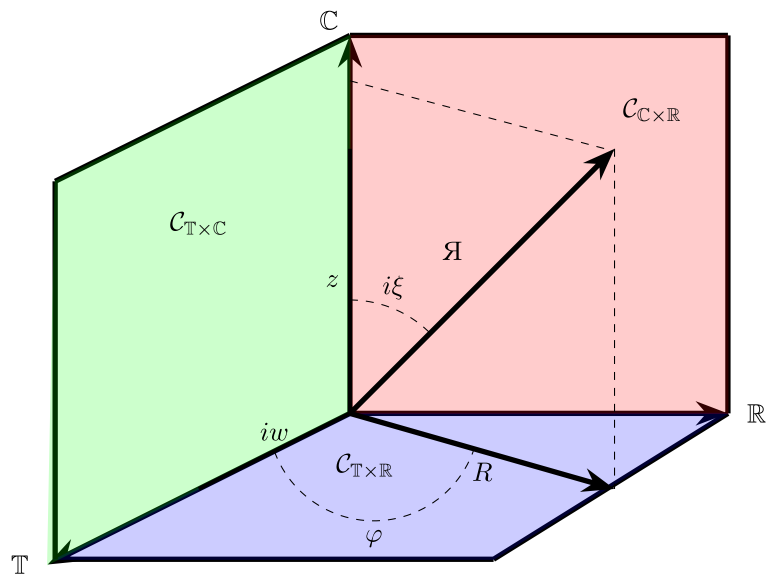
In this section we are going to derive the result that intersects the intuition with the mathematical formalism of the geometry of S_τ.

For such a realization, the formal treatment of the already introduced τ-integral is required – this section will provide enough formalization of the latter until it reaches the domain of Complex Analysis, where the concepts become auto explainable – based on the solid mathematical axioms that already exist.

- **Remark**: For this section, the brief simplification of the degrees of freedom of irreversibility will be adopted $(\mathbb{T}^1; \mathbb{C}^1)$ – and farther transcribed to tensor formalism in $(\mathbb{T}^n; \mathbb{C}^n)$.

5.1 Spherical Hyperbolic Coordinates:

The τ-integral, conceptually, is a field over S_τ; S_τ is regarded as the path in which the field is acting on. The following representation indicates the τ-sphere in hyperbolic coordinates, indicating the $\mathbb{T}; \mathbb{C}; \mathbb{R}$ axes:



- For a τ-variable $q \in \mathbb{T}^1;\ q = z + \tau w$, the transformation of coordinates takes the form:

$$\begin{cases} z = \Re \cosh \xi \\ w = \Re \sinh \xi \end{cases}$$

- $\mid q \mid = \sqrt{z^2 - w^2} = \Re$

- R is the projection of $\Re$ onto the Argand-Gauss plane – the radius of the complex numbers in its polar-form; associated with the angle φ, likewise in Complex Analysis.

- S_τ can be partitioned into three inner circumferences to be analyzed: $S_\tau = \{\{\mathcal{C}_{\mathbb{T}\times\mathbb{C}}\}, \{\mathcal{C}_{\mathbb{T}\times\mathbb{R}}\}, \{\mathcal{C}_{\mathbb{C}\times\mathbb{R}}\}\}$; a remarkable circumference betwixt the partitions of S_τ is $\mathcal{C}_{\mathbb{C}\times\mathbb{R}}$ - being conceptually equivalent to the Argand-Gauss plane.

Therefore, it is clear the necessity to find the Jacobian of such a transformation of variables:

- The Jacobian Matrix for one irreversible dimension takes the form:

$$J(\Re, \xi) = \begin{bmatrix} \dfrac{\partial z}{\partial \Re} & \dfrac{\partial w}{\partial \Re} \\ \dfrac{\partial z}{\partial \xi} & \dfrac{\partial w}{\partial \xi} \end{bmatrix}$$

- It follows that:

$$J(\mathfrak{K}, \xi) = \begin{bmatrix} \cosh \xi & \sinh \xi \\ \mathfrak{K} \sinh \xi & \mathfrak{K} \cosh \xi \end{bmatrix}$$

$$\Rightarrow \det J(\mathfrak{K}, \xi) = \begin{vmatrix} \cosh \xi & \sinh \xi \\ \mathfrak{K} \sinh \xi & \mathfrak{K} \cosh \xi \end{vmatrix} = \mathfrak{K}^2$$

Historical Context I: To compute the area related to a period of a specific partition of S_τ – for purposes of exemplification the period $z = [0, 1]; w = [0, i]$ of $\mathcal{C}_{\mathbb{T}\times\mathbb{C}}$, forming a triangle, would be methodologic to:

1. Calculate the area of the triangle, which would result in a null area.
2. Use **standard integration**:

Let ε be a variable in the τ-space; assume the standard integration applies to it:

$$\int_0^1 f(\varepsilon) d\varepsilon = i \int_0^1 \varepsilon d\varepsilon = i \frac{\varepsilon^2}{2} \bigg|_0^1 = \frac{i}{2}$$

- However, the area of the triangle calculated by $base * height$ and by using integration doesn't match, therefore, what was concluded is that neither of the algorithms for calculating such an area were valid – then – a new algorithm shall be developed, the called (and already commented) τ-integral. At the beginning of the development of novel algorithms for calculating areas inside the τ-sphere, option (1) or (2) were thoughtful to be correct, which wasn't proved to be true.

In the following sections and at the end of the paper the Generalized Residue Theorem is going to be derived, and the algorithm for calculating τ-integrals shall become clear.

5.2 The τ-integral algorithm and the extension for a desirable space $\{S\}$

The algorithm for τ-integration is not closed – it depends strictly of the **space** where it is being realized. A brief introduction to τ-kind spaces will be provided in the following section, however, to find a most complete theoretical derivation, the reader has to guide himself to *Section 7*.

- **<u>Definition IV</u>**: A space $\{S\}$ with an attached variable s - which ranges over it, for τ-algebra is the path in which a function $f(\kappa)$ in a generic space $\{\mathcal{G}\}$ acts as a field. For all considered spaces $\{S\}$ there must be an operation that matches $f(\kappa)$ with the space desirable variable – and such an operation is the inner product.

The inner product betwixt the attached variable s ranging over $\{S\}$, and the function $f(\kappa)$, describes (is equal to) a well-defined function inside the desirable space $\{S\}$ – $f(s)$.

$$f(s) = \langle s|f(\kappa)\rangle$$

- **<u>Theorem IV</u>**: The function in a desired space $\{S\}$ is obtained from the inner product betwixt a generic invariant function $f(\kappa)$ in $\{\mathcal{G}\}$ with the attached variable $s \in S$.

  Proof. Let the reversibility operators $\hat{\Upsilon}_s^{-1}; \hat{\Upsilon}_s$ be defined as linear maps from the generic space onto $\{S\}$ and the inverse, respectively. $\hat{\Upsilon}_s^{-1}$: $\{\mathcal{G}\} \twoheadrightarrow \{S\}; \hat{\Upsilon}_s : \{S\} \twoheadrightarrow \{\mathcal{G}\}$:

$$\hat{\Upsilon}_s^{-1} |f(\kappa)\rangle = f(s)$$
$$; \hat{\Upsilon}_s |f(s)\rangle = f(\kappa)$$
$$\Rightarrow \hat{\Upsilon}_s^{-1} |f(\kappa)\rangle = \langle s|f(\kappa)\rangle \Rightarrow \hat{\Upsilon}_s \hat{\Upsilon}_s^{-1} |f(\kappa)\rangle = \hat{\Upsilon}_s |\langle s|f(\kappa)\rangle\rangle$$

  - Applying *Corollary II* for the space $\{S\}$ (unknown if reversible or not):

$$\Rightarrow \hat{\Upsilon}_s \hat{\Upsilon}_s^{-1} |f(\kappa)\rangle = \hat{\Upsilon}_s |\langle s|f(\kappa)\rangle\rangle = \left| f^{\blacksquare}(\kappa) \right\rangle$$

  - Therefore, if $\left| f^{\blacksquare}(\kappa) \right\rangle$ is a stable function (reversible) in the generic space, then, $\hat{\Upsilon}_s |\langle s|f(\kappa)\rangle\rangle$ is stable too, and therefore reversible.

  But if $\hat{\Upsilon}_s |\langle s|f(\kappa)\rangle\rangle$ is stable, and is defined in the generic space, then, $\langle s|f(\kappa)\rangle$ must be a function (whether reversible or not) inside $\{S\}\blacksquare$ □

<u>Remark</u>: If such a function in the space $\{S\}$ is reversible, then, its irreversible variable (similar to w, in τ-functions) is null, and therefore, by *Theorem II*, $\{S\} \cong \{\mathcal{G}\}$.

The information obtained in *Definition IV* and *Theorem IV* are enough to obtain a concise algorithm for calculating *some integrals in a desirable space $\{S\}$ – **under conditions**:

1. $\forall \kappa \in \{\mathcal{G}\}, \kappa_{cd}$ is the tensor generic variable, which has two elements $|e_c\rangle\,; |e_d\rangle \in \{S\}; |e_d\rangle \in \{\mathcal{G}\}$.

   Analogous in $\mathbb{T}$ (as previously seen): $\forall q \in \mathbb{T}^n, q_{ml}$ is the tensorial τ-variable, which has two elements $|e_m\rangle\,; |e_l\rangle\,; |e_m\rangle \in \mathbb{C}^n; |e_l\rangle \in \mathbb{T}^n$.

2. $\{S\}$ is a proper subset of $\{\mathcal{G}\}$

   Analogous in $\mathbb{T}$ (as previously seen): By *Corollary I* – $\mathbb{C}^n$ is a proper subset of $\mathbb{T}^n$

- Given $\kappa_{cd} \in \{\mathcal{G}\}; \kappa_{cd} = \rho_c\,|e_c\rangle + \lambda_d\,|e_d\rangle$ for any two elements ρ, λ belonging to the proper subset of $\{\mathcal{G}\}$– i.e., $\rho, \lambda \in \{S\}; |e_c\rangle \in \{S\}; |e_d\rangle \in \{\mathcal{G}\}$:

$$\int_{\{S\}} \hat{\Upsilon}_s\,|\langle s|f(\kappa)\rangle\rangle\,ds = \int_{\{S\}} f^{\blacksquare}(\kappa)d\kappa = \int_{\{S\}} f^{\blacksquare}(\kappa(\rho,\lambda))d\kappa$$

- By *Corollary I*:

$$\int_{\{S\}} f^{\blacksquare}(\kappa(\rho,\lambda))d\kappa = \int_{\{S\}} f^{\blacksquare}(\kappa(\rho,\lambda=0))d\kappa = \int_{\{S\}} f^{\blacksquare}(\kappa(\rho))d\kappa = \int_{\{S\}} f^{\blacksquare}(\rho)d\rho$$

5.3 The generalized $\hat{\Upsilon}$ transformation

In the last topics were widely discussed the algebraic properties of the reversibility operator, however, the context requires a most comprehensive definition of the latter.

- **The unique conditions for a reversibility operator to exist are:**

1. Unicity – the reversibility operator must be uniquely defined in a generic space $\{\mathcal{G}\}$ and in its proper subgroup $\{S\}$;
2. The reversibility operator $\hat{\Upsilon}^{-1}$ is a homomorphism, whereas $\hat{\Upsilon}$ is an anti-homomorphism ($\hat{\Upsilon}^{-1}$ maps from the group to its subgroup, whereas $\hat{\Upsilon}$ maps from the subgroup onto the group);
3. Must attend the eigenvalue relation.

However, that are significantly many algebraic operations and operators which do satisfy such conditions. Here below are listed some of them:

→ $\hat{\Upsilon}$ as an integral transformation – similar to Fourier's or Laplace's approaches. Such approach will be treated in more detail in the next section, where the τ-integral transform will be introduced.

→ $\hat{\Upsilon}$ as a multiplication by the Jacobian Transition Matrix – Ӂ – representing a transition inside the τ-sphere domain.

5.4 Application: The algorithm over the Laplace Space $\{\mathcal{L}\}$

In this application, $\hat{Ƴ}$ will be used as an integral transformation, however, an integral transformation inside the Laplace Space – which is commonly recognized as the Laplace Transform.

For a given space $\{S\}=\{\mathcal{L}\};\ y \in \{\mathcal{L}\}; \{\mathcal{L}\}$ is called Laplace Space; the derived algorithm to generic spaces takes the form:

- Let $x \in \mathbb{R};\ y \in \{\mathcal{L}\}$:
  - By condition 1 of generic spaces - $\{\mathcal{L}\}$ is a proper subgroup of $\mathbb{R}, \Rightarrow \{\mathcal{L}\} \subset \mathbb{R}$.
  - It's known that a Laplace-transformed variable, y, does not assume complex values; the Laplace-transform is reversible for well-behaved functions, so, $\mathbb{R}$ has similar elements to $\{\mathcal{L}\}$. Therefore, by *Theorem II* – $\{\mathcal{L}\} \cong \mathbb{R}$.
- Therefore, it's affirmable that $\forall y \in \{\mathcal{L}\},\ y = y_0$ – indicating that y has no other elements (such as complex or irreversible – as discussed above).

$$\int_{\{\mathcal{L}\}} \hat{Ƴ}_y \left|\langle y|f(x)\rangle\right\rangle dy = \int_{\{\mathcal{L}\}} f^{\blacksquare}(y_0)dy_0$$

- If the Laplace-transform is reversible for well-behaved functions, so, $f^{\blacksquare}(y_0) = f(y_0)$, since the function is already in its stable form.
- Then, we get:

$$\int_{\{\mathcal{L}\}} \hat{Ƴ}_y \left|\langle y|f(x)\rangle\right\rangle dy = \int_{\{\mathcal{L}\}} f(y_0)dy_0.$$

6 Dirac-Basis functions and Kernels of the Generalized Transformations

In the following section, using the concepts carefully derived in the latest sections, I shall initially derive a novel way to visualize functions, using artifices provided by distributional calculus, and Bromwich Integral.

Later in this section, once derived the Dirac-Basis for such functions, the target will consist in a generalization regarding the reversibility operator in its integral form for a generic endofunction $f(\kappa)$, as well as the respective kernels for the transformation.

6.1 Dirac-Basis functions

Remark: The Laplace Transform is going to be regarded as an operator $\hat{\mathcal{L}}$

Let $\widetilde{f}(y)$ be a Laplace transform of the function $f(x)$, such that $\widetilde{f}(y) = \hat{\mathcal{L}} \left| f(x) \right\rangle$

Nevertheless, in this section we are worried about the anti-Laplace transform – the reverse path:

$$f(x) = \mathcal{L}^{-1} \left| \widetilde{f}(y) \right\rangle = \frac{1}{2i\pi} \oint_{\widetilde{\Delta} \in \mathbb{C}^1} \widetilde{f}(z) e^{zx} dz$$

- By the definition of the complex integral:

$$\frac{1}{2i\pi} \oint_{\widetilde{\Delta} \in \mathbb{C}^1} \widetilde{f}(z) e^{zx} dz = \frac{1}{2i\pi} \int_{-i\infty+\gamma}^{i\infty+\gamma} \widetilde{f}(y) e^{yx} dy$$

- An important assumption is to set $\gamma = 0$ – that is, assuming that the complex poles we lead with are purely imaginary; such assumption will be explained in the next corollary.

$$\Rightarrow \frac{1}{2i\pi} \int_{-i\infty+\gamma}^{i\infty+\gamma} \widetilde{f}(y) e^{yx} dy \mapsto \frac{1}{2i\pi} \int_{-i\infty}^{i\infty} \widetilde{f}(y) e^{yx} dy$$

- setting $u = \dfrac{y}{i}$:

$$\frac{1}{2i\pi} \int_{-i\infty}^{i\infty} \widetilde{f}(y) e^{yx} dy \mapsto \frac{1}{2\pi} \int_{-\infty}^{\infty} \widetilde{f}(iu) e^{iux} du$$

- At this moment, an important condition must be imposed: $\widetilde{f}\,(iu)$ allows Lorentz Series – that is because I shall use such function in its Taylor Series centered at $iu = p$, from now on:

$$\widetilde{f}(iu) = \sum_{r=-\infty}^{\infty} \widetilde{a}_r (iu - \Pi)^{g_r}$$

- $\widetilde{a}_r$ denotes the coefficient of the Laplace-transformed function - $\widetilde{f}\,(iu)$;
- g_r denotes the coefficient of the rth power related to the variable u;
- $\Pi = ip, \quad \forall p \in \mathcal{C}_{\mathbb{T}\times\mathbb{R}}$ is an imaginary constant – contained in the sub circumference $\mathcal{C}_{\mathbb{T}\times\mathbb{C}}$
  p is delimited in the domain $\mathcal{C}_{\mathbb{T}\times\mathbb{C}}$ due to integral convergence constraints.

- We do our last exchange of variables: $iu - \Pi = iv$

$$\Rightarrow \frac{e^{\Pi x}}{2\pi}\int_{-\infty}^{\infty}\sum_{r=-\infty}^{\infty}\widetilde{a}_r(iv)^{g_r}e^{ivx}dv$$

- By the following identity – where $\delta^{(n)}(M)$ consists of the nth derivative of the Dirac-delta function:

$$\int_{-\infty}^{\infty}\frac{\partial^m}{\partial M^m}e^{iMt}dt = \int_{-\infty}^{\infty}(it)^m e^{iMt}dt = 2\pi\delta^{(m)}(M)$$

– In which $M; m$ are real constants - $M, m \in \mathbb{R}$. Then:

$$\frac{e^{\Pi x}}{2\pi}\int_{-\infty}^{\infty}\sum_{r=-\infty}^{\infty}\widetilde{a}_r(iv)^{g_r}e^{ivx}dv = e^{\Pi x}\sum_{r=-\infty}^{\infty}\widetilde{a}_r\delta^{(g_r)}(x)$$

$$\therefore \underbrace{f}(x) = e^{\Pi x}\sum_{r=-\infty}^{\infty}\widetilde{a}_r\delta^{(g_r)}(x)$$

Remark: The notation for the distributional form of f(x) is denoted by $\underbrace{f}(x)$.

Is remarkable that such a sum has an outside term, in which is a complex exponential – such an exponential will be explored in the next section and be used to derive the Kernel of the τ-integral transform; used to derive the Generalized Residue Theorem for τ-functions.

The referred $\underbrace{f}(x)$ does not consist of the same obtained by the standard Laplace anti-transform – due to its distributional nature. The currently obtained distributional functions – which we are going to call Dirac-Basis functions, exist specifically to solve operations – inasmuch that the latter does not output the same values of the original function obtained by the Laplace anti-transform.

The imaginary constant Π obtained in the process, centralizes the complex function $\widetilde{f}(iu)$ in the point $z = \Pi$ in the sub circumference $\mathcal{C}_{\mathbb{T}\times\mathbb{C}}$. Π can be thought of as a **singularity of the function** $\widetilde{f}(iu)$.

- **Corollary III**: As already proved, $\widetilde{f}(iu)$ is centered at the point p.

In the case $\gamma \neq 0$, the boundaries of integration wouldn't be purely imaginary, then, $\widetilde{f}(iu)$ might be centered at $\Pi + \gamma$; however Π, as discussed, denotes the singularity of the function $\widetilde{f}(iu)$, and when $\gamma \neq 0$, the singularity is displaced γ units, and Π losses its role in denoting a singularity.

6.2 Generalized Integral Transformations

Remark I:In this subsection the operator $\hat{\Upsilon}$ is going to be used to represent an integral transform, respecting the same axioms provided by *Section 4*.
Remark II: The tilde symbol over a function - $\widetilde{f}(s)$ from now on indicates the function in the $\{S\}$ domain, with attached variable s.

When referring to an integral transformation, there are intrinsic imposed conditions that must be remarkably observed:

1. An Integral transformation $\hat{\Upsilon}_s^{-1}$ maps a function $f(\kappa)$ in a generic space $\{\mathcal{G}\}$ to a function $f(s)$ in a subspace $\{S\}$; and $\hat{\Upsilon}_s$ the analogous in the inverse direction.
2. Such a transformation consists in a homomorphism/anti-homomorphism, therefore, it has a kernel K, which consists in elements mapped from $\{\mathcal{G}\}$ onto an identity element in $\{S\}$ – $K = \kappa \in \{\mathcal{G}\} | \hat{\Upsilon}_s^{-1} |f(\kappa)\rangle = e_s$, in which e_s is an identity element in the subspace $\{S\}$.

- Therefore, let $K(\kappa); \widetilde{K}(s)$ be the kernels in $\{\mathcal{G}\}$ and in $\{S\}$, respectively, then the integral transformations $\hat{\Upsilon}_s^{-1} : \{\mathcal{G}\} \underset{K(\kappa)}{\twoheadrightarrow} \{S\}; \hat{\Upsilon}_s : \{S\} \underset{\widetilde{K}(s)}{\twoheadrightarrow} \{\mathcal{G}\}$, take the form:

$$\hat{\Upsilon}_s^{-1} |f(\kappa)\rangle = \sqrt{J} \int_{\{S\}} f(\kappa)K(\kappa)d\kappa = \widetilde{f}(s)$$

$$; \hat{\Upsilon}_s \left|\widetilde{f}(s)\right\rangle = \sqrt{J} \int_{\{\mathcal{G}\}} f(s)\widetilde{K}(s)ds = f(\kappa)$$

Remark: The integral that maps $f(\kappa) \mapsto \widetilde{f}(s)$

- **Corollary IV**: The integral-transformation mapping $f(\kappa) \mapsto \widetilde{f}(s)$ and the inverse transformation $\widetilde{f}(s) \mapsto f(\kappa)$ are evaluated over opposite ordered domains.
  Such fact is due to the selection of properties in which are being displaced from the generic space $\{\mathcal{G}\}$ and its proper subset $\{S\}$; and vice-versa.
- **Theorem V - (The Inversion Interpretation)**: A given **automorphic** transformation $\hat{\Upsilon}_s$ implies:

$$\hat{\Upsilon}_s^{-1} |f(\kappa)\rangle = \sqrt{J} \int_{\{S\}} f(\kappa)K(\kappa)d\kappa = \widetilde{f}(s) = f(\kappa)$$

;

$$\hat{\Upsilon}_s \left| \widetilde{f}(s) \right\rangle = \sqrt{J} \int_{\{\mathcal{G}\}} f(s) \widetilde{K}(s) ds = f(\kappa) = \widetilde{f}(s).$$

;

then, the transformations $\hat{\Upsilon}_s^{-1}; \hat{\Upsilon}_s$, regarding the eigenvalue relation for a generic function $f(\kappa)$, correspond to an inversion and an anti-inversion of $f(\kappa)$, respectively; since $f(\kappa)$ is bijective, $\{\mathcal{G}\}$ **is an algebraic Field**.

Proof. $\hat{\Upsilon}_s$ is automorphic **iff** $\hat{\Upsilon}_s : \{\mathcal{G}\} \underset{K(\kappa)}{\twoheadrightarrow} \{\mathcal{G}\}$, so, $\{\mathcal{G}\} \cong \{S\}$; regarding as an artifice the reverse of *Theorem II* we can affirm, ю $= 1$, then, $\{\mathcal{G}\} \cong \{\mathcal{G}^{\blacksquare}\}$, for some stable generic space $\{\mathcal{G}^{\blacksquare}\}$.

Hence, the dispersion of information must not happen - in exception if $f(\kappa)$ contradicts *Theorem II* in one condition: Not being bijective. Then, a non-bijective generic functiion (unknown if reversible or not - but selecting irreversible in order to do a didatic example), may not have an eigenvalue such that ю$_s = 1$, i.e., the latter might be irreversible. Thus, if the irreversibility still exists, then, there must be an eigenvalue relation which relates the stable form of $f(\kappa)$ with its irreversible form, clearly highlights the singularities (where information is lost), therefore, the reversibility operator must relate the invertibility of $f(\kappa)$, such that, if the latter is invertible and anti-invertible, then ю$_s = 1$ and the total reversibility attends:

$$\hat{\text{Ϻ}}_s \left| f(\kappa) \right\rangle = \text{ю}_s \left| f(\kappa) \right\rangle$$

$$; \hat{\text{Ϻ}}_s = \hat{\Upsilon}_s \hat{\Upsilon}_s^{-1}$$

$$\Rightarrow \hat{\Upsilon}_s^{-1} \left| f(\kappa) \right\rangle = f^{-1}(\kappa); \qquad \hat{\Upsilon}_s \left| f^{-1}(\kappa) \right\rangle = \text{ю}_s f(\kappa)$$

Therefore, if and considering *Theorem II; Theorem III*, which affirm that in addition $\hat{\Upsilon}_s^{-1}; \hat{\text{Ϻ}}_s$ are linear maps over $\{\mathcal{G}\}$ and $\{S\}$; $\{\mathrm{G}\} \cong \{S\}$, we must affirm that $\{\mathcal{G}\}$ is an algebraic Field - since ю$_s = 1$

Such interpretation supplies a cleaner intuition of the power of the eigenvalue relation and reversibility; and proves the theorem $\square$

7 The Tensor Notation and Evaluation of the Generic Integral

Remark: In this section the analysis will be focused on the n-dimensional τ-manifold ($\mathbb{T}^n$); the n-dimensional generic space ($\{\mathcal{G}\}$); their attached variables, q and κ, respectively. From sections 4 and 5:

$$q = \sum_{m=1}^{n} \chi_m \left| e_m \right\rangle + \sum_{l=1}^{n} \psi_l \left| e_l \right\rangle$$

$$; \kappa = \sum_{c=1}^{n} \rho_c \left| e_c \right\rangle + \sum_{d=1}^{n} \lambda_d \left| e_d \right\rangle$$

Relying the definitions, $\mathbb{T}^n$ is the n-dimensional set containing all the points q; n τ-bases. $\forall\, q \in \mathbb{T}^n, \exists\, \chi_m, \psi_l \in \mathbb{C}^{2n} |\, \mathbb{C}^{2n} \subsetneq \mathbb{T}^n$. Each n-dimensional τ-variable, denoted by q_{ml} contains a τ-independent variable χ_m and a τ-dependent variable ψ_l.

From now on, to generalize our interpretation directly, the generic variable κ will be used; in order to return to the τ-domain, the generic space must be defined as being isomorphic to the τ-space - $\{\mathcal{G}\} \cong \mathbb{T}^n$; then, the complex space appears as a consequence of the fact that $\mathbb{C}^{2n} \subsetneq \mathbb{T}^n$.

A generic space $\{\mathcal{G}\}^{\mathcal{X}}$ can be thought of as a generic **irreversible** topological manifold $\mathcal{G}^{\mathcal{X}}$ (the notation of the manifold just changes by the exclusion of the curly brackets), since the latter attends to three basic conditions:
(**Remark**: $\mathcal{G}^{\mathcal{X}}$ is a generic manifold of order $\mathcal{X}$ (whose tensors have order $\mathcal{X}$).)

I – $\mathcal{G}^{\mathcal{X}}$ is a **Hausdorff Space**: $\forall \kappa \in \mathcal{G}^{\mathcal{X}}$, neighbor points κ_1 and κ_2 can be separated into two disjoint subsets U^n and V^n, such that $U^n \cap V^n = \varnothing$; $U^n, V^n \subseteq \mathcal{G}^n$

II – $\mathcal{G}^{\mathcal{X}}$ is locally an Euclidean Space: Each point of $\mathcal{G}^{\mathcal{X}}$ has an assigned homeomorphic element at an open subset $\mathbb{R}^n$.

III – $\mathcal{G}^{\mathcal{X}}$ is second-countable.

(The same conditions are true for τ-manifolds)

Condition I ensures each point in $\mathcal{G}^{\mathcal{X}}$ is uniquely defined; condition II allows the subsequent tensor interpretation of the generic variable by setting that locally, $\mathcal{G}^{\mathcal{X}} \cong \mathbb{R}_1^n \times \mathbb{R}_2^n \times \cdots^{[\mathcal{X}\ \text{times}]} \cdots \times {\mathbb{R}_{\mathcal{X}}}^n$; condition III reaffirms condition II and condition I by ensuring the existence of a countable basis of sets homomorphic to $\mathbb{R}^n$, such that their union forms all the open subsets of $\mathcal{G}^{\mathcal{X}}$.

- By condition II, it is possible to translate a generic variable κ to a order n tensor (or an order n matrix):

$$\kappa = \sum_{c=1}^{n} \rho_c \left|e_c\right\rangle + \sum_{d=1}^{n} \lambda_d \left|e_d\right\rangle \equiv \sum_{c,d}^{n} \begin{bmatrix} \rho_c \\ \lambda_d \end{bmatrix}$$

However, for proceed with the derivation, is essential the comprehension that κ being a matrix denoting both ρ_c and λ_d is not mathematically possible, therefore, we are forced to set a specification: **the pure complex component is unique**, i.e., $\rho_c = 0 \ \forall \ c > 1$.

$$\therefore \kappa = \rho_1 \left|e_1\right\rangle + \sum_{d=1}^{n} \lambda_d \left|e_d\right\rangle$$

- By convention, assume $\rho_1 \left|e_1\right\rangle$ is absorbed by the summation, such that:

$$\rho_1 \left|e_1\right\rangle + \sum_{d=1}^{n} \lambda_d \left|e_d\right\rangle = \sum_{d=1}^{n+1} \lambda_d \left|e_d\right\rangle$$

Such that $\left|e_{d=1}\right\rangle$ is an element of the desired space. Therefore:

$$\sum_{d=1}^{n+1} \lambda_d \left|e_d\right\rangle \equiv \sum_{d}^{n+1} \begin{bmatrix} \lambda_d \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \vdots \\ \lambda_{n+1} \end{bmatrix}$$

- λ_d is a variable contained in the desired space, such that can be broken out in real linear combinations (again, limiting the free variable x to be unique): $\lambda_d = x_d + \sum_{c=1}^{m} y_{dc} \left|e_c\right\rangle$; $\quad x, y$ are real variables that span λ_d; are linearly independent, forming a basis for the latter. Again, we assume x is absorbed in the summation, such that $\lambda_d = \sum_{c=1}^{n+1} y_{dc} \left|e_c\right\rangle$

$$\therefore \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \vdots \\ \lambda_{n+1} \end{bmatrix} \equiv \begin{bmatrix} y_{11} \left|e_1\right\rangle + y_{12} \left|e_2\right\rangle + y_{13} \left|e_3\right\rangle + \ldots + y_{1(m+1)} \left|e_{m+1}\right\rangle \\ y_{21} \left|e_1\right\rangle + y_{22} \left|e_2\right\rangle + y_{23} \left|e_3\right\rangle + \ldots + y_{2(m+1)} \left|e_{m+1}\right\rangle \\ y_{31} \left|e_1\right\rangle + y_{32} \left|e_2\right\rangle + y_{33} \left|e_3\right\rangle + \ldots + y_{3(m+1)} \left|e_{m+1}\right\rangle \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ y_{(n+1)1} \left|e_1\right\rangle + y_{(n+1)2} \left|e_2\right\rangle + y_{(n+1)3} \left|e_3\right\rangle + \ldots + y_{(n+1)(m+1)} \left|e_{m+1}\right\rangle \end{bmatrix}$$

$$\equiv \begin{bmatrix} y_{11} & y_{12} & y_{13} & \cdots & y_{1(m+1)} \\ y_{21} & y_{22} & y_{23} & \cdots & y_{2(m+1)} \\ y_{31} & y_{32} & y_{33} & \cdots & y_{3(m+1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y_{(n+1)1} & y_{(n+1)2} & y_{(n+1)3} & \cdots & y_{(n+1)(m+1)} \end{bmatrix}$$

- y_c are the components in the desired space - they define the rows; λ_d are the generic components - they define the columns of the matrix.

$$\Rightarrow \begin{bmatrix} y_{11} & y_{12} & y_{13} & \cdots & y_{1(m+1)} \\ y_{21} & y_{22} & y_{23} & \cdots & y_{2(m+1)} \\ y_{31} & y_{32} & y_{33} & \cdots & y_{3(m+1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y_{(n+1)1} & y_{(n+1)2} & y_{(n+1)3} & \cdots & y_{(n+1)(m+1)} \end{bmatrix} = \kappa_{dc}$$

- Observe the fact that the generated matrix representing κ_{dc} is the transpose of the tensor κ_{cd} being previously analyzed, therefore, we say κ_{dc} is a **co-tensor**.

- Then, a generic co-tensor κ_{cd} may be represented by a matrix of its real components, as stated by condition II.

7.1 Tensor Generic Integration - Derivation from the Stokes' Generalized Theorem

The Stokes' Generalized Theorem plays an essential role in the field of differential geometry and tensor calculus, generalizing the concepts of rotation and internal/external derivatives from vector fields to manifolds.

The motivation consists of analyzing the generic integral by a tensor viewpoint and then derive similar conditions and properties used in vector calculus to tensor formalism.

Definition V: Let $\mathcal{G}^{\mathcal{X}}$ be the manifold in which a generic integration with respect to an assigned variable κ is being realized:

$$\int_{\mathcal{G}} f(\kappa) d\kappa$$

Using of the tensor formalism, we define the same integral as being:

$$\int_{\mathcal{G}^{\mathcal{X}}} \mathcal{H}(\kappa_{cd}) \mathscr{D} \kappa_{dc}$$

- $\mathcal{H}(\kappa_{cd}) = \mathcal{H}_{cd}$ is the tensor field (a matrix functional) contained in a space $\mathscr{H}^{\mathcal{X}} = \{\hbar_{cd}\}_{c,d=1}^{\mathcal{X}}$; $\hbar_{cd}$ denotes each element of the tensor function $\mathcal{H}(\kappa_{cd})$.

- $\mathscr{D}\kappa_{dc}$ is the co-tensor differential, such that $\mathscr{D}\kappa_{dc} = \begin{bmatrix} dy_{11} & dy_{12} & \dots & dy_{1(m+1)} \\ dy_{21} & dy_{22} & \dots & dy_{2(m+1)} \\ \vdots & \vdots & \ddots & \vdots \\ dy_{(n+1)1} & dy_{(n+1)2} & \dots & dy_{(n+1)(m+1)} \end{bmatrix}$.

Thus, the integral takes the form:

$$\int_{\mathcal{G}^{\mathcal{X}}} \mathcal{H}(\kappa_{cd})\mathscr{D}\kappa_{dc} = \int_{\mathcal{G}^{\mathcal{X}}} \begin{bmatrix} \hbar_{11} & \hbar_{12} & \dots & \hbar_{1(n+1)} \\ \hbar_{21} & \hbar_{22} & \dots & \hbar_{2(n+1)} \\ \vdots & \vdots & \ddots & \vdots \\ \hbar_{1(n+1)} & \hbar_{2(n+1)} & \dots & \hbar_{(m+1)(n+1)} \end{bmatrix} \begin{bmatrix} dy_{11} & dy_{12} & \dots & dy_{1(m+1)} \\ dy_{21} & dy_{22} & \dots & dy_{2(m+1)} \\ \vdots & \vdots & \ddots & \vdots \\ dy_{(n+1)1} & dy_{(n+1)2} & \dots & dy_{(n+1)(m+1)} \end{bmatrix}$$

7.2 Conservative Generic Tensor Fields

Conservative fields in vector calculus represent a specific group of functions that attend to additional conditions, which significantly simplifies interpretations about the rotation and behavior of such a field. In the analysis of generic integrals it helps the understanding of how singularities induced by a tensor field are distributed along the path, as well as implications in the rotation of such fields.

$$\int_{\mathcal{G}^{\mathcal{X}}} \mathcal{H}(\kappa_{cd})\mathscr{D}\kappa_{dc} = \int_{\mathcal{G}^{\mathcal{X}}} \begin{bmatrix} \hbar_{11} & \hbar_{12} & \dots & \hbar_{1(n+1)} \\ \hbar_{21} & \hbar_{22} & \dots & \hbar_{2(n+1)} \\ \vdots & \vdots & \ddots & \vdots \\ \hbar_{1(n+1)} & \hbar_{2(n+1)} & \dots & \hbar_{(m+1)(n+1)} \end{bmatrix} \begin{bmatrix} dy_{11} & dy_{12} & \dots & dy_{1(m+1)} \\ dy_{21} & dy_{22} & \dots & dy_{2(m+1)} \\ \vdots & \vdots & \ddots & \vdots \\ dy_{(n+1)1} & dy_{(n+1)2} & \dots & dy_{(n+1)(m+1)} \end{bmatrix}$$

$$= \int_{\mathcal{G}^{\mathcal{X}}} \begin{bmatrix} \sum_{k=1}^{n+1} \hbar_{1k} dy_{k1} & \sum_{k=1}^{n+1} \hbar_{1k} dy_{k2} & \dots & \sum_{k=1}^{n+1} \hbar_{1k} dy_{k(n+1)} \\ \sum_{k=1}^{n+1} \hbar_{2k} dy_{k1} & \sum_{k=1}^{n+1} \hbar_{2k} dy_{k2} & \dots & \sum_{k=1}^{n+1} \hbar_{2k} dy_{k(n+1)} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k=1}^{n+1} \hbar_{(m+1)k} dy_{k1} & \sum_{k=1}^{n+1} \hbar_{(m+1)k} dy_{k2} & \dots & \sum_{k=1}^{n+1} \hbar_{(m+1)k} dy_{k(n+1)} \end{bmatrix}$$

- Then, since all the summations above are equal to zero, then $\mathcal{H}_{cd}$ is a conservative tensor field over the generic manifold $\mathcal{G}^{\mathcal{X}}$ - this is the first condition of a conservative tensor field:

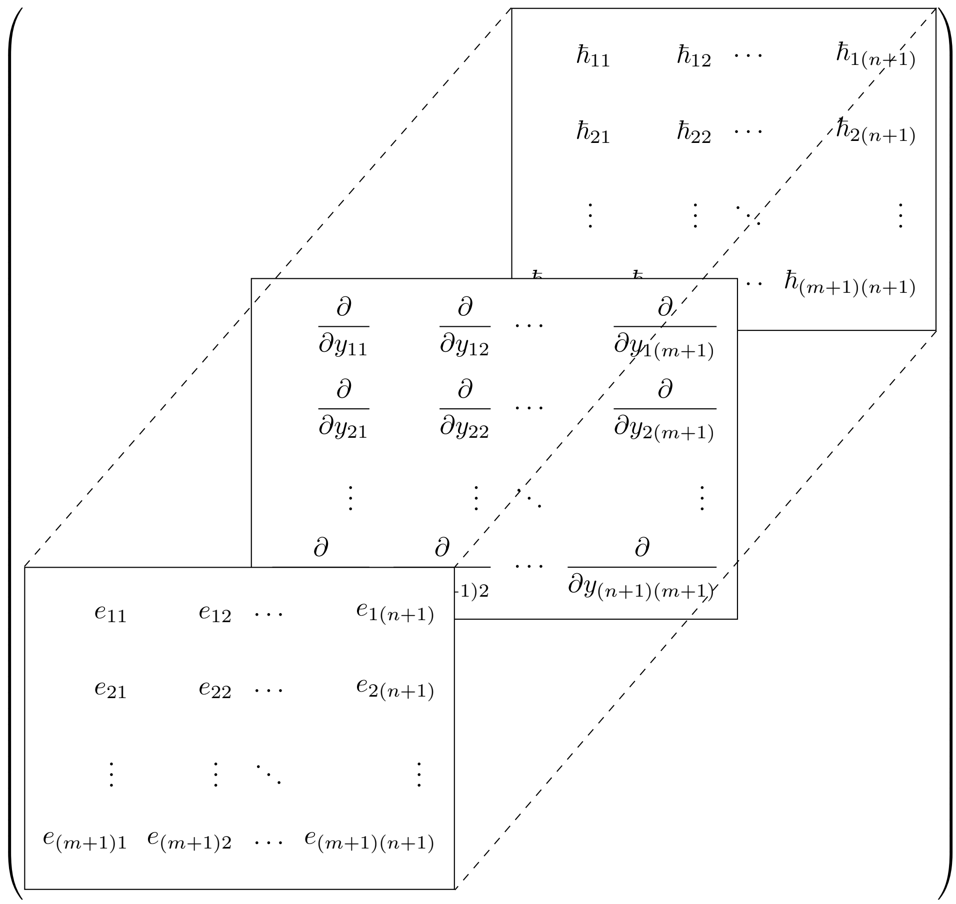
$$\Rightarrow \sum_{k=1}^{n+1} \hbar_{(m+1)k} dy_{k(n+1)} = 0 \;\; \forall \; (m+1), (n+1); \;\; (1,1) \leq (m+1, n+1)$$

- The second condition lies on the rotation. To interpret rotation, it's required to use the gradient co-tensor:

$$\nabla_{dc} = \begin{bmatrix} \dfrac{\partial}{\partial y_{11}} & \dfrac{\partial}{\partial y_{12}} & \cdots & \dfrac{\partial}{\partial y_{1(m+1)}} \\ \dfrac{\partial}{\partial y_{21}} & \dfrac{\partial}{\partial y_{22}} & \cdots & \dfrac{\partial}{\partial y_{2(m+1)}} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial}{\partial y_{(n+1)1}} & \dfrac{\partial}{\partial y_{(n+1)2}} & \cdots & \dfrac{\partial}{\partial y_{(n+1)(m+1)}} \end{bmatrix}$$

- Thus, the curl of a tensor field $\mathcal{H}_{cd}$ is defined as follows:

$$\nabla_{dc}\times\mathcal{H}_{cd} = \det$$



- Well, it's clear that such notation should not be used.
- Therefore, the context requires the introduction of the dense tensor notation:

7.2.1 The Dense Tensor Notation

The dense tensor/co-tensor is nothing more than an isomorphism of tensors (*to be proved in Theorem VI*), which reduces each row of the matrix to a unique vector element, so then, the dense tensor is a matrix of vectors:

For example, for a given a matrix X with elements $x_{(m+1)(n+1)}$, let a function $\Phi(X) = \not{X}$ represent a functional acting from a matrix to its dense form ($\not{X}$ is the notation to represent a dense matrix), then :

$$\Phi(X) = \Phi\left(\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1(n+1)} \\ x_{21} & x_{22} & \cdots & x_{2(n+1)} \\ \vdots & \vdots & \ddots & \vdots \\ x_{(m+1)1} & x_{(m+1)2} & \cdots & x_{(m+1)(n+1)} \end{bmatrix}\right) = \not{X} = \begin{bmatrix} \vec{\not{x}}_1 \\ \vec{\not{x}}_2 \\ \vdots \\ \vec{\not{x}}_{m+1} \end{bmatrix}$$

<u>Theorem VI</u>: Let $\Phi(X)$ be the function that carries a generic tensor to its dense form, then, Φ is an isomorphism from $X \mapsto \not{X}$; and $\vec{\not{x}}_{m+1}$ is an isomorphic element of the $(m+1)$th generic element of a generic variable.

Proof. Let the $(m+1)$th dense component of a generic matrix X be defined as follows:

$$\begin{aligned} \vec{\not{x}}_{m+1} &= (\not{x}_1 \, |\not{e}_1\rangle + \not{x}_2 \, |\not{e}_2\rangle + \cdots + \not{x}_n \, |\not{e}_n\rangle + \not{x}_{n+1} \, |\not{e}_{n+1}\rangle)_{m+1} \\ &= \left(\sum_{c=1}^{n+1} \not{x}_c \, |\not{e}_c\rangle\right)_{m+1} \end{aligned}$$

Equivalently, breaking out in $(m+1)$ elements, we get:

$$\begin{aligned} \left(\sum_{c=1}^{n+1} \not{x}_c \, |\not{e}_c\rangle\right)_{m+1} &= \sum_{d=1}^{m+1} \left(\sum_{c=1}^{n+1} \not{x}_c \, |\not{e}_c\rangle\right)_d |\not{e}_d\rangle \\ &= \sum_{d=1}^{m+1} \sum_{c=1}^{n+1} \not{x}_{cd} \, |\not{e}_c\rangle \, |\not{e}_d\rangle \end{aligned}$$

Regarding $\mathcal{G}^{\mathcal{X}} \cong \mathbb{R}_1^n \times \mathbb{R}_2^n \times \cdots^{[\mathcal{X} \text{ times}]} \cdots \times \mathbb{R}_{\mathcal{X}}^n$, it's possible to assign each element $\not{x}_{cd} \, |\not{e}_c\rangle \, |\not{e}_d\rangle$ to a dth generic variable, such that each element $\not{x}_c \, |\not{e}_c\rangle$ denotes a variable in the desired manifold S, i.e., inverting the process done in this section hitherto. Therefore, $\not{X} \cong X$, since X is obtained as a translation of a generic variable onto an order two tensor.

□

Therefore, relying the main discussion, the curl of the tensor field - $\nabla_{dc} \times \mathcal{H}_{cd}$ - can be translated to a curl of the dense tensor field $\not{\mathcal{H}}_{cd}$ by the product rule of isomorphisms (see *Theorem II*):

$$\Phi(\nabla_{dc} \times \mathcal{H}_{cd}) = \Phi(\nabla_{dc}) \times \Phi(\mathcal{H}_{cd}) = \not{\nabla}_{dc} \times \not{\mathcal{H}}_{cd}.$$

Such interpretation mitigates the problem of the dimensions of the curl matrix, however, does not change the fact that we must handle a manifold of order $\mathcal{X}$, i.e., the rotation must be analyzed for each pair of permutations of the total dimensions, similar to what was done at Section 5 regarding the sub circumferences of S_τ.

7.2.2 Condition 3 of Conservative Generic Tensor Fields

The translation of the third condition for conservative vector fields must be carefully regarded for generic tensor fields - that's due to the irreversible behavior observed at such more general fields.

Therefore, I judged it deserves a specific sub subsection due to its broad applications.

The condition lies on the topic of closed rotations, in which the point of the beginning of the rotation coincides with the end point. Specifically, we regard a closed contour integral of a conservative field $\mathcal{H}$ over $\mathcal{G}^{\mathcal{X}}$.

Closed rotations, specially, happens when there is a complete rotation of all the components $\hbar_{cd}$ of the tensor field.

From our previous evaluations, we considered a generic function (eventually interpreted as a tensor field) whose complete reversibility happens if it attends the eigenvalue relation for some transformation $\hat{\Upsilon}$; is holomorphic (i.e., angularly invariant in relation to $\hat{\Upsilon}$); is bijective. If the conditions are fully attended, thus, such field is reversible, i.e., does not create singularities throughout its domain.
Such function does have the closed contour integral equals to zero, and that's the case in which the third condition for conservative vector fields is naturally translated to conservative tensor fields.

Therefore, a **stable generic function** $\mathcal{H}^{\blacksquare}_{cd}$ does have the closed contour integral zero:

$$\oint_{\mathcal{G}^{\mathcal{X}}} \mathcal{H}^{\blacksquare}_{cd} \mathscr{D}\kappa_{dc} = \mathbf{0}$$

The following considerations, though, are concerned for when the generic function is not completely reversible, i.e., the latter is not stable, and creates singularities throughout its path, and thus, a closed rotation may not end up to the same point, although it looks to be. Such condition is more profoundly explored at the following Subsection 7.3, where the Generalized Stokes' Theorem is used to geometrically interpret unstable rotations.

7.3 Implications of the Generalized Stokes' Theorem on Generic Tensors

The upcoming formalism requires a manipulation of the Generalized Stokes' Theorem regarding generic spaces - in which whether or not might be an irreversible manifold. For such interpretation, the explored concept of rotation in irreversible manifolds shall be recaptured.

To provide the Generalized Stokes' Theorem enough artifices to be discussed, the concept of covariant irreversible derivative shall be introduced, and for this, also the concept of planar cycles:

Definition VI (Planar Cycles & Order $\mathcal{X}$ Boundaries): Let $\mathcal{G}^{\mathcal{X}}$ be a generic manifold with a topology $\mathscr{T}$ given by the conditions stated in the last subsection. **A planar cycle** around the boundary $\nabla\mathcal{G}^{\mathcal{X}}$ (see remarks at *Notations and Conventions* section) is defined as a complete turn over an axis $\vartheta^{\mathcal{Z}} \subseteq \mathrm{Int}(\mathcal{G}^{\mathcal{X}}) \subseteq \mathcal{G}^{\mathcal{X}}$, such that $\mathcal{Z}$ denotes the order of the axis ϑ; $1 \leq \mathcal{Z} \leq \mathcal{X}$.

A planar cycle state is defined according to the elements of the axis of rotation $\vartheta^{\mathcal{Z}} \subseteq \mathcal{G}^{\mathcal{X}}$, such that for each different combination of elements a new planar cycle state is defined.

$\rightarrow$ Remarkable planar cycle states:

- When $\mathrm{Int}(\mathcal{G}^{\mathcal{X}}) = \mathcal{G}^{\mathcal{X}}$, then $\mathcal{G}^{\mathcal{X}}$ is an open manifold, and consequently contains none of its boundary points.
- If $\mathcal{Z} = \mathcal{X}$, then $\vartheta^{\mathcal{Z}} = \vartheta^{\mathcal{X}}$, i.e., the axis possesses the same order of the manifold - and therefore the same elements - implying an entire rotation of the latter.

Then, the Generalized Stokes' Theorem states:

Conclusion and Prospections

Such preprint ends up providing enough concepts and novel purposes for future enhancements and connections with applicable mathematics; some of them are listed in the following topics, according to the chronologic order of short-term ambitions and long-term ambitions.

- Short-Term Ambitions:

- Rigorous derivation of the Generalized Residue Theorem (as promised throughout the paper).
- Application of Dirac-Basis functions regarding the scape paths of the τ-sphere to define Dirac-Barriers.
- Application of the Generalized Stokes' Theorem to evaluate generic integrations. In such interpretation, the generic variable κ is broken out in its real components, then, setting $\{\mathcal{G}\} \cong \mathbb{R}^n$ we regard κ as a matrix of its real components, which is tensor of order n. Then, the function $f(\kappa)$ denotes a tensor field over the manifold $\mathcal{G}$. The Generalized Stokes' Theorem analyzes the relation between internal and external derivatives of the manifold $\mathcal{G}$ when regarding a tensor field $f(\kappa)$.

- Long-Term Ambitions:

- τ-dynamics approach - the dynamical process of dispersion of information and singularities over the τ-manifold, discussing topics such as the irreversible continuity equation.
- Applications in physics, regarding Dirac-Basis functions as potentials generated from $\mathbb{T} \to \mathbb{C}$, exploring the diversity of functions (potentials) that can be generated using Dirac-Basis. For specific analysis of dynamical physical phenomena, a parameter "ϟ" shall be introduced, playing a similar role to the time parameter, who guides the dynamics.
- Regarding the physical extension, wave equations; field equations describing the hyperbolic nature of $\mathbb{T}$; Navier-Stokes'-type equations shall be derived, connecting the mathematical abstraction completely onto the physical world.

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