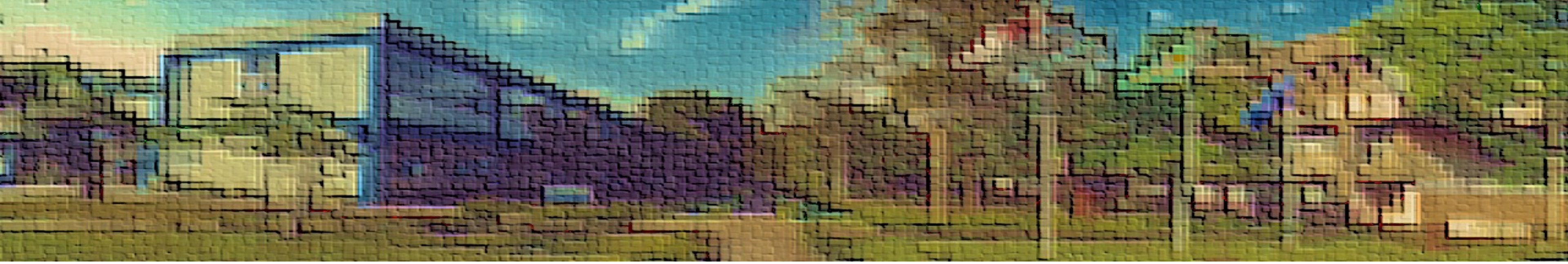

MAC317
Introdução ao Processamento de Sinais Digitais

Prof. Marcel P. Jackowski

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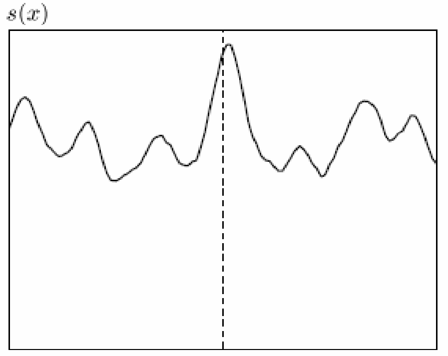
Aula #6: Transformada discreta de Fourier (Parte 2)

Fourier

Theory developed by Joseph Fourier (1768-1830)

The Fourier transform of a signal s(x) yields its *frequency spectrum S*(*k*)

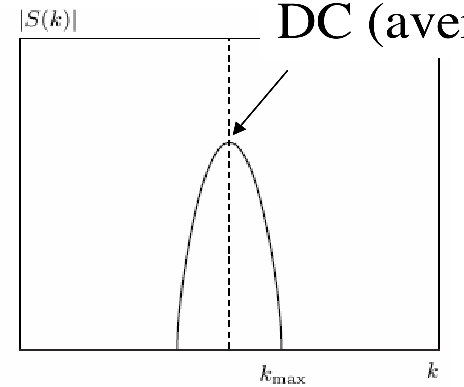


$s(x)$

forward transform

$$S(k) = F\{s(x)\} = \int_{-\infty}^{+\infty} s(x) e^{-2\pi i k x} dx$$

DC (average) term



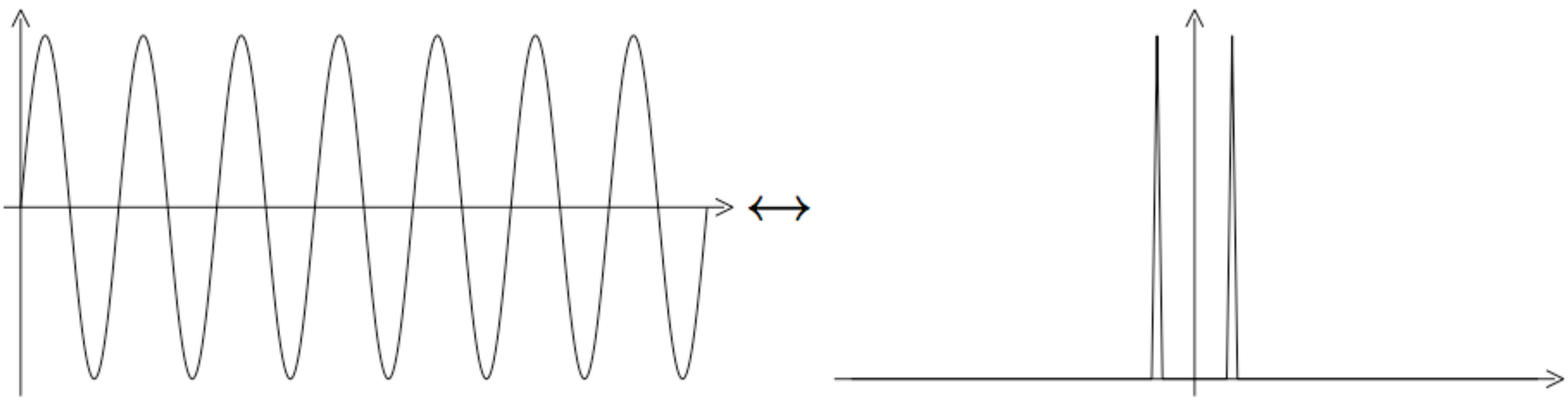
$S(k)$

inverse transform

$$s(x) = F^{-1}\{S(k)\} = \int_{-\infty}^{+\infty} S(k) e^{2\pi i k x} dk$$

Exemplo de sinal: seno

Single sine wave $f(x) = \sin(\omega x)$:



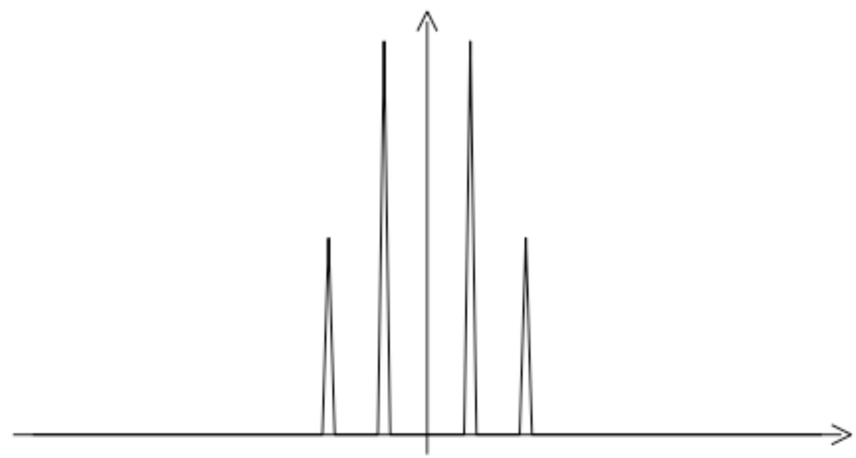
$f(x)$ $\quad$ $|F(\omega)|$

Exemplo de sinal senoidal

Sum of two sine waves

$f(x) = \sin(\omega x) + 0.5\sin(2\omega x)$:

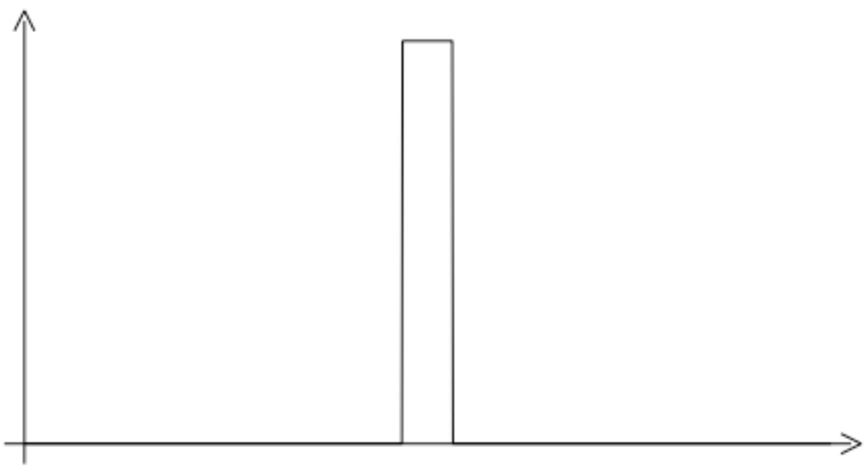
$\longleftrightarrow$



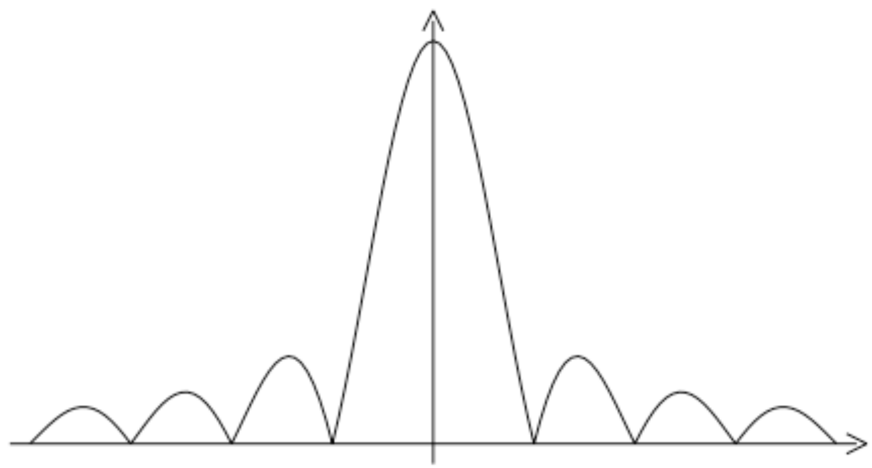
f(x) **|F(ω)|**

Função Rect

Box function:



$\leftrightarrow$

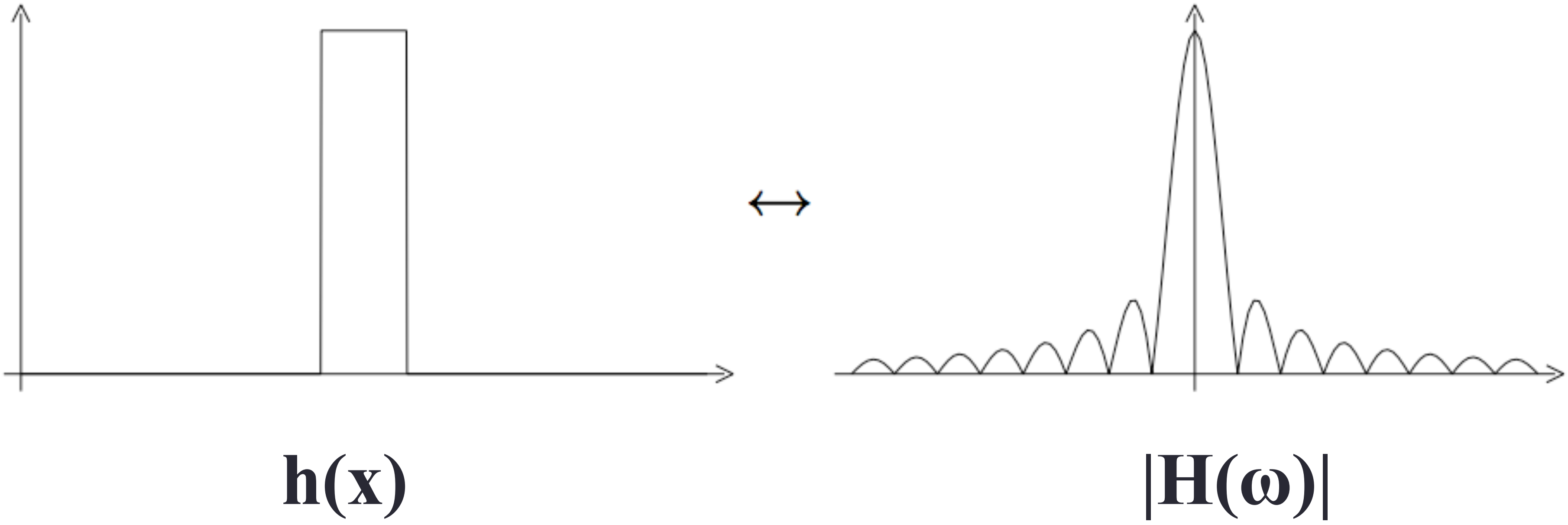


$h(x)$

$|H(\omega)|$

Função Rect

Wider box function:



Generalização

The Fourier transform generalizes to higher dimensions

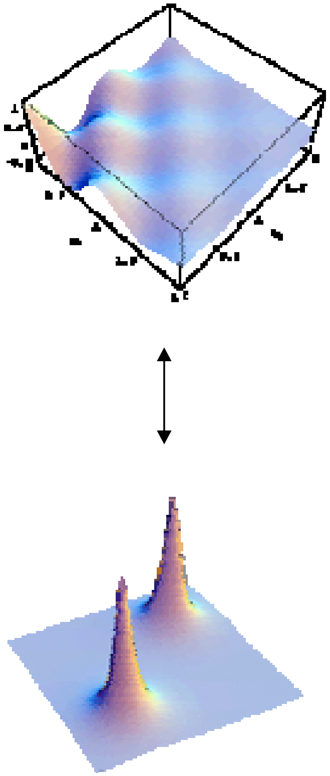
Consider the 2D case:

forward transform

$$S(k,l) = F\{s(x,y)\} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} s(x,y)e^{-2\pi i(kx+ly)}dxdy$$

inverse transform

$$s(x,y) = F^{-1}\{S(k,l)\} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} S(k,l)e^{2\pi i(kx+ly)}dkdl$$



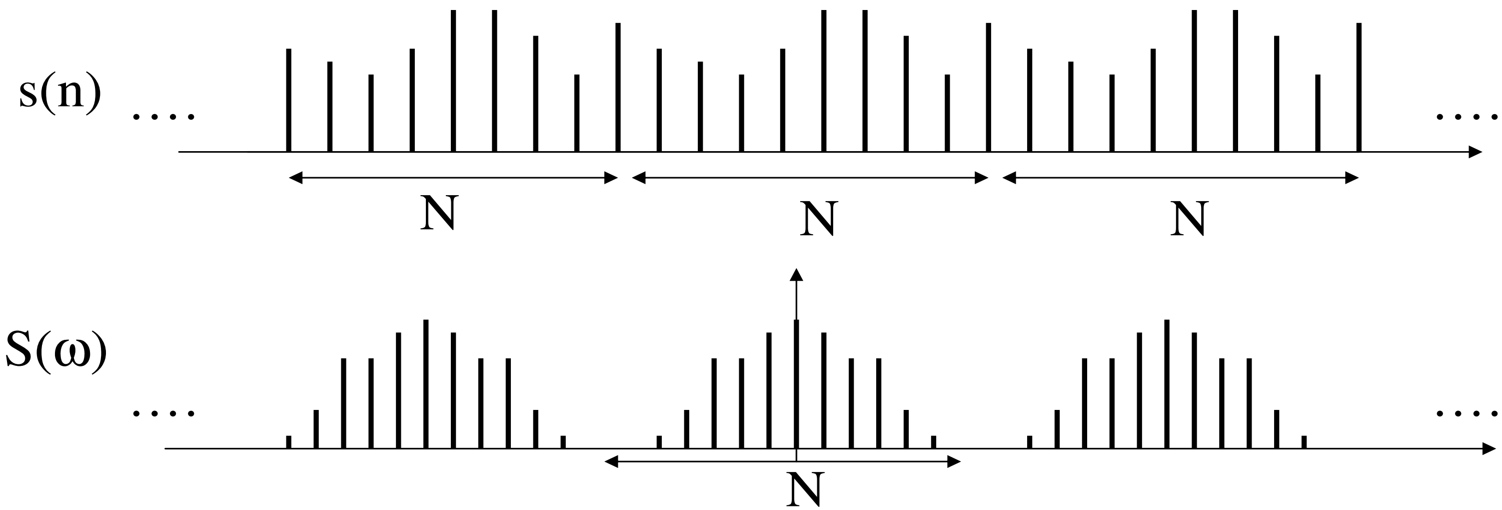
Transformada discreta

Discrete Fourier Transform (DFT)

- assumes that the signal is discrete and finite

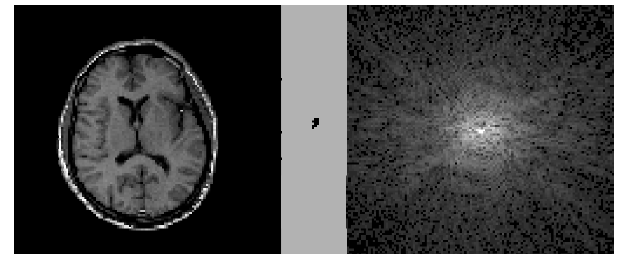
$$S(k) = \sum_{n=0}^{N-1} s(n) e^{\frac{-i2\pi kn}{N}} \qquad s(n) = \frac{1}{N} \sum_{n=0}^{N-1} S(k) e^{\frac{i2\pi kn}{N}}$$

- now we have only N samples, and we can calculate N frequencies
- the frequency spectrum is now discrete, and it is periodic in N



Exemplo

The 2D transform:



$$S(k,l)=\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}s(n,m)e^{\frac{-i2\pi(kn+lm)}{NM}}$$

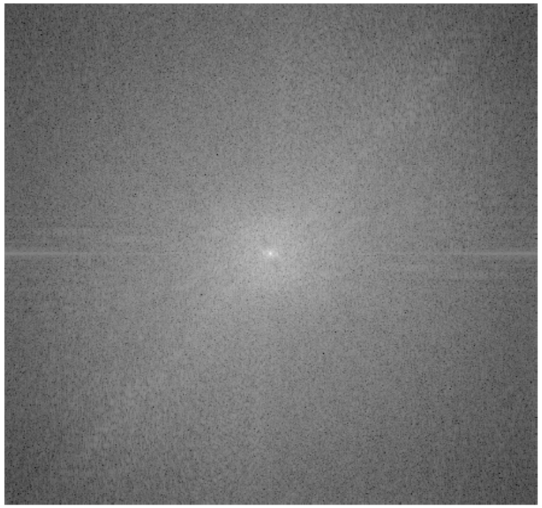
$$s(n,m)=\frac{1}{NM}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}S(k,l)e^{\frac{i2\pi(kn+lm)}{NM}}$$

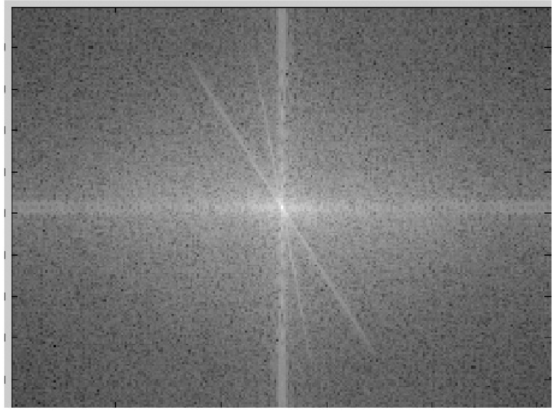
Separability:

$$S(k,l)=\frac{1}{NM}\sum_{m=0}^{M-1}e^{\frac{-i2\pi lm}{M}}P(k,m) \quad \text{where } P(k,m)=\sum_{n=0}^{N-1}s(n,m)e^{\frac{-i2\pi kn}{N}}$$

$$s(n,m)=\frac{1}{NM}\sum_{l=0}^{M-1}e^{\frac{-i2\pi lm}{M}}p(n,l) \quad \text{where } \mathrm{p}(n,l)=\sum_{k=0}^{N-1}S(n,m)e^{\frac{-i2\pi kn}{N}}$$

Exemplos





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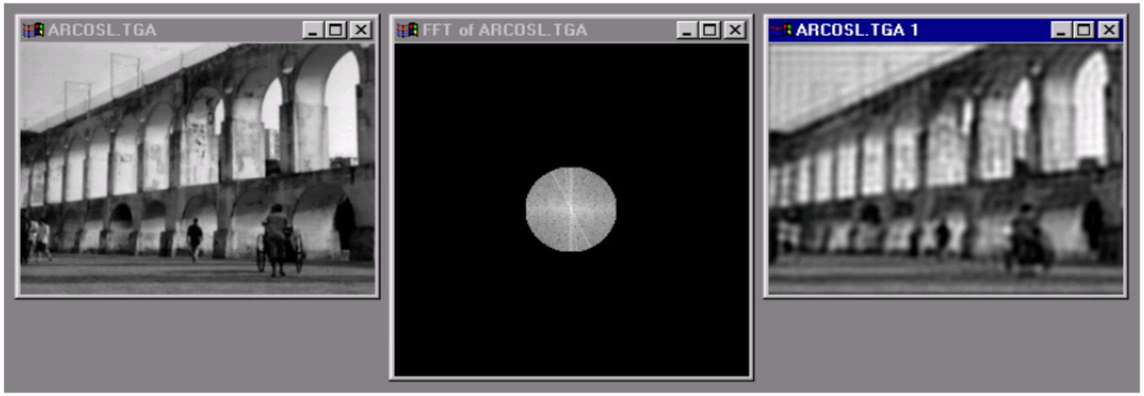
Modificações no espectro

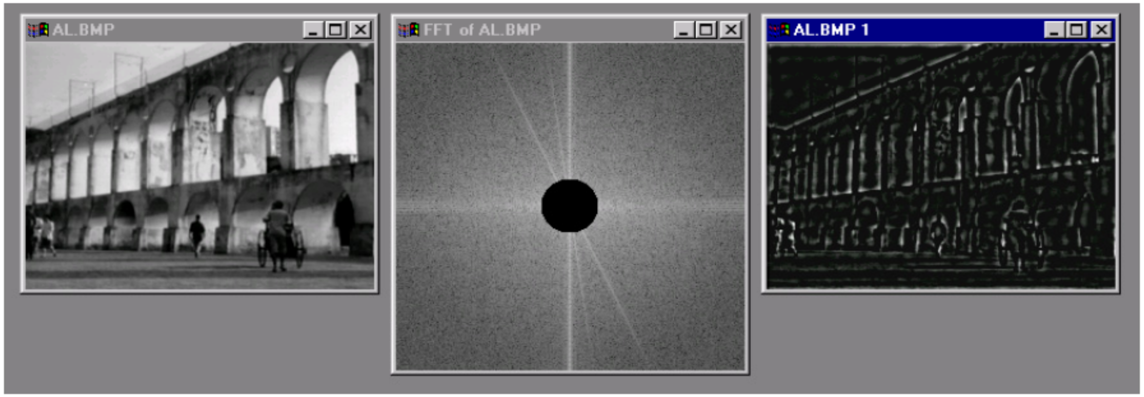
(a) Lower frequencies (close to origin) give overall structure

(b) Higher frequencies (periphery) give detail (sharp edges)

(a)

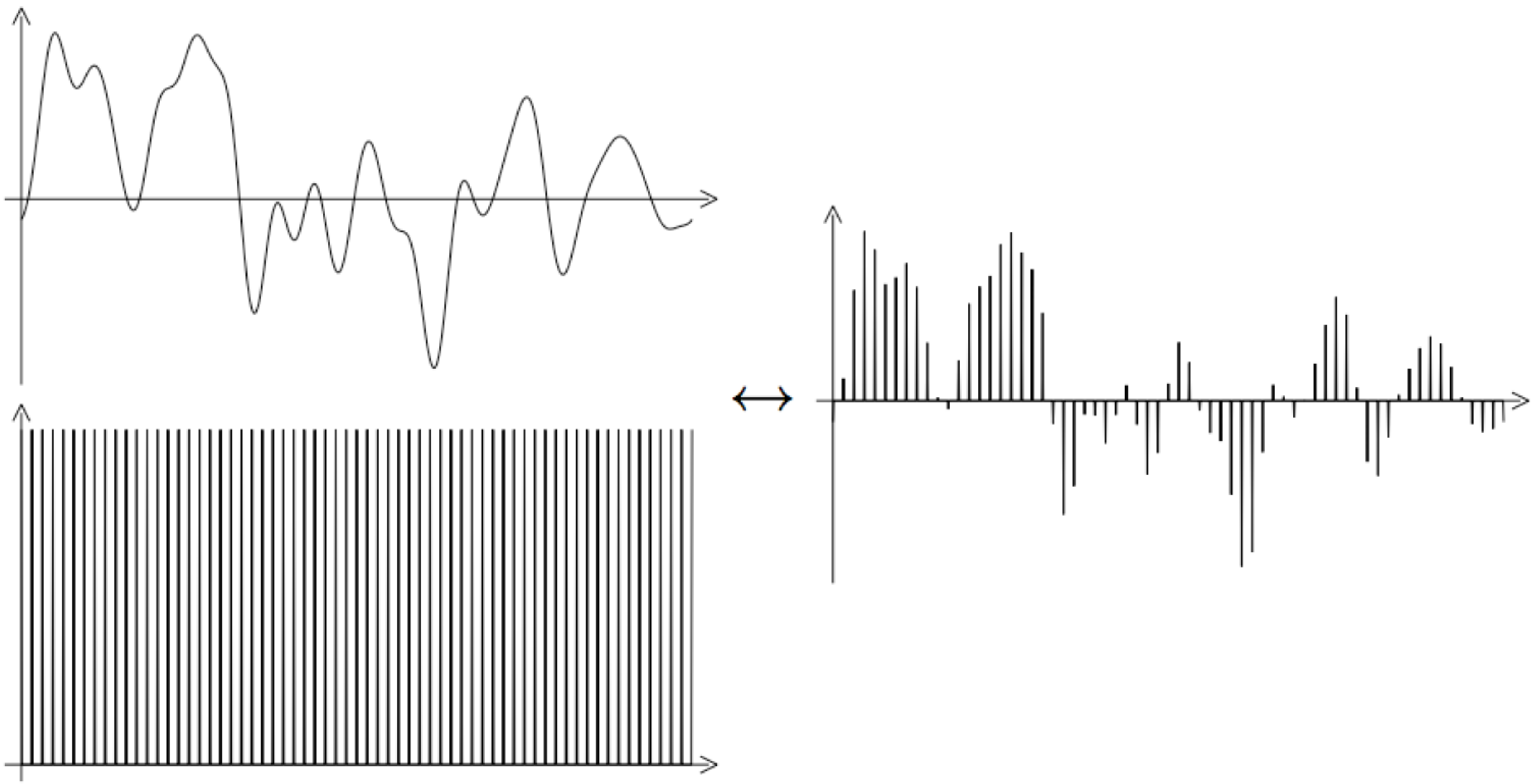


(b)



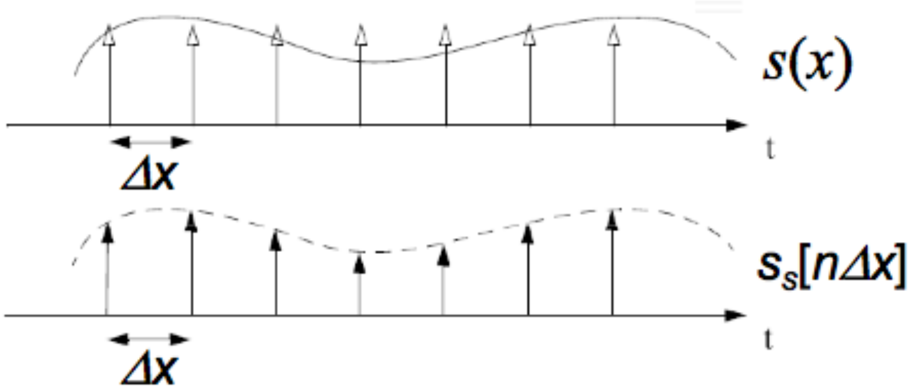
A. Efros, CMU

Amostragem



Filtro “pente”

- a continuous signal *s*(*x*) is measured at fixed instances spaced apart by an interval Δx
- the data points so obtained form a discrete signal $s_s[n\Delta x] = s_s(n\Delta x)$
- here, Δx is called the *sampling period (distance)*, and K = 1/Δx the *sampling frequency*



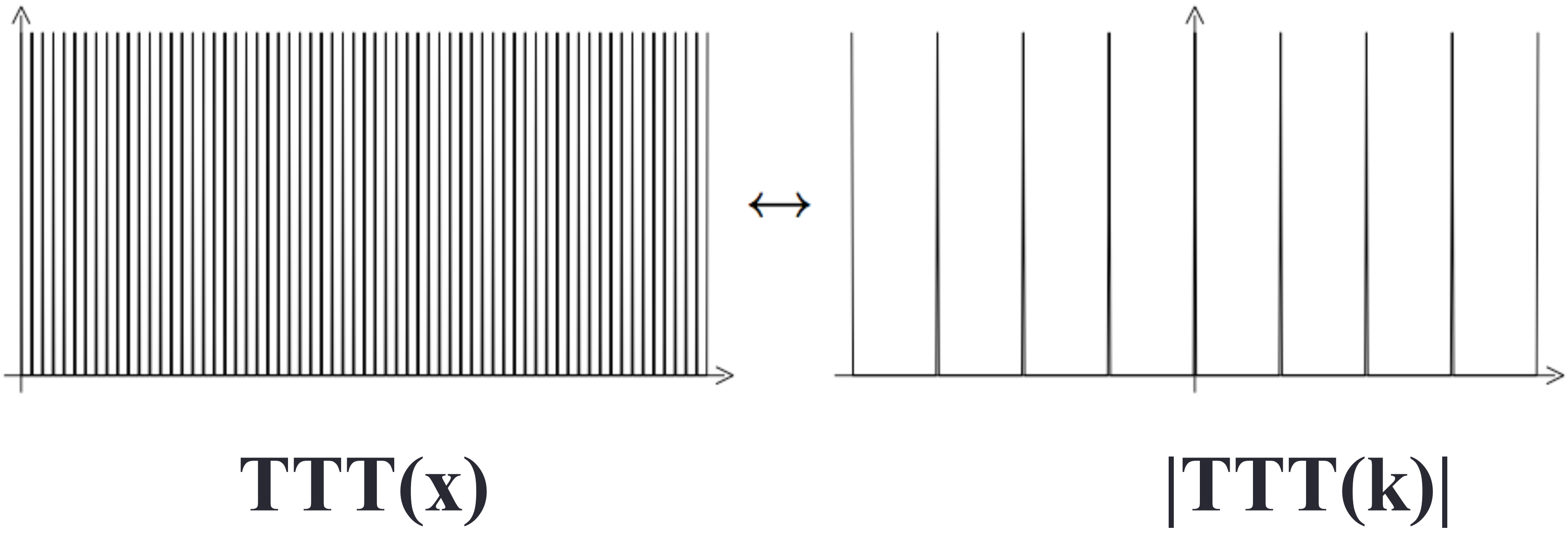
Sampling is the multiplication of the signal with an impulse train:

$$s_s(x) = s(x) \cdot \mathrm{TTT}(x)$$

$$\mathrm{TTT}(x) = \sum_{n=-\infty}^{+\infty} \delta(x - n\Delta x), \quad \mathrm{TTT}(x) \text{ is the comb function}$$

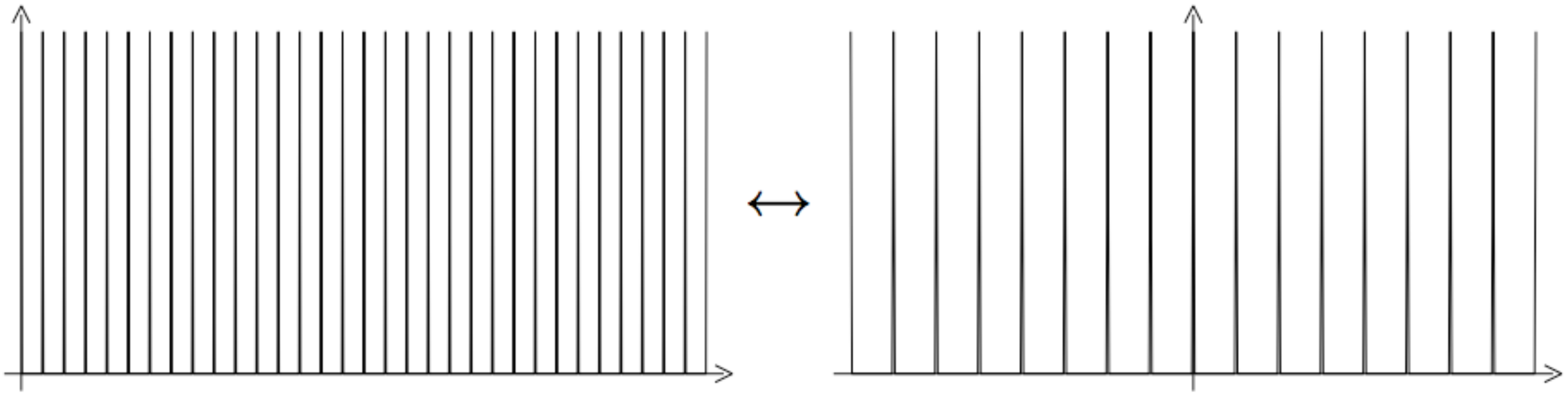
Filtro ”pente”

Comb function:



Filtro “pente”

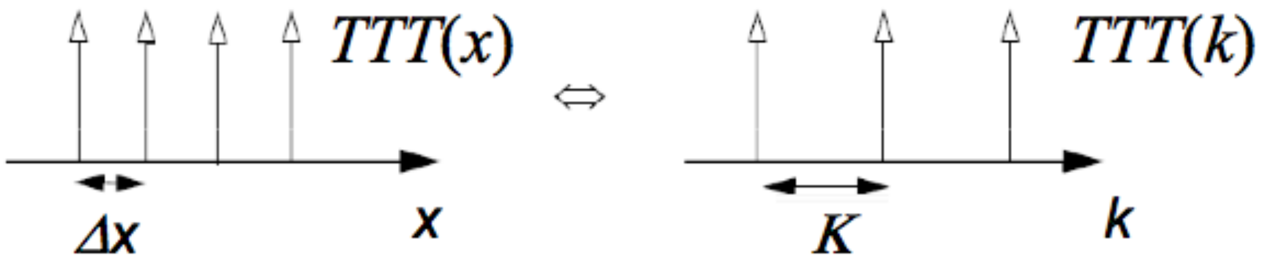
Wider comb function:



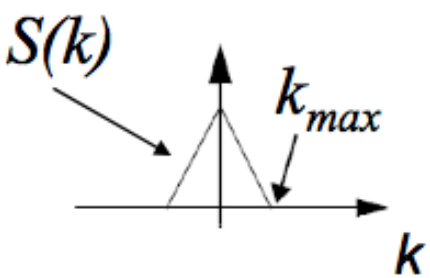
TTT(x) $\leftrightarrow$ |TTT(k)|

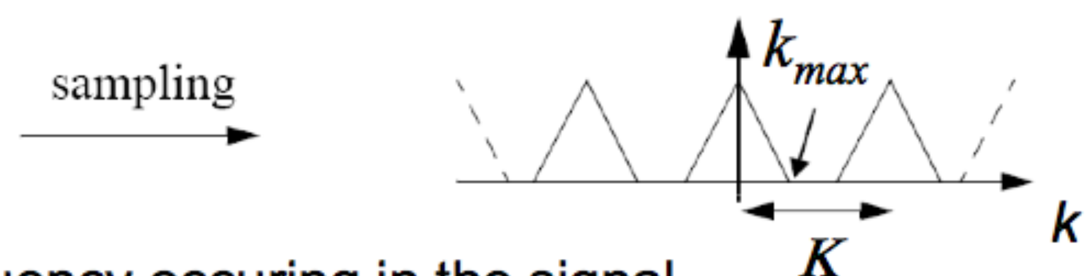
Amostragem no espaço Fourier

$$S_s(k) = S(k) * F\{\mathrm{TTT}(x)\}, \quad \text{where } F\{\mathrm{TTT}(x)\} = K \sum_{l=-\infty}^{+\infty} \delta(k - lK)$$



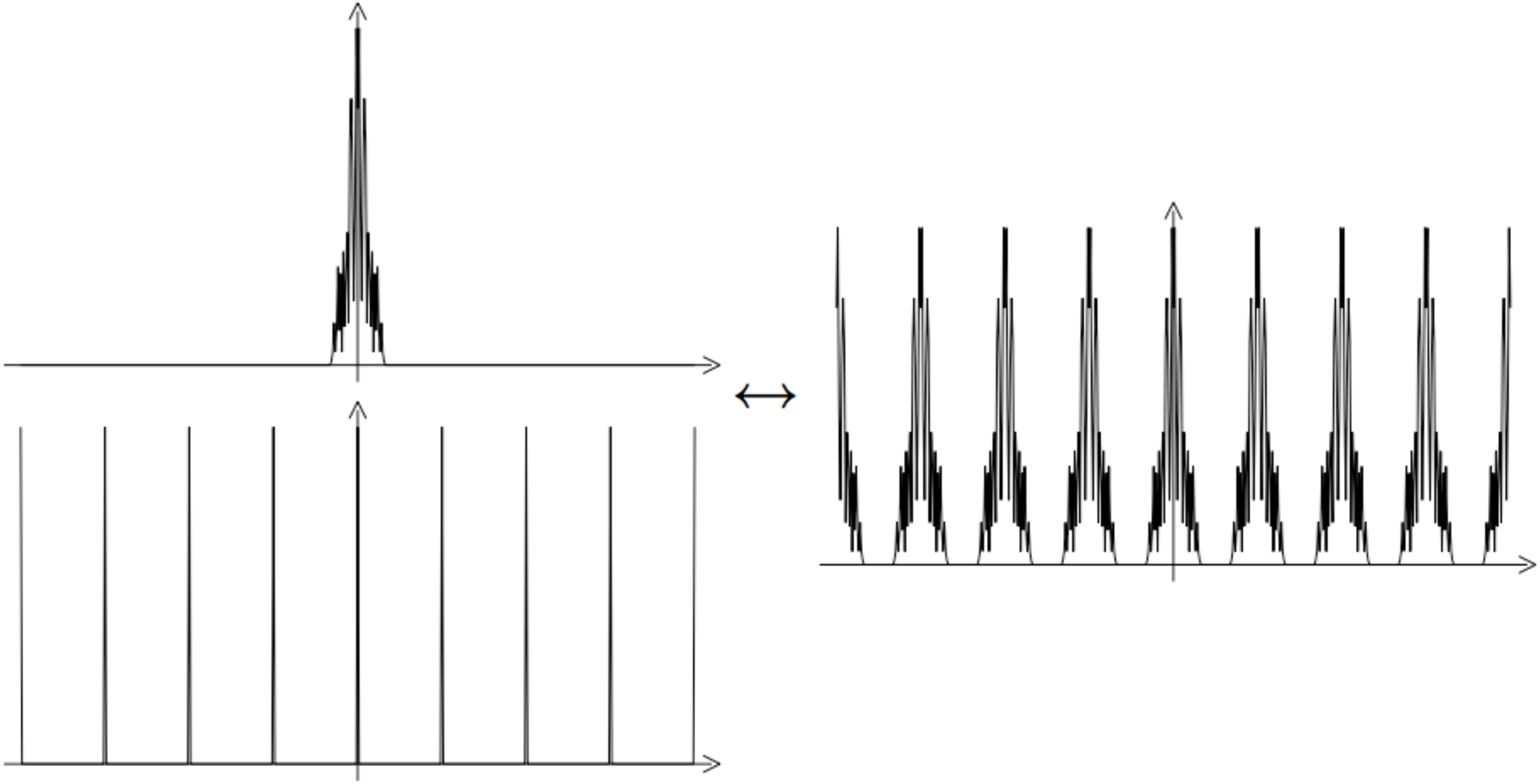
- the smaller Δx the wider K (recall the Fourier scaling theorem)
- sampling (the convolution of $TTT(k)$ and $S(k)$) replicates the signal spectrum $S(k)$ at integer multiples of sampling frequency K



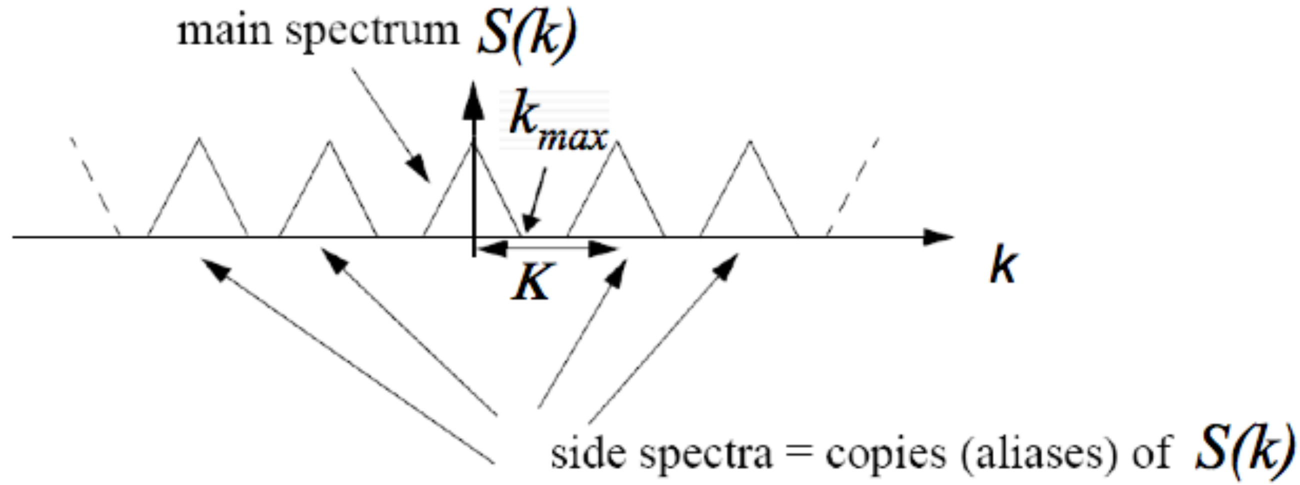


- k_{max} is maximum frequency occuring in the signal

Amostragem no espaço Fourier

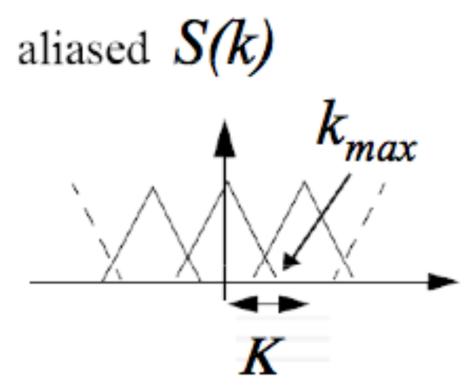


"Aliasing"



However, if we choose $K < 2\, k_{max}$ the aliases overlap and we get *aliasing*

- what does aliasing look like?



Frequência de Nyquist

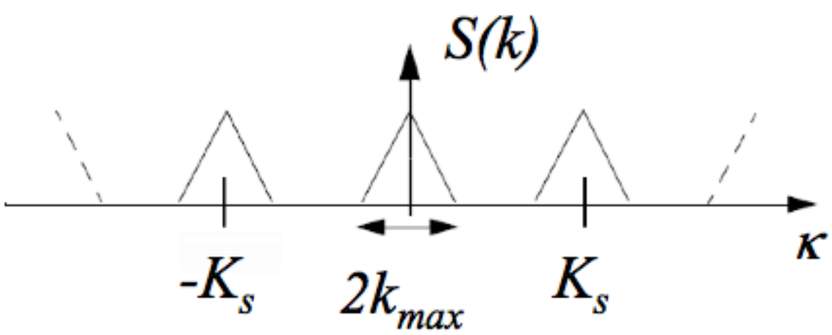
$$K > K_s = 2 \cdot k_{\max}, \quad K_s \text{ is the } \textit{Nyquist rate}$$

In other words:

- the samples only uniquely define the signal if:

$$S(k) = 0 \quad \forall |k| > k_{\max}$$

$$\frac{1}{\Delta x} > 2k_{\max} = K_s$$



- this assumes that the signal is band-limited ($S(k)$=0 above K_s

"Antialising"

Usually signals are not band-limited

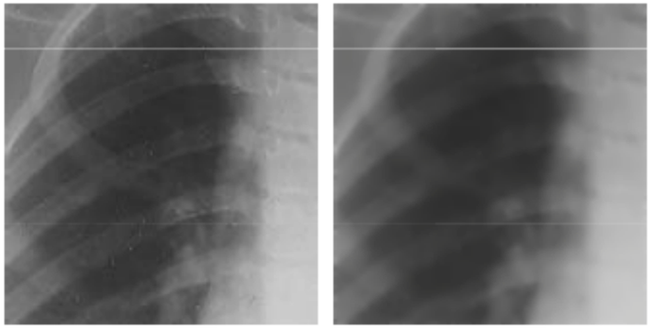
- recall the infinite spectrum of a sharp edge (for example: a bone)

To prevent the inevitable aliasing we must perform anti-aliasing before sampling the signal

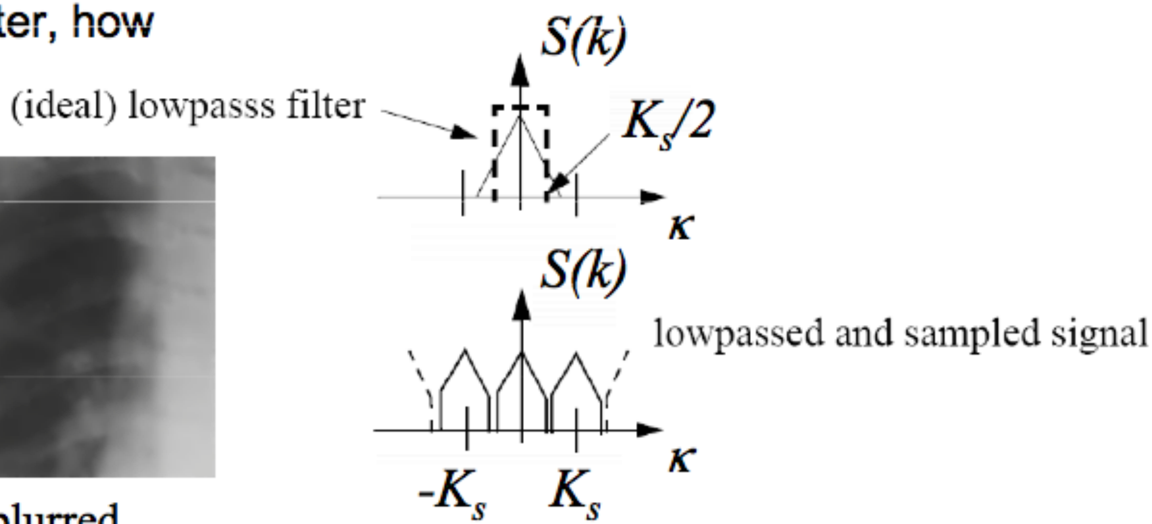
- for example: when digitizing a radiograph of a bone or a chest

Anti-aliasing is done by low-pass filtering (blurring)

- band-limit the signal *prior* to sampling
- we shall see later, how

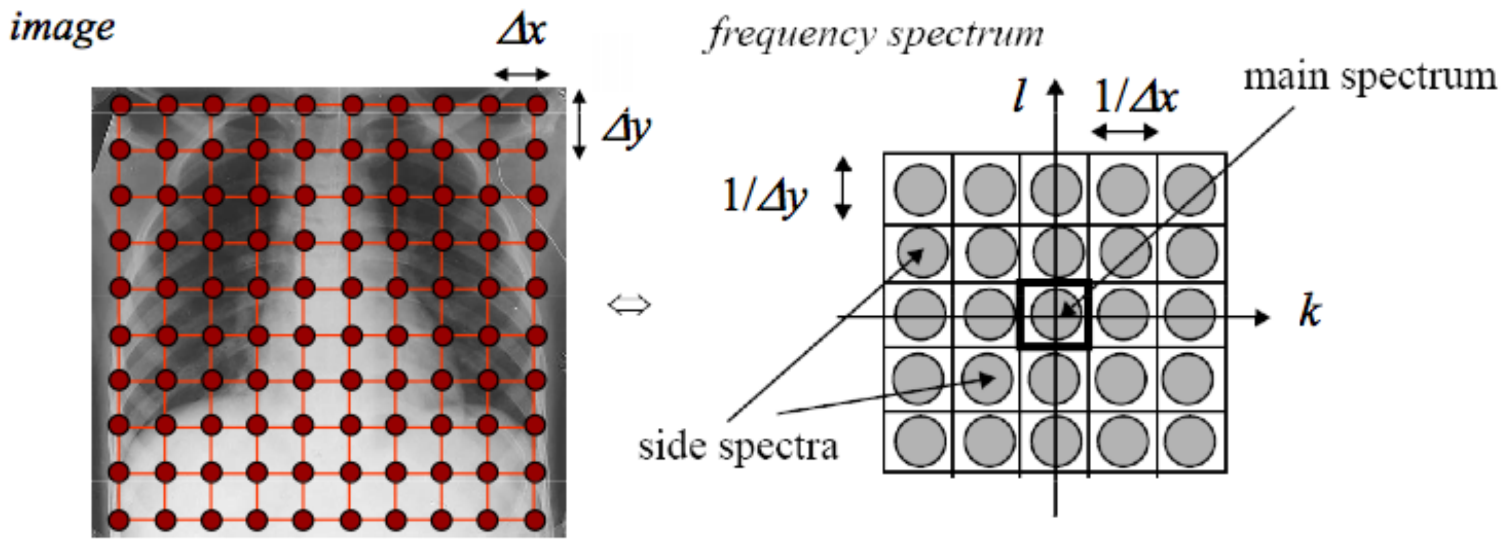


original blurred



Generalização

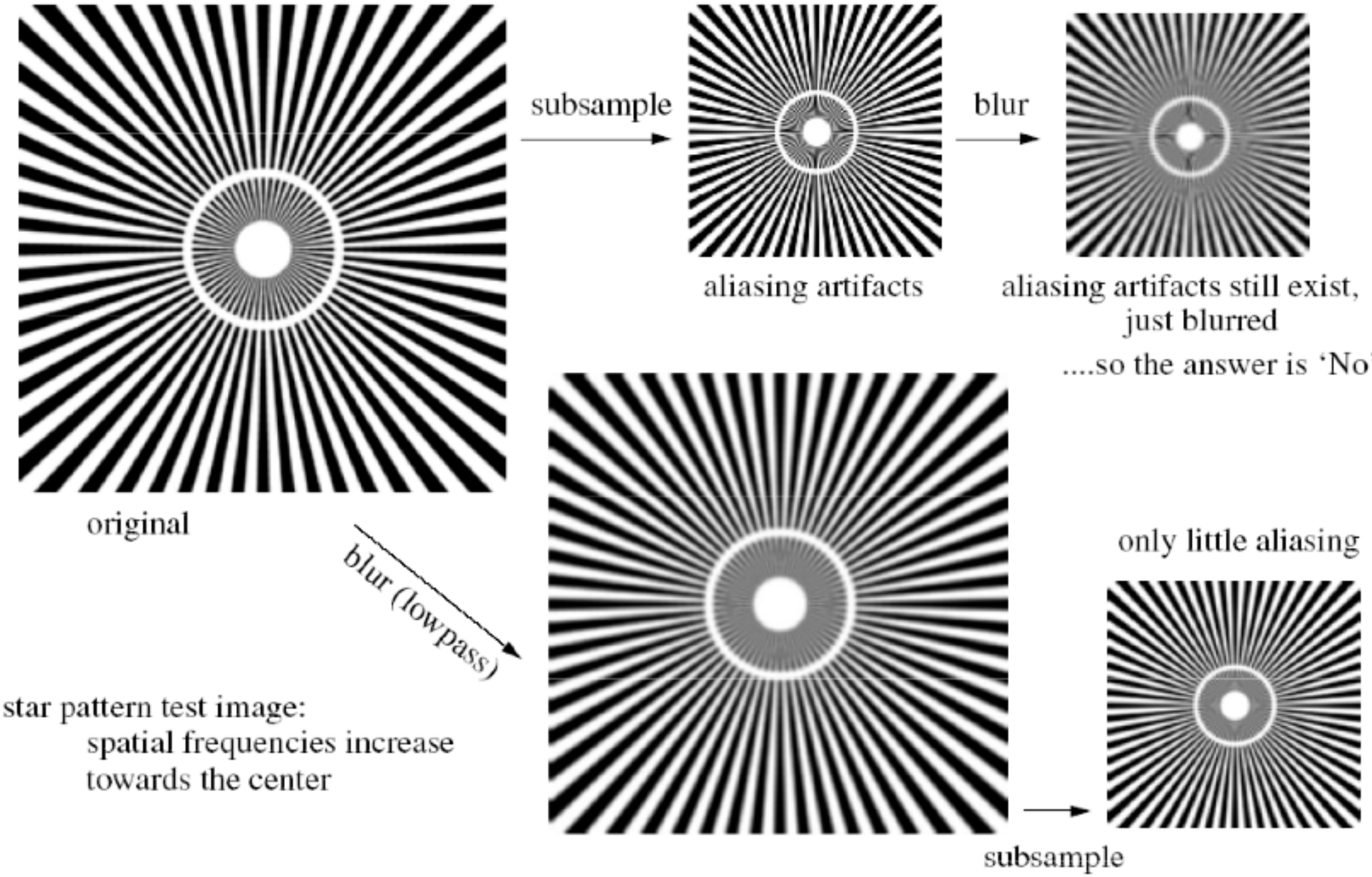
All of these concepts readily extend to higher dimensions



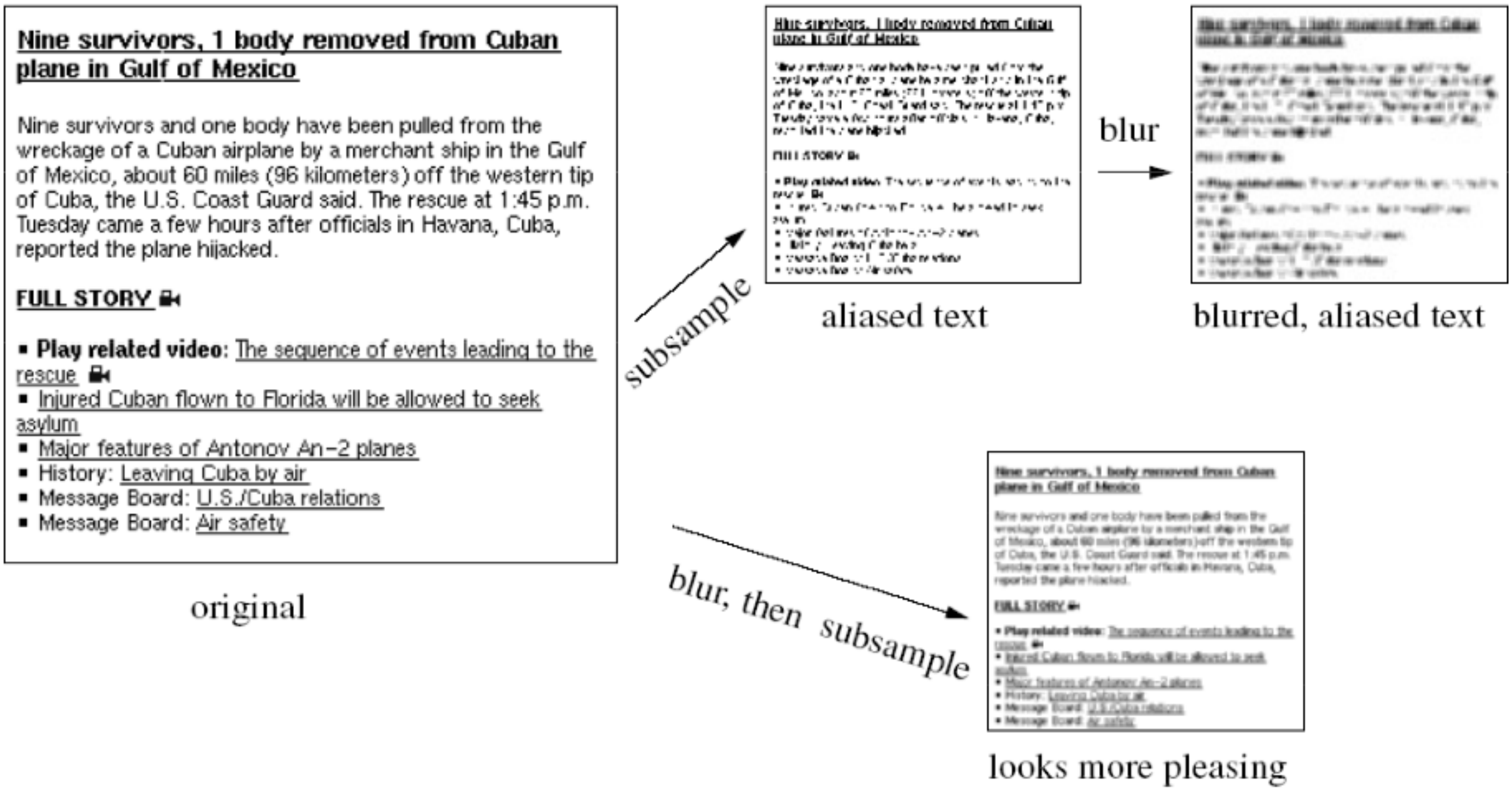
Main spectrum ($S(k,l)$) must fit into the center box to prevent overlap with side-spectra (and aliasing)

$$\frac{1}{\Delta x} > 2 \cdot k_{x\max} \qquad \frac{1}{\Delta y} > 2 \cdot k_{y\max}$$

Exemplo


subsample
blur
aliasing artifacts
aliasing artifacts still exist,
just blurred
....so the answer is 'No'
original
blur (lowpass)
only little aliasing
star pattern test image:
spatial frequencies increase
towards the center
subsample

Exemplo



We observe: Anti-aliasing (i.e., blurring, lowpassing) must be applied before sampling

Resposta de sistema

Now assume the input is a complex sinusoid with $Ae^{2\pi ikx}$ then:

for now, assume $\varphi=0$

$$s_0(x) = \int_{-\infty}^{+\infty} Ae^{2\pi ik(x-\xi)} h(\xi)d\xi$$

$$= Ae^{2\pi ikx} \int_{-\infty}^{+\infty} e^{-2\pi ik\xi} h(\xi)d\xi$$

$$= Ae^{2\pi ikx} H$$

H is called the *Fourier Transform* of *h*(*x*):

$$H = \int_{-\infty}^{+\infty} e^{-2\pi ik\xi} h(\xi)d\xi$$

- *H* is also often called the *transfer function* or *filter*

Observações

H scales, and maybe phase-shifts, the input sinusoid S_i

In essence, we have now two alternative representations:

- determine the effect of L on s_i by convolution with h: $s_i * h$
- determine the effect of L on s_i by multiplication with H: $S_i \cdot H$

$$s_i * h \leftrightarrow S_i \cdot H$$

Since convolution is expensive for wide h, the multiplication may be cheaper

- but we need to perform the Fourier transforms of s_i and h

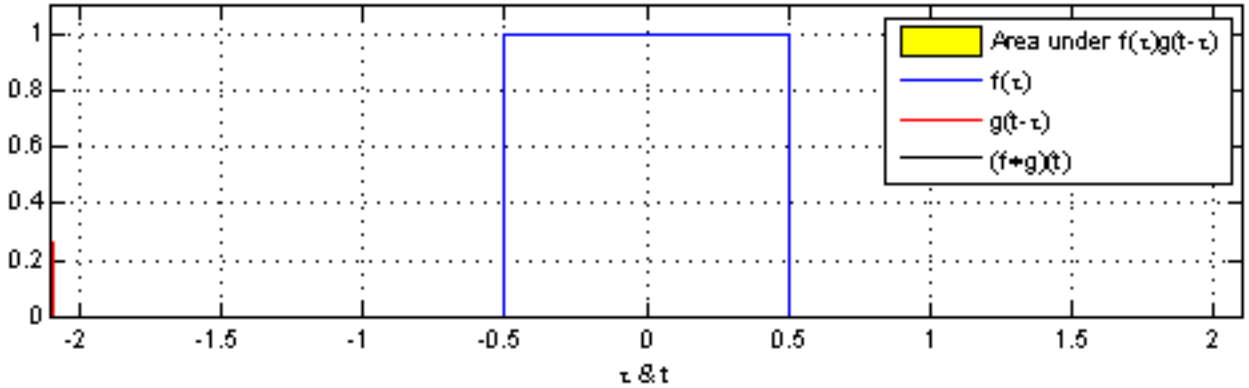
Convolução

The expression

$$s_o(x) = \int_{-\infty}^{+\infty} s_i(\xi) h(x-\xi) d\xi = s_i * h$$

is called *convolution*, defined as:

$$s_1(x) * s_2(x) = \int_{-\infty}^{+\infty} s_1(\xi) s_2(x-\xi) d\xi = \int_{-\infty}^{+\infty} s_1(x-\xi) s_2(\xi) d\xi$$



Sinais 2-D

$$s_1(x,y) * s_2(x,y)$$

$$= \iint_{-\infty}^{+\infty} s_1(x-\xi, y-\zeta)\, s_2(\xi,\zeta)\, \mathrm{d}\xi\, \mathrm{d}\zeta,$$

- Propriedades:
  - $s_1 * s_2 = s_2 * s_1$ (comutativa)
  - $(s_1 * s_2) * s3 = (s_1 * s_2) * s_3$ (associativa)
  - $s_1 * (s_2 + s_3) = s_1 * s_2 + s_1 * s_3$ (distributiva)

Resposta de sistema

Now assume the input is a complex sinusoid with $Ae^{2\pi ikx}$ then:

for now, assume $\varphi=0$

$$s_0(x) = \int_{-\infty}^{+\infty} Ae^{2\pi ik(x-\xi)} h(\xi) d\xi$$

$$= Ae^{2\pi ikx} \int_{-\infty}^{+\infty} e^{-2\pi ik\xi} h(\xi) d\xi$$

$$= Ae^{2\pi ikx} H$$

H is called the *Fourier Transform* of *h*(*x*):

$$H = \int_{-\infty}^{+\infty} e^{-2\pi ik\xi} h(\xi) d\xi$$

- *H* is also often called the *transfer function* or *filter*

Observações

H scales, and maybe phase-shifts, the input sinusoid S_i

In essence, we have now two alternative representations:

- determine the effect of *L* on s_i by convolution with *h*: $s_i * h$
- determine the effect of *L* on s_i by multiplication with *H*: $S_i \cdot H$

$$s_i * h \leftrightarrow S_i \cdot H$$

Since convolution is expensive for wide *h*, the multiplication may be cheaper

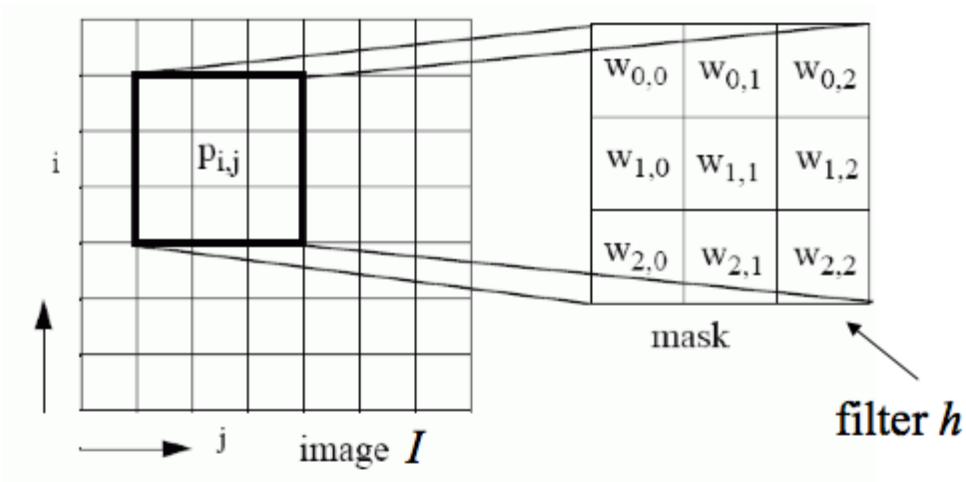
- but we need to perform the Fourier transforms of s_i and *h*

Filtragem espacial

We say *discrete filters* since they operate on a discretized signal, the image

- to implement discrete filters we use discrete convolution

$h * I$



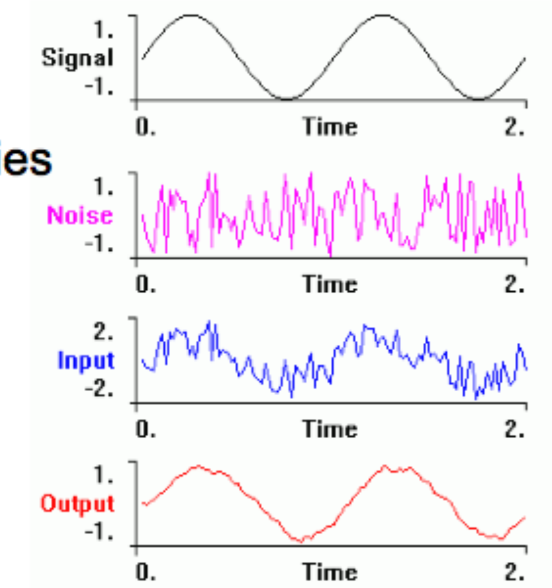
Procedure:

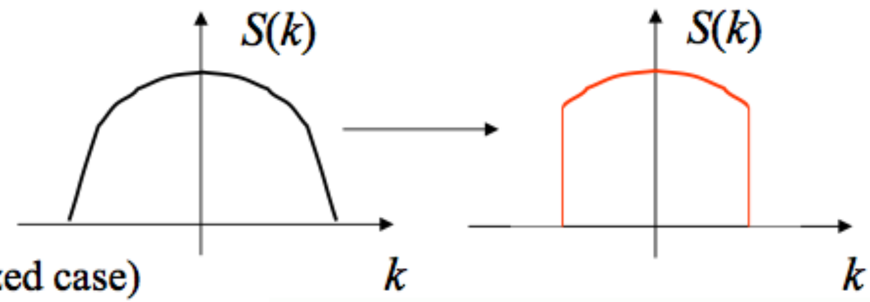
- place a weight matrix or *mask* at each pixel location p_{ij}
- this mask weighs the pixel's neighborhood and determines the output pixel's value
- important: do not replace the computed values into the original image, but write to an output image

Suavização

Smoothing (averaging):

- also called *low-passing:* keeps the low frequencies, but reduces the high frequencies
- removes noise and jagged edges
- but also blurs the signal

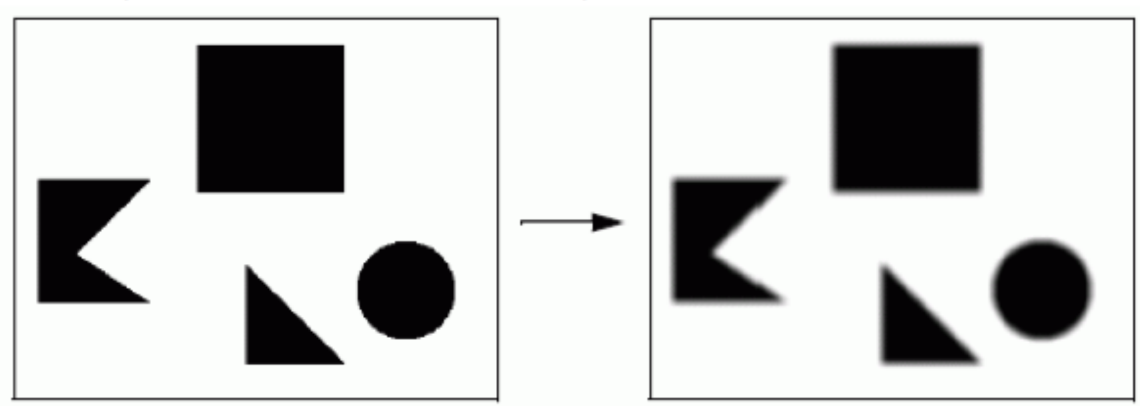




(idealized case)

1/9 ×

1	1	1
1	1	1
1	1	1



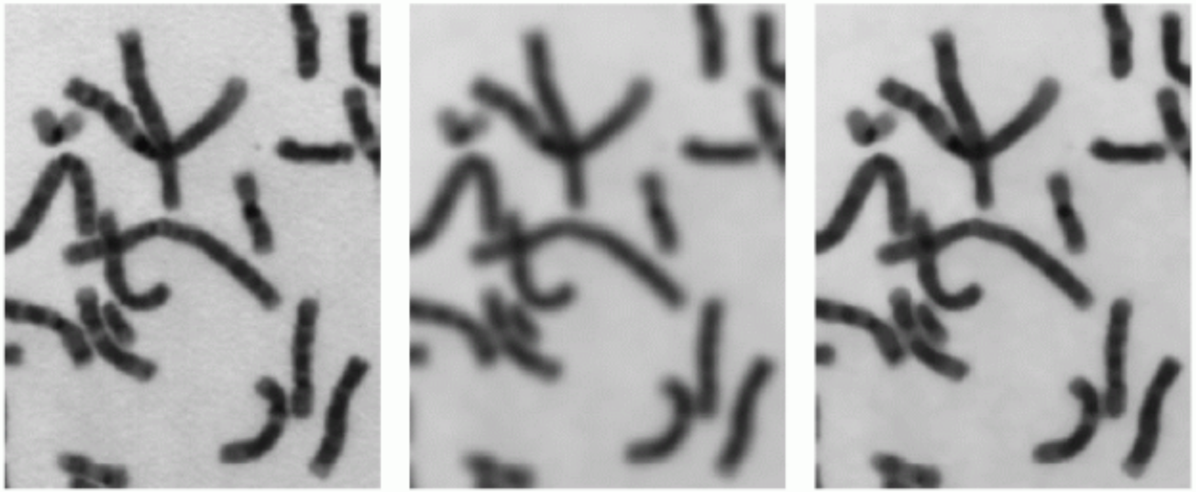
Mediana

A non-linear filter, best used to remove speckle noise

- a regular smoothing filter would blur the speckles (and the signal)
- the median filter will elimate the speckle and leave the signal as is

Procedure:

- convolve with a mask as usual
- but this time, for each mask position, sort the values under the mask
- pick the median and write to the output image
- the speckle pixel will be an outlier and not be selected as the median



original smoothed median filtered

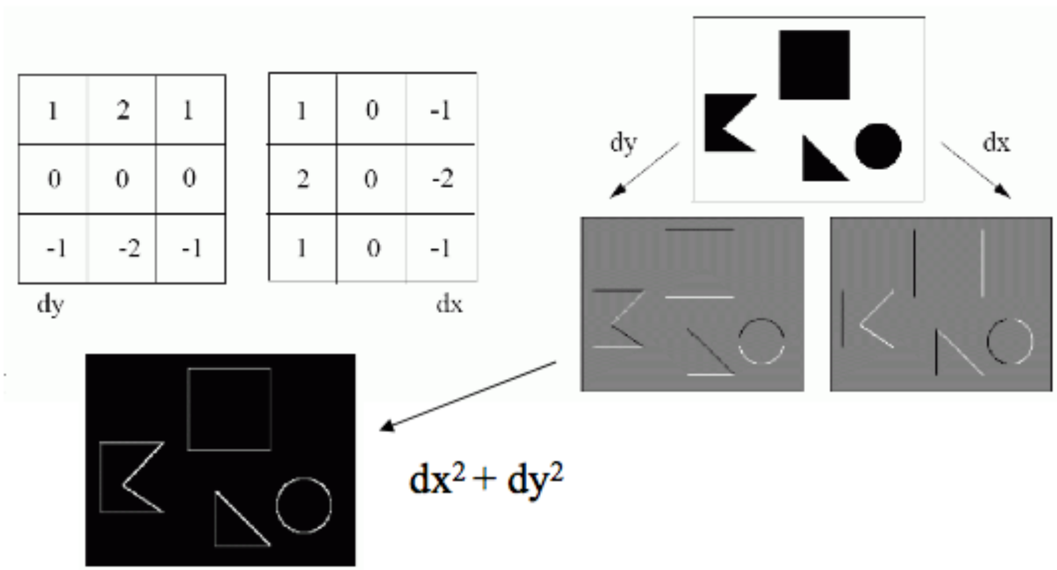
Filtro passa-alta

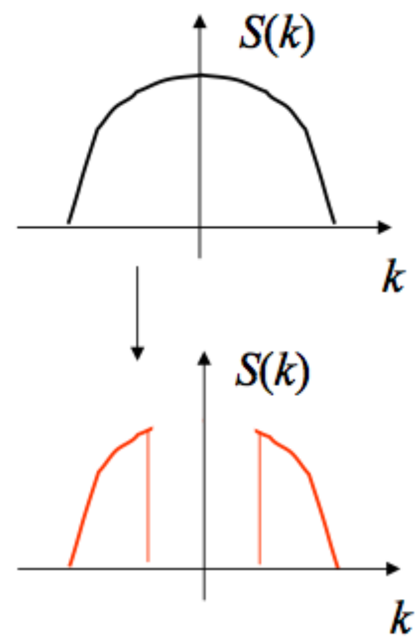
Edge detector / enhancer:

$$\nabla I = \nabla h * I \qquad \text{first derivative (gradient)}$$

$$\nabla^2 I = \nabla^2 h * I \qquad \text{second derivative (Laplacian)}$$

- also called *high-passing:* keeps the high frequencies, but reduces the low frequencies
- enhances edges and contrast
- but also enhances noise and jagged edges



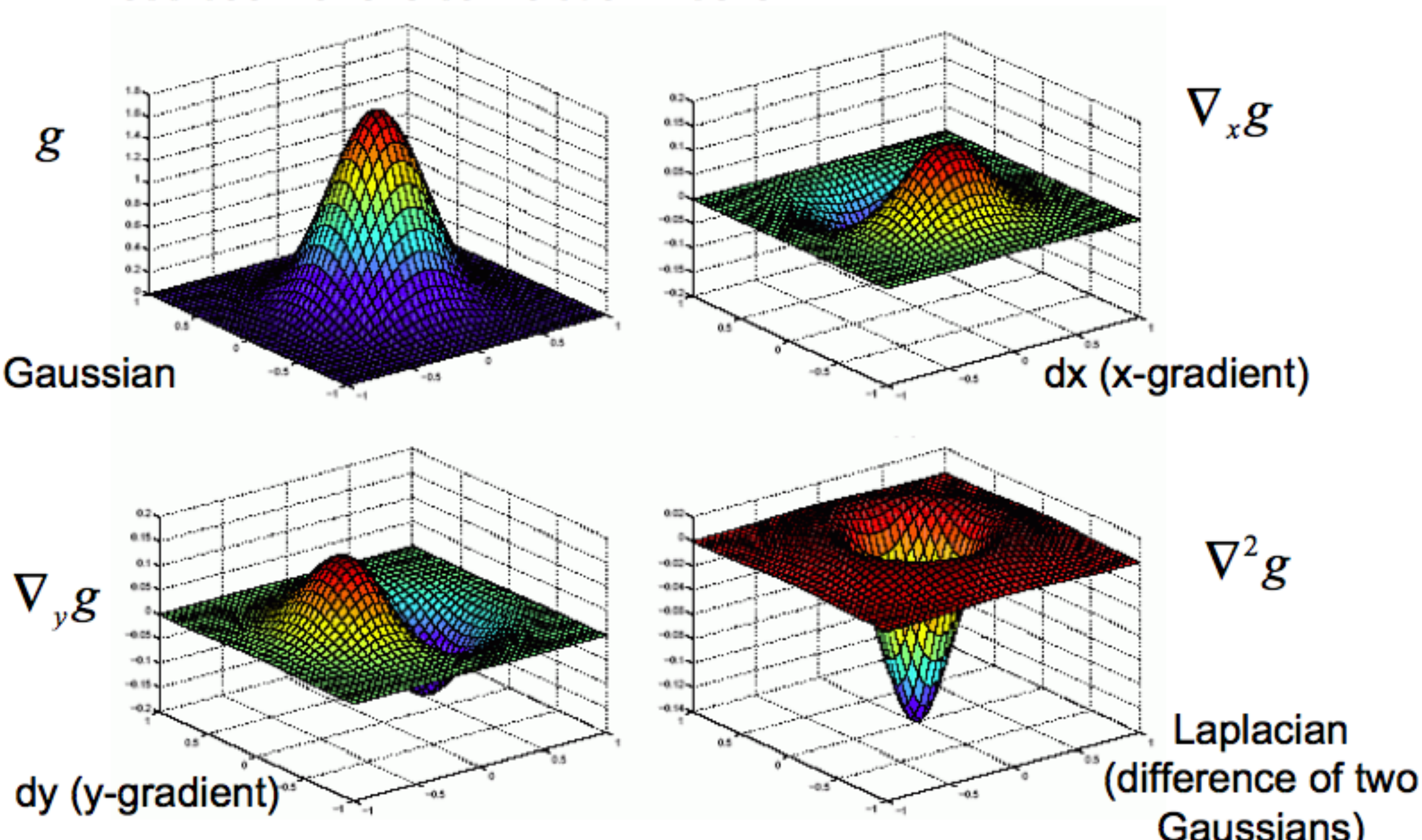


(idealized case)

Função Gaussiana

The Gaussian kernel is a popular filter function

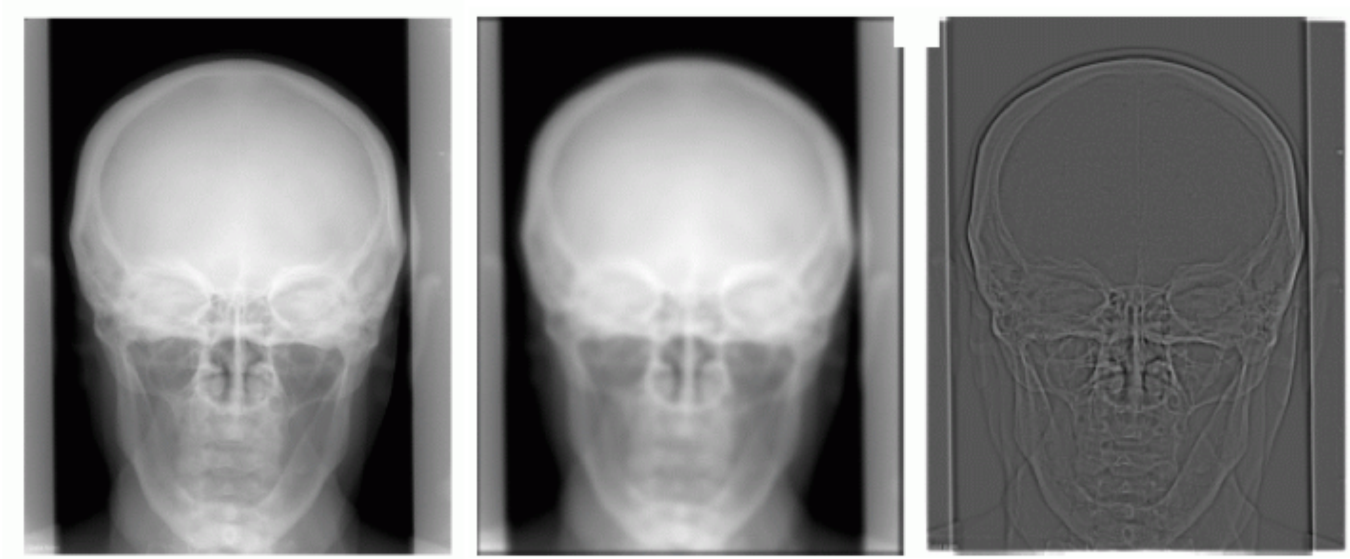
- see book for 3x3 convolution masks



Multiplas operações

Several useful effects can be achieved by subsequent filtering with different masks (kernels) and/or multi-image operations

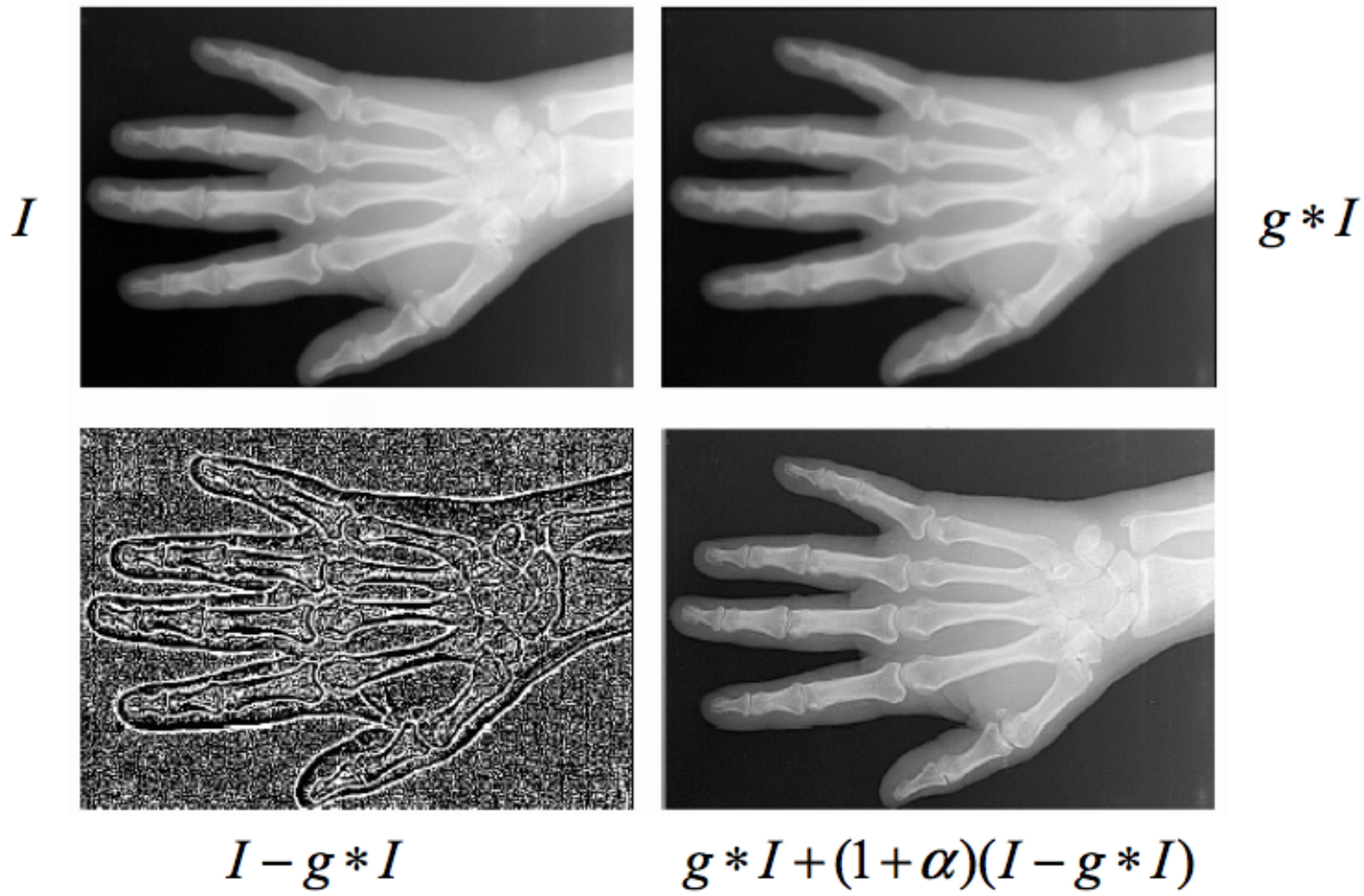
Subtracting a smoothed image from the original image leaves the edges (the high frequencies):



original I | smoothed $g * I$ | original - smoothed $I - g * I$

"Sharpening"

Places the enhanced edges on top of a smoothed original



Alterações locais e globais

Windowing enhances contrast only for a specific range of grey levels (not sensitive to edges)

- strong edges with already good contrast are further enhanced

Edge enhancement (such as sharp masking) only boosts features within a certain frequency band

- this frequency band is determined by filter size -- features outside that band are not enhanced (cannot see many scales at the same time)
- all grey value variations (within that band) are enhanced, even if they already had good contrast

original small filter: small detail large filter: large-scale variations

Tarefa de casa

- Exercício-programa 1