Sinais bidimensionais

- Podem ser expressos através da expressão:

$$e^{i(\alpha x+\beta y)} = e^{2\pi i(px+qy)},$$

where p and q are frequencies in the x and y directions.

$$\begin{aligned} f_{\alpha,\beta}(x,y) &= e^{i(\alpha x+\beta y)} = e^{i\alpha x}e^{i\beta y} \\ &= \cos(\alpha x+\beta y) + i\sin(\alpha x+\beta y) \\ &= \cos(\alpha x)\cos(\beta y) - \sin(\alpha x)\sin(\beta y) \\ &\quad + i(\sin(\alpha x)\cos(\beta y) + \cos(\alpha x)\sin(\beta y)) \end{aligned}$$

Fourier

forward transform $S(k) = F\{s(x)\} = \int_{-\infty}^{+\infty} s(x)e^{-2\pi ikx}dx$

inverse transform $s(x) = F^{-1}\{S(k)\} = \int_{-\infty}^{+\infty} S(k)e^{2\pi ikx}dk$

sistema

- Um sistema transforma um sinal de entrada (i.e. excitacão) em um sinal de saída (i.e. resposta)

$$s_o = \mathcal{L}\{s_i\},$$

Sistemas lineares

System response L: $s_o = L\{s_i\}$

- might be a function of time t or space x

$s_o(t) = L\{s_i(t)\}$ or $s_o(x) = L\{s_i(x)\}$

Finding the mathematical relationship between in- and output is called *modeling*

Linear systems fulfill *superposition principle*:

$L\{c_1s_1 + c_2s_2\} = c_1L\{s_1\} + c_2L\{s_2\} \quad \forall c_1, c_2 \in \Re$

where s_1, s_2 are arbitrary signals

- for example, consider an amplifier with gain A:

$$\begin{aligned} L\{c_1s_1 + c_2s_2\} &= A(c_1s_1 + c_2s_2) \\ &= c_1As_1 + c_2As_2 = c_1L\{s_1\} + c_2L\{s_2\} \end{aligned}$$

Sistemas não lineares

- Muitos sistemas do mundo real possuem natureza não-linear

$$\begin{aligned} \mathcal{L}\{c_1s_1 + c_2s_2\} &= (c_1s_1 + c_2s_2)^2 \\ &\neq (c_1s_1)^2 + (c_2s_2)^2 \end{aligned}$$

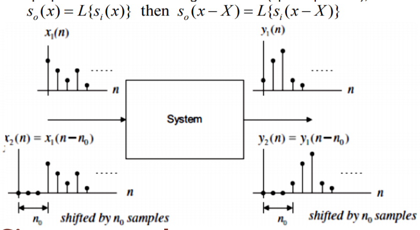
- Porém podem ser aproximados por uma soma ponderada de sinais.

Invariância espacial e temporal

Time-invariance (shift-invariance = LSI):

- properties of L do not change over time (spatial position), that is:

$s_o(x) = L\{s_i(x)\}$ then $s_o(x-X) = L\{s_i(x-X)\}$



Sistema causal

- A causal system is the one in which the output $y(n)$ at time n depends only on the current input $x(n)$ at time n, and its past input sample values such as $x(n-1)$, $x(n-2)$, ... Otherwise, if a system output depends on future input values such as $x(n+1)$, $x(n+2)$, ... the system is noncausal.
- The noncausal system cannot be realized in real time.
  - Determine whether the following systems are causal or not.

$y(n) = 0.5x(n) + 2.5x(n-2)$, for $n \geq 0$

$y(n) = 0.25x(n-1) + 0.5x(n+2) - 0.4y(n-1)$ for $n \geq 0$

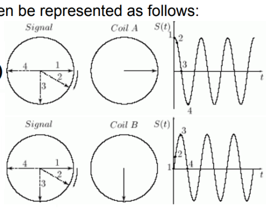
Representação do sinal

To improve SNR, we use two coils, one aligned with the *x*-axis and one aligned with the *y*-axis (*quadrature* scheme)

- the detected signal can then be represented as follows:

$$s_x(t) = Ae^{-\frac{t}{T_2}}\cos(-\omega_0 t)$$

$$s_y(t) = Ae^{-\frac{t}{T_2}}\sin(-\omega_0 t)$$



thus, coil x gives the real part and coil y the imaginary part of a complex-valued signal:

$$s(t) = Ae^{-\frac{t}{T_2}}e^{-i\omega_0 t}$$

Critério de Nyquist

Definition: The Nyquist frequency is ½ the sampling frequency (1/Ts)

Frequencies above the Nyquist frequency appear as aliases

No aliases appear if the function being sampled has no frequencies above the Nyquist frequency

Antialiasing

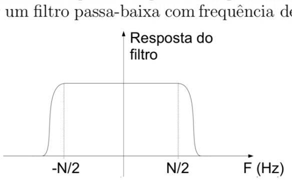
Simple idea:

Remove frequencies above the Nyquist frequency before sampling

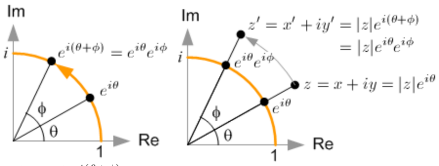
How? Filtering before sampling

Filtragem

Na conversão analógico $\rightarrow$ digital é necessário garantir que a entrada contenha apenas frequências representáveis, o que é feito por um filtro passa-baixa com frequência de corte $\frac{N}{2}$ Hz:



Rotações em 2D



$$\begin{aligned} z' &= |z'|e^{i(\theta+\phi)} \\ &= |z|e^{i(\theta+\phi)} \quad (\because |z'| = |z|) \\ &= |z|e^{i\theta}e^{i\phi} \\ &= (x+iy)e^{i\phi} \quad (\because z = |z|e^{i\theta} = x+iy) \\ &= (x+iy)(\cos\phi + i\sin\phi) \\ &= \cos\phi\cdot x - \sin\phi\cdot y + i(\sin\phi\cdot x + \cos\phi\cdot y) \end{aligned}$$

$$\boxed{\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}}$$

Espaços vetoriais

The vector space $\mathbb{R}^N$ consists of vectors $\mathbf{x}$ of the form

$$\mathbf{x} = (x_1, x_2, \ldots, x_N), \qquad (1.6)$$

where the x_k are all real numbers. Vector addition and scalar multiplication are defined component by component as

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_N + y_N), \quad c\mathbf{x} = (cx_1, cx_2, \ldots, cx_N),$$

where $\mathbf{y} = (y_1, y_2, \ldots, y_N)$ and $c \in \mathbb{R}$. The space $\mathbb{R}^N$ is appropriate when we work with sampled audio or other one-dimensional signals. If we allow the x_k in (1.6) and scalar c to be complex numbers, then we obtain the vector space $\mathbb{C}^N$. That $\mathbb{R}^N$ or $\mathbb{C}^N$ satisfy the properties of a vector space (with addition and scalar multiplication as defined) follows easily, with zero vector $\mathbf{0} = (0, 0, \ldots, 0)$ and additive inverse $(-x_1, -x_2, \ldots, -x_n)$ for any vector $\mathbf{x}$.

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) | x_i \in \mathbb{R}, i = 1, 2, \ldots, n\}$$
$$\mathbb{C}^n = \{(x_1, x_2, \ldots, x_n) | x_i \in \mathbb{C}, i = 1, 2, \ldots, n\}$$

Um espaço vetorial sobre $\mathbb{R}$ (ou $\mathbb{C}$) é um conjunto V munido das operações de soma (entre elementos de V) e multiplicação de elementos de V por escalares (em $\mathbb{R}$ ou em $\mathbb{C}$) com as seguintes propriedades:

1. Fecho por adição: $\forall u, v \in V$ a soma $u + v$ está bem definida e pertence a V;
2. Fecho por multiplicação por escalar: $\forall u \in V$ e $\forall \alpha \in \mathbb{R}$ (ou $\forall \alpha \in \mathbb{C}$) temos αu bem definido e pertence a V;
3. A soma e o produto por escalar satisfazem as propriedades algébricas abaixo $\forall a, b \in \mathbb{R}$ (ou $\mathbb{C}$) e $\forall u, v \in V$:

(a) $u + v = v + u$ (comutatividade)

(b) $(u + v) + w = u + (v + w)$ (associatividade)

(c) $\exists$ um vetor $\mathbf{0}$ t.q. $u + \mathbf{0} = \mathbf{0} + u = u$ (elemento neutro da soma)

(d) $\forall u \in V$ $\exists$ um vetor w t.q. $u + w = \mathbf{0}$ (elemento inverso da soma)

(e) $(ab)u = a(bu)$

(f) $(a + b)u = au + bu$

(g) $a(u + v) = au + av$

(h) $1u = u$

$$M_{m,n}(\mathbb{R}) = \mathbb{R}^{m\times n} = \left\{\begin{pmatrix} a_11 & \ldots & a_1n \\ a_21 & \ldots & a_2n \\ \vdots & \ddots & \\ a_n1 & \ldots & a_nn \end{pmatrix} a_{i,j} \in \mathbb{R}, i = 1, \ldots, m, j = 1, \ldots, n\right\}$$

$$M_{m,n}(\mathbb{C}) = \mathbb{C}^{m\times n} = \left\{\begin{pmatrix} a_11 & \ldots & a_1n \\ a_21 & \ldots & a_2n \\ \vdots & \ddots & \\ a_n1 & \ldots & a_nn \end{pmatrix} a_{i,j} \in \mathbb{C}, i = 1, \ldots, m, j = 1, \ldots, n\right\}$$

TABLE 1.1 Discrete Signal Models and Uses

Notation	Vector Space Description
$\mathbb{R}^N$	$\{\mathbf{x} = (x_1, \ldots, x_N) : x_i \in \mathbb{R}\}$, finite sampled signals
$\mathbb{C}^N$	$\{\mathbf{x} = (x_1, \ldots, x_N) : x_i \in \mathbb{C}\}$, analysis of sampled signals
$L^\infty(\mathbb{N})$ or ℓ^∞	$\{\mathbf{x} = (x_0, x_1, \ldots) : x_i \in \mathbb{R}, \ \lvert x_i\rvert \leq M$ for all $i \geq 0\}$ bounded, sampled signals, infinite time
$L^2(\mathbb{N})$ or ℓ^2	$\{\mathbf{x} = (x_0, x_1, \ldots) : x_i \in \mathbb{R}$ or $x_i \in \mathbb{C}, \sum_k \lvert x_k\rvert^2 < \infty\}$ sampled signals, finite energy, infinite time
$L^2(\mathbb{Z})$	$\{\mathbf{x} = (\ldots, x_{-1}, x_0, x_1, \ldots) : x_i \in \mathbb{R}$ or $x_i \in \mathbb{C}, \sum_k \lvert x_k\rvert^2 < \infty\}$ sampled signals, finite energy, bi-infinite time
$M_{m,n}(\mathbb{R})$	Real $m \times n$ matrices, sampled rectangular images
$M_{m,n}(\mathbb{C})$	Complex $m \times n$ matrices, analysis of images

$$\begin{aligned} \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\ i\sin x &= ix - i\frac{x^3}{3!} + i\frac{x^5}{5!} - i\frac{x^7}{7!} + \cdots \\ e^{ix} = \cos x + i\sin x &= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!} + \cdots \end{aligned}$$

$$\boxed{\begin{aligned} &\lambda = 2\pi/|\omega| \\ &q = l/\lambda = \omega/2\pi \\ &(\text{so } \omega = 2\pi q) \end{aligned}} \quad \boxed{\begin{aligned} &e^{2\pi iqk/N} = e^{2\pi i\tilde{q}k/N} \\ &1 = \left(e^{2\pi i(\tilde{q}-q)/N}\right)^k \\ &\tilde{q} - q = mN \end{aligned}}$$

$$\mathbf{E}_{N,k} = \begin{bmatrix} e^{2\pi ik0/N} \\ e^{2\pi ik1/N} \\ \vdots \\ e^{2\pi ik(N-1)/N} \end{bmatrix}, \quad \mathbf{E}_{N,k}(m) = e^{2\pi ikm/N}$$