Analysis of Irreversible Phenomena via τ-Manifold: A Formal τ-Analysis Approach

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Section 0 – Introduction:

Irreversibility is a fundamental topic in both pure and applied sciences, including mathematics, mathematical physics, and complex systems. Despite extensive efforts by leading researchers, traditional methods often rely on approximations or computationally intensive algorithms. Consequently, many irreversible phenomena — such as entropy evolution or complex economic fluctuations — remain only partially understood.
In this preprint I will introduce a novel formalism for treating irreversibility not merely as a unavoidable loss of information, but as a structured domain – a τ manifold, systematically designed to contain and control information dissipation.

To formalize an irreversible process, we introduce a unit τ satisfying:

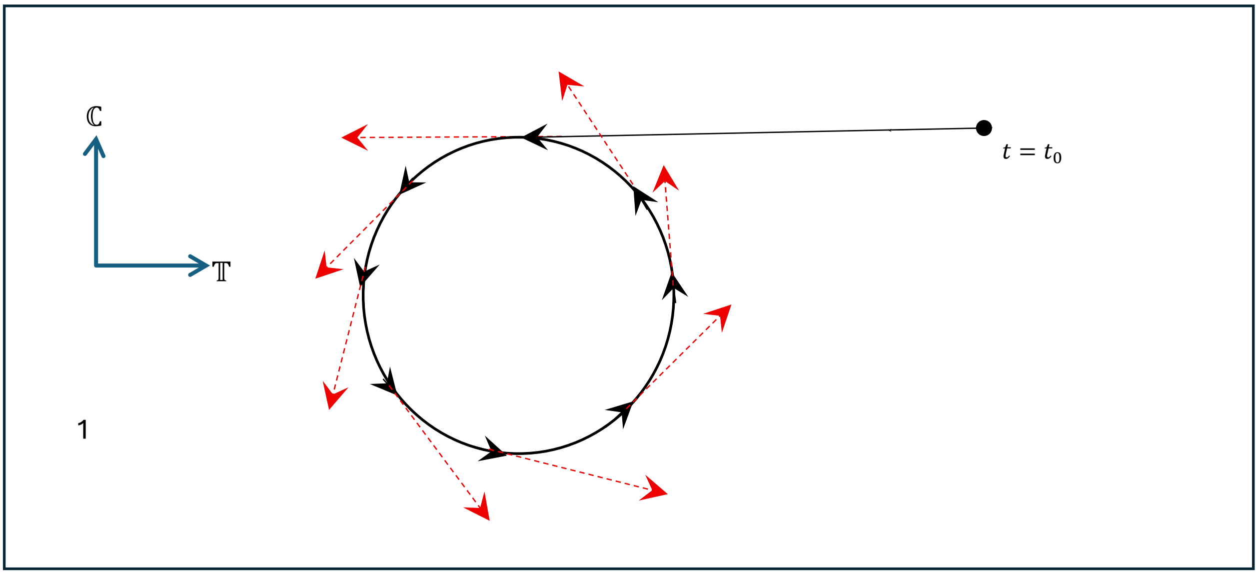
$$\sqrt{\tau} = -1$$

Squaring both sides yields $\tau = 1$, however the original negative sign is lost in the squaring process, indicating an intrinsic loss of information. This property exemplifies the irreversible nature encoded in τ and motivates its use as a foundational unit within the τ-manifold.

Section 1 – The τ-Sphere:

To follow the chronological sequence of the theory, for an advance on mathematical abstraction I shall introduce the idea of time-current. The time is one of the most logical and direct ideals to comprehend irreversibility, in the sight that, a change in some parameter in some point in a given time position, may affect future events chaotically.

The interpretation of time-current is directly connected with the definition of circulation - described as follows:

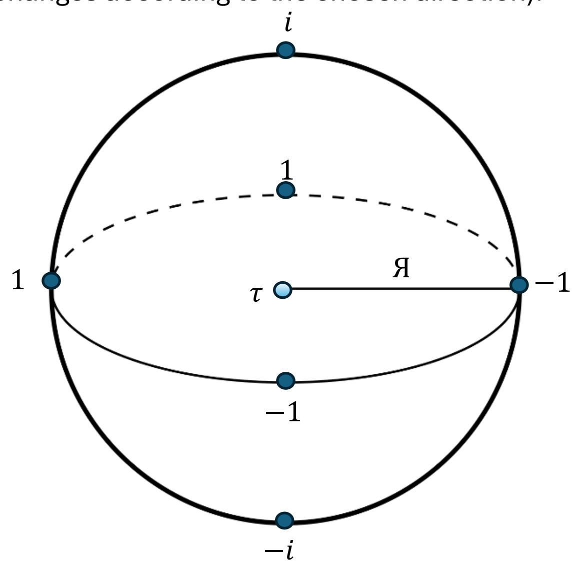


Once a perturbation at t_0 is done, the time current takes the path towards a point t; the path walked by the current passes along a circumference, which represents geometrically the τ space. According to the degree of reversibility of the analyzed system, the current scapes by an angle ξ compared with the original route, and such an angle describes the degree of irreversibility of the system. By convention, because the degree of irreversibility is not periodic, the angle ξ is denoted to be hyperbolic, $-\infty < \xi < \infty$, such that, when $\xi = 0$, there is absence of deviation, and when $\xi \to \infty$ the complete irreversibility happens (in hyperbolic coordinates it's similar to affirm orthogonality).

- ***Definition I***: The τ-sphere - S_τ, is a remarkable artifice to interpret the τ-space geometry: It consists of poles representing the supremum of S_τ:

- $\sup(S_\tau) = \begin{cases} \pm i \ \boldsymbol{latitudinally} \\ \pm 1 \ \boldsymbol{longitudinally} \end{cases}$

The previously seen circulation consists in a sub circumference of S_τ. Inside the sphere the pure τ domain is evident; the center of the sphere is the τ-unit, far from a distance Я of the vertexes (that changes according to the chosen direction).



- ***Theorem I:*** *The τ-unit $\{\tau\}$ is a basis for S_τ and $\mathbb{C}$ concomitantly, but not for $\mathbb{R}$.*

Let $\sqrt{\tau} = -1$, then:
1) $\sqrt[4]{\tau} \equiv i \in \mathbb{C}$

2) $\tau^2 \neq 1$, otherwise, information dispersion occurs.

Therefore, rational powers of the τ-unit are represented as complex numbers, but the same is not true for real numbers.

Section 2 – Mathematical Framework:

This section will be entirely dedicated to analyzing what these simple properties of S_τ imply in a more systematic mathematical framework.

The current situation requires an introduction of operators; their function in τ-analysis is extremely important and provides a direct and clear interpretation and application using the concepts explored hitherto.

Let Ṫ: $\mathbb{C} \rightarrow \mathbb{T}$ be the called τ-inversion operator, which diverse meanings depending the analyzed situations – I shall present some of them in detail further.

For every operator, there must be a function in which the same can be applied:

Remark: We are going to use Dirac notation for inner products; bra's ⟨ | and ket's | ⟩, due to many similarities our approach has with Quantum Mechanics.

Let $\boldsymbol{q} = \boldsymbol{z} + \boldsymbol{\tau w};\ z, w \ \in \mathbb{C}$ be a τ-variable – this is, a well-defined variable inside S_τ. Then, let $\boldsymbol{f_\tau(q)}$ be a function $f_\tau: \mathbb{T} \rightarrow \mathbb{T}$. The following process will be, with caution, to adopt a similar approach to what is done in Complex Analysis and derive precisely the two kinds of τ-functions:

1) **τ-trivial functions**
2) **τ-analytic functions.**

The τ-integration:

- For a well behaved τ-function $f_\tau: \mathbb{T} \rightarrow \mathbb{T}$ in which is integrable; differentiable by each one of its components (z, w); continuous under the period of integration we define a τ-integral of f_τ as follows:

$$\oint_{S_\tau} f_\tau(q)dq$$

Where f_τ is identified as a **field**, and S_τ as a **path** in which the field acts.

1) Definition of τ-trivial functions:

- Given $f_\tau(q) = \text{Ո}(\mathrm{z,w}) + \tau\text{Ᵽ}(\mathrm{z,w})$:

$$\oint_{S_\tau} f_\tau(q)dq = \oiint \langle \text{Ո}(\mathrm{z,w}) + \tau\text{Ᵽ}(\mathrm{z,w}) | dz + \tau \mathrm{dw} \rangle$$

- By Stokes Theorem we get:

$$= \oiint \langle \text{Ո}(\mathrm{z,w}) + \tau\text{Ᵽ}(\mathrm{z,w}) | dz + \tau \mathrm{dw} \rangle = \oiint (\nabla_q \times f_\tau) dzdw$$

$$= \oiint \frac{\partial \text{Ᵽ}(\mathrm{z,w})}{\partial \mathrm{z}} - \frac{\partial \text{Ո}(\mathrm{z,w})}{\partial w} dzdw$$

- τ-trivial functions are those in which $\frac{\partial \text{Ᵽ}(\mathrm{z,w})}{\partial \mathrm{z}} = \frac{\partial \text{Ո}(\mathrm{z,w})}{\partial w}$, such that the τ-integral is zero.

τ-trivial functions led to instantaneous understanding about the rotation the field f_τ enforces – the rotation is zero, therefore the angle ξ has an absence of deviation, then, τ-trivial functions represent the called irreversibility static functions.

2) Definition of τ-analytic functions:

- Let $q = z + \tau w;\ f_\tau(q) = f_\tau(q(z,w))$, then:

$$\oint_{S_\tau} f_\tau(q)dq = \oint_{S_\tau} f_\tau(q(z,w)dq$$

- Analyzing a specific case – when $w = 0 \Rightarrow q = z \in \mathbb{C}$ – we are led to the Corollary:

- ***Corollary I***: *The case $w = 0$ implies $q = z$, then $f_\tau(q(z,w)) = f_\tau\big(q(z,0)\big) = f_\tau(q(z))$, however, $q = z$, therefore, $\boldsymbol{f_\tau(q(z,0))} \mapsto \boldsymbol{f(z)}$ – implying the **absence of the influence of $\boldsymbol{S_\tau}$**. Such an absence implies a remarkable simplification to the standard τ-integral:*

$$\oint_{S_\tau} f_\tau(q(z,0))dq \mapsto \oint_{\Gamma\in\mathbb{C}} f(z)dz$$

- Therefore, such an integral is on the complex domain – which is clearly reversible.
- This concludes that the complex space $\mathbb{C}$ **is a proper subset of the $\boldsymbol{\tau}$-space** $\mathbb{T}$: Given $q \in \mathbb{T};\ \boldsymbol{q = z + \tau w},\ w = 0$ implies $q = z$; z is a variable in the complex space.
  $\therefore z \in \mathbb{C};\ \boldsymbol{z = q(z, w = 0)} \in \mathbb{T} \Rightarrow \mathbb{C} \subset \mathbb{T} \Rightarrow \mathbb{C}$ is a proper subset of $\mathbb{T}$**.**

Now, the context requires the introduction of the **reversibility operator ($\hat{\Upsilon}$)**: Such an operator takes a function in the complex manifold $f(z): \mathbb{C} \to \mathbb{C}$ and maps the latter in the τ-manifold - $\hat{\Upsilon}: \mathbb{C} \to \mathbb{T}$. Symmetrically, $\hat{\Upsilon}^{-1}$ is defined as being a mapping of $\mathbb{T}$ **onto** $\mathbb{C}$ – $\hat{\Upsilon}^{-1}: \mathbb{T} \to \mathbb{C}$.

The junction of $\hat{\Upsilon}$ and $\hat{\Upsilon}^{-1}$ forms the operator $\hat{\mathfrak{K}} = \hat{\Upsilon}\hat{\Upsilon}^{-1}$. Applying the new definitions to the latter analyzed case – $w = 0$: $\hat{\Upsilon}\hat{\Upsilon}^{-1}\left|f_\tau(q(z,0))\right\rangle = \hat{\mathfrak{K}}\left|f_\tau(q(z,0))\right\rangle = \mathcal{I}\left|f_\tau(q(z,0))\right\rangle$; $\mathcal{I}$ is called the identity eigenvalue. In such a case the irreversible component of the τ-variable, w, is zero, so then, $f_\tau(q(z,0)) \mapsto f(z)$, and behaves statically with respect to dissipation of information.

For generalize the operator approach for an arbitrary well-behaved endofunction f_τ is crucial the addition of the variable w – which carries the irreversibility. Then, the influence of the reversibility operator under a potentially irreversible τ-function requires a **scale of measurement of the dissipation** – an eigenvalue labeled ю, such that:

- $\hat{\mathfrak{K}}|f_\tau(q(z,w))\rangle = ю|f_\tau(q(z,w))\rangle$

Such a relation is well known in linear algebra – the eigenvalue relation. τ-functions that respect the eigenvalue relation (those which are eigenfunctions of ю) are said to be **$\boldsymbol{\tau}$-analytic**.

- The reversibility operators can be recognized as **homomorphisms betwixt $\mathbb{C}$ and** $\mathbb{T}$; and this relation, assuming that Υ consists in a mapping, is explored in Theorem *II:*

- ***Theorem II (Structural reversibility condition)****: Let $\hat{\Upsilon} : \mathbb{C}^1 \to \mathbb{T}^1$ a mapping from $\mathbb{C}^1$* ***onto*** *$\mathbb{T}^1$ - then $\hat{\Upsilon}$ is an anti-homomorphism from $\mathbb{C}$ to $\mathbb{T}$ and $\hat{\Upsilon}^{-1}$ is a homomorphism from $\mathbb{T}$ to $\mathbb{C}$, and, since the eigenvalue $ю = 1$, then $\mathbb{C}^1 \cong \mathbb{T}^1$.*
    - ***Remark: The notation $A \cong B$ for two sets denotes "A is isomorphic to B"***

Let $f_\tau(q)\colon \mathbb{C}^1 \to \mathbb{C}^1; g_\tau\colon \mathbb{C}^1 \to \mathbb{C}^1 \;\Rightarrow\; \hat{\Upsilon}^{-1}|f_\tau(q(z,w))\rangle = f(z);\; \hat{\Upsilon}^{-1}|g_\tau(q(z,w))\rangle = g(z) \Rightarrow$

$\hat{Ѫ}\begin{vmatrix} f_\tau \\ g_\tau \end{vmatrix}\rangle = ю\begin{vmatrix} f_\tau \\ g_\tau \end{vmatrix}\rangle$. The bijectivity of the operator is defined exclusively when $ю = 1 \;\Rightarrow\; \hat{Ѫ}\begin{vmatrix} f_\tau \\ g_\tau \end{vmatrix}\rangle = \begin{vmatrix} f_\tau \\ g_\tau \end{vmatrix}\rangle \Rightarrow \hat{Ѫ} = \hat{\mathrm{I}}$, otherwise:

$Ѫ = \hat{\Upsilon}\hat{\Upsilon}^{-1}\colon \mathbb{T}^1 \to \mathbb{C}^1 \to \mathbb{T}^{\blacksquare 1}$, i.e., $\hat{Ѫ}\colon \mathbb{T}^1 \to \mathbb{T}^{\blacksquare 1}$, unless $ю = 1$ – the unique case in which $\mathbb{T}^1 \cong \mathbb{T}^{\blacksquare 1}$, which proves the statement two.

- **$\mathbb{T}^{\blacksquare}$ is called the stable subset of $\mathbb{T}$, and such a relation shall be explained in the next section.**

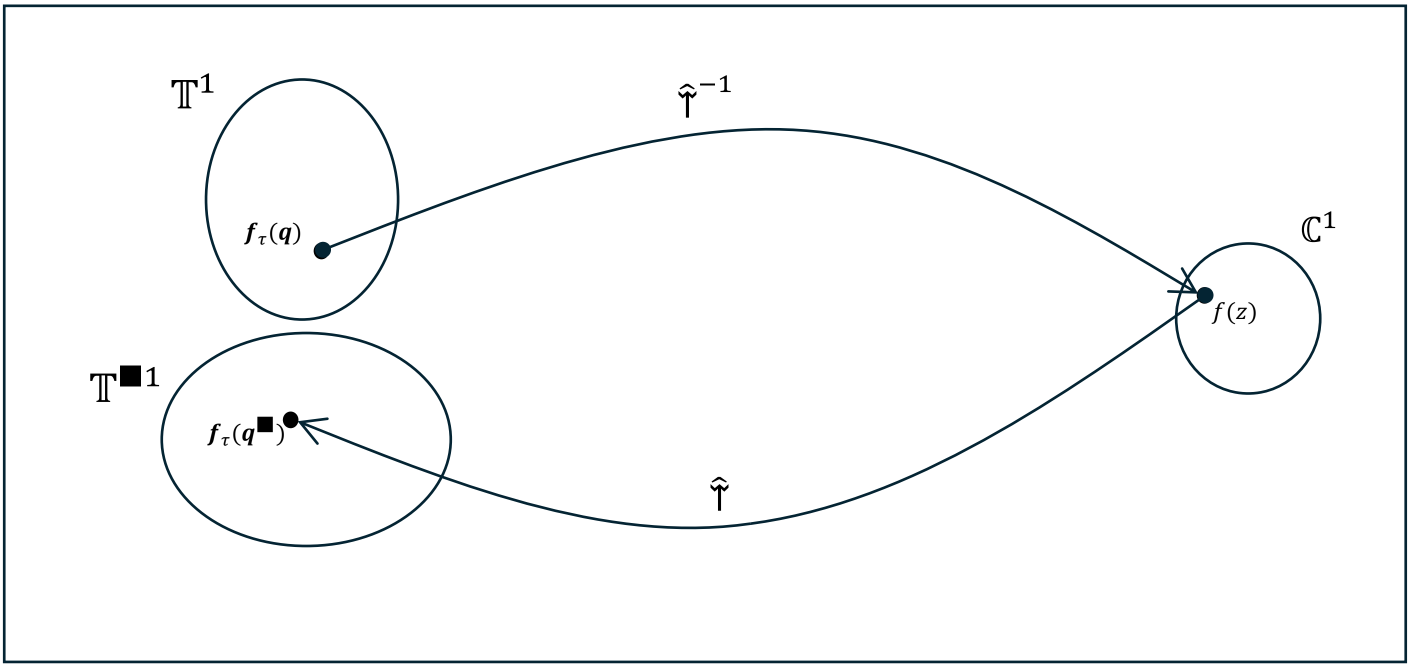
; Since $\hat{\Upsilon}$ returns to a different set of the which $\hat{\Upsilon}^{-1}$ has departed; and as *Corollary I* affirms that $\mathbb{C}$ is a proper subset of $\mathbb{T}$, then:

1) $\hat{\Upsilon}^{-1}\colon \mathbb{T}^1 \overset{\Upsilon^{-1}}{\mapsto\!\!\!\!\twoheadrightarrow} \mathbb{C}^1$ – ($\hat{\Upsilon}^{-1}$ is a homomorphism from $\mathbb{T}^1$ to $\mathbb{C}^1$)

2) $\hat{\Upsilon}\colon \mathbb{C}^1 \overset{\Upsilon}{\mapsto\!\!\!\!\twoheadrightarrow} \mathbb{T}^{\blacksquare 1}$ – ($\hat{\Upsilon}$ is an anti-homomorphism from $\mathbb{C}^1$ to $\mathbb{T}^{\blacksquare 1}$)

Which proves statement one, and completes the proof ■

- Below is a schematic representation of *Theorem II*:



- ***Remark: $f_\tau(q)$ is an endofunction in $\mathbb{T}^1$, while $f_\tau(q^{\blacksquare})$ is an endofunction in $\mathbb{T}^{\blacksquare 1}$.***

- ***Corollary II***: For a well-behaved endofunction $f_\tau(q(z,w))$, its stable form, $f_\tau^{\blacksquare}(q(z,w))$, under a τ-integration is equivalent to the complex integration of $f_\tau^{\blacksquare}(q(z,0)) = f^{\blacksquare}(z)$:

$$\Rightarrow \quad \oint_{S_\tau} \hat{Ӿ}|f_\tau(q(z,w))\rangle \, dq = \oint_{S_\tau} f_\tau^{\blacksquare}(q(z,w))dq = \oint_{S_\tau} f_\tau^{\blacksquare}(q(z))dq$$

- By *Corollary I*:

$$\oint_{S_\tau} f_\tau^{\blacksquare}(q(z))dq = \oint_{\Gamma \in \mathbb{C}} f^{\blacksquare}(z)dz.$$

Section 3 – Degrees of Freedom of Irreversibility & Multiple τ-Dimensions:

A diversity of observable phenomena – most of them, have a complex system of dependencies of variables – and therefore a more entangled chain of irreversible factors; that when blended, transforms a predictable system into pseudo-chaotic or even chaotic systems.

The main goal of this section is to develop a higher understanding of pseudo-chaotic systems, and how the latter can be broken out into isolated and simpler systems, that yet connect with each other.

Pseudo-chaotic systems appear chaotic only because **repeated anti-homomorphisms** distort a reversible structure.

Anti-homomorphism and the codification of irreversibility:

- ***Definition II***: Let $\hat{\Upsilon}; \hat{\Upsilon}^{-1}$ represent an anti-homomorphism and homomorphism, respectively (as proved in *Theorem II*); $\hat{\Upsilon}: \mathbb{C}^1 \twoheadrightarrow \mathbb{T}^1; \Upsilon^{-1}: \mathbb{T}^1 \twoheadrightarrow \mathbb{C}^1$; $Ӿ = \hat{\Upsilon}\hat{\Upsilon}^{-1}$ and a τ-function $f_\tau(q) = ᑎ(z,w) + \tau Ᵽ(\mathrm{z}, \mathrm{w})$. To analyze the nth degree of freedom of irreversibility of $f_\tau(q)$ there is a necessity of extend the -space from $\mathbb{T}^1$ to $\mathbb{T}^n$. Then, each component of f_τ takes the following form:

$$Ᵽ = \sum_{\sigma=1}^{n} ɰ_\sigma |\, e_\sigma\rangle$$

$$ᑎ = \sum_{j=1}^{n} \mathfrak{h}_j |\, e_j\rangle$$

; the previously defined eigenvalue relation takes the form:

$$\begin{pmatrix} \hat{Ѫ}_1 & 0 & \cdots & 0 & 0 \\ 0 & \hat{Ѫ}_2 & \cdots & 0 & 0 \\ 0 & 0 & \hat{Ѫ}_3 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \hat{Ѫ}_{\mathrm{n}} \end{pmatrix} \left| \begin{matrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_n \end{matrix} \right\rangle = Ю \left| \begin{matrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_n \end{matrix} \right\rangle$$

Such a definition led us to comprehend why *Theorem II* limited the degree of freedom of $\hat{ϒ}$ to $n = 1$ – for $n > 1$ it is necessary to analyze each eigenvalue component by itself, with respect to its respective $\hat{Ѫ}_{\mathrm{n}}$.

τ-holomorphic functions and the Dispersion Derivative:

The cartesian representation of a τ-function with n degrees of freedom is not trivial – in the sight that $3n$ dimensions would be necessary to graph the latter. For this subtopic of Section 3, we will follow strictly the derivation made by Cauchy of the conditions for a holomorphic function, however, extending it to τ-functions.

In order to keep the clearness and the geometric interpretation of τ-functions, the approach to be used will have much in common with the complex analysis:

- Let $q^{\blacksquare} \in \mathbb{T}^n;\ q^{\blacksquare} = \sum_{m=1}^{n} \chi_m^{\blacksquare} |\, e_m\rangle + \sum_{l=1}^{n} \psi_l^{\blacksquare} |\, e_l\rangle$

$$Ո = \sum_{j=1}^{n} \mathfrak{h}_j |\, e_j\rangle$$

$$Ᵽ = \sum_{\sigma=1}^{n} ɰ_\sigma |\, e_\sigma\rangle$$

$$\Rightarrow \partial Ո = \frac{\partial}{\partial \chi_m^{\blacksquare}} \sum_{j=1}^{n} \mathfrak{h}_j |\, e_j\rangle\, d\chi_m^{\blacksquare} + \frac{\partial}{\partial \psi_l^{\blacksquare}} \sum_{j=1}^{n} \mathfrak{h}_j |\, e_j\rangle d\psi_l^{\blacksquare}$$

$$= \sum_{j=1}^{n} \frac{\partial \mathfrak{h}_j}{\partial \chi_m^{\blacksquare}} |\, e_j\rangle d\chi_m^{\blacksquare} + \sum_{j=1}^{n} \frac{\partial \mathfrak{h}_j}{\partial \psi_l^{\blacksquare}} |\, e_j\rangle d\psi_l^{\blacksquare}$$

$$;\ \partial Ᵽ = \frac{\partial}{\partial \chi_m^{\blacksquare}} \sum_{\sigma=1}^{n} ɰ_\sigma |\, e_\sigma\rangle\, d\chi_m^{\blacksquare} + \frac{\partial}{\partial \psi_l^{\blacksquare}} \sum_{\sigma=1}^{n} ɰ_\sigma |\, e_\sigma\rangle d\psi_l^{\blacksquare}$$

$$= \sum_{\sigma=1}^{n} \frac{\partial ɰ_\sigma}{\partial \chi_m^{\blacksquare}} |\, e_\sigma\rangle d\chi_m^{\blacksquare} + \sum_{\sigma=1}^{n} \frac{\partial ɰ_\sigma}{\partial \psi_l^{\blacksquare}} |\, e_\sigma\rangle d\psi_l^{\blacksquare}.$$

$$\Rightarrow \partial Ո_{\mathrm{j}} = \begin{bmatrix} \frac{\partial \mathfrak{h}_j}{\partial \chi_m^{\blacksquare}} & \frac{\partial \mathfrak{h}_j}{\partial \psi_l^{\blacksquare}} \end{bmatrix} \times \begin{bmatrix} d\chi_m^{\blacksquare} \\ d\psi_l^{\blacksquare} \end{bmatrix}$$

$$\Rightarrow \partial Ᵽ_\sigma = \begin{bmatrix} \frac{\partial ɰ_\sigma}{\partial \chi_m^{\blacksquare}} & \frac{\partial ɰ_\sigma}{\partial \psi_l^{\blacksquare}} \end{bmatrix} \times \begin{bmatrix} d\chi_m^{\blacksquare} \\ d\psi_l^{\blacksquare} \end{bmatrix}$$

$$\therefore \begin{bmatrix}\partial \text{Ո}_j \\ \partial \text{Ᵽ}_\sigma\end{bmatrix} = \underbrace{\begin{bmatrix}\dfrac{\partial \text{ɧ}_j}{\partial \chi_m^{\blacksquare}} & \dfrac{\partial \text{ɧ}_j}{\partial \psi_l^{\blacksquare}} \\ \dfrac{\partial \text{ɰ}_\sigma}{\partial \chi_m^{\blacksquare}} & \dfrac{\partial \text{ɰ}_\sigma}{\partial \psi_l^{\blacksquare}}\end{bmatrix}}_{\text{Ӂ}} \times \begin{bmatrix} d\chi_m^{\blacksquare} \\ d\psi_l^{\blacksquare}\end{bmatrix}$$

- Ӂ is the matrix of the derivatives of the components of $\partial \text{Ո}_j; \partial \text{Ᵽ}_\sigma$ – called the **Jacobian Transition Matrix**.

The conditions for derivable functions in the τ-space is provided by multiplying the τ-variable $\boldsymbol{q_{ml}}$ **by its stable differential form** $\boldsymbol{dq_{ml}}^{\blacksquare}$;(the indexes $m;l$ indicate the mth complex component and the lth τ-component):

Let $\mathrm{q}_{ml} = \chi_m |\, e_m\rangle + \psi_l |\, e_l\rangle$; $q_{ml}{}^{\blacksquare} = \chi_m^{\blacksquare} |\, e_m\rangle + \psi_l^{\blacksquare} |\, e_l\rangle$

$$q\ dq^{\blacksquare} = (\chi_m |\, e_m\rangle + \psi_l |\, e_l\rangle)\big(d\chi_m^{\blacksquare} |\, e_m\rangle + d\psi_l^{\blacksquare} |\, e_l\rangle\big)$$

$$= \chi_m |\, e_m\rangle d\chi_m^{\blacksquare} |\, e_m\rangle + \chi_m |\, e_m\rangle d\psi_l^{\blacksquare} |\, e_l\rangle + \psi_l |\, e_l\rangle d\chi_m^{\blacksquare} |\, e_m\rangle + \psi_l |\, e_l\rangle d\psi_l^{\blacksquare} |\, e_l\rangle$$

$$= \psi_l d\psi_l^{\blacksquare} |\, e_l\rangle + \chi_m d\chi_m^{\blacksquare} |\, e_m\rangle^2 + (\chi_m d\psi_l^{\blacksquare} |\, e_l\rangle + \psi_l d\chi_m^{\blacksquare} |\, e_l\rangle) |\, e_m\rangle$$

- ***Remark***: $|\, e_m\rangle^2 \equiv \tau^2$, which is remained as τ^2, but treated as an outside element of the τ-vector, as demonstrates the organization of the elements above.

$$\Rightarrow q\ dq^{\blacksquare} = \begin{bmatrix}\psi_l & \tau^2\chi_m \\ \chi_m & \psi_l\end{bmatrix} \times \begin{bmatrix} d\chi_m^{\blacksquare} \\ d\psi_l^{\blacksquare}\end{bmatrix}$$

$$\therefore \begin{cases} \begin{bmatrix}\partial \text{Ո}_j \\ \partial \text{Ᵽ}_\sigma\end{bmatrix} = \begin{bmatrix}\dfrac{\partial \text{ɧ}_j}{\partial \chi_m^{\blacksquare}} & \dfrac{\partial \text{ɧ}_j}{\partial \psi_l^{\blacksquare}} \\ \dfrac{\partial \text{ɰ}_\sigma}{\partial \chi_m^{\blacksquare}} & \dfrac{\partial \text{ɰ}_\sigma}{\partial \psi_l^{\blacksquare}}\end{bmatrix} \times \begin{bmatrix} d\chi_m^{\blacksquare} \\ d\psi_l^{\blacksquare}\end{bmatrix} \\ q\ dq^{\blacksquare} = \begin{bmatrix}\psi_l & \tau^2\chi_m \\ \chi_m & \psi_l\end{bmatrix} \times \begin{bmatrix} d\chi_m^{\blacksquare} \\ d\psi_l^{\blacksquare}\end{bmatrix} \end{cases}$$

- ***Definition III***: $f_\tau(q)$ is holomorphic **iff** $q\ dq^{\blacksquare} \equiv \begin{bmatrix}\partial \text{Ո}_j \\ \partial \text{Ᵽ}_\sigma\end{bmatrix}$.

That's clear, because the matrix of the derivatives of f_τ must be Ӂ, then:

$$\Rightarrow q\ dq^{\blacksquare} \equiv \begin{bmatrix}\partial \text{Ո}_j \\ \partial \text{Ᵽ}_\sigma\end{bmatrix} \Leftrightarrow \begin{bmatrix}\psi_l & \tau^2\chi_m \\ \chi_m & \psi_l\end{bmatrix} = \begin{bmatrix}\dfrac{\partial \text{ɧ}_j}{\partial \chi_m^{\blacksquare}} & \dfrac{\partial \text{ɧ}_j}{\partial \psi_l^{\blacksquare}} \\ \dfrac{\partial \text{ɰ}_\sigma}{\partial \chi_m^{\blacksquare}} & \dfrac{\partial \text{ɰ}_\sigma}{\partial \psi_l^{\blacksquare}}\end{bmatrix}$$

$\Rightarrow \psi_l = \frac{\partial \text{ɧ}_j}{\partial \chi_m^{\blacksquare}}$

$; \tau^2 \chi_m = \frac{\partial \text{ђ}_j}{\partial \psi_l^{\blacksquare}} \Rightarrow \chi_m = \tau^2 \frac{\partial \text{ђ}_j}{\partial \psi_l^{\blacksquare}}$

$; \psi_l = \frac{\partial \text{щ}_\sigma}{\partial \psi_l^{\blacksquare}}$

$; \chi_m = \frac{\partial \text{щ}_\sigma}{\partial \chi_m^{\blacksquare}}$

$\therefore$

$$\boxed{1)\ \frac{\partial \text{ђ}_j}{\partial \chi_m^{\blacksquare}} = \frac{\partial \text{щ}_\sigma}{\partial \psi_l^{\blacksquare}} \qquad ; \qquad 2)\ \tau^2 \frac{\partial \text{ђ}_j}{\partial \psi_l^{\blacksquare}} = \frac{\partial \text{щ}_\sigma}{\partial \chi_m^{\blacksquare}}}$$

- Here, one can easily assimilate the conditions for holomorphic τ-functions with the τ-trivial functions' condition, although the same are deduced in a completely distinctive approach. However, there is the necessity to point out that such conditions are independent of each other and are not the same assumption.

Application: Normalization of f_τ:

- The τ-function is a remarkable artifice to decode dispersion of information – highlighted when the latter is in $\mathbb{T}^n$, then, the dispersion is broken-out in an nth dimension vector field. However, questions such as if any τ-function is allowed, and if which of the latter have a real meaning need to be retorted:

- In the following derivation, notations such as $\xi_\alpha; \xi_\beta$ will represent two different scape-angles from the τ-sphere; $\text{Ħ}(q)$ used to denote a well-behaved function in the τ-sphere – which is τ-integrable; τ-analytic and endomorphic. $\hat{\text{Ѫ}}_\alpha; \hat{\text{Ѫ}}_\beta$ are going to be used to represent transformations of the τ-functions depending on $\alpha; \beta$, respectively; $\text{ю}_\alpha; \text{ю}_\beta$ the eigenvalues.

- Given two functions $\text{Ħ}_\alpha(q); \text{Ħ}_\beta(q)$ in hyperbolic coordinates - $\text{Ħ}(\text{Я}, \xi_\alpha); \text{Ħ}(\text{Я}, \xi_\beta)$, their inner product is defined as follows:

$$\langle \text{ю}_\alpha \text{Ħ}(\text{Я}, \xi_\alpha) | \text{ю}_\beta \text{Ħ}(\text{Я}, \xi_\beta) \rangle = \text{ю}_\alpha \text{ю}_\beta \delta_{\alpha\beta}$$

; $\delta_{\alpha\beta}$ is the Kronecker Delta, defined as follows: $\delta_{\alpha\beta} = \begin{cases} 1, \alpha = \beta \\ 0, \alpha \neq \beta \end{cases}$.

Therefore, the inner product is just different of zero when the angles are equal. Then:

$$\langle \text{ю}_\alpha \text{Ħ}(\text{Я}, \xi_\alpha) | \text{ю}_\beta \text{Ħ}(\text{Я}, \xi_\beta) \rangle = \oiint \overline{\left(\text{ю}_\alpha \text{Ħ}(\text{Я}, \xi_\alpha)\right)} \left(\text{ю}_\beta \text{Ħ}(\text{Я}, \xi_\beta)\right) d\text{Я} d\xi = \text{ю}_\alpha \text{ю}_\beta \delta_{\alpha\beta}$$

$$\Rightarrow \oiint \overline{\left(\text{ю}_\alpha \text{Ħ}(\text{Я}, \xi_\alpha)\right)} \left(\text{ю}_\beta \text{Ħ}(\text{Я}, \xi_\beta)\right) d\text{Я} d\xi \overset{\textit{eigenvalue}}{=} \oiint \overline{\left(\hat{\text{Ѫ}}_\alpha | \text{Ħ}(\text{Я}, \xi_\alpha)\rangle\right)} \left(\hat{\text{Ѫ}}_\beta | \text{Ħ}(\text{Я}, \xi_\beta)\rangle\right) d\text{Я} d\xi$$

- ***Theorem IV***: If $\text{Ħ}_\alpha{}^{\blacksquare}$ and are $\text{Ħ}_\beta{}^{\blacksquare}$ are the stable form of $\text{Ħ}_\alpha; \text{Ħ}_\beta$, respectively; $\hat{\text{Ϯ}}^{-1}$ and $\hat{\text{Ϯ}}$ are homomorphisms and anti-homomorphisms, respectively ($\hat{\text{Ϯ}}_\alpha^{-1}, \hat{\text{Ϯ}}_\beta^{-1}: \mathbb{T}^n \twoheadrightarrow \mathbb{C}^n; \hat{\text{Ϯ}}_\alpha, \hat{\text{Ϯ}}_\beta: \mathbb{C}^n \twoheadrightarrow \mathbb{T}^n$) – (as proved in *Theorem II*), then the stable function of the product of Ħ_α and Ħ_β is also stable.

If $\hat{\Upsilon}^{-1}$ and $\hat{\Upsilon}$ are homomorphisms and anti-homomorphisms, respectively, then:

$$\hat{\Upsilon}^{-1}|\,\varsigma\Phi\rangle = \hat{\Upsilon}^{-1}|\,\varsigma\rangle\hat{\Upsilon}^{-1}|\,\Phi\rangle$$
$$;\,\hat{\Upsilon}|\,\varsigma\Phi\rangle = \hat{\Upsilon}|\,\Phi\rangle\hat{\Upsilon}|\,\varsigma\rangle$$
For any two τ-functions $\varsigma;\ \Phi \in \mathbb{T}^n$.

$$\Rightarrow \hat{\Upsilon}\hat{\Upsilon}^{-1}|\,\varsigma\Phi\rangle = \hat{\Upsilon}\Big|\ \hat{\Upsilon}^{-1}|\,\varsigma\rangle\hat{\Upsilon}^{-1}|\,\Phi\rangle\Big\rangle = \hat{\Upsilon}\hat{\Upsilon}^{-1}|\,\Phi\rangle\hat{\Upsilon}\hat{\Upsilon}^{-1}|\,\varsigma\rangle$$

$$\Rightarrow \hat{Ѫ}|\,\varsigma\Phi\rangle = \hat{Ѫ}|\,\Phi\rangle\hat{Ѫ}|\,\varsigma\rangle$$

- Applying $Ꙟ_\alpha;\ Ꙟ_\beta$:

$\hat{Ѫ}|\ Ꙟ_\alpha Ꙟ_\beta\rangle = Ѫ|\ Ꙟ_\beta\rangle\hat{Ѫ}|\ Ꙟ_\alpha\rangle = Ꙟ_\beta{}^{\blacksquare}Ꙟ_\alpha{}^{\blacksquare}$, which is stable.

This completes our proof, and provides the necessary information to proceed with the main derivation ■

- Applying *Theorem IV* to the double integral:

$$\oiint \overline{\left(\hat{Ѫ}_\alpha|\ Ꙟ(Я,\xi_\alpha)\rangle\right)}\left(\hat{Ѫ}_\beta|\ Ꙟ(Я,\xi_\beta)\rangle\right)dЯd\xi = \oiint \overline{Ꙟ^{\blacksquare}(Я,\xi_\beta)}Ꙟ^{\blacksquare}(Я,\xi_\alpha)dЯd\xi$$

- By *Corollary I; Corollary II*:

$$\boxed{\oiint \overline{Ꙟ^{\blacksquare}(Я,\xi_\beta)}Ꙟ^{\blacksquare}(Я,\xi_\alpha)dЯd\xi = \oiint_{\Gamma\in\mathbb{C}} \overline{Ꙟ^{\blacksquare}(Я,\xi_\beta)}Ꙟ^{\blacksquare}(Я,\xi_\alpha)dЯd\xi = ю_\alpha ю_\beta \delta_{\alpha\beta}}$$

- And for $\beta = \alpha$:

$$\oiint_{\Gamma\in\mathbb{C}} \overline{Ꙟ^{\blacksquare}(Я,\xi_\alpha)}Ꙟ^{\blacksquare}(Я,\xi_\alpha)dЯd\xi = \oiint_{\Gamma\in\mathbb{C}} \left|Ꙟ^{\blacksquare}(Я,\xi_\alpha)\right|^2 dЯd\xi = ю_\alpha^2$$

- The index α can be removed without any loss of generality:

$$\boxed{\therefore\ \oiint_{\Gamma\in\mathbb{C}} \left|Ꙟ^{\blacksquare}(Я,\xi)\right|^2 dЯd\xi = ю^2}$$

Section 4 – The τ-line integral:

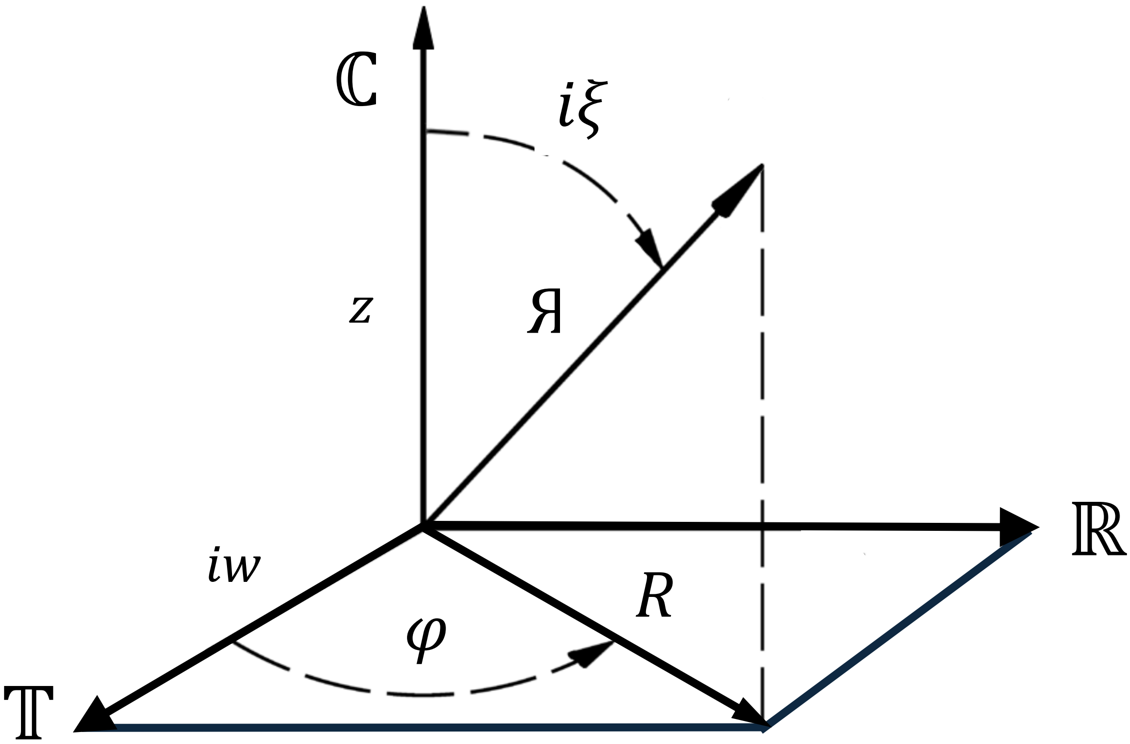
In this section we are going to derive the result that intersects the intuition with the mathematical formalism of the geometry of S_τ.

For such a realization, the formal treatment of the already introduced τ-integral is required – this section will provide enough formalization of the latter until it reaches the domain of Complex Analysis, where the concepts become auto explainable – based on the solid mathematical axioms that already exist.

- ***<u>Remark: For this section, the brief simplification of the degrees of freedom of irreversibility will be adopted ($\mathbb{T}^1$; $\mathbb{C}^1$) – and farther transcribed to tensor formalism.</u>***

Spherical Hyperbolic Coordinates:

The τ-integral, conceptually, is a field over S_τ, and S_τ is regarded as the path in which the field is acting on. The following representation indicates the τ-sphere in hyperbolic coordinates, indicating the $\mathbb{T}$; $\mathbb{C}$; $\mathbb{R}$ axes:



- For a τ-variable $q \in \mathbb{T}^1$; $q = z + \tau w$, the transformation of coordinates takes the form:

$$\begin{cases} z = Я cosh(\xi) \\ w = Я sinh(\xi) \end{cases}$$

- $|q| = \sqrt{z^2 - w^2} = Я$
- R is the projection of Я onto the Argand-Gauss plane – the radius of the complex numbers in its polar-form; associated with the angle φ, likewise in Complex Analysis.
- S_τ can be partitioned into three inner circumferences to be analyzed: $S_\tau = \{\{в_{\mathbb{T}\times\mathbb{C}}\}, \{в_{\mathbb{T}\times\mathbb{R}}\}, \{в_{\mathbb{C}\times\mathbb{R}}\}\}$; a remarkable circumference betwixt the partitions of S_τ is $в_{\mathbb{C}\times\mathbb{R}}$ – because the same is framework of Complex Analysis.

Therefore, it is clear the necessity to find the Jacobian of such a transformation of variables:

- The Jacobian Matrix for one irreversible dimension takes the form:

$$J(Я,\xi)=\begin{bmatrix}\frac{\partial z}{\partial Я} & \frac{\partial \mathrm{w}}{\partial Я}\\ \frac{\partial \mathrm{z}}{\partial \xi} & \frac{\partial \mathrm{w}}{\partial \xi}\end{bmatrix}$$

- It follows that:

$$J(Я,\xi)=\begin{bmatrix}cosh(\xi) & sinh(\xi)\\ Яsinh(\xi) & Яcosh(\xi)\end{bmatrix}$$

$$\Rightarrow det[J(Я,\xi)]=\begin{vmatrix}cosh(\xi) & sinh(\xi)\\ Яsinh(\xi) & Яcosh(\xi)\end{vmatrix}=Я^2$$

- ***Historical Context I***: In order to compute the area related to a period of a specific partition of S_τ – for purposes of exemplification the period $z=[0,1]; w=[0,i]$ of $C_{\mathbb{T}\times\mathbb{R}}$, forming a triangle, would be methodologic to:

1) Calculate the area of the triangle, which would result in a null area.
2) Use **standard integration**:

$$\int_0^1 f(q)\,dq=i\int_0^1 q\,dq=i\frac{q^2}{2}\bigg|_0^1=\frac{i}{2}$$

  - However, the area of the triangle calculated by $base*height$ and by using integration doesn't match, therefore, what was concluded is that neither of the algorithms for calculating such an area were valid – then – a new algorithm shall be developed, the called (and already commented) τ-integral.
  - At the beginning of the development of novel algorithms for calculating areas inside the τ-sphere, option (1) or (2) were thoughtful to be correct, which wasn't proved to be true.

The τ-integral algorithm and the extension to a desirable space $\{S\}$:

The algorithm for τ-integration is not closed – it depends strictly of the **space** where it is being realized. A brief introduction to τ-kind spaces will be provided in the following section, however, to find a most complete theoretical derivation, the reader has to guide himself to *Section [X]*.

- ***Definition IV***: A space $\{S\}$ with an attached variable s - which ranges over it, for τ-algebra is the path in which a function $f(\kappa)$ in a generic space $\{Ϛ\}$ acts as a field. For all considered spaces $\{S\}$ there must be an operation that matches $f(\kappa)$ with the space desirable variable – and such an operation is the inner product.

The inner product betwixt the attached variable s ranging over $\{S\}$, and the function $f(\kappa)$, describes (is equal to) a well-defined function inside the desirable space $\{S\}$ – $f(s)$.

$$\Rightarrow f(s)=\langle s|f(\kappa)\rangle\ .$$

- ***Theorem V***: The function in a desired space $\{S\}$ is obtained from the inner product betwixt a generic invariant function $f(\kappa)$ in $\{\varsigma\}$ with the attached variable $s \in \{S\}$.

- Let the reversibility operators $\hat{\Upsilon}_s^{-1}; \hat{\Upsilon}_s$ be defined as linear maps from the generic space onto $\{S\}$ and the inverse, respectively. ($\hat{\Upsilon}_s^{-1}: \{\varsigma\} \mapsto\!\!\!\!\rightarrow \{S\};\ \hat{\Upsilon}_s: \{S\} \mapsto\!\!\!\!\rightarrow \{\varsigma\}$):

$$\hat{\Upsilon}_s^{-1} |\, f(\kappa)\rangle = f(s);\ \hat{\Upsilon}_s |\, f(s)\rangle = f(\kappa)$$

$$\Rightarrow \hat{\Upsilon}_s^{-1} |\, f(\kappa)\rangle = \langle s|f(\kappa)\rangle \ \Rightarrow \hat{\Upsilon}_s\hat{\Upsilon}_s^{-1} |\, f(\kappa)\rangle = \hat{\Upsilon}_s \big|\, \langle s|f(\kappa)\rangle\big\rangle$$

- Applying *Corollary II* for the space {S} (unknown if reversible or not):

$$\Rightarrow \hat{\Upsilon}_s\hat{\Upsilon}_s^{-1} |\, f(\kappa)\rangle = \hat{\Upsilon}_s \big|\, \langle s|f(\kappa)\rangle\big\rangle = |\, f^{\blacksquare}(\kappa)\rangle$$

- Therefore, if $|\, f^{\blacksquare}(\kappa)\rangle$ is a stable function (reversible) in the generic space, then, $\hat{\Upsilon}_s \big|\, \langle s|f(\kappa)\rangle\big\rangle$ is stable too, and therefore reversible.

But if $\hat{\Upsilon}_s \big|\, \langle s|f(\kappa)\rangle\big\rangle$ is stable, and is defined in the generic space, then, $\langle s|f(\kappa)\rangle$ must be a function (whether reversible or not) inside $\{S\}$ ■

Remark: If such a function in the space $\{S\}$ is reversible, then, its irreversible variable (similar to w, in τ-functions) is null, and therefore, by Theorem II, $\{S\} \cong \{\varsigma\}$.

The information obtained in *Definition IV* and *Theorem V* are enough to obtain a concise algorithm for calculating *some integrals in a desirable space $\{S\}$ – **under conditions:**

- Assume variables in a generic space $\{\varsigma\}$ are regarded to be as the previously treated τ-variable (will be provided a general definition for generic spaces together with its equivalence in $\mathbb{T}$):

1) $\forall \kappa \in \{\varsigma\}, \kappa_{cd}$ is the tensorial generic variable, which has two elements $|\, e_c\rangle;\ |\, e_d\rangle; |\, e_c\rangle \in \{S\};\ |\, e_d\rangle \in \{\varsigma\}$.
   - ***Analogous in $\mathbb{T}$ (as previously seen)***: $\forall q \in \mathbb{T}^{\boldsymbol{n}}, q_{ml}$ is the tensorial τ-variable, which has two elements $|\, e_m\rangle; |\, e_l\rangle; |\, e_m\rangle \in \mathbb{C}^n; |\, e_l\rangle \in \mathbb{T}^{\boldsymbol{n}}$.
2) $\{S\}$ is a proper subset of $\{\varsigma\}$
   - ***Analogous in $\mathbb{T}$ (as previously seen)***: By *Corollary I* – $\mathbb{C}^n$ is a proper subset of $\mathbb{T}^{\mathrm{n}}$.

- Given $\kappa_{cd} \in \{\varsigma\}; \kappa_{cd} = \rho_c |\, e_c\rangle + \lambda_d |\, e_d\rangle$ for two elements ρ, λ belonging to the proper subset of $\{\varsigma\}$ – i.e., $\rho, \lambda \in \{S\}; |\, e_c\rangle \in \{S\};\ |\, e_d\rangle \in \{\varsigma\}$:

$$\Rightarrow \int_{\{S\}} \hat{\Upsilon}_s \big|\, \langle s|f(\kappa)\rangle\big\rangle ds = \int_{\{S\}} f^{\blacksquare}(\kappa)\, d\kappa = \int_{\{S\}} f^{\blacksquare}(\kappa(\rho,\lambda))\, d\kappa = \int_{\{S\}} f^{\blacksquare}(\kappa(\rho,\lambda = 0))d\kappa$$
$$= \int_{\{S\}} f^{\blacksquare}(\kappa(\rho))d\kappa = \int_{\{S\}} f^{\blacksquare}(\rho)d\rho$$

The generalized $\hat{\Upsilon}$ transformation:

Were widely discussed the algebraic properties of the reversibility operator, however, the context requires a most comprehensive definition of the latter.

- The unique conditions for a reversibility operator to exist are:

1) Unicity – the reversibility operator must be uniquely defined in a generic space $\{\varsigma\}$ and in its proper subgroup $\{S\}$;
2) The reversibility operator $\hat{\Upsilon}^{-1}$ is a homomorphism, whereas $\hat{\Upsilon}$ is an anti-homomorphism ($\hat{\Upsilon}^{-1}$ maps from the group to its subgroup, whereas $\hat{\Upsilon}$ maps from the subgroup onto the group);
3) Must attend the eigenvalue relation.

However, that are significantly many algebraic operations and operators which do satisfy such conditions. Here below are listed some of them:

- $\hat{\Upsilon}$ as an integral transformation – similar to Fourier's or Laplace's approaches. Such approach will be treated in more detail in the next section, where the τ-integral transform will be introduced.
- $\hat{\Upsilon}$ as a multiplication by the Jacobian Transition Matrix – Ӂ – representing a transition inside the τ-sphere domain.

Application: The algorithm over the Laplace Space $\{\mathcal{L}\}$:

In this application, $\hat{\Upsilon}$ will be used as an integral transformation, however, an integral transformation inside the Laplace Space – which is commonly recognized as the Laplace Transform.

For a given space $\{S\} = \{\mathcal{L}\};\ y \in \{\mathcal{L}\}$; $\{\mathcal{L}\}$ is called Laplace Space; the derived algorithm to generic spaces takes the form:

- Let $x \in \mathbb{R};\ y \in \{\mathcal{L}\}$:
  - By condition **1** of generic spaces - $\{\mathcal{L}\}$ is a proper subgroup of $\mathbb{R}$, $\Rightarrow$ $\{\mathcal{L}\} \subset \mathbb{R}$.
  - It's known that a Laplace-transformed variable, y, does not assume complex values; the Laplace-transform is reversible for well-behaved functions, so, $\mathbb{R}$ has similar elements to $\{\mathcal{L}\}$. Therefore, by *Theorem II* – $\{\mathcal{L}\} \cong \mathbb{R}$.

- Therefore, it's affirmable that $\forall y \in \{\mathcal{L}\}, y = y_0$ – indicating that y has no other elements (such as complex or irreversible – as discussed above)

$$\int_{\{\mathcal{L}\}} \hat{\Upsilon}_y |\ \langle y|f(x)\rangle\rangle dy = \int_{\{\mathcal{L}\}} f^{\blacksquare}(y_0)dy_0$$

- If the Laplace-transform is reversible for well-behaved functions, so, $f^{\blacksquare}(y_0) = f(y_0)$, since the function is already in its stable form.

- Then, we get:

$$\int_{\{\mathcal{L}\}} \hat{\Upsilon}_y | \langle y | f(x) \rangle \rangle dy = \int_{\{\mathcal{L}\}} f(y_0) dy_0 \,.$$

Section 5 – Dirac-Basis functions and Kernels of the Generalized Transformations:

In the following section, using the concepts carefully derived in the latest sections, I shall initially derive a novel way to visualize functions, using artifices provided by distributional calculus, and Bromwich Integral.

Later in this section, once derived the Dirac-Basis for such functions, the objective will consist in a generalization regarding the reversibility operator in its integral form for a generic function $f(\kappa)$, as well as the respective kernels for the transformation.

Dirac-Basis functions:

- ***Remark: The Laplace Transform is going to be regarded as an operator $\widehat{\mathcal{L}}$.***

Let $\tilde{f}(y)$ be a Laplace transform of the function $f(x)$, such that $\tilde{f}(y) = \hat{\mathcal{L}} | f(x) \rangle$.

Nevertheless, in this section we are worried about the anti-Laplace transform – the reverse path:

$$f(x) = \hat{\mathcal{L}}^{-1} | \tilde{f}(y) \rangle = \frac{1}{2i\pi} \oint_{\tilde{\Delta} \epsilon \mathbb{C}^1} \tilde{f}(z) e^{zx} \, dz$$

- By the definition of the complex integral:

$$\frac{1}{2i\pi} \oint_{\tilde{\Delta} \epsilon \mathbb{C}^1} \tilde{f}(z) e^{zx} \, dz = \frac{1}{2i\pi} \int_{-i\infty+\gamma}^{i\infty+\gamma} \tilde{f}(y) e^{yx} dy$$

- An important assumption is to set $\gamma = 0$ – that is, assuming that the complex poles we lead with are purely imaginary; and that will be the case. Therefore, $\gamma = 0$.

$$\Rightarrow \frac{1}{2i\pi} \oint_{\tilde{\Delta} \epsilon \mathbb{C}^1} \tilde{f}(z) e^{zx} \, dz = \frac{1}{2i\pi} \int_{-i\infty+\gamma}^{i\infty+\gamma} \tilde{f}(y) e^{yx} dy \mapsto \frac{1}{2i\pi} \int_{-i\infty}^{i\infty} \tilde{f}(y) e^{yx} dy$$

- Setting $u = \frac{y}{i}$:

$$\frac{1}{2i\pi} \int_{-i\infty}^{i\infty} \tilde{f}(y) e^{yx} dy \mapsto \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(iu) e^{iux} du$$

- At this moment, an important condition must be imposed: $\tilde{f}(iu)$ allows Lorentz Series – that is because I shall use such function in its Taylor Series centered at $iu = p$, from now on:

$$\tilde{f}(iu) = \sum_{r}^{\infty} \tilde{a}_r \,(iu - p)^{g_r}$$

- $\tilde{a}_r$ denotes the coefficient of the Laplace-transformed function - $\tilde{f}(iu)$;
- g_r denotes the coefficient of the rth power related to the variable u;
- ip is an imaginary constant – contained in the sub circumference $ϐ_{\mathbb{T}\times\mathbb{C}}$

- We do our last exchange of variables: $iu - ip = iv$

$$\Rightarrow \frac{e^{ipx}}{2\pi} \int_{-\infty}^{\infty} \sum_{r}^{\infty} \tilde{a}_r \,(iv)^{g_r} e^{ivx} dv$$

- By the following identity – where $\delta^{(n)}(M)$ consists of the nth derivative of the Dirac-delta function:

$$\int_{-\infty}^{\infty} \frac{\partial^n}{\partial M^n} e^{iMt} dt = \int_{-\infty}^{\infty} (it)^n \, e^{iMt} dt = 2\pi\delta^{(n)}(M)$$

$$\Rightarrow \frac{e^{ipx}}{2\pi} \int_{-\infty}^{\infty} \sum_{r}^{\infty} \tilde{a}_r \,(iv)^{g_r} e^{ivx} dv = e^{ipx} \sum_{r}^{\infty} \tilde{a}_r \, \delta^{(g_r)}(x)$$

$$\boxed{\Rightarrow \underbrace{f}(x) = e^{ipx} \sum_{r}^{\infty} \tilde{a}_r \, \delta^{(g_r)}(x)}$$

- ***Remark: The notation for the distributional form of $f(x)$ is $\underbrace{f}(x)$.***

Is remarkable that such a sum has an outside term, in which is a complex exponential – such an exponential will be explored in the next section and be used to derive the Kernel of the τ-integral transform; used to derive the Generalized Residue Theorem for τ-functions.

The referred $f(x)$ does not consist of the same obtained by the standard Laplace anti-transform – due to its distributional nature. Distributional functions – which we are going to call Dirac-Basis functions, exist specifically to solve operations – in the sight that the latter does not output the same values of the original function obtained by the Laplace anti-transform.

The imaginary constant ip obtained in the process, centralizes the complex function $\tilde{f}(iu)$ in the point $z = ip$ in the sub circumference $ϐ_{\mathbb{T}\times\mathbb{C}}$. The point p can be thought of as a

singularity of the function $\tilde{f}(iu)$, whose behavior will be deeply defined in the next section.

Generalized Integral Transformations:

Remark: In this subsection the operator $\hat{\Upsilon}$ is going to be used to represent an integral transform, respecting the same axioms provided by Section 4:

When referring to an integral transformation, there are intrinsic imposed conditions that must be remarkably observed:

1) An Integral transformation $\hat{\Upsilon}_s^{-1}$ maps a function $f(\kappa)$ in a generic space $\{\varsigma\}$ to a function $f(s)$ in a subspace $\{S\}$; and $\hat{\Upsilon}_s$ the analogous in the inverse direction.
2) Such a transformation consists in a homomorphism/anti-homomorphism, therefore, it has a kernel K, which consists in elements mapped from $\{\varsigma\}$ onto an identity element in $\{S\} - K = \{\kappa \in \{\varsigma\} | \hat{\Upsilon}_s^{-1} | f(\kappa)\rangle = e_s\}$; e is an identity element in $\{S\}$.

- Therefore, let $K(\kappa)$ *and* $K(s)$ be kernels in $\{\varsigma\}$ and in $\{S\}$, respectively, then the integral transformations $\hat{\Upsilon}_s^{-1}: \{\varsigma\} \mapsto\!\!\!\!\twoheadrightarrow \{S\}$; $\hat{\Upsilon}_s: \{S\} \mapsto\!\!\!\!\twoheadrightarrow \{\varsigma\}$; take the form:

$$\hat{\Upsilon}_s^{-1} | f(\kappa)\rangle = \sqrt{J} \int_{\{\varsigma\}} \mathrm{f}(\kappa) K(\kappa) d\kappa = f(s)$$

$$; \hat{\Upsilon}_s | f(s)\rangle = \sqrt{J} \int_{\{S\}} \mathrm{f}(s) K(s) ds = f^{\blacksquare}(\kappa)$$

- J is the Jacobian of the transformation.
  - Observe that $\hat{\Upsilon}_s^{-1}$ is calculated over the generic space $\{\varsigma\}$, whereas $\hat{\Upsilon}_s$ over the desired space $\{S\}$.
- To proceed, the algorithm previously used in this section to derive Dirac-basis functions, can be generalized for an arbitrary transform $\hat{\Upsilon}_s$:

$$\therefore \hat{\Upsilon}_s \hat{\Upsilon}_s^{-1} | f(\kappa)\rangle = \sqrt{J} \hat{\Upsilon}_s | f(s)\rangle = J \int_{\{S\}} \mathrm{f}(s) K(s) ds = f^{\blacksquare}(\kappa)$$

- $f(s)$ can be written in Lorentz Series; $\breve{\alpha}_\mu$ is the μth coefficient of such expansion:

$$\mathrm{f}(s) = \sum_{\mu}^{\infty} \breve{\alpha}_\mu (s - p)^\mu$$

$$\Rightarrow J \int_{\{S\}} \mathrm{f}(s) K(s) ds = J \int_{\{S\}} \sum_{\mu}^{\infty} \breve{\alpha}_\mu (s - p)^\mu K(s) ds = f^{\blacksquare}(\kappa)$$

- ***Corollary III***: Analogously to the case of the evaluation of the Bromwich Integral, the integration of the factor $(s-p)^{\mu}$ together with the kernel $K(s)$ over the space $\{S\}$ fragments the latter into a product of two functions of κ inside $\{\varsigma\}$:

$$J(s-p)^{\mu}K(s) \mapsto \eta_p(\kappa)\Omega^{\mu}(\kappa)$$

- $\eta_p(\kappa)$ denotes a function that preserves the singularity p in the process of anti-transforming $f(s)$.
- $\Omega^{\mu}(\kappa)$ denotes a special function in $\{\varsigma\}$, which will be called the basis of $f(\kappa)$ in its distributional form.

Proceeding with this approach, the distributional function in $\{\varsigma\}$, $\underbrace{f}(\kappa)$, is defined as follows:

$$J\int_{\{S\}}\sum_{\mu}^{\infty}\breve{\alpha}_{\mu}(s-p)^{\mu}\,K(s)ds = J\sum_{\mu}^{\infty}\breve{\alpha}_{\mu}\eta_p(\kappa)\Omega^{\mu}(\kappa) = \eta_p(\kappa)\sum_{\mu}^{\infty}\breve{\alpha}_{\mu}\Omega^{\mu}(\kappa) = \underbrace{f}^{\blacksquare}(\kappa)$$

$$\Rightarrow \underbrace{f}^{\blacksquare}(\kappa) = \eta_p(\kappa)\sum_{\mu}^{\infty}\breve{\alpha}_{\mu}\Omega^{\mu}(\kappa)$$

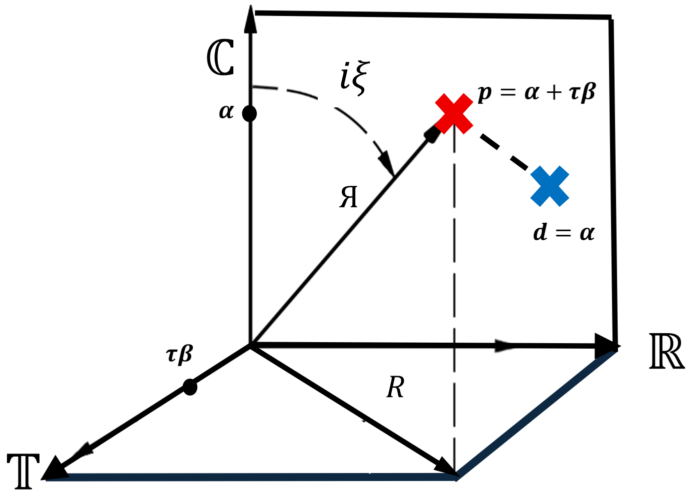
- ***Theorem VI***: The function that carries the singularity p in the reversion process $\hat{\Upsilon}_s: \{S\} \twoheadrightarrow \{\varsigma\} - \eta_p(\kappa)$, corresponds to the **kernel of the generalized integral transformation – $\boldsymbol{K(\kappa)}$,** for all $f(s)$ that allow Lorentz Series and can be represented using Ω-basis.

Section 6 – The Generalized Residue Theorem:

Once both the generic and desired functions and their respective spaces $\{\varsigma\};\ \{S\}$ were already introduced in the last two sections, as well as the algorithms for generalized integration and reversibility operators, the next requirement to be treated about is the Residue Theorem, most specifically, the generalized Residue Theorem.

The Residue Theorem sets the pillars of Complex Analysis, nevertheless its applications are restricted to the complex domain.

- ***Definition V***: Regard that S_{τ} can be partitioned into three sub circumferences (as treated in the last section): $S_{\tau} = \{\{\beta_{\mathbb{T}\times\mathbb{C}}\}, \{\beta_{\mathbb{T}\times\mathbb{R}}\}, \{\beta_{\mathbb{C}\times\mathbb{R}}\}\}$. Let $p;\ d$ be singularities in S_{τ} and $\beta_{\mathbb{T}\times\mathbb{C}}$, respectively. Below is the geometric representation of these statements:



- Like in the geometric representation above, the singularity $x = \alpha$ is regarded as being the projection on $\mathsf{б}_{\mathbb{T}\times\mathbb{C}}$ of the singularity $p = \alpha + \tau\beta$ in S_τ.
- To find a generalized residue theorem, such an interpretation allows to fragment the problem for each partition of the τ-sphere, such that the problem consists of find:

1) A residue theorem for $\mathsf{б}_{\mathbb{T}\times\mathbb{C}}$;
2) A residue theorem for $\mathsf{б}_{\mathbb{T}\times\mathbb{R}}$;
3) A residue theorem for $\mathsf{б}_{\mathbb{C}\times\mathbb{R}}$.

Furthermore, the residue theorem for $\mathsf{б}_{\mathbb{C}\times\mathbb{R}}$ consists of the standard residue theorem, which is solved. Then, to complete the generalization, there are just two sub circumferences left to work in: $\mathsf{б}_{\mathbb{T}\times\mathbb{C}}$ and $\mathsf{б}_{\mathbb{T}\times\mathbb{R}}$.

Following the sequence of the last section, I shall first derive the Residue theorem for the τ-space, and then, use it as a basis to construct such a theorem for any arbitrary generic function $f(\kappa)$:

Initially, I shall prove *Theorem VII*, which generalizes the holomorphic conditions for an order t monomial in $\mathbb{T}^n$:

Theorem VII: Let q be a variable in the n dimensional τ-space - $\mathbb{T}^n$, then, according to the previous derived conditions for τ-holomorphic functions, q^t is holomorphic, $\forall t \in \mathbb{R}$.

Therefore, as previously assumed in the derivation of the holomorphic conditions for τ-functions:

$$f_\tau(q) = \text{Ꞥ} + \tau\text{Ᵽ} = \sum_{j=1}^{n} \mathfrak{h}_j | e_j\rangle + \sum_{\sigma=1}^{n} \text{ɰ}_\sigma | e_\sigma\rangle$$

- $\mathfrak{h}_j$ is the complex part of f_τ;
- ɰ_σ is the τ-part of f_τ.

To keep the clearness of notation (such notations shall be in a table at page 1), let us work with the tensorial τ-function – $f_\tau^{j\sigma}(q)$:

$$\Rightarrow\ f_{\tau}^{j\sigma}(q) = \text{ɧ}_j|\, e_j\rangle + \text{ɰ}_\sigma|\, e_\sigma\rangle$$

- Assume ɧ_j; ɰ_σ are functions of stable variables $\chi_m^{\blacksquare}$; $\psi_l^{\blacksquare}$ – which are assumed to have predictable singularities, as well as being reversible. Then:
- Assume that the variable q can be written as a linear combination of the stable variables – $q(\chi_m^{\blacksquare}, \psi_l^{\blacksquare}) = \chi_m^{\blacksquare}|\, e_m\rangle + \psi_l^{\blacksquare}|\, e_l\rangle$; regard $|\, e_m\rangle$ the complex component and $|\, e_l\rangle$ the τ-component.

$$f_{\tau}^{j\sigma}(q) = \text{ɧ}_j|\, e_j\rangle + \text{ɰ}_\sigma|\, e_\sigma\rangle = f_{\tau}^{j\sigma}(q) = \text{ɧ}_j(\chi_m^{\blacksquare}, \psi_l^{\blacksquare})|\, e_j\rangle + \text{ɰ}_\sigma(\chi_m^{\blacksquare}, \psi_l^{\blacksquare})|\, e_\sigma\rangle$$

$$f_{\tau}^{j\sigma}(q) = q^t = \left(\chi_m^{\blacksquare}|\, e_m\rangle + \psi_l^{\blacksquare}|\, e_l\rangle\right)^t$$

- Using the binomial expansion:

$$q^t = \left(\chi_m^{\blacksquare}|\, e_m\rangle + \psi_l^{\blacksquare}|\, e_l\rangle\right)^t = \sum_{\lambda=0}^{t} \binom{t}{\lambda} {\chi_m^{\blacksquare}}^{t-\lambda}|\, e_m\rangle^{t-\lambda} {\psi_l^{\blacksquare}}^{\lambda}|\, e_l\rangle^{\lambda}$$

- As done before, consider $|\, e_l\rangle^2 \equiv \tau^2$, however, τ^2 is not an element of the τ-vector. Therefore, it is possible to divide the sum above in two, such that one has basis $|\, e_m\rangle$, whereas the other has basis $|\, e_l\rangle$.

$$\sum_{\lambda=0}^{t} \binom{t}{\lambda} {\chi_m^{\blacksquare}}^{t-\lambda}|\, e_m\rangle^{t-\lambda} {\psi_l^{\blacksquare}}^{\lambda}|\, e_l\rangle^{\lambda} =$$

$$= \sum_{\mathrm{h}=0}^{\left\lfloor\frac{t}{2}\right\rfloor} \binom{\left\lfloor\frac{t}{2}\right\rfloor}{2\mathrm{h}} \chi_m^{\blacksquare t-2\mathrm{h}}\,|\, e_m\rangle^{t-2\mathrm{h}} {\psi_l^{\blacksquare}}^{2\mathrm{h}}|\, e_l\rangle^{2\mathrm{h}} + \sum_{\mathrm{v}=0}^{\left\lfloor\frac{t}{2}\right\rfloor} \binom{\left\lfloor\frac{t}{2}\right\rfloor}{2\mathrm{v}+1} \chi_m^{\blacksquare t-(2\mathrm{v}+1)}\,|\, e_m\rangle^{t-(2\mathrm{v}+1)} {\psi_l^{\blacksquare}}^{2\mathrm{v}+1}|\, e_l\rangle^{2\mathrm{v}+1}$$

- In such a moment it is possible to relate $\text{ɧ}_j(\chi_m^{\blacksquare}, \psi_l^{\blacksquare})$ to the first sum; and $\text{ɰ}_\sigma(\chi_m^{\blacksquare}, \psi_l^{\blacksquare})$ to the second, such that:

$$\begin{bmatrix} \text{ɧ}_j \\ \text{ɰ}_\sigma \end{bmatrix} = \begin{bmatrix} \displaystyle\sum_{\mathrm{h}=0}^{\left\lfloor\frac{t}{2}\right\rfloor} \binom{\left\lfloor\frac{t}{2}\right\rfloor}{2\mathrm{h}} \chi_m^{\blacksquare t-2\mathrm{h}}\,|\, e_m\rangle^{t-2\mathrm{h}} {\psi_l^{\blacksquare}}^{2\mathrm{h}}|\, e_l\rangle^{2\mathrm{h}} \\ \displaystyle\sum_{\mathrm{v}=0}^{\left\lfloor\frac{t}{2}\right\rfloor} \binom{\left\lfloor\frac{t}{2}\right\rfloor}{2\mathrm{v}+1} \chi_{\mathrm{m}}^{\blacksquare t-(2\mathrm{v}+1)}\,|\, e_m\rangle^{t-(2\mathrm{v}+1)} {\psi_l^{\blacksquare}}^{2\mathrm{v}+1}|\, e_l\rangle^{2\mathrm{v}+1} \end{bmatrix}$$

- Applying the condition for holomorphic τ-functions:

$$1)\ \frac{\partial \text{ɧ}_j}{\partial \chi_m^{\blacksquare}} = \frac{\partial \text{ɰ}_\sigma}{\partial \psi_l^{\blacksquare}}$$

$$\Rightarrow \frac{\partial \text{ɧ}_j}{\partial \chi_m^{\blacksquare}} = \sum_{\mathrm{h}=0}^{\left\lfloor\frac{t}{2}\right\rfloor} \binom{\left\lfloor\frac{t}{2}\right\rfloor}{2\mathrm{h}} (t-2\mathrm{h}) \chi_m^{\blacksquare t-(2\mathrm{h}+1)}\,|\, e_m\rangle^{t-2\mathrm{h}} {\psi_l^{\blacksquare}}^{2\mathrm{h}}|\, e_l\rangle^{2\mathrm{h}}$$

$$; \frac{\partial \text{ɰ}_\sigma}{\partial \psi_l^{\blacksquare}} = \sum_{\mathrm{v}=0}^{\left\lfloor \frac{t}{2} \right\rfloor} \binom{\left\lfloor \frac{t}{2} \right\rfloor}{2\mathrm{v}+1} \chi_{\mathrm{m}}^{\blacksquare t-(2\mathrm{v}+1)} \mid e_m\rangle^{t-(2\mathrm{v}+1)} (2\mathrm{v}+1) \psi_l^{\blacksquare 2\mathrm{v}} \mid e_l\rangle^{2\mathrm{v}+1}$$

- With careful analysis, it is possible to observe that one expression equals to the other, satisfying the first condition.
- For the second condition:

$$2)\ \ \tau^2 \frac{\partial \mathfrak{h}_j}{\partial \psi_l^{\blacksquare}} = \frac{\partial \text{ɰ}_\sigma}{\partial \chi_m^{\blacksquare}}$$

$$\tau^2 \frac{\partial \mathfrak{h}_j}{\partial \psi_l^{\blacksquare}} \equiv \mid e_m\rangle^2 \frac{\partial \mathfrak{h}_j}{\partial \psi_l^{\blacksquare}} \sum_{\mathrm{h}=0}^{\left\lfloor \frac{t}{2} \right\rfloor} \binom{\left\lfloor \frac{t}{2} \right\rfloor}{2\mathrm{h}} \chi_m^{\blacksquare t-2\mathrm{h}} \mid e_m\rangle^{t-2\mathrm{h}} (2\mathrm{h}) \psi_l^{\blacksquare 2\mathrm{h}\text{-}1} \mid e_l\rangle^{2(\mathrm{h}+1)}$$

$$; \frac{\partial \text{ɰ}_\sigma}{\partial \chi_m^{\blacksquare}} = \sum_{\mathrm{v}=0}^{\left\lfloor \frac{t}{2} \right\rfloor} \binom{\left\lfloor \frac{t}{2} \right\rfloor}{2\mathrm{v}+1} (t-(2\mathrm{v}+1)) \chi_{\mathrm{m}}^{\blacksquare t-(2\mathrm{v}+1)} \mid e_m\rangle^{t-(2\mathrm{v}+1)} \psi_l^{\blacksquare 2\mathrm{v}+1} \mid e_l\rangle^{2\mathrm{v}+1}$$

- Which also are equivalent, over a rigid analysis.

Thus, the conditions of holomorphy are satisfied, then, $f_\tau(q) = q^t$ is a holomorphic function ■

Such theorem proved, led to a significant simplification of the problem:

Regard $f_\tau(q)$ in its Lorentz Series form:

$$f_\tau(q) = \sum_{r=-\infty}^{\infty} \varepsilon_r (q-p)^r$$

- p is the singularity to be treated about – a constant that does not change the results obtained in *Theorem VII* (it can be easily proved by absorbing p as a complex constant in $\chi_m^{\blacksquare}$; the latter remains being stable by *Theorem IV*).

The main goal is to, from the influence of the singularity p, obtain a residue theorem which provides help to solve τ-integrals. Therefore, consider that the objective is to evaluate the following integral in $\mathbb{T}^2$:

- From the last theorem we obtain:

$$\oint_{S_\tau} f_\tau(q) dq = \oint_{S_\tau} \sum_{r=-\infty}^{\infty} \varepsilon_r (q-p)^r \, dq$$

$$= \oint_{S_\tau} \sum_{r=1}^{\infty} \varepsilon_{\text{-}r} (q-p)^{\text{-}r} \, dq$$

Such reduction of the Lorentz Series to a sum of the negative powers of the series centered on the singularity is already known in Complex Analysis – such functions are called meromorphic. A τ-meromorphic function is regarded as holomorphic at every point of its domain – except at $\mathrm{q} = \mathrm{p}$ – where the singularity is contained.

- ***<u>Definition VI</u>***: τ-holomorphic endofunctions preserve the scape angle ξ. Such effect is called the angular conservation of τ-holomorphic functions; if the angle ξ is not changed over the range of such functions, they are said to be reversible.

A τ-meromorphic function also does a conservation of the scape angle – except at the singularity, where such angle abruptly changes.