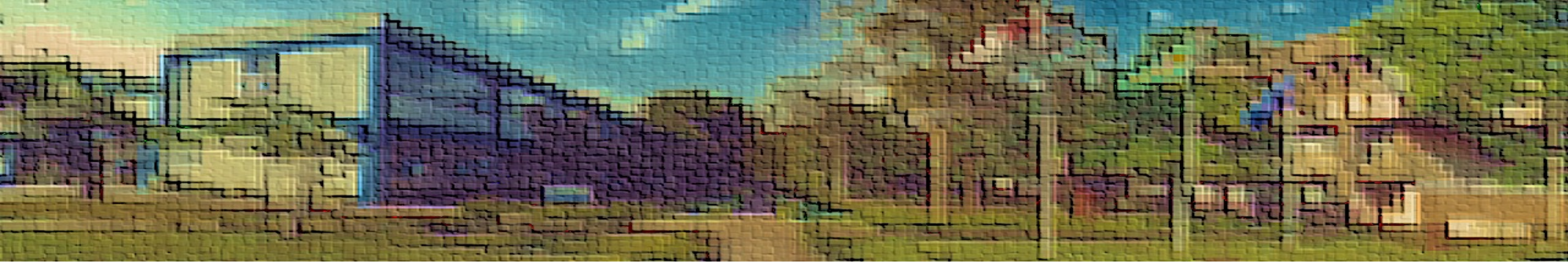

MAC317
Introdução ao Processamento de Sinais Digitais

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Aula #5: Transformada discreta de Fourier

Tempo/espaço vs. frequência

- Many operations are easier, both computationally and conceptually, in the frequency domain.
- We transform the time domain signal to the frequency domain, perform the operation of interest there, and then transform the altered signal back to the time domain.

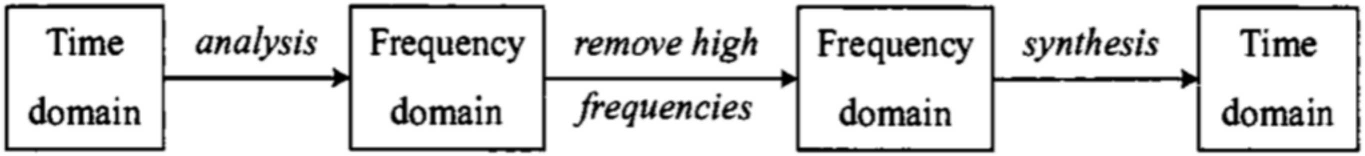
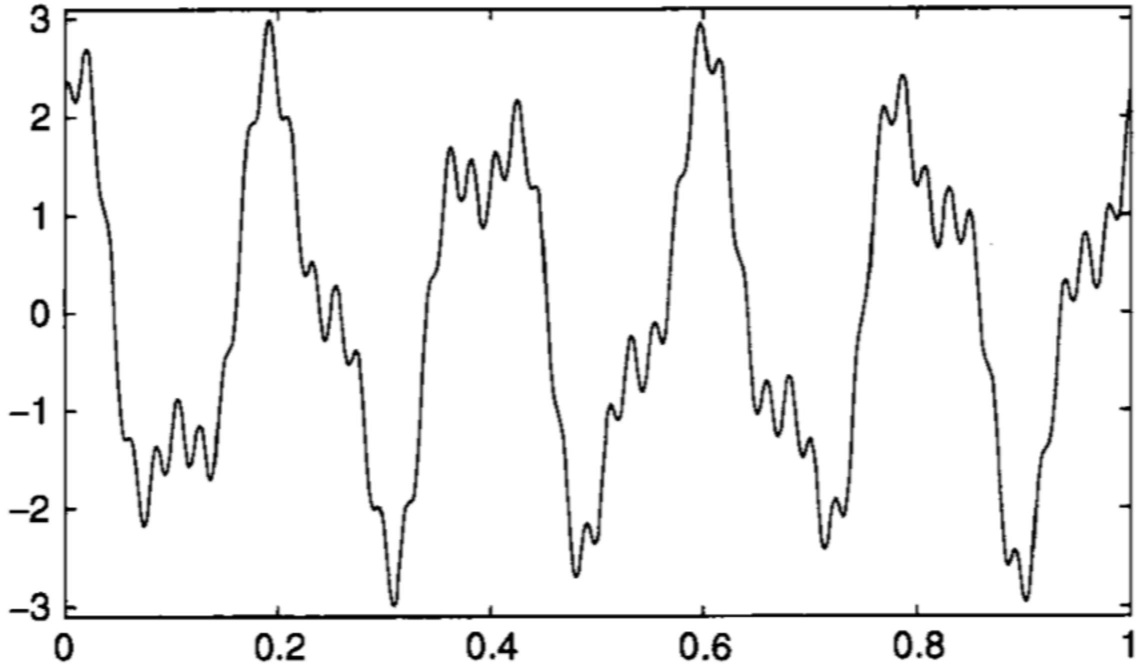


FIGURE 2.1 Processing a signal in the frequency domain.

Exemplo



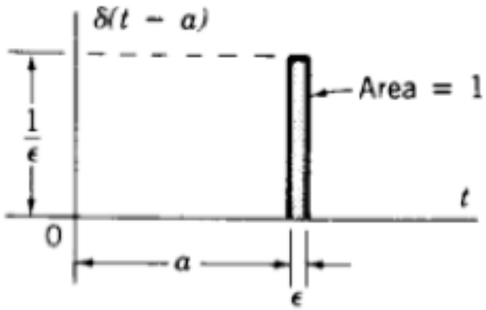
$$x(t) = 2.0\cos(2\pi \cdot 5t) + 0.8\sin(2\pi \cdot 12t) + 0.3\cos(2\pi \cdot 47t)$$

Função impulso

- Também conhecida como função Dirac ou função delta
- Modela a propriedade de uma fonte puntual tendo largura infinitesimal e área (volume) unitária

$$\delta(x - x_0) = 0 \quad \text{for } x \neq x_0,$$

$$\int_{-\infty}^{+\infty} \delta(x - x_0)\, \mathrm{d}x = 1,$$



Função impulso em 2D

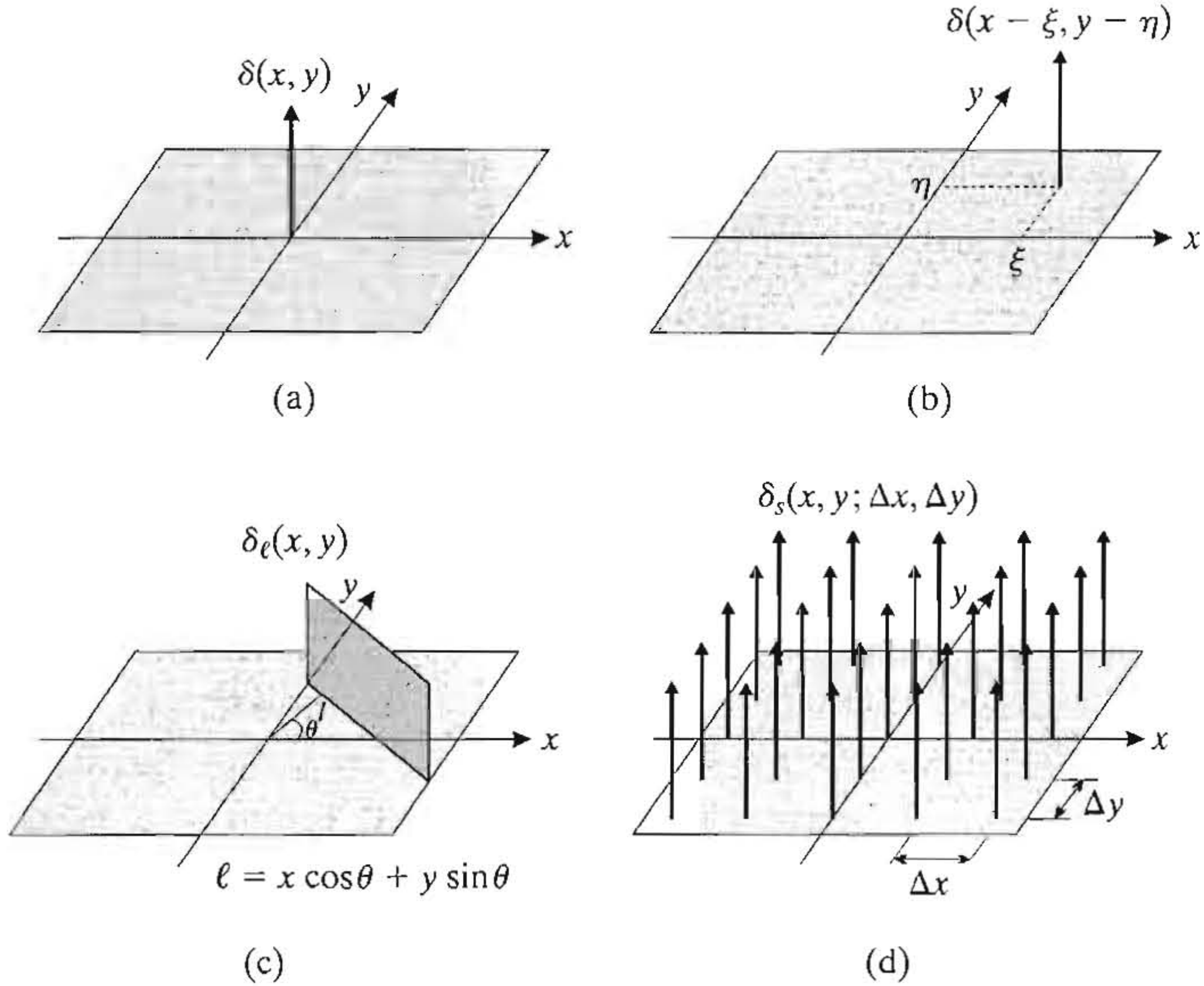


Figure 2.3
Signals derived from the point impulse: (a) point impulse $\delta(x, y)$, (b) shifted point impulse $\delta(x - \xi, y - \eta)$, (c) line impulse $\delta_\ell(x, y)$, and (d) sampling function $\delta_s(x, y; \Delta x, \Delta y)$.

Filtro "peneira"

sifting let $s(x)$ be continuous at $x = x_0$, then

$$\int_{-\infty}^{+\infty} s(x)\,\delta(x - x_0)\,\mathrm{d}x = s(x_0);$$

scaling

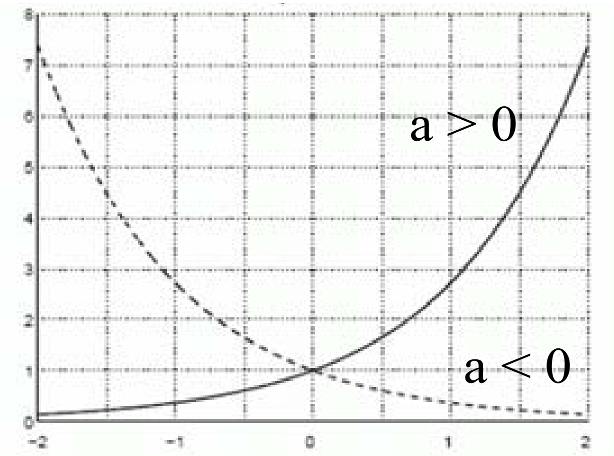
$$\int_{-\infty}^{+\infty} A\,\delta(x)\,\mathrm{d}x = A,$$

this is a special case of sifting.

$$\iint_{-\infty}^{+\infty} s(\vec{r})\,\delta(\vec{r} - \vec{r}_0)\,\mathrm{d}\vec{r} = s(\vec{r}_0).$$

Sinais importantes

Exponential *exp*

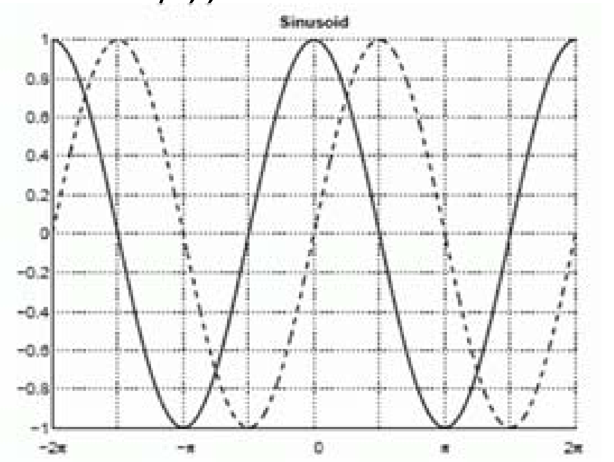


$$\exp(ax) = e^{ax}$$

- when *a* > 0 then *exp* increases with increasing *x*
- when *a* < 0 then *exp* approximates 0 with increasing *x*

Complex exponential / sinusoid:

$$Ae^{i(2\pi kx+\phi)} = A(\cos(2\pi kx+\phi)+i\sin(2\pi kx+\phi))$$



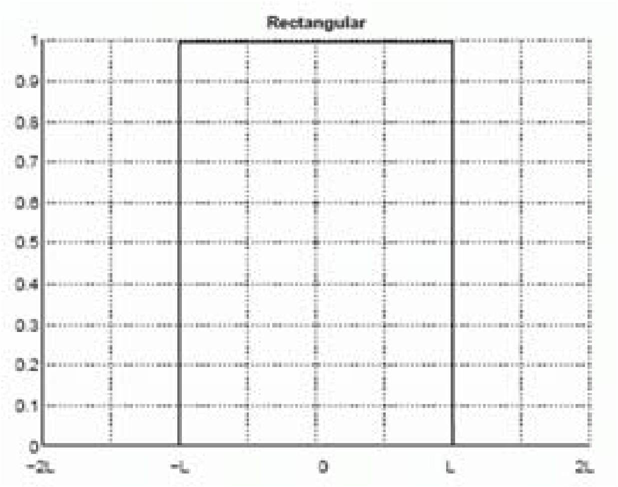
As before

- the *cos* term is the signal's real part
- the *sin* term is the signal's imaginary part
- *A* is the amplitude, φ the phase shift, *k* determines the frequency

Funções adicionais

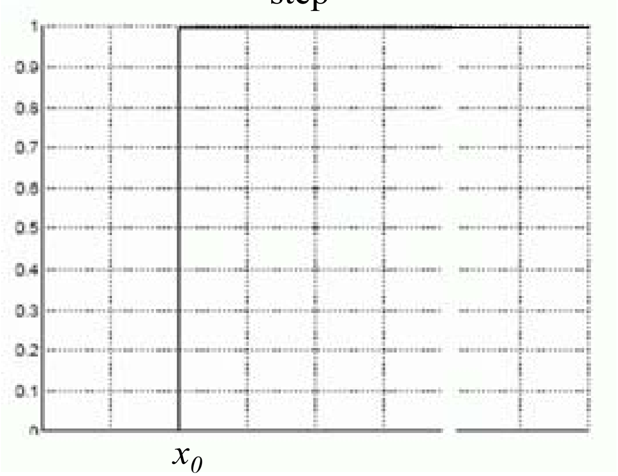
Rectangular function:

$$
\begin{aligned}
\Pi\left(\frac{x}{2L}\right) &= 1 \quad \text{for} \quad |x| < L \\
&= \frac{1}{2} \quad \text{for} \quad |x| = L \\
&= 0 \quad \text{for} \quad |x| > L
\end{aligned}
$$



Step function:

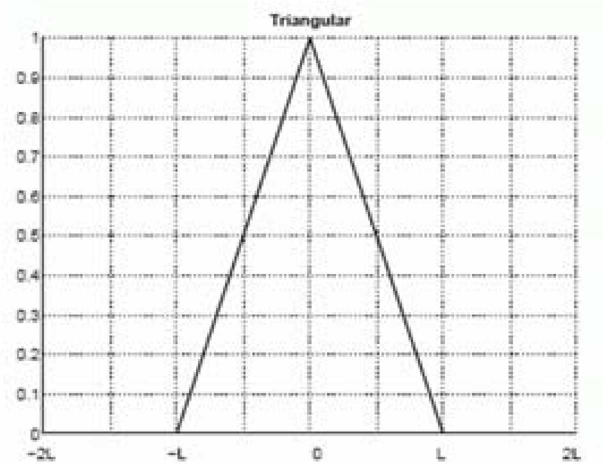
$$
\begin{aligned}
u(x - x_0) &= 0 \quad \text{for} \quad x < x_0 \\
&= \frac{1}{2} \quad \text{for} \quad x = x_0 \\
&= 1 \quad \text{for} \quad x > x_0
\end{aligned}
$$



Funções adicionais

Triangular function:

$$Tri\left(\frac{x}{2L}\right) = 1 - \frac{|x|}{L} \qquad \text{for} \quad |x| < L$$
$$= 0 \qquad \text{for} \quad |x| > L$$



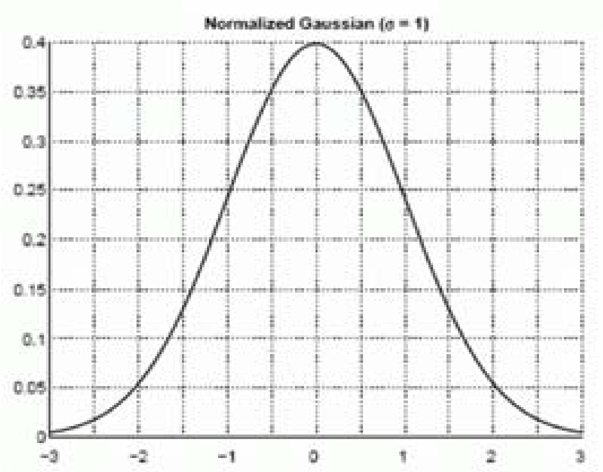
Normalized Gaussian:

$$G_n(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

μ is the mean

σ is the standard deviation

- normalized means that the integral for all *x* is 1

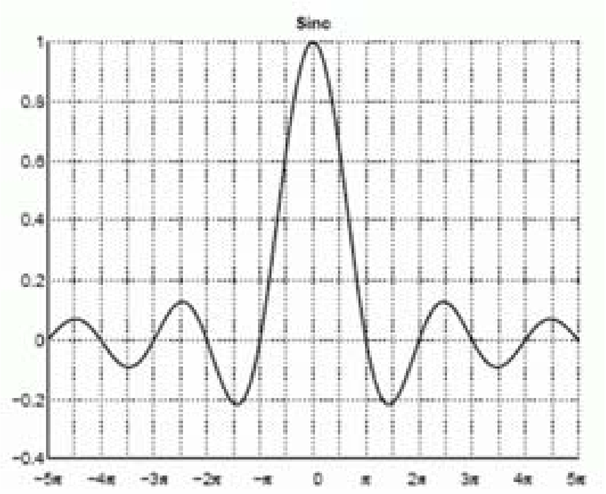


Funções adicionais

Sinc function:

$$sinc(x) = \frac{\sin(\pi x)}{\pi x}$$

- sinc(0) = 1 (L'Hopital's rule)



Separabilidade

- Um sinal f(x,y) é separável se existirem dois sinais unidimensionais $f_1(x)$ e $f_2(y)$ tais que:
  - $f(x,y) = f_1(x)\, f_2(y)$
- São limitados, pois podem somente modelar variações independentes em cada variável
- São apropriados em certas situações, pois são mais simples de operar do que funções em 2D ou 3D.

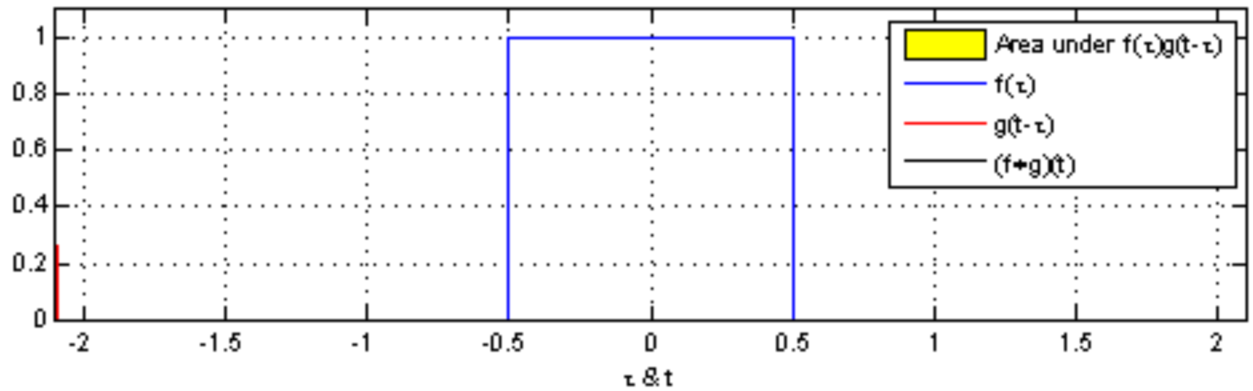
Convolução

The expression

$$s_o(x) = \int_{-\infty}^{+\infty} s_i(\xi)h(x-\xi)d\xi = s_i * h$$

is called *convolution*, defined as:

$$s_1(x) * s_2(x) = \int_{-\infty}^{+\infty} s_1(\xi)s_2(x-\xi)d\xi = \int_{-\infty}^{+\infty} s_1(x-\xi)s_2(\xi)d\xi$$



Sinais 2-D

$$s_1(x,y) * s_2(x,y)$$

$$= \iint_{-\infty}^{+\infty} s_1(x-\xi, y-\zeta)\, s_2(\xi,\zeta)\, \mathrm{d}\xi\, \mathrm{d}\zeta,$$

- Propriedades:
  - $s_1 * s_2 = s_2 * s_1$ (comutativa)
  - $(s_1 * s_2) * s3 = (s_1 * s_2) * s_3$ (associativa)
  - $s_1 * (s_2 + s_3) = s_1 * s_2 + s_1 * s_3$ (distributiva)

Resposta de sistema

Now assume the input is a complex sinusoid with $Ae^{2\pi ikx}$ then:

$$s_0(x) = \int_{-\infty}^{+\infty} Ae^{2\pi ik(x-\xi)} h(\xi) d\xi$$

for now, assume $\varphi=0$

$$= Ae^{2\pi ikx} \int_{-\infty}^{+\infty} e^{-2\pi ik\xi} h(\xi) d\xi$$

$$= Ae^{2\pi ikx} H$$

H is called the *Fourier Transform* of *h*(*x*):

$$H = \int_{-\infty}^{+\infty} e^{-2\pi ik\xi} h(\xi) d\xi$$

- *H* is also often called the *transfer function* or *filter*

Observações

H scales, and maybe phase-shifts, the input sinusoid S_i

In essence, we have now two alternative representations:

- determine the effect of L on s_i by convolution with h: $s_i * h$
- determine the effect of L on s_i by multiplication with H: $S_i \cdot H$

$$s_i * h \leftrightarrow S_i \cdot H$$

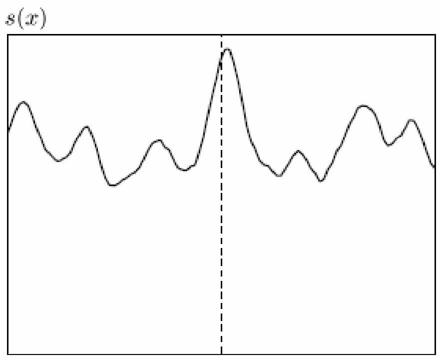
Since convolution is expensive for wide h, the multiplication may be cheaper

- but we need to perform the Fourier transforms of s_i and h

Fourier

Theory developed by Joseph Fourier (1768-1830)

The Fourier transform of a signal s(x) yields its *frequency spectrum S(k)*

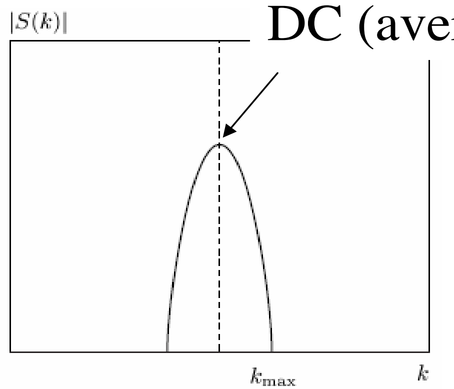


$s(x)$

forward transform

$$S(k) = F\{s(x)\} = \int_{-\infty}^{+\infty} s(x) e^{-2\pi ikx} dx$$

DC (average) term



$S(k)$

inverse transform

$$s(x) = F^{-1}\{S(k)\} = \int_{-\infty}^{+\infty} S(k) e^{2\pi ikx} dk$$

Tutorial visual de FT

- https://youtu.be/spUNpyF58BY

Generalização

The Fourier transform generalizes to higher dimensions

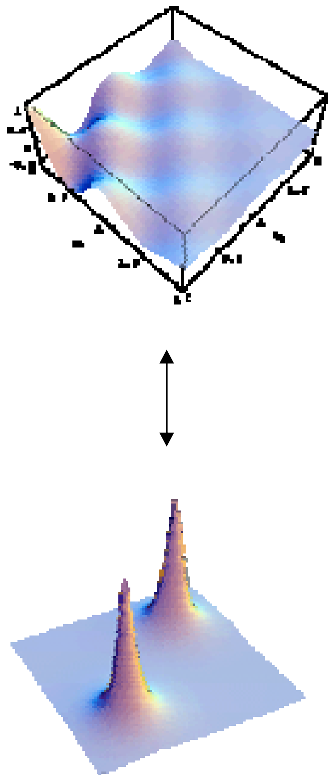
Consider the 2D case:

forward transform

$$S(k,l) = F\{s(x,y)\} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} s(x,y)e^{-2\pi i(kx+ly)}dxdy$$

inverse transform

$$s(x,y) = F^{-1}\{S(k,l)\} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} S(k,l)e^{2\pi i(kx+ly)}dkdl$$

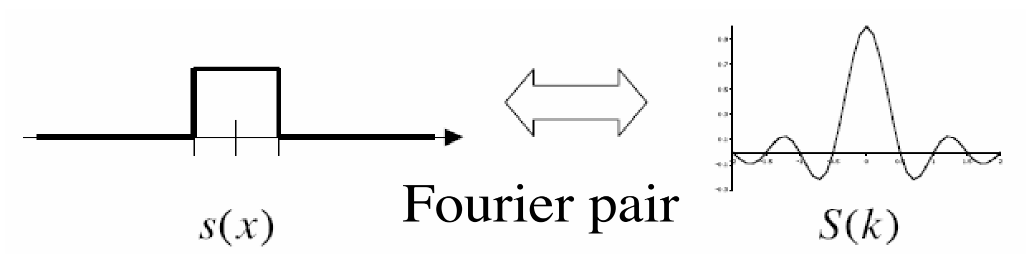


Exemplo

$$S(k) = F\{A\Pi(\frac{x}{2L})\} = \int_{-\infty}^{+\infty} A\Pi(\frac{x}{2L}) e^{-i2\pi kx} dx = \int_{-L}^{+L} A e^{-i2\pi kx} dx$$

$$= -\frac{A}{2\pi ki}(e^{-i2\pi kL} - e^{i2\pi kL}) = \frac{A}{2\pi k} 2\sin(2\pi kL)$$

$$= 2AL \text{ sinc}(2\pi kL)$$



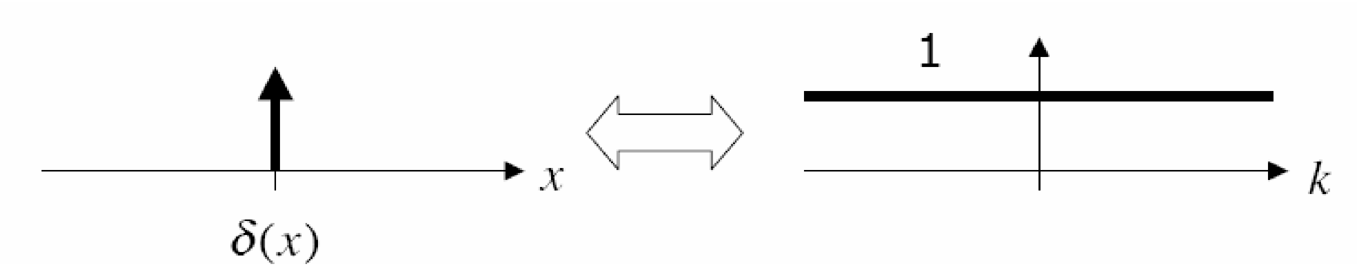
We see that a finite signal in the *x*-domain creates an infinite signal in the *k*-domain (the frequency domain)

- the same is true vice versa

Função impulso (Dirac)

For s(*x*)=δ(*x*):

$$S(k) = F\{\delta(x)\} = \int_{-\infty}^{+\infty} \delta(x) e^{-i2\pi kx} dx = e^{-i2\pi k0} = 1$$



Recall that the Dirac is an extremely thin rect function

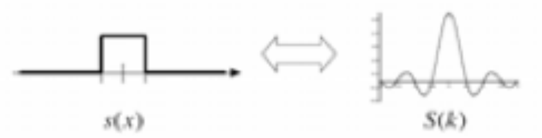
- the frequency spectrum is therefore extremely broad (1 everywhere)

This illustrates a key feature of the Fourier Transform:

- the narrower the *s*(*x*), the wider the *S*(*k*)
- sharp objects need higher frequencies to represent that sharpness

Propriedades

Scaling:



$$S(k) = 2AL\ \mathrm{sinc}(2\pi kL)$$

$$F\{s(ax)\} = \frac{1}{|a|} S(\frac{k}{a})$$

$a>1$ shrinks s
$a<1$ stretches s

Linearity: $F\{as_1(x) + bs_2(x)\} = F\{as_1(x)\} + F\{bs_2(x)\}$

Translation: $F\{s(x - x_0)\} = S(k)e^{-2\pi i x_0 k}$

phase shift

Convolution: $F\{s_1(x) * s_2(x)\} = S_1(k) \cdot S_2(k)$

$$F\{s_1(x) \cdot s_2(x)\} = S_1(k) * S_2(k)$$

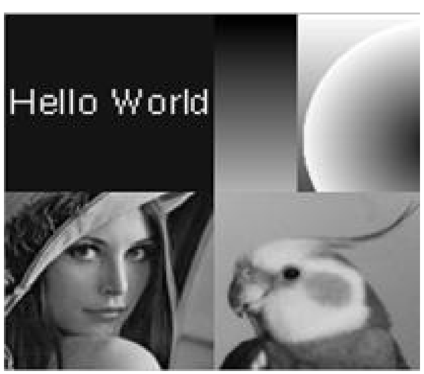
Influência na função de transferência

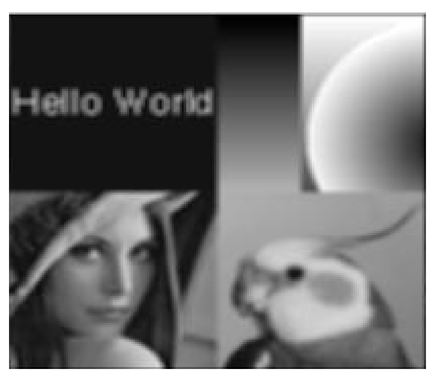
$$s_0(x) = \int_{-\infty}^{+\infty} S_i(k) e^{2\pi i k x} H(k)\,dk$$

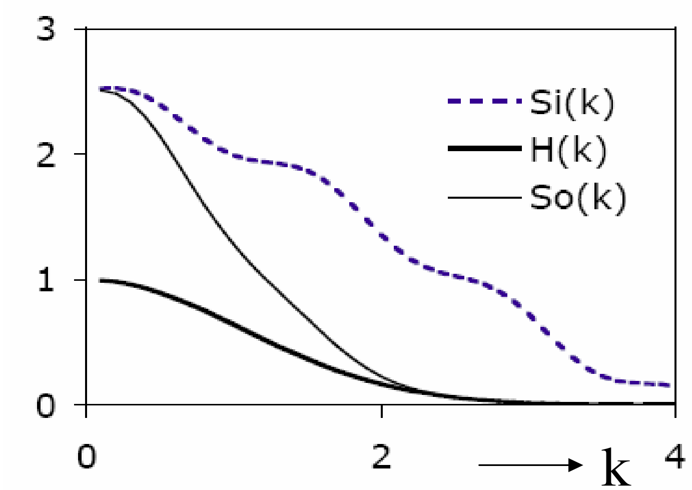
$$s_o(x) = s_i(x) * h(x) \leftrightarrow S_i(k) \cdot H(k) = S_o(k)$$

Let's look at a concrete example:

- *H* is a *lowpass (blurring) filter*: it reduces the higher frequencies of *S* more than the lower ones

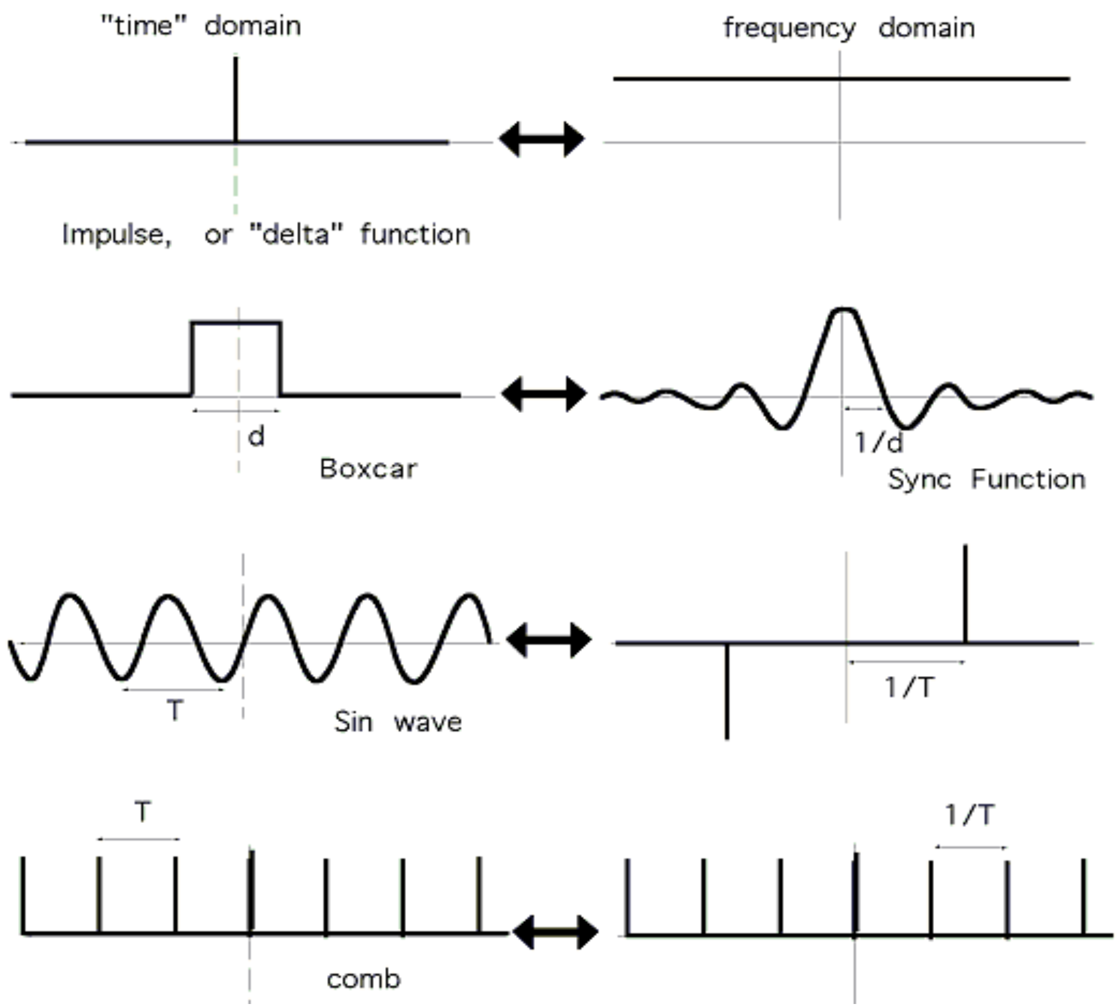






after application of *H*

Pares importantes



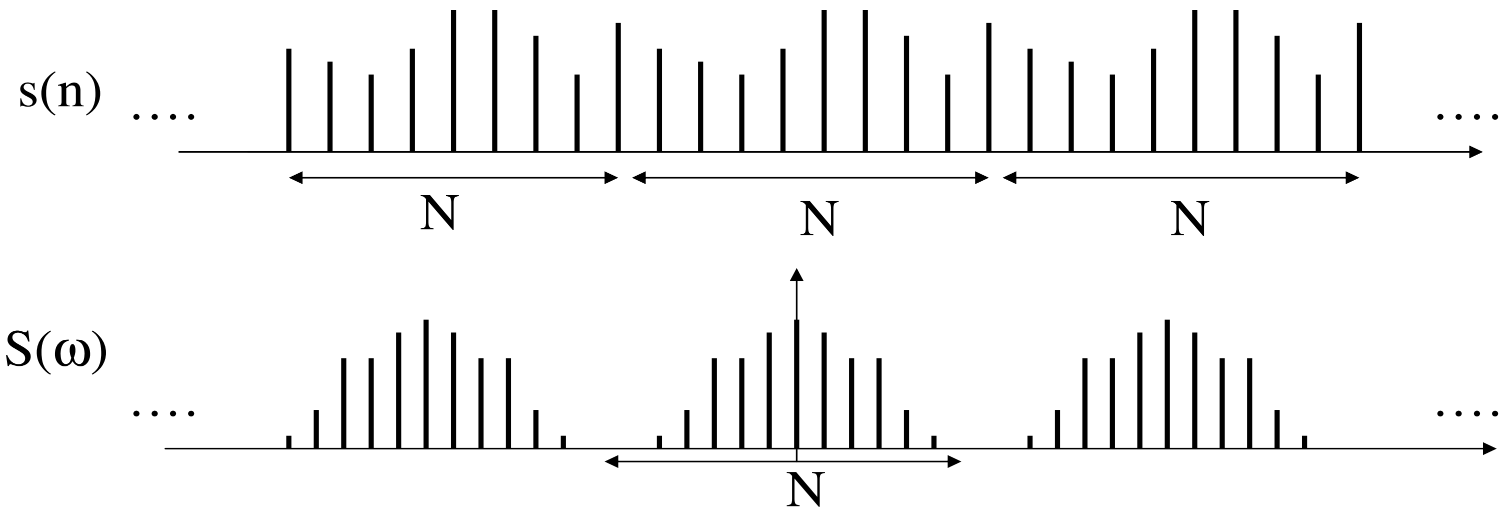
Transformada discreta

Discrete Fourier Transform (DFT)

- assumes that the signal is discrete and finite

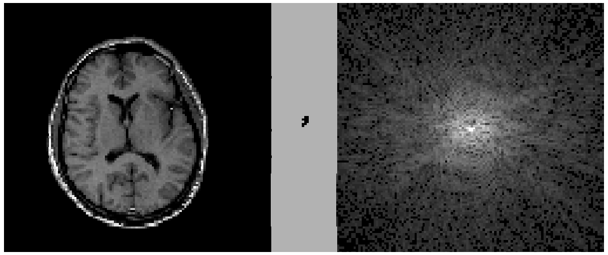
$$S(k) = \sum_{n=0}^{N-1} s(n) e^{\frac{-i2\pi kn}{N}} \qquad s(n) = \frac{1}{N} \sum_{n=0}^{N-1} S(k) e^{\frac{i2\pi kn}{N}}$$

- now we have only N samples, and we can calculate N frequencies
- the frequency spectrum is now <u>discrete</u>, and it is <u>periodic</u> in N



Exemplo

The 2D transform:



$$S(k,l)=\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}s(n,m)e^{\frac{-i2\pi(kn+lm)}{NM}}$$

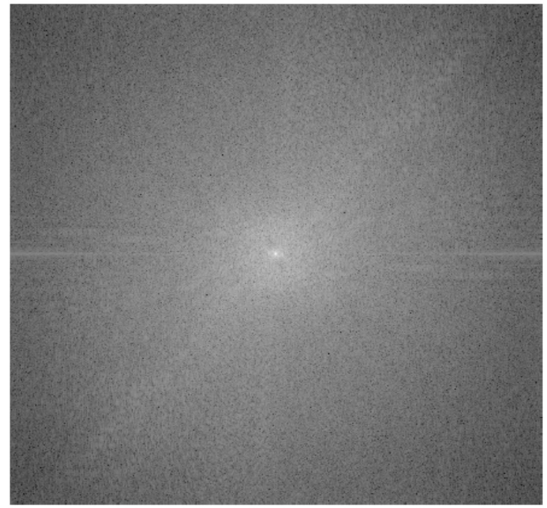
$$s(n,m)=\frac{1}{NM}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}S(k,l)e^{\frac{i2\pi(kn+lm)}{NM}}$$

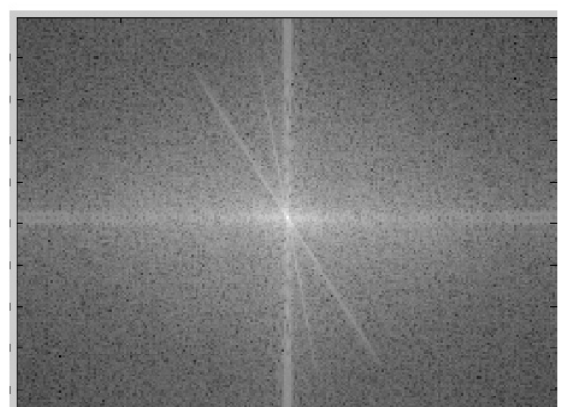
Separability:

$$S(k,l)=\frac{1}{NM}\sum_{m=0}^{M-1}e^{\frac{-i2\pi lm}{M}}P(k,m) \quad \text{where } P(k,m)=\sum_{n=0}^{N-1}s(n,m)e^{\frac{-i2\pi kn}{N}}$$

$$s(n,m)=\frac{1}{NM}\sum_{l=0}^{M-1}e^{\frac{-i2\pi lm}{M}}p(n,l) \quad \text{where p}(n,l)=\sum_{k=0}^{N-1}S(n,m)e^{\frac{-i2\pi kn}{N}}$$

Exemplos





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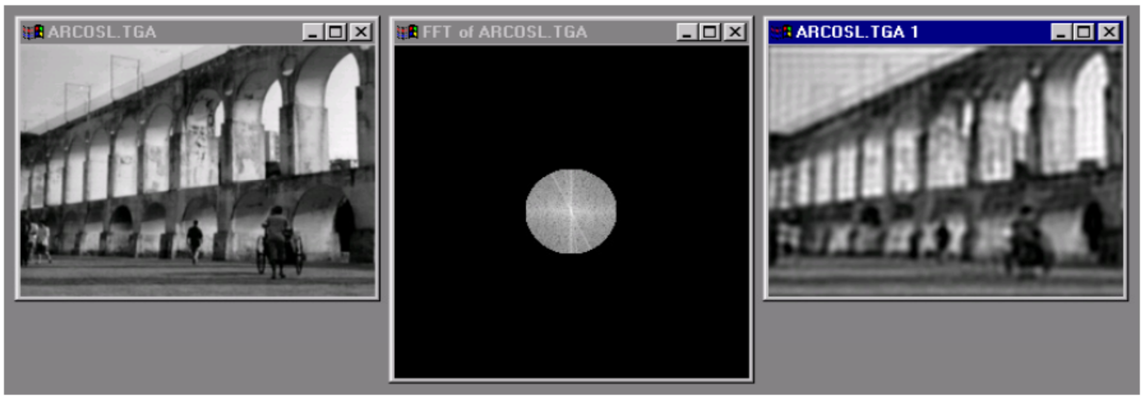
Modificações no espectro

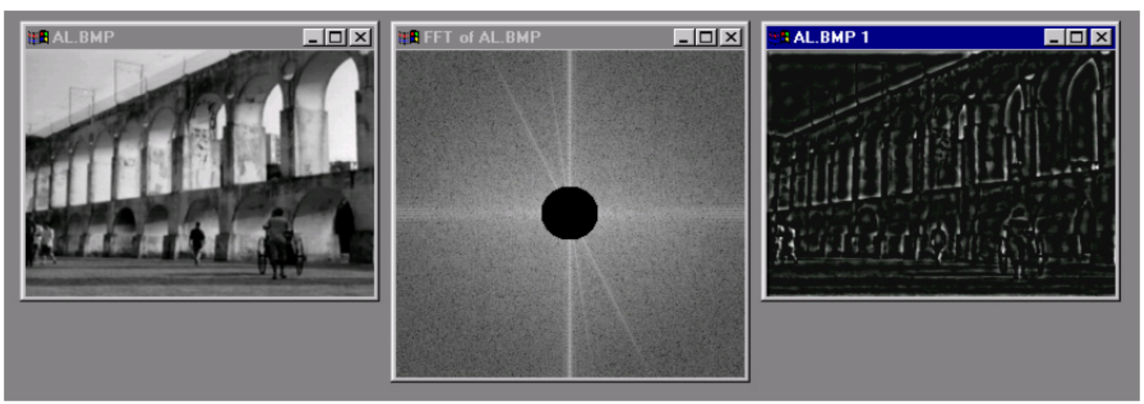
(a) Lower frequencies (close to origin) give overall structure

(b) Higher frequencies (periphery) give detail (sharp edges)

(a)

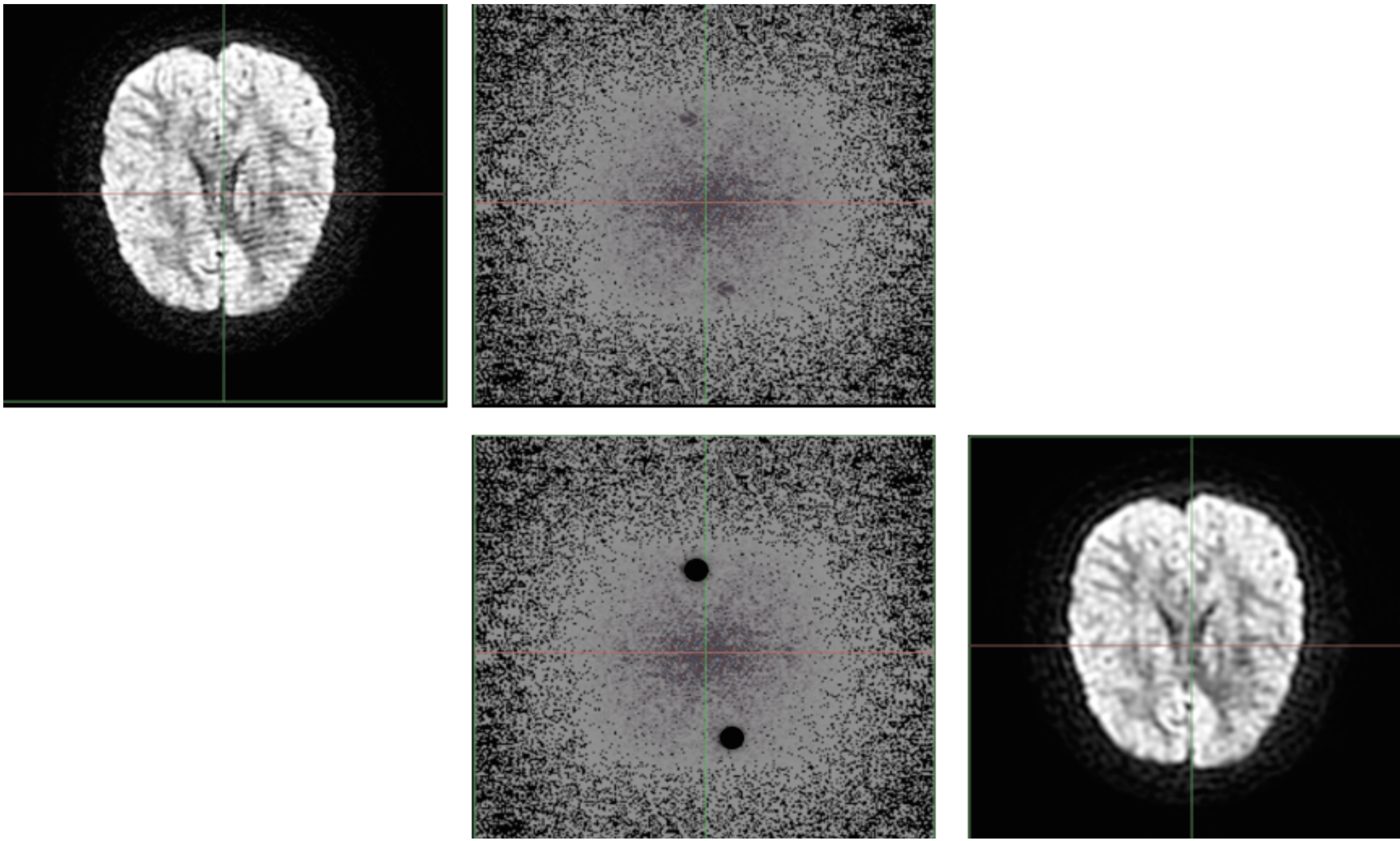


(b)



A. Efros, CMU

Remoção de ruído



Tarefa de casa

- Leitura do livro-texto até a seção 2.4