Chapter 1
Numerical simulation of the electrical power system model

In this Chapter, we will explain a numerical simulation method for the electrical power system model described by a nonlinear differential-algebraic equation system. A policy for creating a structured numerical simulation environment will also be shown. The structure of this Chapter is as follows. First, in Section **??**, we will explain the difficulty of the calculation time response of the electrical power system model. Next, in Section **??**, we will explain the power flow calculation, which is a process that numerically searches the equilibrium of the electrical power system model. In Section **??**, we will discuss a method to determine constants for a load model and steady values for the internal state of generators to be consistent with the steady values of the voltage and electric power of the bus bars determined by the power flow calculation. In Section **??**, we will explain a calculation method for the time response to changes in the initial value, the load, and ground fault, and also present actual time response calculations. Finally, in Section **??**, we will mathematically analyze a steady current as a synchronization phenomenon of the bus bar voltage as an evolving topic.

1 To calculate time response of the electrical power system

1.1 Difficulties in calculating time response

In Chapter **??**, we explained that, as a generator model and a load model were combined by a power grid model, a mathematical model of the entire electrical power system can be described with a nonlinear differential-algebraic equation system. Therefore, the time response of the electrical power system can be obtained by numerical integration of the differential-algebraic equation with appropriately chosen initial values and external inputs. However, in numerical simulation of the electrical power system model, properties unique to an electrical power system must be considered:

- Unless external inputs are correctly set, supply and demand will not be in balance; thus, the steady value of the frequency deviation does not become 0, and the rotor argument of the generators will continue to change.
- Since there are innumerable combinations of external input values to achieve 0 for the steady value of the frequency deviation, realistically valid external input values must be chosen.
- Values for the voltage phasor and current phasor of the bus bars must be selected so that they are consistent as dependent variables to state variables of generators.

Therefore, simply using the differential-algebraic equation solver on MATLAB cannot correctly execute the numerical simulation of an electrical power system. This is one of the factors that makes calculation of the time response for the electrical power system difficult.

1.2 Calculation steps

The standard calculation method for the time response of the electrical power system model described with a nonlinear differential-algebraic equation can be divided into the following three steps.

(A) To specify the state of an electrical power system in which the supply and demand to be analyzed are in equilibrium,the admittance matrix determined by the power grids is used to calculate the value of the current phasor and voltage phasor for all bus bars in a steady state.
(B) To be consistent with the determined steady values of the current phasor and voltage phasor of the bus bars, the steady values of internal voltage and the rotor argument of each generator, the external input to the generators, and the impedance of each load are back-calculated.
(C) With the state of an electrical power system, in which the demand calculated in the Steps A and B are in equilibrium, as the initial value, various sizes of disturbances are caused, such as perturbation applied to the internal state of generators, ground fault of bus bar voltage, and change in load parameters, and the time response is calculated.

From the perspective of control systems engineering, Step A can be understood as “determining one equilibrium point from many equilibrium points to perform numerical analysis”. As discussed in Section **??**, when the steady values for the current phasor and voltage phasor of all bus bars are provided, the steady value of the internal state of generators, external input, and load parameters that achieve this always exist. As such, calculating the steady values of the current phasor and voltage phasor for all bus bars is equivalent to calculating the equilibrium point of the electrical power system model expressed by a differential-algebraic equation. In electrical power system engineering, this process is called **power flow calculation**.

Obtaining the steady values of the current phasor and voltage phasor of each bus bar is mathematically equivalent to obtaining the steady values of the active reactive power supplied from each equipment to the bus bars.

In Step B, to be consistent with the steady values of the current phasor and voltage phasor of the bus bars, the steady value of the internal state of the generators, external input, and load parameters are back-calculated. In this Step, one must pay attention to the load parameters; in other words, the mathematical model of load "is determined by back calculating from the result of the power flow calculation in Step A". For example, if setting the load connected to bus bars in the constant impedance model, the impedance of the load is back-calculated as a value where the current phasor of the bus bars obtained in the power flow calculation is divided by the voltage phasor. If setting load model parameters as desired values, in Step C, the time response to load parameter fluctuations is calculated while changing the load parameters to desired values.

Finally, in Step C, the time response of the electrical power system model is calculated under various conditions depending on the purpose. For example, if the external input to each generator and load parameter calculated in Steps A and B are set as constants in the model, and the time response is calculated with the initial value appropriate for generators, the internal state of the generators asymptotically converges to the steady state calculated in Step B over time. However, to set valid initial values such that asymptotic convergence is established, as in Steps A and B, the equilibrium point that serves as the reference for the analysis must be calculated first. The calculated equilibrium point must be stable in an appropriate sense.

2 Power flow calculation that numerically searches the steady stat

In this Section, we discuss the outline of the power flow calculation that searches the steady state of the electrical power system that was explained as Step A in Section **??** and the implementation method by MATLAB. Below, the distribution of the current phasor and voltage phasor of the bus bars:

$$\begin{bmatrix} \boldsymbol{I}_1(t) \\ \vdots \\ \boldsymbol{I}_N(t) \end{bmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{Y}_{11} & \cdots & \boldsymbol{Y}_{1N} \\ \vdots & \ddots & \vdots \\ \boldsymbol{Y}_{N1} & \cdots & \boldsymbol{Y}_{NN} \end{bmatrix}}_{\boldsymbol{Y}} \begin{bmatrix} \boldsymbol{V}_1(t) \\ \vdots \\ \boldsymbol{V}_N(t) \end{bmatrix} \tag{1}$$

that satisfies the given admittance matrix $\boldsymbol{Y}$:

$$\big(|\boldsymbol{I}_1(t)|, \angle\boldsymbol{I}_1(t), |\boldsymbol{V}_1(t)|, \angle\boldsymbol{V}_1(t), \ldots |\boldsymbol{I}_N(t)|, \angle\boldsymbol{I}_N(t), |\boldsymbol{V}_N(t)|, \angle\boldsymbol{V}_N(t)\big) \tag{2}$$

is called **power flow distribution** at time t. Each current phasor and voltage phasor changes with time, and from the laws of physics of current and voltage, these must satisfy Equation **??** at arbitrary time t.

The power flow calculation is a calculation process that obtains “one of the steady power flow distributions”. Here, a steady power flow distribution means that the following holds for a constant current phasor $\boldsymbol{I}_i^\star$ and voltage phasor $\boldsymbol{V}_i^\star$ for all bus bars i:

$$\boldsymbol{I}_i(t) = \boldsymbol{I}_i^\star, \qquad \boldsymbol{V}_i(t) = \boldsymbol{V}_i^\star, \qquad \forall t \geq 0$$

Using the definitions of active power and reactive power supplied to the bus bars:

$$P_i(t) + \boldsymbol{j}Q_i(t) = \boldsymbol{V}_i(t)\overline{\boldsymbol{I}}_i(t) \tag{3}$$

current phasor is cancelled. In this manner, we can see that simultaneous equations of Equation **??** are equivalent to:

$$\begin{cases} P_1(t) + \boldsymbol{j}Q_1(t) = \displaystyle\sum_{j=1}^{N} \overline{\boldsymbol{Y}}_{1j}|\boldsymbol{V}_1(t)||\boldsymbol{V}_j(t)|e^{\boldsymbol{j}\left(\angle\boldsymbol{V}_1(t)-\angle\boldsymbol{V}_j(t)\right)} \\ \qquad\vdots \\ P_N(t) + \boldsymbol{j}Q_N(t) = \displaystyle\sum_{j=1}^{N} \overline{\boldsymbol{Y}}_{Nj}|\boldsymbol{V}_N(t)||\boldsymbol{V}_j(t)|e^{\boldsymbol{j}\left(\angle\boldsymbol{V}_N(t)-\angle\boldsymbol{V}_j(t)\right)} \end{cases} \tag{4}$$

Therefore, based on the context, the distribution of active power, reactive power, and voltage power that satisfies Equation **??**:

$$\big(P_1(t), Q_1(t), |\boldsymbol{V}_1(t)|, \angle\boldsymbol{V}_1(t), \ldots, P_N(t), Q_N(t), |\boldsymbol{V}_N(t)|, \angle\boldsymbol{V}_N(t)\big) \tag{5}$$

is also called the power flow distribution of time t.

2.1 Outline of the power flow calculation

Let us explain the characteristics of the power flow calculation using a simple example consisting of two bus bars.

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The power flow calculation of the electrical power system model consisting of two bus bars] Let us consider an electrical power system consisting of two bus bars in Figure **??**. We assume that a load or generator is connected to each bus bar, but the type of equipment does not have to be specified in the power flow calculation.

The admittance of a transmission line that connects two bus bars is $\boldsymbol{y} \in \mathbb{C}$. When employing a basic transmission line model, the following relationship is established between the voltage phasor and current phasor of the bus bars:

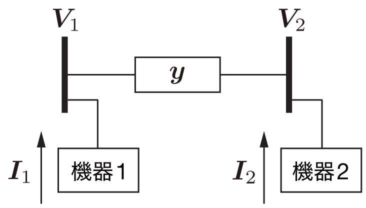

Fig. 1 Power system model consisting of two bus bars

$$\begin{bmatrix} I_1^\star \\ I_2^\star \end{bmatrix} = \begin{bmatrix} y & -y \\ -y & y \end{bmatrix} \begin{bmatrix} V_1^\star \\ V_2^\star \end{bmatrix} \tag{6}$$

However, to clarify that these are steady state values, we added "$\star$". Here, if the current phasor is cancelled using the relationship of Equation **??**, simultaneous equations equivalent to Equation **??** can be obtained as follows by using the active power, reactive power, and voltage phasor under a steady state:

$$\begin{cases} P_1^\star + jQ_1^\star = \overline{y}\left(|V_1^\star|^2 - |V_1^\star||V_2^\star|e^{j(\angle V_1^\star - \angle V_2^\star)}\right) \\ P_2^\star + jQ_2^\star = \overline{y}\left(|V_2^\star|^2 - |V_1^\star||V_2^\star|e^{j(\angle V_2^\star - \angle V_1^\star)}\right) \end{cases} \tag{7a}$$

The objective of the power flow calculation is to determine a pair of:

$$(P_1^\star, Q_1^\star, |V_1^\star|, \angle V_1^\star, P_2^\star, Q_2^\star, |V_2^\star|, \angle V_2^\star)$$

that satisfies these simultaneous equations. Conductance and susceptance of a transmission line are expressed as:

$$g := \mathrm{Re}[y], \qquad b := \mathrm{i}[y]$$

At this time, if we consider an equation related to the real and imaginary parts of Equation **??**, four simultaneous equations are obtained:

$$\begin{cases} P_1^\star = g|V_1^\star|^2 - g|V_1^\star||V_2^\star|\cos\angle V_{12}^\star - b|V_1^\star||V_2^\star|\sin\angle V_{12}^\star \\ P_2^\star = g|V_2^\star|^2 - g|V_1^\star||V_2^\star|\cos\angle V_{21}^\star - b|V_1^\star||V_2^\star|\sin\angle V_{21}^\star \\ Q_1^\star = -b|V_1^\star|^2 + b|V_1^\star||V_2^\star|\cos\angle V_{12}^\star - g|V_1^\star||V_2^\star|\sin\angle V_{12}^\star \\ Q_2 = -b|V_2^\star|^2 + b|V_1^\star||V_2^\star|\cos\angle V_{21}^\star - g|V_1^\star||V_2^\star|\sin\angle V_{21}^\star \end{cases} \tag{7b}$$

where: $\angle V_{ij}^\star$ is $\angle V_i^\star - \angle V_j^\star$ The phase of the voltage phasor is only meaningful in terms of the difference value; thus, the number of variables that must be practically determined is seven. Therefore, there are three variables of freedom in Equation **??**. A simple method to determine these values, for example, is to specify appropriate values for the three variables of $(|V_1^\star|, |V_2^\star|, \angle V_{12}^\star)$. In this manner, the remaining $(P_1^\star, P_2^\star, Q_1^\star, Q_2^\star)$ are determined by simply calculating the right side of Equation

??. However, while the voltage phasor of each bus bar can be arbitrarily set with this method, the electric power supplied to each bus bar and the electric power that is consumed cannot be arbitrarily set. In other words, we cannot specify which load bus bar consumes electric power and which generator bus supplies electric power. Therefore, to execute numerical simulation under a realistic setting, in many cases, the value of the voltage phasor must be appropriately set to achieve the specified active and reactive power values.

For example, let us consider obtaining $(|\boldsymbol{V}_1^\star|, |\boldsymbol{V}_2^\star|, \angle \boldsymbol{V}_{12}^\star)$ to achieve:

$$P_1^\star = 1, \qquad P_2^\star = -1 \tag{8}$$

Where the active power supplied to bus bars is balanced. This is equivalent to obtaining the voltage phasor distribution, when the active power supplied by the equipment connected to bus bar 1, and the active power consumed by the equipment connected to bus bar 2, are both 1 under a steady state. When formulas related to $P_1^\star$ and $P_2^\star$ of Equation **??** are added, the sum of $P_1^\star$ and $P_2^\star$ is 0; thus,

$$\begin{aligned} 0 &= g\left\{|\boldsymbol{V}_1^\star|^2 + |\boldsymbol{V}_2^\star|^2 - 2|\boldsymbol{V}_1^\star||\boldsymbol{V}_2^\star| \cos \angle \boldsymbol{V}_{12}^\star\right\} \\ &= g\left\{\left(|\boldsymbol{V}_1^\star| - |\boldsymbol{V}_2^\star|\right)^2 + 2|\boldsymbol{V}_1^\star||\boldsymbol{V}_2^\star|\left(1 - \cos \angle \boldsymbol{V}_{12}^\star\right)\right\} \end{aligned}$$

is obtained. Please note that in a realistic power flow distribution, the phase difference of the voltage phasor of a bus bar $\angle \boldsymbol{V}_{12}^\star$ is within a range of $\pm\frac{\pi}{2}$. As such, if the conductance g of a real part of $\boldsymbol{y}$ of the transmission line is not 0, the voltage phasor that satisfies this Equation must always satisfy:

$$|\boldsymbol{V}_1^\star| = |\boldsymbol{V}_2^\star|, \qquad \angle \boldsymbol{V}_{12}^\star = 0 \tag{9}$$

However, Equation **??** means that $P_1^\star$ and $P_2^\star$ are both 0. Therefore, as long as g is not 0, a steady power flow distribution that satisfies Equation **??** cannot be realized. This indicates that, since transmission loss occurs due to the conductance component (resistance component) of the transmission line, setting of Equation **??** does not balance the supply and demand of electric power in the entire system. Thus, attention must be paid such that, if "all active power and reactive power are set to specific values, variables that satisfy Equation (**??**) might not exist."

Below, for the sake of simplification, let us assume that the conductance g of the transmission line is 0. At this time, we also assume that the capacitance to ground is sufficiently small, and susceptance b is negative. Since:

$$P_1^\star = -b|\boldsymbol{V}_1^\star||\boldsymbol{V}_2^\star| \sin \angle \boldsymbol{V}_{12}^\star, \qquad P_2^\star = b|\boldsymbol{V}_1^\star||\boldsymbol{V}_2^\star| \sin \angle \boldsymbol{V}_{12}^\star$$

distribution of the voltage phasor must be equal to $P_1^\star$ and $-P_2^\star$. For example, if the value of Equation **??** is specified, by specifying the absolute value of the voltage phasor:

$$|\boldsymbol{V}_1^\star| = \sqrt{\tfrac{2}{|b|}}, \qquad |\boldsymbol{V}_2^\star| = \sqrt{\tfrac{2}{|b|}}$$

the phase difference is determined as:

$$\angle \boldsymbol{V}_{12}^\star = \frac{\pi}{6}$$

Since values of more than three variables are already determined, the reactive power value is automatically determined:

$$Q_1^\star = 2 - \sqrt{3}, \qquad Q_2^\star = 2 - \sqrt{3}$$

As shown in the Example **??**, a process for determining one set of $4N$ constants that satisfies $2N$ simultaneous equations under a steady state:

$$\begin{cases} P_1^\star + \boldsymbol{j}Q_1^\star = \sum_{j=1}^{N} \overline{\boldsymbol{Y}}_{1j} |\boldsymbol{V}_1^\star||\boldsymbol{V}_j^\star| e^{\boldsymbol{j}(\angle \boldsymbol{V}_1^\star - \angle \boldsymbol{V}_j^\star)} \\ \vdots \\ P_N^\star + \boldsymbol{j}Q_N^\star = \sum_{j=1}^{N} \overline{\boldsymbol{Y}}_{Nj} |\boldsymbol{V}_N^\star||\boldsymbol{V}_j^\star| e^{\boldsymbol{j}(\angle \boldsymbol{V}_N^\star - \angle \boldsymbol{V}_j^\star)} \end{cases} \tag{10}$$

$$\left(P_1^\star, Q_1^\star, |\boldsymbol{V}_1^\star|, \angle \boldsymbol{V}_1^\star, \ldots, P_N^\star, Q_N^\star, |\boldsymbol{V}_N^\star|, \angle \boldsymbol{V}_N^\star, \right) \tag{11}$$

for the admittance matrix of the given power grid $\boldsymbol{Y}$ is the power flow calculation. However, the phase of the voltage phasor is only meaningful in terms of the relative value; thus, $(4N - 1)$ variables must be practically determined. As discussed in Section **??**, the power flow calculation can be interpreted as a process that obtains the equilibrium point of the electrical power system model that is able to balance the supply and demand for the entire system. In this process, the properties of the equipment, such as the generator and load, are not considered, whereas only the steady values of the input and output for each bus bar are obtained. Therefore, please note that, by performing the calculation of Step B in Section **??**, one equilibrium point related to the internal state of a differential-algebraic equation system is determined. Calculation of Step B will be discussed in Section **??**.

2.2 Numerical search method for steady power flow distribution

In general, Institute of Electrical Engineers of Japan standard models [**?**], such as IEEE 39 bus bar system model [**?**], and IEEE 68 bus bar system model [**?**], that provide reference values for the electric power consumed by each load bus bar and the electric power supplied to each generator bus, in addition to the impedance of each transmission line, are provided in a datasheet. By specifying $2N$ variables based on these reference values, the remaining variables can be numerically searched.

The data sheet usually provides the active power and reactive power consumed by each load bus bar, the active power supplied by each generator bus, and the absolute value of the voltage phasor for these bus bars. Therefore, by using these values, $2N$ steady values can be specified in advance. However, as shown in Example **??**, if the steady value of the active power is specified for all bus bars in advance, the impact of transmission loss leads to none of the remaining variables satisfying the simultaneous equations of Equation **??**. For example, if the active and reactive power of some load bus bars are set to steady values that differ from those in the data sheet, the steady values of the active and reactive power supplied to generator buses will change, and the steady values of the electric power flowing in the power grid and bus bar voltage phasor will also change; thus, the total value for transmission lines in the entire system will also change. Therefore, if the steady values of active power for all generator buses is specified ahead of time, simultaneous equations of Equation **??** are usually unsolvable.

A typical method of solving this problem is to introduce one special generator bus, called a **slack bus**. With the slack bus, instead of specifying the active power, the phase of the voltage phasor is specified. At this time, only the relative values of the phase of the voltage phasor in each bus has meaning; thus, the steady value of the phase in the slack bus being set to 0 does not cause generality to be lost. As a result, the active power in the slack bus is automatically determined such that it would be consistent to the value of transmission loss in the entire system. The above steps can be summarized as follows:

(a) Based on the data sheet, the value of $(|\boldsymbol{V}_{i_0}^\star|, \angle \boldsymbol{V}_{i_0}^\star)$ is specified for the slack bus, the value of $(P_i^\star, |\boldsymbol{V}_i^\star|)_{i \in \mathcal{I}_\mathrm{G} \setminus \{i_0\}}$ is specified for other generator buses, and the value of $(P_i^\star, Q_i^\star)_{i \in \mathcal{I}_\mathrm{L}}$ is specified for the load bus bar.
(b) Other variables are numerically searched to satisfy simultaneous equations of Equation **??**.

where $\mathcal{I}_\mathrm{G}$ is a subscript set of the generator buses, $\mathcal{I}_\mathrm{L}$ is a subscript set of the load bus bars, and $i_0 \in \mathcal{I}_\mathrm{G}$ is subscript of the slack bus. Bus bars without equipment are handled as load bus bars where the consumed active and reactive power are 0.

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Table 1 Data sheet and tidal current calculation results (1)

	Bus 1	Bus 2	Bus 3
$P_i^\star$	0.5	−3	
$Q_i^\star$		0	
$\lvert \boldsymbol{V}_i^\star \rvert$	2		2
$\angle \boldsymbol{V}_i^\star$			0

(a) Data sheet

	Bus 1	Bus 2	Bus 3
$P_i^\star$			2.5006
$Q_i^\star$	0.0157		0.1388
$\lvert \boldsymbol{V}_i^\star \rvert$		1.9969	
$\angle \boldsymbol{V}_i^\star$	−0.0490	−0.0596	

(b) Tidal current calculation results

Table 2 Data sheet and tidal flow calculation results (2)

	Bus 1	Bus 2	Bus 3
$P_i^\star$		−3	0.5
$Q_i^\star$		0	
$\lvert \boldsymbol{V}_i^\star \rvert$	2		2
$\angle \boldsymbol{V}_i$	0		

(a) Data sheet

	Bus 1	Bus 2	Bus 3
$P_i^\star$	2.5158		
$Q_i^\star$	−0.0347		0.1759
$\lvert \boldsymbol{V}_i^\star \rvert$		1.9918	
$\angle \boldsymbol{V}_i^\star$		−0.0538	−0.0419

(b) Tidal current calculation results

Power flow calculations based on the datasheet] Let us consider the electrical power system model consisting of 3 bus bars discussed in Example **??**. For the admittance matrix of the power grid of Equation **??**, the admittance values of the two transmission lines are set as:

$$\boldsymbol{y}_{12} = 1.3652 - \boldsymbol{j}11.6041, \qquad \boldsymbol{y}_{23} = -\boldsymbol{j}10.5107 \tag{12}$$

Please note that the conductance (real part of $\boldsymbol{y}_{23}$) of the transmission line that connects bus bar 2 and bus bar 3 is set to 0, and the transmission loss of active power by this transmission line is 0.

First, let us consider a case where bus bar 1 is the generator bus, bus bar 2 is the load bus bar, and bus bar 3 is the slack bus. Specifically, we assume that the value in **??**(a) is specified for each bus bar. At this time, the sets of variables that satisfy simultaneous equations in Equation **??** are obtained as in **??**(b): Transmission loss in this case is:

$$P_1^\star + P_2^\star + P_3^\star = 6.2562 \times 10^{-4}, \qquad Q_1^\star + Q_2^\star + Q_3^\star = 1.5450 \times 10^{-1}$$

The ratio of transmission loss for the electric power for consumption for the load is extremely low at about 0.02%. The reason for the transmission loss of active power being small is as follows: While the active power consumed by the load of bus bar 2 is 3 [pu], the generator of the bus bar supplies most of it, ca. 2.5 [pu], while the generator of bus bar 1 only supplies 0.5 [pu]. In other words, the majority of active power consumed by bus bar 2 is supplied through the transmission line on the right where there is no transmission loss.

Next, assuming that bus bar 1 is the slack bus, bus bar 2 is the load bus, and bus bar 3 is the generator bus, we specify variables for each bus bar as in **??**(a). At this time, the result of the power flow calculation is shown in **??**(b). Transmission loss in this case is:

$$P_1^\star + P_2^\star + P_3^\star = 1.5826 \times 10^{-2}, \qquad Q_1^\star + Q_2^\star + Q_3^\star = 1.4120 \times 10^{-1}$$

In this case, the ratio of transmission loss for consumed electric power of the load is about 0.52%. Compared to the previous Example, the transmission loss of active power is high. This is because the majority of active power consumed by bus bar 2 was supplied by the left transmission line where transmission loss occurs. In contrast, it can be seen that the reactive power loss is smaller than in the previous Example.

We can see that the power flow distribution obtained from Example **??** varies the size of the transmission loss for the entire system. Generally, when supplying electric power for consumption using a transmission line with a high conductance component (resistance component), the transmission loss of active power becomes large. When the transmission loss is larger, the power generation necessary to supply the same active power for consumption also increases; thus, the economic cost of power generation cost also increases.

On the other hand, one must be aware of the fact that a steady power flow distribution with low economic cost does not necessarily mean a highly stable equilibrium point. Therefore, it is important to search for a better equilibrium point while considering the trade-off related to economy and stability for practical application. Such a better equilibrium point search process is called **optimal power flow calculation** in electrical power system engineering. In this book, the relationship between the selection of an equilibrium point and stability is discussed in Chapter **??** and Chapter **??**.

2.3 The relationship between the admittance matrix and transmission loss*‡

Below, for a typical power grid consisting of N bus bars, we derive a mathematical expression of transmission loss related to an arbitrary power flow distribution. If the real and imaginary parts of the admittance matrix $\boldsymbol{Y}$, the conductance matrix G and the susceptance matrix B, are symmetrical, they can be rewritten as follows using appropriate constants $\phi_{ij} = \phi_{ji}$ and $\psi_{ij} = \psi_{ji}$:

$$G_{ij} = \begin{cases} \sum_{j=1}^{N} \phi_{ij}, & i = j \\ -\phi_{ij}, & i \neq j \end{cases} \qquad B_{ij} = \begin{cases} -\sum_{j=1}^{N} \psi_{ij}, & i = j \\ \psi_{ij}, & i \neq j \end{cases} \tag{13}$$

However, G_{ij} and B_{ij} are (i, j)th elements of the conductance matrix G and the susceptance matrix B, respectively. Equation **??**is equivalent to the definition below:

$$\phi_{ii} := \sum_{j=1}^{N} G_{ij}, \qquad \phi_{ij} := -G_{ij}, \qquad \psi_{ii} := -\sum_{j=1}^{N} B_{ij}, \qquad \psi_{ij} := B_{ij}$$

Using this expression, the following facts are shown.

Theorem 1.1 (Expression of transmission loss through bus bar voltage phasor) *For Equation* **??***, we define the transmission loss of active power and reactive power for the entire system:*

$$L_P(t) := P_1(t) + \cdots P_N(t), \qquad L_Q(t) := Q_1(t) + \cdots Q_N(t) \tag{14}$$

These transmission losses are obtained with:

$$\begin{aligned} L_P(t) &= \sum_{i=1}^{N} \phi_{ii} |\boldsymbol{V}_i(t)|^2 + \sum_{i=1}^{N} \sum_{j=i+1}^{N} \phi_{ij} W\big(\boldsymbol{V}_i(t), \boldsymbol{V}_j(t)\big) \\ L_Q(t) &= \sum_{i=1}^{N} \psi_{ii} |\boldsymbol{V}_i(t)|^2 + \sum_{i=1}^{N} \sum_{j=i+1}^{N} \psi_{ij} W\big(\boldsymbol{V}_i(t), \boldsymbol{V}_j(t)\big) \end{aligned} \tag{15}$$

However, the following is true:

$$W(\boldsymbol{V}_i, \boldsymbol{V}_j) := \big(|\boldsymbol{V}_i| - |\boldsymbol{V}_j|\big)^2 + 2|\boldsymbol{V}_i||\boldsymbol{V}_j|\big\{1 - \cos(\angle \boldsymbol{V}_i - \angle \boldsymbol{V}_j)\big\}$$

Proof For simplification, time t is omitted. Also, $\angle \boldsymbol{V}_i - \angle \boldsymbol{V}_j$ is expressed as $\angle \boldsymbol{V}_{ij}$. From Equation **??**, for $\boldsymbol{Y}_{ij} = G_{ij} + \boldsymbol{j} B_{ij}$, because of $\boldsymbol{Y}_{ij} = \boldsymbol{Y}_{ji}$, the following is true:

$$\begin{aligned} \sum_{i=1}^{N} (P_i + \boldsymbol{j} Q_i) &= \sum_{i=1}^{N} \overline{\boldsymbol{Y}}_{ii} |\boldsymbol{V}_i|^2 + \sum_{i=1}^{N} \sum_{j=i+1}^{N} \overline{\boldsymbol{Y}}_{ij} |\boldsymbol{V}_i||\boldsymbol{V}_j| \left(e^{\boldsymbol{j} \angle \boldsymbol{V}_{ij}} + e^{\boldsymbol{j} \angle \boldsymbol{V}_{ji}} \right) \\ &= \sum_{i=1}^{N} \overline{\boldsymbol{Y}}_{ii} |\boldsymbol{V}_i|^2 + 2 \sum_{i=1}^{N} \sum_{j=i+1}^{N} \overline{\boldsymbol{Y}}_{ij} |\boldsymbol{V}_i||\boldsymbol{V}_j| \cos \angle \boldsymbol{V}_{ij} \end{aligned}$$

Therefore, by using the notations G_{ij} and B_{ij} of Equation **??**, the following is obtained:

$$\begin{aligned} L_P &= \sum_{i=1}^{N} \left(\sum_{j=1}^{N} \phi_{ij} \right) |\boldsymbol{V}_i|^2 - 2 \sum_{i=1}^{N} \sum_{j=i+1}^{N} \phi_{ij} |\boldsymbol{V}_i||\boldsymbol{V}_j| \cos \angle \boldsymbol{V}_{ij} \\ L_Q &= \sum_{i=1}^{N} \left(\sum_{j=1}^{N} \psi_{ij} \right) |\boldsymbol{V}_i|^2 - 2 \sum_{i=1}^{N} \sum_{j=i+1}^{N} \psi_{ij} |\boldsymbol{V}_i||\boldsymbol{V}_j| \cos \angle \boldsymbol{V}_{ij} \end{aligned}$$

If we focus on the first term of L_P, we can see that $\phi_{ij} = \phi_{ji}$; thus,

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\phi_{ij}|\boldsymbol{V}_i|^2=\sum_{i=1}^{N}\phi_{ii}|\boldsymbol{V}_i|^2+\sum_{i=1}^{N}\sum_{j=i+1}^{N}\phi_{ij}|\boldsymbol{V}_i|^2+\sum_{i=1}^{N}\sum_{j=i+1}^{N}\phi_{ij}|\boldsymbol{V}_j|^2$$

If we rewrite the first term of L_P using this, we can obtain:

$$\begin{aligned}L_P&=\sum_{i=1}^{N}\phi_{ii}|\boldsymbol{V}_i|^2+\sum_{i=1}^{N}\sum_{j=i+1}^{N}\phi_{ij}\left(|\boldsymbol{V}_i|^2+|\boldsymbol{V}_j|^2-2|\boldsymbol{V}_i||\boldsymbol{V}_j|\cos\angle\boldsymbol{V}_{ij}\right)\\&=\sum_{i=1}^{N}\phi_{ii}|\boldsymbol{V}_i|^2+\sum_{i=1}^{N}\sum_{j=i+1}^{N}\phi_{ij}\left\{\left(|\boldsymbol{V}_i|-|\boldsymbol{V}_j|\right)^2+2|\boldsymbol{V}_i||\boldsymbol{V}_j|(1-\cos\angle\boldsymbol{V}_{ij})\right\}\end{aligned}$$

Therefore, L_P can be expressed with the format of Equation **??**. With the same steps, L_Q can be expressed with the format of Equation **??**.

Theorem **??** shows that the discussion of electrical power loss in the electrical power system of two bus bars shown in the Example **??** can be generalized in the same manner as an electrical power system consisting of an arbitrary number of bus bars. Here, if we express the admittance matrix $\boldsymbol{Y}$ as in Section **??**:

$$\boldsymbol{Y}=\boldsymbol{Y}_0+\boldsymbol{j}\,\mathsf{diag}(b_i)_{i\in\{1,\dots,N\}}$$

$\boldsymbol{Y}_0\mathbb{1}$ is 0; in other words, the sum of all rows for the real and imaginary parts of $\boldsymbol{Y}_0$ is zero; thus, it becomes:

$$\phi_{ii}=0,\qquad\psi_{ii}=-b_i$$

Since the conductance of each transmission line is non-negative and the susceptance is negative, ϕ_{ij} and ψ_{ij} for all $i\neq j$ is non-negative. Therefore, as long as ϕ_{ij} is not 0, loss will always occur when transmitting active power between a bus bar i and a bus bar j. Thus, the transmission loss of active power in the entire system $L_P(t)$ is positive for an arbitrary time t. Similarly, if b_i is sufficiently small; in other words, if the capacitance to ground of a transmission line is sufficiently small, loss of reactive power $L_Q(t)$ is also positive. This means that increasing the capacitance to ground of a transmission line and connecting equipment with capacitance properties to a bus bar will reduce the loss of reactive power.

3 The steady state of generators that achieves a desired supply of electric power

In this Section, we explain a method to back calculate the steady value of the internal state of the generators, external input, and load parameters so that the numbers are consistent with the power flow calculation result, which was explained in Step B of Section **??**.

3.1 The steady state of generators that achieves a desired supply of electric power

Let us consider a generator model that inputs the voltage phasor in Section **??**. For the sake of simplification, subscripts i have been omitted:

$$\begin{cases} \dot{\delta} = \omega_0 \Delta\omega \\ M\Delta\dot{\omega} = -D\Delta\omega - P + P_{\mathrm{mech}} \\ \tau \dot{E} = -\frac{X}{X'}E + \left(\frac{X}{X'} - 1\right)|\boldsymbol{V}|\cos(\delta - \angle \boldsymbol{V}) + V_{\mathrm{field}} \end{cases} \tag{16}$$

Here, if the active power and reactive power are the outputs:

$$P = \frac{|\boldsymbol{V}|E}{X'}\sin(\delta - \angle\boldsymbol{V}), \qquad Q = \frac{|\boldsymbol{V}|E}{X'}\cos(\delta - \angle\boldsymbol{V}) - \frac{|\boldsymbol{V}|^2}{X'} \tag{17}$$

The purpose here is to obtain the steady values for the internal state of the generators and external inputs that are consistent with the absolute values of the active power, reactive power, and voltage phasor of the bus bars to which the generators are connected as a result of the power flow calculation. Specifically, when expressing sets of the absolute value for the active power, reactive power, and voltage phasor provided as $(P^\star, Q^\star, |V^\star|, \angle\boldsymbol{V}^\star)$, the objective is to obtain the steady value of the internal state of the generators $(\delta^\star, E^\star)$ and the steady value of the external input $(P^\star_{\mathrm{mech}}, V^\star_{\mathrm{field}})$ that satisfies the simultaneous equations of:

$$\begin{cases} P^\star = \frac{|V^\star|E^\star}{X'}\sin(\delta^\star - \angle\boldsymbol{V}^\star), \\ Q^\star = \frac{|V^\star|E^\star}{X'}\cos(\delta^\star - \angle\boldsymbol{V}^\star) - \frac{|V^\star|^2}{X'}, \\ 0 = -P^\star + P^\star_{\mathrm{mech}}, \\ 0 = -\frac{X}{X'}E^\star + \left(\frac{X}{X'} - 1\right)|V^\star|\cos(\delta^\star - \angle\boldsymbol{V}^\star) + V^\star_{\mathrm{field}} \end{cases} \tag{18}$$

The simultaneous equations of Equation **??** are equations related to the equilibrium point when the steady value of frequency deviation $\Delta\omega$ in Equation **??** is 0. Also, the provided $(P^\star, Q^\star, |V^\star|, \angle\boldsymbol{V}^\star)$ corresponds to the input and output of the generator model for each bus bar.

If we perform a concrete calculation of the steady value of the internal state of the generators that satisfies Equation **??**, the following is obtained:

$$\delta^\star = \angle \boldsymbol{V}^\star + \mathrm{arctan}\left(\frac{P^\star}{Q^\star + \frac{|\boldsymbol{V}^\star|^2}{X'}} \right),$$
$$E^\star = \frac{X'}{|\boldsymbol{V}^\star|}\sqrt{\left(Q^\star + \frac{|\boldsymbol{V}^\star|^2}{X'}\right)^2 + (P^\star)^2} \tag{19a}$$

The steady values of mechanical torque and field voltage are:

$$P^\star_{\mathrm{mech}} = P^\star,$$
$$V^\star_{\mathrm{field}} = \frac{\frac{X}{|\boldsymbol{V}^\star|}\left\{ \left(Q^\star + \frac{|\boldsymbol{V}^\star|^2}{X'}\right)\left(Q^\star + \frac{|\boldsymbol{V}^\star|^2}{X}\right) + (P^\star)^2 \right\}}{\sqrt{\left(Q^\star + \frac{|\boldsymbol{V}^\star|^2}{X'}\right)^2 + (P^\star)^2}} \tag{19b}$$

Please refer to Section **??** for these derivation processes.

3.2 Load parameters that achieve the desired electric power consumption

The method of setting the parameters matching the desired power consumption for the load model described in the **??** section is explained. If we cancel the current phasor using Equation **??**, the constant impedance model is written as:

$$P + \boldsymbol{j}Q = -\frac{|\boldsymbol{V}|^2}{\overline{z}^\star_{\mathrm{load}}} \tag{20}$$

However, the subscripts of bus bars i were omitted for simplification. This can be interpreted as the load constant impedance model when current phasor $\boldsymbol{V}$ is the input and active power P and reactive power Q are the output. If the values of active power and reactive power determined by the power flow calculation are expressed as $P^\star$ and $Q^\star$, and the voltage phasor and absolute value are expressed as $|\boldsymbol{V}^\star|$, the real part (resistance) and the imaginary part (reactance) of load impedance $z^\star_{\mathrm{load}}$ can be obtained as:

$$\mathrm{Re}[z^\star_{\mathrm{load}}] = -\frac{P^\star|\boldsymbol{V}^\star|^2}{(P^\star)^2 + (Q^\star)^2}, \qquad \mathrm{i}[z^\star_{\mathrm{load}}] = -\frac{Q^\star|\boldsymbol{V}^\star|^2}{(P^\star)^2 + (Q^\star)^2} \tag{21}$$

Similarly, the constant current model can be written as:

$$P + \boldsymbol{j}Q = \overline{\boldsymbol{I}}^\star_{\mathrm{load}}|\boldsymbol{V}| \tag{22}$$

Thus, the real part and the imaginary part of the load current parameters can be obtained as:

$$\mathsf{Re}[\boldsymbol{I}_{\text{load}}^{\star}] = \frac{P^{\star}}{|\boldsymbol{V}^{\star}|}, \qquad \mathrm{i}[\boldsymbol{I}_{\text{load}}^{\star}] = -\frac{Q^{\star}}{|\boldsymbol{V}^{\star}|}$$

Since the constant power model is:

$$P + \boldsymbol{j}Q = P_{\text{load}}^{\star} + \boldsymbol{j}Q_{\text{load}}^{\star} \tag{23}$$

Clearly, the parameters are:

$$P_{\text{load}}^{\star} = P^{\star}, \qquad Q_{\text{load}}^{\star} = Q^{\star}$$

If we use these parameters in the load model, the power flow distribution obtained with the power flow calculation can be steadily achieved.

3.3 Mathematical relationship between the internal state of generators and input/output[‡]

Below, we mathematically analyze the relationship between the internal state of the generators, the active and reactive power provided by the bus bars, and the voltage phasor of the bus bars. Here, we use the generator model handled in Section **??**. For simplification, subscripts i have been omitted:

$$\begin{cases} \dot{\delta} = \omega_0 \Delta\omega \\ M\Delta\dot{\omega} = -D\Delta\omega - P + P_{\text{mech}} \\ \tau\dot{E} = -\frac{X_{\text{d}}}{X'_{\text{d}}}E + \left(\frac{X_{\text{d}}}{X'_{\text{d}}} - 1\right)|\boldsymbol{V}|\,\mathsf{cos}(\delta - \angle\boldsymbol{V}) + V_{\text{field}} \end{cases} \tag{24}$$

Here, if the active power and reactive power are the output:

$$\begin{aligned} P &= \frac{|\boldsymbol{V}|E}{X'_{\text{d}}}\,\mathsf{sin}(\delta - \angle\boldsymbol{V}) - \left(\frac{1}{X'_{\text{d}}} - \frac{1}{X_{\text{q}}}\right)|\boldsymbol{V}|^2\,\mathsf{sin}(\delta - \angle\boldsymbol{V})\,\mathsf{cos}(\delta - \angle\boldsymbol{V}), \\ Q &= \frac{|\boldsymbol{V}|E}{X'_{\text{d}}}\,\mathsf{cos}(\delta - \angle\boldsymbol{V}) - |\boldsymbol{V}|^2\left(\frac{\mathsf{cos}^2(\delta - \angle\boldsymbol{V})}{X'_{\text{d}}} + \frac{\mathsf{sin}^2(\delta - \angle\boldsymbol{V})}{X_{\text{q}}}\right) \end{aligned} \tag{25}$$

If X'_{d} and X_{q} are equal and X', and X_{d} are substituted with X, it becomes consistent with the generator model handled in Section **??**. If the current phasor is output:

$$\begin{aligned} |\boldsymbol{I}|\cos(\delta - \angle \boldsymbol{I}) &= \frac{|\boldsymbol{V}|}{X_{\mathrm{q}}}\sin(\delta - \angle \boldsymbol{V}), \\ |\boldsymbol{I}|\sin(\delta - \angle \boldsymbol{I}) &= \frac{E - |\boldsymbol{V}|\cos(\delta - \angle \boldsymbol{V})}{X'_{\mathrm{d}}} \end{aligned} \tag{26}$$

Please note that Equation **??** and Equation **??** are equivalent outputs; in other words, for the given arbitrary $(\delta, E, |\boldsymbol{V}|, \angle \boldsymbol{V})$, there is a one-to-one relationship for (P, Q) and $(|\boldsymbol{I}|, \angle \boldsymbol{I})$.

For this generator model, the following facts are presented.

Lemma 1.1 (The relationship between the internal state of generators and input/output) *If we consider Equation* **??** *as simultaneous equations related to $\delta - \angle \boldsymbol{V}$ and E, the solution is provided with:*

$$\delta - \angle \boldsymbol{V} = \arctan\left(\frac{P}{Q + \frac{|\boldsymbol{V}|^2}{X_{\mathrm{q}}}}\right), \tag{27a}$$

$$E = \frac{\frac{X'_{\mathrm{d}}}{|\boldsymbol{V}|}\left\{\left(Q + \frac{|\boldsymbol{V}|^2}{X_{\mathrm{q}}}\right)\left(Q + \frac{|\boldsymbol{V}|^2}{X'_{\mathrm{d}}}\right) + P^2\right\}}{\sqrt{\left(Q + \frac{|\boldsymbol{V}|^2}{X_{\mathrm{q}}}\right)^2 + P^2}} \tag{27b}$$

*where $|\boldsymbol{V}| \neq 0$. In contrast, if we consider Equation (**??**) simultaneous equations related to P and Q, the solution is given with Equation* **??**.

Proof Translated with DeepL First, we derive Equation **??** from Equation **??**. Multiplying P by $\cos(\delta - \angle \boldsymbol{V})$ and Q by $\sin(\delta - \angle \boldsymbol{V})$ and taking the difference

$$P\cos(\delta - \angle \boldsymbol{V}) - Q\sin(\delta - \angle \boldsymbol{V}) = \frac{|\boldsymbol{V}|^2}{X_{\mathrm{q}}}\sin(\delta - \angle \boldsymbol{V})$$

is obtained. Dividing both sides by $\cos(\delta - \angle \boldsymbol{V})$, we obtain the Equation **??**. Next, the relation of Euqtion **??** is shown. Rewriting P and Q in $\boldsymbol{I}$ using the Equation **??**, the Equation **??** is equivalently transformed to**??**. This is also equivalent to

$$|\boldsymbol{V}|e^{\boldsymbol{j}(\delta - \angle \boldsymbol{V})} = E - X'_{\mathrm{d}}|\boldsymbol{I}|\sin(\delta - \angle \boldsymbol{I}) + \boldsymbol{j}X_{\mathrm{q}}|\boldsymbol{I}|\cos(\delta - \angle \boldsymbol{I}) \tag{28}$$

is equivalent to Expressing the relation in complex numbers in Equation **??** is

$$\frac{e^{\boldsymbol{j}(\delta - \angle \boldsymbol{V})} - e^{-\boldsymbol{j}(\delta - \angle \boldsymbol{V})}}{e^{\boldsymbol{j}(\delta - \angle \boldsymbol{V})} + e^{-\boldsymbol{j}(\delta - \angle \boldsymbol{V})}} = \underbrace{\frac{P}{Q + \frac{|\boldsymbol{V}|^2}{X_{\mathrm{q}}}}}_{\alpha}\boldsymbol{j}$$

given that it is

$$e^{-\boldsymbol{j}(\delta-\angle \boldsymbol{V})} = \frac{1-\alpha \boldsymbol{j}}{1+\alpha \boldsymbol{j}} e^{\boldsymbol{j}(\delta-\angle \boldsymbol{V})}$$

can be found. Therefore, an equivalent variant of the Equation **??**

$$|\boldsymbol{I}|e^{\boldsymbol{j}(\delta-\angle \boldsymbol{I})} = \frac{P+\boldsymbol{j}Q}{|\boldsymbol{V}|} e^{\boldsymbol{j}(\delta-\angle \boldsymbol{V})} \tag{29}$$

by considering its complex conjugation:

$$|\boldsymbol{I}|e^{-\boldsymbol{j}(\delta-\angle \boldsymbol{I})} = \frac{P-\boldsymbol{j}Q}{|\boldsymbol{V}|} \cdot \frac{1-\alpha \boldsymbol{j}}{1+\alpha \boldsymbol{j}} e^{\boldsymbol{j}(\delta-\angle \boldsymbol{V})} \tag{30}$$

We obtain From Equation **??** and Equation **??**:

$$\begin{aligned} |\boldsymbol{I}| \sin(\delta - \angle \boldsymbol{I}) &= \frac{1}{|\boldsymbol{V}|} \cdot \frac{\alpha P + Q}{1+\alpha \boldsymbol{j}} e^{\boldsymbol{j}(\delta-\angle \boldsymbol{V})}, \\ |\boldsymbol{I}| \cos(\delta - \angle \boldsymbol{I}) &= \frac{1}{|\boldsymbol{V}|} \cdot \frac{P - \alpha Q}{1+\alpha \boldsymbol{j}} e^{\boldsymbol{j}(\delta-\angle \boldsymbol{V})} \end{aligned}$$

We can see that P and Q can be used in the equation G and Q. By substituting these into Equation**??** and rewriting $\boldsymbol{I}$ in terms of P and Q, when the relationship in Equation **??** holds, the Equation**??** becomes

$$E = \frac{X'_{\mathrm{d}}}{|\boldsymbol{V}|} \left\{ \left(Q + \frac{|\boldsymbol{V}|^2}{X_{\mathrm{q}}}\right)\left(Q + \frac{|\boldsymbol{V}|^2}{X'_{\mathrm{d}}}\right) + P^2 \right\} \frac{Q + \frac{|\boldsymbol{V}|^2}{X_{\mathrm{q}}} - \boldsymbol{j}P}{\left(Q + \frac{|\boldsymbol{V}|^2}{X_{\mathrm{q}}}\right)^2 + P^2} e^{\boldsymbol{j}(\delta-\angle \boldsymbol{V})} \tag{31}$$

is equivalent to where, from the relation in Equation **??**, it can be shown that:

$$Q + \frac{|\boldsymbol{V}|^2}{X_{\mathrm{q}}} - \boldsymbol{j}P = \left| Q + \frac{|\boldsymbol{V}|^2}{X_{\mathrm{q}}} - \boldsymbol{j}P \right| e^{-\boldsymbol{j}(\delta-\angle \boldsymbol{V})}$$

Furthermore,

$$|E| = E, \qquad \left(Q + \frac{|\boldsymbol{V}|^2}{X_{\mathrm{q}}}\right)^2 + P^2 = \left| Q + \frac{|\boldsymbol{V}|^2}{X_{\mathrm{q}}} - \boldsymbol{j}P \right|^2$$

and thus the relation in Equation **??** is obtained.

Following the reverse procedure, we derive the expression **??** from the expression (**??**). Using the relationship in **??**, E in **??** can be rewritten by E in **??**. As mentioned above, when the relation **??** holds, the expression**??** is equivalent to the expression**??**. □

The lemma **??** shows that there is a one-to-one relationship between a variable set $(\delta - \angle \boldsymbol{V}, E)$ and a set (P, Q). Specifically, it shows that the internal state of generators (δ, E) can be uniquely back-calculated from the input/output of $(|\boldsymbol{V}|, \angle \boldsymbol{V})$

and (P, Q). Please note that the relationship of Equation (**??**) holds for arbitrary time t regardless of the steady state or transient state.

The following theorem gives a relationship that holds between the input, output, and internal state under a steady state of generators.

Theorem 1.2 (The relationship between the internal state of generators and input/output under a steady state) *Let us consider the generator models of Equation* **??** *and Equation* **??**. *For some constants,* $|\boldsymbol{V}^\star|$, $\Delta\omega^\star$, $\angle\boldsymbol{V}^\star$, $P^\star$, *and* $Q^\star$, *let us assume that inputs of mechanical torque and field voltage are constants given by:*

$$\begin{aligned} P_{\mathrm{mech}}(t) &= D\Delta\omega^\star + P^\star, \\ V_{\mathrm{field}}(t) &= \frac{\frac{X_{\mathrm{d}}}{|\boldsymbol{V}^\star|}\left\{\left(Q^\star + \frac{|\boldsymbol{V}^\star|^2}{X_{\mathrm{q}}}\right)\left(Q^\star + \frac{|\boldsymbol{V}^\star|^2}{X_{\mathrm{d}}}\right) + (P^\star)^2\right\}}{\sqrt{\left(Q^\star + \frac{|\boldsymbol{V}^\star|^2}{X_{\mathrm{q}}}\right)^2 + (P^\star)^2}} \end{aligned} \tag{32a}$$

And the inputs of the voltage phasor of the bus bars are determined by:

$$|\boldsymbol{V}(t)| = |\boldsymbol{V}^\star|, \qquad \angle\boldsymbol{V}(t) = \omega_0\Delta\omega^\star t + \angle\boldsymbol{V}^\star \tag{32b}$$

At this time, the rotor argument, frequency deviation, and internal voltage have:

$$\begin{aligned} \delta(t) &= \angle\boldsymbol{V}(t) + \mathrm{arctan}\left(\frac{P^\star}{Q^\star + \frac{|\boldsymbol{V}^\star|^2}{X_{\mathrm{q}}}}\right), \\ \Delta\omega(t) &= \Delta\omega^\star, \\ E(t) &= \frac{\frac{X'_{\mathrm{d}}}{|\boldsymbol{V}^\star|}\left\{\left(Q^\star + \frac{|\boldsymbol{V}^\star|^2}{X_{\mathrm{q}}}\right)\left(Q^\star + \frac{|\boldsymbol{V}^\star|^2}{X'_{\mathrm{d}}}\right) + (P^\star)^2\right\}}{\sqrt{\left(Q^\star + \frac{|\boldsymbol{V}^\star|^2}{X_{\mathrm{q}}}\right)^2 + (P^\star)^2}} \end{aligned} \tag{33}$$

as the steady solutions. The active and reactive power supplied to bus bars are constants given by:

$$P(t) = P^\star, \qquad Q(t) = Q^\star \tag{34}$$

Proof First, under the input of Equation (**??**), we show that Equation **??** is a solution of the differential Equation of Equation **??**, and that Equation **??** holds for the output. If we consider Equation **??** as equations for $P^\star$ and $Q^\star$, as shown in the Complement**??**, their solutions given by:

$$\begin{aligned}
P^\star &= \frac{|\boldsymbol{V}^\star|E(t)}{X'_\mathrm{d}} \sin\left(\delta(t) - \angle \boldsymbol{V}(t)\right) \\
&\quad - \left(\frac{1}{X'_\mathrm{d}} - \frac{1}{X_\mathrm{q}}\right)|\boldsymbol{V}^\star|^2 \sin\left(\delta(t) - \angle \boldsymbol{V}(t)\right)\cos\left(\delta(t) - \angle \boldsymbol{V}(t)\right), \\
Q^\star &= \frac{|\boldsymbol{V}^\star|E(t)}{X'_\mathrm{d}} \cos\left(\delta(t) - \angle \boldsymbol{V}(t)\right) \\
&\quad - |\boldsymbol{V}^\star|^2\left(\frac{\cos^2\left(\delta(t) - \angle \boldsymbol{V}(t)\right)}{X'_\mathrm{d}} + \frac{\sin^2\left(\delta(t) - \angle \boldsymbol{V}(t)\right)}{X_\mathrm{q}}\right)
\end{aligned}$$

This implies Equation **??**.

Next, under the input of Equation (**??**), we check that Equation **??** is a solution of the differential Equation of Equation**??**. The differential equations for δ and $\delta\omega$ in Equation**??** are equivalents:

$$\frac{M}{\omega_0}\ddot{\delta}(t) + \frac{D}{\omega_0}\dot{\delta}(t) + P(t) - P_\mathrm{mech}(t) = 0$$

Substituting $P_\mathrm{mech}(t)$ in Equation**??**, $P(t)$ in Equation**??** and $\delta(t)$ in Equation**??**, this differential equation is satisfied from the relation in Equation**??** We can see that this differential equation is satisfied. Similarly, by substituting $V_\mathrm{field}(t)$ in Equation**??**, $\delta(t) - \angle \boldsymbol{V}(t)$ and $E(t)$ in Equation**??**, the differential equation for E in Equation**??** is satisfied.

However,

$$\cos\left(\arctan\left(\frac{P^\star}{Q^\star + \frac{|\boldsymbol{V}^\star|^2}{X_\mathrm{q}}}\right)\right) = \frac{Q^\star + \frac{|\boldsymbol{V}^\star|^2}{X_\mathrm{q}}}{\sqrt{\left(Q^\star + \frac{|\boldsymbol{V}^\star|^2}{X_\mathrm{q}}\right)^2 + (P^\star)^2}}$$

Theorem **??** shows, the mechanical torque and field voltage values necessary to achieve the voltage phasor, active power, and reactive power of bus bars determined by the power flow calculations $(P^\star_\mathrm{mech}, V^\star_\mathrm{field})$, and the steady behavior of the internal state (δ, E) at the time. With Equation **??**, the phase of the voltage phasor is not a constant. However, since $\Delta\omega^\star$ that expresses the steady state of the frequency deviation is a constant that should usually be set to 0, only the value $\angle \boldsymbol{V}^\star$ determined by the power flow calculation has a practical meaning. Clearly, when $\Delta\omega^\star$ is 0:

$$P_\mathrm{mech}(t) = P^\star, \qquad \delta(t) = \angle \boldsymbol{V}^\star + \arctan\left(\frac{P^\star}{Q^\star + \frac{|\boldsymbol{V}^\star|^2}{X_\mathrm{q}}}\right)$$

Based on the above discussions, even if the dynamic characteristics of the generators are not considered, and $(P^\star, Q^\star, |\boldsymbol{V}^\star|, \angle \boldsymbol{V}^\star)$ is determined by the power flow calculation, a consistent $(P^\star_\mathrm{mech}, V^\star_\mathrm{field})$ can be uniquely back-calculated. With this result, Equation (**??**) can be derived.

Furthermore, the next Theorem provides an equivalence relationship related to the input/output and the internal state that hold under a steady state of generators.

Theorem 1.3 (Equivalence generators related to input/output and the internal state)
For the generator models of Equation **??** *and Equation* **??***, conditions necessary for the following:*

$$\frac{d^2\delta}{dt^2}(t) = 0, \quad \frac{dE}{dt}(t) = 0, \quad \frac{dP_{\text{mech}}}{dt}(t) = 0, \quad \frac{dV_{\text{field}}}{dt}(t) = 0 \tag{35}$$

are true for all $t \geq 0$*.*

$$\frac{dP}{dt}(t) = 0, \quad \frac{dQ}{dt}(t) = 0, \quad \frac{d|\boldsymbol{V}|}{dt}(t) = 0, \quad \frac{d^2\angle\boldsymbol{V}}{dt^2}(t) = 0 \tag{36}$$

When Equation **??** *or Equation* **??** *is true, the following is true:*

$$\Delta\omega(t) = \frac{1}{\omega_0}\frac{d\angle\boldsymbol{V}}{dt}(t) \tag{37}$$

which is a constant.

Proof Translated with DeepL First, if the expression **??** holds, then the expression **??** holds. In the formula **??**, $\Delta\omega$, E, P_{mech}, V_{field} are all constants, so P You can see that $|\boldsymbol{V}|\cos(\delta - \angle\boldsymbol{V})$ is a constant. Therefore, from the two equations **??**, we can see that Q and $|\boldsymbol{V}|\sin(\delta - \angle\boldsymbol{V})$ are also constants. Also,

$$|\boldsymbol{V}|^2\cos^2(\delta - \angle\boldsymbol{V}) + |\boldsymbol{V}|^2\sin^2(\delta - \angle\boldsymbol{V}) = |\boldsymbol{V}|^2$$

and since the left side is constant, $|\boldsymbol{V}|$ is also constant. Furthermore, in the first relation of Equation (**??**), the derivatives of $\angle\boldsymbol{V}$ and δ are equal at any order, since the right side is constant. Therefore,

$$\frac{d^2\angle\boldsymbol{V}}{dt^2} = \frac{d^2\delta}{dt^2} = 0$$

is obtained. Next, we show that if the Equation **??** holds, then the Equation **??** holds. From Equation (**??**), we know that E is constant and that the second derivative of δ is 0, that is, $\delta\omega$ is constant. Therefore, in Equation **??**, P and $|\boldsymbol{V}|\cos(\delta - \angle\boldsymbol{V})$ are constants, which means that P_{mech} and V_{field} are constants Equation **??** is evident from the fact that the derivatives of $\angle\boldsymbol{V}$ and δ are equal. □

As Theorem **??** shows, an external input to generators $(P_{\text{mech}}, V_{\text{field}})$ being a constant, the internal state (δ, E) being in a steady state, and input to and output from bus bar s$(P, Q, |V|, \angle V)$ being in a steady state, are equivalent. Therefore, the steps of the power flow calculations that determine the active power, reactive power, and voltage phasor of each bus bar are, assuming that the external input of all generators are in a steady state, mathematically equivalent to searching the steady state of the entire electrical power system; in other words, searching the equilibrium point of the electrical power system model.

4 Calculating time response of the electrical power system model

In this Section, we explain a method to calculate the time response of the electric power system model explained as Step C in Section **??**. We will also numerically analyze the behavior of the electrical power system against several disturbances.

4.1 Initial response

With the steps shown in Section **??** and Section **??**, one equilibrium point for the electrical power system is found as a steady state where the frequency deviation of all generators is 0. Specifically, if the bus bar variables determined in the power flow calculation are used to set the initial value of the internal state shown by Theorem **??** and the steady value of the external input for each generator model, and to set the constants back-calculated in Section **??** for each load model, the electrical power system model expressed by a differential-algebraic equation system is balanced by a given steady power flow distribution. Specifically, if the obtained equilibrium point is stable in an appropriate sense, even if there are some perturbations for the internal state of the generators, the internal state of the electrical power system model asymptotically converges to the original steady value. Let us confirm this fact with the following example:

Table 3 Steady-state values of generators for the tidal flow calculation results of ??

i	$P^{\star}_{\mathrm{mech}i}$ [pu]	$V^{\star}_{\mathrm{filed}i}$ [pu]	$\delta^{\star}_i$ [rad]	$\Delta\omega^{\star}_i$ [pu]	$E^{\star}_i$ [pu]
1	0.5000	2.0442	0.0670	0	2.0210
3	2.5006	2.5062	0.3870	0	2.2097

Table 4 Steady-state values of generators for the tidal flow calculation results of ??

i	$P^{\star}_{\mathrm{mech}i}$ [pu]	$V^{\star}_{\mathrm{filed}i}$ [pu]	$\delta^{\star}_i$ [rad]	$\Delta\omega^{\star}_i$ [pu]	$E^{\star}_i$ [pu]
1	2.5158	2.7038	0.5356	0	2.3069
3	0.5000	2.1250	0.0390	0	2.0654

[

Initial response of the electrical power system model] Let us consider the electrical power system model consisting of three bus bars handled in the Example **??**. Let us consider a situation where the generator model of Equation **??** is connected to bus bar 1 and bus bar 3, and a load model with constant impedance of Equation **??** is connected to bus bar 2. For the generator model, we use the constants for generator 1 and generator 2 presented in **??**.

For the two results of the power flow calculation in **??** and **??**, we calculate the steady values of mechanical torque, field voltage, rotor argument, and interval voltage for generator 1 and generator 2 based on Equation (**??**). The calculated steady values are shown in **??** and **??**. Similarly, if we back calculate the load impedance based on Equation **??**, it can be obtained as in the first row of **??** and **??**. Please note that, depending on the result of the power flow calculation, the internal state of the generators, the steady value of the external input, and the load impedance vary.

Let us perturb the steady value of the internal state of the generators as the initial value. Specifically:

$$\begin{bmatrix} \delta_1(0) \\ \delta_3(0) \end{bmatrix} = \begin{bmatrix} \delta_1^\star + \frac{\pi}{6} \\ \delta_3^\star \end{bmatrix}, \qquad \begin{bmatrix} E_1(0) \\ E_3(0) \end{bmatrix} = \begin{bmatrix} E_1^\star + 0.1 \\ E_3^\star \end{bmatrix} \tag{38}$$

The initial value of the frequency deviation is 0. Figure **??** shows the initial value response of the frequency deviation corresponding to the two power flow calculations at this time. The solid blue line represents the frequency deviation of generator 1, while the broken red line is the frequency deviation of generator 3. With either one of the steady power flow distributions, oscillations with a similar frequency were generated. Since the system frequency was set to 60 [Hz], a frequency deviation of 0.015 [pu] is equal to 0.9 [Hz].

Table 5 Impedance value of load [pu]

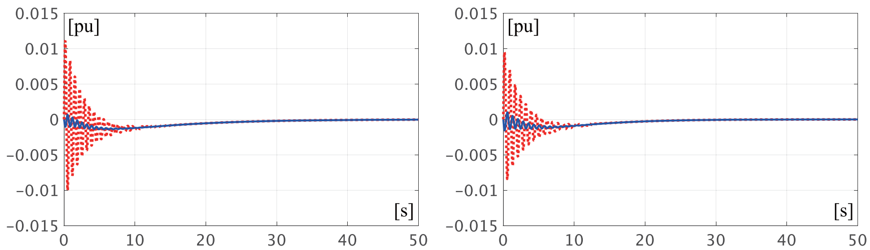
$\mathrm{Re}[z_{\mathrm{load2}}^\star]$	$\mathrm{i}[z_{\mathrm{load2}}^\star]$
1.3293	0
1.3426	0
1.3160	0

(a) **??**

Table 6 [pu]

$\mathrm{Re}[z_{\mathrm{load2}}^\star]$	$\mathrm{i}[z_{\mathrm{load2}}^\star]$
1.3224	0
1.3356	0
1.3092	0

(a) **??**



(a) **??**

(b) **??**

Fig. 2 $\Delta\omega_1 \Delta\omega_3$

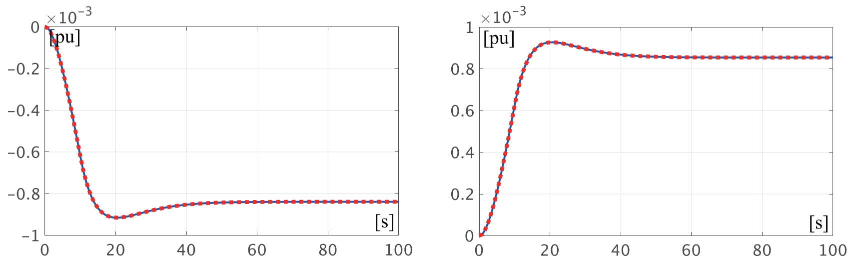
4.2 Response to parameter fluctuations of the load model

If we change even one constant of the load model relative to the electrical power system in a steady power flow distribution, the voltage phasor and current phasor of all bus bars will usually change. At this time, since the supply and demand of electric power for the entire system will usually become unbalanced, unless mechanical torque is appropriately corrected depending on the given field voltage value, the frequency deviation of each generator does not converge to 0. Let us confirm this.

[

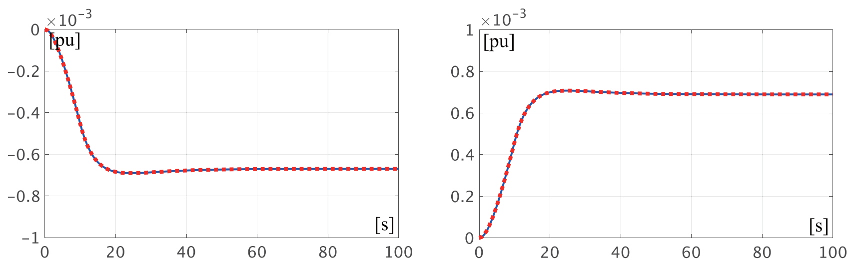
Response time of the electrical power system model to changes in load impedance] Let us calculate the response time of frequency deviation when load impedance changes in the same setting as the Example **??**. Specifically, we calculate the response time by either increasing or decreasing the load resistance, which was back-calculated from the result of the power flow calculation, by 1%. The load impedance values after increase and decrease are shown in the second and third rows of **??** and **??**.

Calculation results are shown in Figure **??** and Figure **??**. The solid blue line is the frequency deviation of generator 1, and the broken red line is the frequency deviation of generator 3. In any case, the frequency deviation of two generators changes in sync. If the resistance increases, consumption of the active power usually increases; thus, the frequency of the generators decreases. The trend is opposite if the resistance decreases. Since the mechanical torque of the generators is set for the value before the resistance changes, the supply and demand of the active power becomes unbalanced, and the steady value of the frequency deviation becomes nonzero. Changes in the load resistance are about the same for Figure **??** and Figure **??**, but the values of generated frequency deviation are different. This indicates that how one chooses the equilibrium point of the electrical power system changes the sensitivity (stability) to disturbance.



(a) If resistance is increased by 1% (b) If the resistance is reduced by 1%

Fig. 3 Time response of angular frequency deviation to changes in load
(Steady current state of **??**, line type is the same as **??**.)



(a) If resistance is increased by 1% (b) If the resistance is reduced by 1%

Fig. 4 Time response of angular frequency deviation to changes in load
(Steady current state of **??**, line type is the same as **??**.)

The result in Example **??** was different from that in Example **??**, and th frequency deviation of the generators have nonzero steady values. To make this frequency deviation 0, the mechanical torque of the generators or the value of the field voltage must be adjusted appropriately. In contrast, making the steady value of frequency deviation 0 means balancing the supply and demand of active power; thus, usually, frequency is regulated using a control algorithm that adjusts the mechanical torque of the generators. However, it is unrealistic and difficult to accurately measure changes in all loads. Thus, a feedback system is necessary, where the value of mechanical torque achieves the supply-demand balance through a control operation based on an integrator. Details will be discussed in Section **??** and Section **??**.

4.3 Response by ground fault

4.3.1 What is ground fault?

Triggers, such as a contact between an object and transmission line, and lightning, cause an electric circuit to come into contact with the ground, discharging significant current to the ground. This phenomenon is called a **ground fault**. If a ground fault continues for a long time, connected equipment and facilities could become damaged; thus, in an electrical power system, there is a device that detects ground faults and blocks the ground fault current rapidly. If the block is released after eliminating the ground fault current, the electrical power system can return to operating in the same way as before the ground fault.

The time it takes to detect a ground fault and eliminate the same is usually 50 [ms] to 100 [ms]. Ground fault current that flows into the ground during this time becomes a severe disturbance that causes significant fluctuations in the state of an electrical power system. Disturbances caused by ground faults are not modeled as external input, but as "a fluctuation that switches to a different electrical power system model while the ground fault continues".

4.3.2 Formulation of bus bar ground fault

Ground faults generated in bus bars discussed in this book are modeled assuming that the voltage phasor of the bus bars while the ground fault continues converges to 0. Below, we explain that ground faults are generated in bus bars 1 without losing generality in the electrical power system model consisting of N bus bars. It is also assumed that ground faults are generated at time 0 [s] and continue to time t_0 [s]. At this time, the power flow distribution in Equation **??** usually satisfies the algebraic equation of Equation **??** between $t < 0$ and $t \geq t_0$, but during $t \in [0, t_0)$ when the ground fault continues, the power flow distribution must satisfy the algebraic equation of:

$$\begin{bmatrix} \boldsymbol{I}_2(t) \\ \vdots \\ \boldsymbol{I}_N(t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{Y}_{22} & \cdots & \boldsymbol{Y}_{2N} \\ \vdots & \ddots & \vdots \\ \boldsymbol{Y}_{N2} & \cdots & \boldsymbol{Y}_{NN} \end{bmatrix} \begin{bmatrix} \boldsymbol{V}_2(t) \\ \vdots \\ \boldsymbol{V}_N(t) \end{bmatrix} \tag{39a}$$

excluding equations related to the current phasor of the ground fault bus bar $\boldsymbol{I}_1(t)$, and constraints for the voltage phasor of the ground fault bus bar:

$$|\boldsymbol{V}_1(t)| = 0 \tag{39b}$$

For bus bars without ground faults or when there is no ground fault, the current phasor and voltage phasor follow a differential equation or algebraic equation that are expressed as a model of each equipment. This is the same as the time without ground fault. In other words, this is equal to using the following two models:

- When there is no ground fault, a normal electrical power system model is sued.
- When the ground fault is continuing, an electrical power system model is used with the bus bars with a ground fault and any equipment and transmission lines connected to these bus bars are removed.

Current phasor $\boldsymbol{I}_1(t)$ flowing from equipment 1 to bus bars 1 during a ground fault is determined from dynamic characteristics. Specifically, if the equipment is a generator, at the time $t \in [0, t_0)$, the internal state of the generator develops over time according to a differential equation where $|\boldsymbol{V}_1(t)|$ is 0:

$$\begin{cases} \dot{\delta}_1 = \omega_0 \Delta\omega_1 \\ M_1 \Delta\dot{\omega}_1 = -D_1 \Delta\omega_1 + P_{\mathrm{mech}1} \\ \tau_1 \dot{E}_1 = -\frac{X_1}{X_1'} E_1 + V_{\mathrm{field}1} \end{cases} \tag{40}$$

At this time, current phasor $\boldsymbol{I}_1(t)$ is given with the following as an output of generators:

$$|\boldsymbol{I}_1(t)| = \frac{E_1(t)}{X_1'}, \qquad \angle \boldsymbol{I}_1(t) = \delta_1(t) - \frac{\pi}{2}$$

Similarly, if equipment 1 is a load model with constant impedance, $|\boldsymbol{V}_1(t)|$ of Equation **??** is 0; thus, $|\boldsymbol{I}_1(t)|$ is also 0. Other load models are basically the same, but with a load model with constant electric power, $|\boldsymbol{I}_1(t)|$ becomes infinitely large; thus, in a numerical simulation, one must be careful of numerical instability. The ground fault current phasor from bus bars 1 to the ground can be expressed as:

$$\boldsymbol{I}_1'(t) := \boldsymbol{I}_1(t) - \sum_{j=2}^{N} \boldsymbol{Y}_{1j} \boldsymbol{V}_j(t), \qquad t \in [0, t_0)$$

The value of the ground fault current phasor does not have to be calculated to execute a numerical simulation of the electrical power system model. On the other hand, if generators are connected to the ground fault bus bar, it is important to calculate the internal state of generators for the time when the ground fault is eliminated t_0; thus, the differential equation of Equation **??** must be solved. For the internal state of the electric power system model at the time of ground fault; in other words, for the internal state of each generator at the initial time, a value for arbitrary power flow distribution can be set. In this book, we set an appropriate value for the steady power flow distribution obtained as a result of the power flow calculation. During ground faults, and when ground faults are eliminated, the internal state of each generator is continuous, but the voltage phasor and current phasor of each bus bar change discontinuously.

Calculation of the time response to the above bus bar ground fault can be summarized in the following steps:

(a) Variable values in the steady power flow distribution obtained with the power flow calculation are used as the initial value of each generator.
(b) During time $t \in [0, t_0)$ when a ground fault is continuing, time development is calculated with the electrical power system model where bus bars with ground fault and equipment and transmission lines connected to these bus bars are removed.
(c) If generators are connected to bus bars with ground fault, the bus bar voltage phasor is set to 0 during the time of continuing ground fault $t \in [0, t_0)$, and time development of the internal state of the generators is calculated.
(d) The value of the internal state of each generator at the time t_0 is set, and time development after eliminating ground fault is calculated using the normal electrical power system model with all equipment connected.

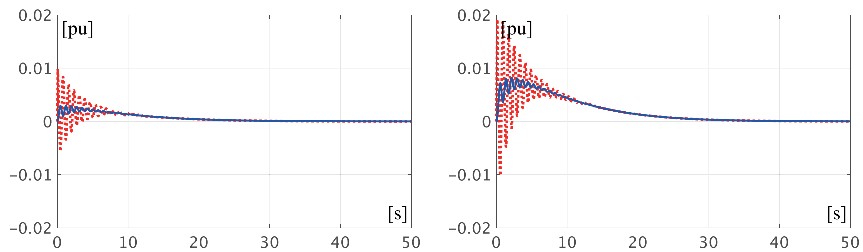
With these steps, let us calculate the time response of the bus bar ground fault.

[

Time response of the electrical power system model to bus bar ground fault] Let us calculate the time response of frequency deviation to ground fault of bus bars under the same setting as the Example **??** and Example **??**. Specifically, we set the two steady power flow distributions, where the electrical power system was obtained with the power flow calculation, as the initial values, and calculate the time response when the ground fault occurs in bus bar 1. For comparison, we consider two situations: when the time it takes for the ground fault to be eliminated is 50 [ms] and 100 [ms].

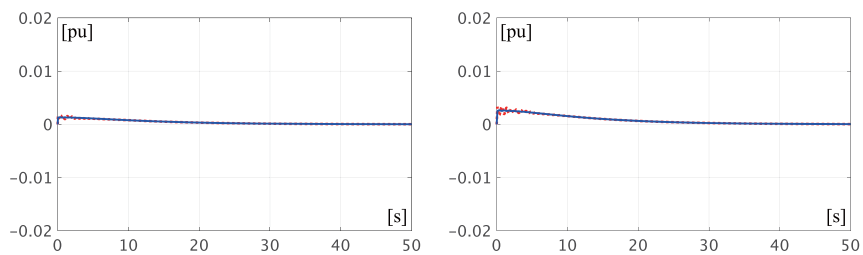
The calculation result is shown in Figure **??** and Figure **??**. The solid blue line is the frequency deviation of generator 1, and the broken red line is the frequency deviation of generator 3. Figure **??** shows that the oscillation of the frequency of generator 3 is large under the steady power flow distribution of **??**. Specifically, when the duration of the ground fault is 100 [ms], oscillation is even larger. The reason oscillation of generator 3 is large is because, as shown in **??**, while the inertia constant of generator 1 is large at 100 [s], the inertia constant of generator 3 is small at 12 [s]. In other words, oscillation of generators with large inertia causes large oscillations in generators with small inertia.

On the other hand, with the steady power flow distribution of Table **??**, frequency oscillation is small even for the same ground fault (Figure **??**). The reason for this is that, with the steady power flow distribution of **??**, the small inertia of generator 3 provided the majority of active power consumed by the load. In contrast, with the steady power flow distribution of **??**, the large inertia of generator 1 provided the majority of active power. Generally, the sensitivity of synchronous generators to disturbance increases as they more closely approach the electric power provided. With the steady power flow distribution of **??**, the stability of generator 3 with small inertia is relatively high; thus, the sensitivity of frequency deviation to ground faults is low.



(a) Ground fault at 50 [ms] (b) Ground fault at 100 [ms]

Fig. 5 Time response of angular frequency deviation to ground fault
(Steady current state of **??**, line type is the same as **??**)



(a) Ground fault at 50 [ms] (b) Ground fault at 100 [ms]

Fig. 6 Time response of angular frequency deviation to ground fault
(Steady current state of **??**, line type is the same as **??**)

The Example **??** shows that the stability of the electrical power system model to ground faults changes according to the steady power flow distribution (equilibrium point). Specifically, with the Example **??**, the steady power flow distribution of **??** was superior in terms of transmission loss, while in the Example **??**, the steady power flow distribution of **??** is superior in terms of system stability against ground faults. These examples show the importance of considering the trade-off with economics and stability and searching for the desirable equilibrium point. Ground faults do not only occur to specific bus bars or transmission lines, and can potentially occur in various grounds. Thus, the stability of the entire electrical power system should be improved in an appropriate sense. To that end, the control mechanism is explained in Section **??** and Section **??**.

The results of simulations from Example **??** to Example **??** show that, if the system reaches a steady state without diverging from the internal state, “frequency deviation of all generators converge to the same value”. A similar result was also observed in Example **??**. Synchronization of this frequency in the steady power flow distribution is a universal phenomenon in the electrical power system model. With frequency control, which will be discussed in Section **??** and Section **??**, the control algorithm is constructed assuming that the frequency automatically synchronizes with the steady power flow distribution.

5 Synchronization of bus bar voltage in a steady power flow distribution‡

In this Section, we mathematically discuss synchronization phenomenon of the frequency observed in Section **??** from the viewpoint of graph structure for the power grid. Based on Theorem **??**, the following definition is introduced.

Definition 1.1 (Synchronization of the steady power flow distribution and bus bar voltage) Let us consider an electrical power system model wherein a group of equipment is combined with simultaneous equations of the Equation **??**. For all bus bars i, when:

$$\frac{dP_i}{dt}(t) = 0, \quad \frac{dQ_i}{dt}(t) = 0, \quad \frac{d|\boldsymbol{V}_i|}{dt}(t) = 0, \quad \frac{d^2\angle\boldsymbol{V}_i}{dt^2}(t) = 0 \tag{41}$$

is true for all $t \geq 0$, the electrical power system is said to be in the **steady power flow distribution**. [1]

When the electrical power system is in the steady power flow distribution, and:

$$\frac{d\angle\boldsymbol{V}_i}{dt}(t) = \frac{d\angle\boldsymbol{V}_j}{dt}(t) \tag{42}$$

is true, bus bar i and bus bar j are **synchronized** in the steady power flow distribution.

As shown in Theorem **??**, for generator buses, the Equation **??** being true and the internal state of generators and external inputs being in a steady state are equivalent. In addition, if a set of arbitrarily selected bus bars (i, j) is synchronized according to the definition **??**, it is concluded that the frequency deviation of all generators will converge to the same value. The fact that Equation **??** holds for any bus bars means that:

$$\frac{d|\boldsymbol{I}_i|}{dt}(t) = 0, \qquad \frac{d^2\angle\boldsymbol{I}_i}{dt^2}(t) = 0, \qquad \frac{d\angle\boldsymbol{I}_i}{dt}(t) = \frac{d\angle\boldsymbol{V}_i}{dt}(t)$$

is true for current phasor $\boldsymbol{I}_i$. This can be easily confirmed since:

$$|P_i(t) + \boldsymbol{j}Q_i(t)| = |\boldsymbol{V}_i(t)||\boldsymbol{I}_i(t)|, \qquad \angle(P_i(t) + \boldsymbol{j}Q_i(t)) = \angle\boldsymbol{V}_i(t) - \angle\boldsymbol{I}_i(t)$$

[1] Please note that the mathematical definition of this “steady power flow distribution” is unique to this book and is not typically used in electrical power system engineering.

A set of adjacent bus bars connected to the bus bar i with a transmission line is expressed as $\mathcal{N}_i$. In other words:

$$\mathcal{N}_i := \left\{ j : \boldsymbol{Y}_{ij} \neq 0, \quad j \neq i \right\}$$

The number of adjacent bus bars connected to the bus bar i with a transmission line is $|\mathcal{N}_i|$. This is called " **degree** of bus bar i is $|\mathcal{N}_i|$". We can use the following expression assuming that the electrical power system model is in the steady power flow distribution,

$$\angle \boldsymbol{V}_i(t) = \Omega_i t + \phi_i$$

However, Ω_i and ϕ_i are constant. At this time, electric power balance equation for bus bar i in Equation **??** is:

$$P_i + \boldsymbol{j} Q_i = \sum_{j=1}^{N} \overline{\boldsymbol{Y}}_{ij} |\boldsymbol{V}_i| |\boldsymbol{V}_j| e^{\boldsymbol{j} \left\{ (\Omega_i - \Omega_j) t + \phi_i - \phi_j \right\}}$$

If we assume a steady power flow distribution, the absolute value of the active power, reactive power, and bus bar voltage phasor provided to the bus bar i are all constants. If these are expressed as $P_i^\star$, $Q_i^\star$, and $|\boldsymbol{V}_i^\star|$, this electric power balance equation can be rewritten as:

$$\underbrace{\frac{1}{|\mathcal{N}_i|} \sum_{j \in \mathcal{N}_i} r_{ij} e^{\boldsymbol{j} (\Omega_{ij} t + \Phi_{ij})}}_{\boldsymbol{C}_i(t)} = z_i \tag{43}$$

However, the frequency difference of the voltage phasor for bus bars i and j is expressed as:

$$\Omega_{ij} := \Omega_i - \Omega_j$$

r_{ij}, Φ_{ij}, and z_i are all constants, and defined by:

$$r_{ij} := |\boldsymbol{V}_i^\star| |\boldsymbol{V}_j^\star| |\boldsymbol{Y}_{ij}|, \qquad \Phi_{ij} := \phi_i - \phi_j - \angle \boldsymbol{Y}_{ij},$$
$$z_i := \frac{\left\{ P_i^\star - \mathrm{Re}[\boldsymbol{Y}_{ii}] |\boldsymbol{V}_i^\star|^2 + \boldsymbol{j} \left(Q_i^\star + \mathrm{i}[\boldsymbol{Y}_{ii}] |\boldsymbol{V}_i^\star|^2 \right) \right\}}{|\mathcal{N}_i|}$$

Below, let us think about Ω_{ij} being 0 for all $j \in \mathcal{N}_i$ as an equation that expresses the synchronicity with adjacent bus bars from Equation **??**; in other words, let us think about deriving:

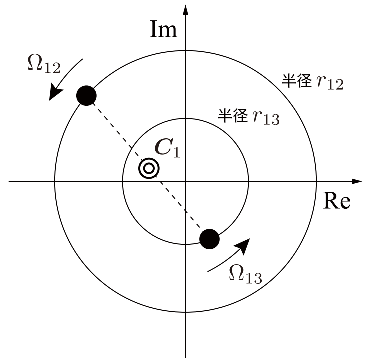


Fig. 7 When bus bar 1 is connected to bus bars 2 and 3

$$\Omega_i = \Omega_j, \qquad \forall j \in \mathcal{N}_i \tag{44}$$

Here, Equation **??** expresses that "the center of gravity $\boldsymbol{C}_i(t)$ of $|\mathcal{N}_i|$ points that move at a constant velocity on the circumference of a circle with the radius of r_{ij} with the origin as the center with the initial phase of Φ_{ij} and the angular velocity of Ω_{ij} is invariable on a point z_i on a complex plane". Figure **??** shows the relationship when bus bar 1 is connected to bus bar 2 and bus bar 3. Focusing on this fact, the following result can be derived.

Lemma 1.2 (Synchronization of bus bars derived from the balance equation of electric power)
*Let us consider $\boldsymbol{C}_i(t)$ of Equation **??** for real constants r_{ij}, Ω_i, Ω_j, and Φ_{ij}, where $r_{ij} > 0$. At this time, if $|\mathcal{N}_i| = 1$, $\boldsymbol{C}_i(t)$ being a constant that is not dependent on t is equivalent to Equation **??**. In addition, if $|\mathcal{N}_i| = 2$, $\boldsymbol{C}_i(t)$ being a constant that is not dependent on t is equivalent to Equation **??** being true, or to:*

$$\Omega_{j_1} = \Omega_{j_2}, \qquad r_{ij_1} = r_{ij_2}, \qquad |\Phi_{ij_1} - \Phi_{ij_2}| = \pi \tag{45}$$

*where $\mathcal{N}_i = \{j_1, j_2\}$. Furthermore, if $|\mathcal{N}_i| = 3$, $\boldsymbol{C}_i(t)$ being a constant that is not dependent on t is equivalent to Equation **??** being true, or Equation **??** being true for $j_3 \in \mathcal{N}_i$ satisfying $\Omega_i = \Omega_{j_3}$, where $\mathcal{N}_i \setminus \{j_3\} = \{j_1, j_2\}$.*

$$\Omega_{j_1} = \Omega_{j_2} = \Omega_{j_2}, \qquad \sum_{j \in \mathcal{N}_i} r_{ij} e^{\boldsymbol{j}\Phi_{ij}} = 0 \tag{46}$$

Proof Translated with DeepL By applying the complementary problem**??** at the end of the chapter, we can show the facts for the cases $|\mathcal{N}_i| = 1$ and $|\mathcal{N}_i| = 2$. Thus, in the following we consider the case $|\mathcal{N}_i| = 3$. For simplicity of notation, let $j \in \{1, 2, 3\}$ and denote r_{ij}, Φ_{ij}, Ω_i and $\boldsymbol{C}_i$ as r_j, Φ_j, Ω_0 and $\boldsymbol{C}_0$ respectively. First, let's consider

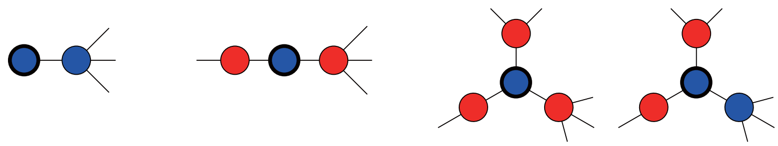


Fig. 8

the following case,

$$\Omega_j \neq \Omega_0, \qquad \forall j \in \{1, 2, 3\} \tag{47}$$

From the complement **??**, when the formula**??** holds, the formula**??** is equivalent to:

$$\Omega_1 = \Omega_2 = \Omega_3, \qquad \sum_{j=1}^{3} r_j e^{\boldsymbol{j}\Phi_j} = 0$$

This implies that the expression **??**

Next, consider the case where the expression **??** does not hold, i.e., where $\Omega_0 = \Omega_j$ for some $j \in \{1, 2, 3\}$. In particular, consider the case where $\Omega_0 = \Omega_3$ without loss of generality from symmetry with respect to j. In this case

$$\boldsymbol{C}_0(t) = \frac{1}{3}\left\{ r_3 e^{\boldsymbol{j}\Phi_3} + \sum_{j=1}^{2} r_j e^{\boldsymbol{j}\{(\Omega_0 - \Omega_j)t + \Phi_j\}} \right\}$$

$t|\mathcal{N}_i| = 2\, \boldsymbol{C}_0(t)t$**??**

$$\Omega_1 = \Omega_2, \qquad r_1 = r_2, \qquad |\Phi_1 - \Phi_2| = \pi$$

□

Lemma **??** shows that when the degree of the bus bar of interest is 1; in other words, for end-point bus bars such as Figure **??**(a), bus bars adjacent to such bus bars synchronize. In addition, if the degree of the bus bar of interest (the node shown with a thick line) is 2; in other words, for bus bars in a chain such as Figure **??**(b), at least on bus bar on each side, are synchronized. Furthermore, when the degree of the bus bar of interest is 3; meaning, a bus bar of a node is connected with three transmission lines as in Figure **??**(c), at least one adjacent bus bar synchronizes with the bus bar of interest. This means that there is no situation where only three arbitrarily chosen bus bars are synchronized and one remaining bus bar is not synchronized, or no set of bus bars is synchronized.

Fig. 9 Synchronization of bus bars in a power grid with a tree structure

When the degree of the bus bar of interest is four or higher, the same analysis can be performed. However, since the obtained conditions could be a higher-order equation related to Ω_i and Ω_j, or there are multiple combinations where only some adjacent bus bars that are arbitrarily selected synchronize, describing the equivalent conditions for synchronization is usually cumbersome. However, for the bus bar of interest i and $|\mathcal{N}_i|$ adjacent bus bars, if some of the $|\mathcal{N}_i| - 1$ adjacent bus bars synchronize with the bus bar i, the remaining one adjacent bus bar usually synchronizes.

By combining conditions with the degree of three or less shown in lemma **??**, even when a bus bar with degree of four or more is included in the power grid, all bus bars could be synchronized. For example, the next fact can be shown.

Theorem 1.4 (Synchronization of bus bars in a tree-structured power grid) *With simultaneous equations of Equation* **??***, let us consider an electrical power system model in which a group of equipment is connected. When the graph of the power grid has a tree structure, all bus bars are synchronized under a steady power flow distribution.*

Proof Translated with DeepL Note the bus bar at the endpoint indicated by the bold line in Figure **??**(a). Since the order of the bus bar is 1, the adjacent bus bars are synchronous with the bus bars of the endpoints. Next, if the bus bar next to the endpoint is in the chain path, the degree of the bus bar is 2, so at least both of its adjacent bus bars are synchronized. By repeating this process, as in Figure **??**(a), synchronization of all the bus bars is shown in the chain path connecting the endpoint and the bus bars of nodes of degree 3 or more.

Similarly, since all the bus bars in a node of degree 3 or more from another endpoint are synchronous, we see that all the bus bars in all chain paths connected to the bus bar in the node indicated by the bold line, as in Figure **??**, are all synchronous. Repeating this argument, it can be shown that all the bus bars composing the tree are synchronous. □

There is no limit to the degree of bus bars in a power grid with a tree structure shown in Theorem **??**. As such, even if a bus bar with degree of four or higher is included in the power grid, synchronization of all bus bars could be deduced based only on the information from the graph structure. Similarly, the following fact can be shown.

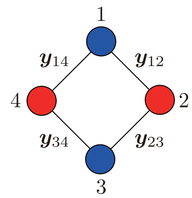


Fig. 10 4

Theorem 1.5 (Synchronization of bus bars in a power grid with a ring structure) *With simultaneous equations of Equation **??**, let us consider an electrical power system model where a group of equipment is connected. When the graph of a power grid has a ring structure, and the total number of bus bars is odd, all bus bars synchronize under a steady power flow distribution.*

Proof Focusing on one bus bar shows the synchronization of the bus bars at both ends of it. By repeating this process, synchronization of all bus bars is shown when the total number of bus bars is odd. □

Theorem **??** indicates that, in a power grid with a ring structure, if the total number of bus bars is odd, regardless of the admittance of each power grid, all bus bars synchronize. On the other hand, if the total number of bus bars is even, unless using additional information, such as admittance of transmission lines, synchronization of all bus bars cannot be concluded even for a power grid with a ring structure. In the Example **??** discussed later, even if the total number of bus bars is even, as long as there is some information such as admittance, all bus bars can be synchronized.

The next example is an application of Lemma **??** to a power grid that includes a bus bar with degree of three.

[

Synchronization of four bus bars in a power grid with a ring structure] Let us consider synchronization of bus bars under a steady power flow distribution in a power grid with a ring structure shown in Figure **??**. Since the number of bus bars is even, synchronization of bus bars cannot be shown using only the information of the graph structure as in Theorem **??**. However, as it is shown with red and blue in Figure **??**, synchronization of alternating bus bars is shown by Lemma **??**. Therefore, for all bus bars to show synchronization, all that is necessary is to show that one of the conditions of Equation **??** is not satisfied for at least one bus bar.

The admittance of the transmission line that connects bus bar i and bus bar j is expressed as $\boldsymbol{y}_{ij}$. At this point, the center condition of Equation **??** for each bus bar can be rewritten as:

$$
\begin{aligned}
|\boldsymbol{V}_2^\star||\boldsymbol{y}_{12}| &= |\boldsymbol{V}_4^\star||\boldsymbol{y}_{14}|, \qquad & |\boldsymbol{V}_1^\star||\boldsymbol{y}_{12}| &= |\boldsymbol{V}_3^\star||\boldsymbol{y}_{23}|,\\
|\boldsymbol{V}_2^\star||\boldsymbol{y}_{23}| &= |\boldsymbol{V}_4^\star||\boldsymbol{y}_{34}|, \qquad & |\boldsymbol{V}_3^\star||\boldsymbol{y}_{34}| &= |\boldsymbol{V}_1^\star||\boldsymbol{y}_{14}|
\end{aligned}
$$

If we express this as a matrix:

$$\underbrace{\begin{bmatrix} 0 & |\boldsymbol{y}_{12}| & 0 & -|\boldsymbol{y}_{14}| \\ -|\boldsymbol{y}_{12}| & 0 & |\boldsymbol{y}_{23}| & 0 \\ 0 & -|\boldsymbol{y}_{23}| & 0 & |\boldsymbol{y}_{34}| \\ |\boldsymbol{y}_{14}| & 0 & -|\boldsymbol{y}_{34}| & 0 \end{bmatrix}}_{S} \begin{bmatrix} |\boldsymbol{V}_1^\star| \\ |\boldsymbol{V}_2^\star| \\ |\boldsymbol{V}_3^\star| \\ |\boldsymbol{V}_4^\star| \end{bmatrix} = 0$$

For conditions necessary for a positive vector that satisfy this Equation $(|\boldsymbol{V}_1^\star|, \ldots, |\boldsymbol{V}_4^\star|)$ to exist, S on the left side must be nonsingular. Because of the sparse structure of the column vector, for S to be non-regular; the following must be true:

$$|\boldsymbol{y}_{12}||\boldsymbol{y}_{34}| = |\boldsymbol{y}_{14}||\boldsymbol{y}_{23}| \tag{48}$$

Therefore, unless the admittance matrix satisfies these conditions, all bus bars synchronize. If Equation **??** is satisfied, the absolute value of the desired voltage phasor can be determined; thus, the condition necessary for $(|\boldsymbol{V}_1^\star|, \ldots, |\boldsymbol{V}_4^\star|)$ that satisfies the central condition of Equation **??** to exist is the Equation **??**.

Next, the condition of the right side of Equation **??** can be written as below if we focus on bus bar 1 and bus bar 3:

$$|\phi_2 - \phi_4 + \angle \boldsymbol{y}_{12} - \angle \boldsymbol{y}_{14}| = \pi, \qquad |\phi_2 - \phi_4 + \angle \boldsymbol{y}_{23} - \angle \boldsymbol{y}_{34}| = \pi$$

Similarly, if we focus on bus bar 2 and bus bar 4, the following can be obtained:

$$|\phi_1 - \phi_3 + \angle \boldsymbol{y}_{12} - \angle \boldsymbol{y}_{23}| = \pi, \qquad |\phi_1 - \phi_3 + \angle \boldsymbol{y}_{14} - \angle \boldsymbol{y}_{34}| = \pi$$

Generally, when capacitance to ground is sufficiently small, the real part of admittance, the conductance component, was non-negative, while the imaginary part, the susceptance component, was negative. In other words:

$$\angle \boldsymbol{y}_{ij} \in \left[-\frac{\pi}{2}, 0\right]$$

If we focus on this, the condition necessary for $(\phi_1, \ldots, \phi_4)$ that satisfies the above conditions to exist is derived as:

$$\angle \boldsymbol{y}_{12} - \angle \boldsymbol{y}_{14} = \angle \boldsymbol{y}_{23} - \angle \boldsymbol{y}_{34} \tag{49}$$

Therefore, as long as the admittance matrix does not satisfy this condition, all bus bars synchronize.

The above discussion shows that the conditions necessary for at least one set of bus bars that do not synchronize under a steady power flow distribution to exist are Equation **??** and Equation **??**. These two conditions indicate that only when the power grid of Figure **??** has specific symmetry related to the admittance value, a condition arises where only alternating bus bars synchronize.

The Example **??** shows that the condition of Equation **??** presents specific symmetry related to the admittance of the power grid. In a practical application, all bus bars synchronizing under a steady power flow distribution for a sparse grid where each bus has a low degree is a universal fact as long as there is no specific symmetry in the graph structure. Actually, as far as the author is aware, synchronization of all bus bars under a steady power flow distribution has been numerically confirmed for all parameters of the electric power system model that are set to realistic values.

6 Method to implement the power flow calculation

Translated with DeepL We consider building a numerical simulation environment for power system analysis and control. In order to correctly implement numerical simulation of a large and complex power system model, object-oriented thinking and programming techniques that describe the model as a group of modules divided by function are useful. In this section, we describe an implementation method for tidal current calculation based on this way of thinking.

6.1 How to solve algebraic equations

For the power flow calculation, simultaneous equations of Equation **??** must be solved. In this Section, we will first use a simple problem to show a method to search the solution of algebraic equations with MATLAB .

[

Searching for a solution for algebraic equations] Let us consider a set of (x, y) that satisfies algebraic equations through numerical calculation:

$$x^2 - y = 0 \tag{50a}$$

$$x^2 + y^2 - 2 = 0 \tag{50b}$$

Please note that there are two solutions: $(-1, 1)$ and $(1, 1)$. `fsolve` of `optimization toolbox` is a convenient command that solves algebraic equations in MATLAB . To use this command, function $f(x, y)$ must be implemented, where algebraic equations of Equation (**??**) are expressed as follows:

$$f(x, y) = 0$$

When this function is implemented with a file name of `func_ex1.m`, it is called program **??**.

[count,title=func_ex1.m]

```
function out = func_ex1(x_in)

x = x_in(1);
y = x_in(2);

out = zeros(2, 1);
out(1) = x^2 - y;
out(2) = x^2 + y^2 - 2;

end
```

Next, if a program is written to execute `fsolve` using this function and search for a solution for algebraic equations, it becomes program **??**: [count,title=main_ex1.m]

```
options = optimoptions('fsolve', 'Display', 'iter');
x0 = [0.1; 0.5];
x_sol = fsolve(@func_ex1, x0, options)
```

Here `options` in the first row sets the optimum option that converges $f(x, y)$ to 0. With this example, the optimization process is shown. `x0 = [0.1; 0.5];` in the second row shows the initial value at the time of optimization with convergence calculation. When we execute this program, the following result is obtained:

```
Iteration  Func-count     f(x)
    0          3           3.2677
    1          6         0.282554

    4         15      3.60447e-14



x_sol =
    1.0000
    1.0000
```

Based on this result, with repetitions, the value of $f(x)$ becomes small, and ultimately, a solution is found where $f(x)$ becomes almost 0. The solution sought here is $(1, 1)$. This value is an actual solution to algebraic equations, but in a numerical solution that uses `fsolve`, not all solutions are obtained, as only one solution is obtained.

Next, let us look at a situation where another solution is obtained. The next result is obtained when the second row of the program **??** is changed to `x0 = [-0.1; 0.5];`.

```
 Iteration  Func-count     f(x)
     0           3         3.2677
     1           6       0.282554

     4          15     3.60447e-14



x_sol =

   -1.0000
    1.0000
```

We can see that the solution $(-1, 1)$ is obtained. As such, with the numerical solution of non-linear algebraic equations, the solution varies based on the initial value. As a reference, let us look at the result of executing the program **??** with the initial value in the second row set as `x0 = [-0.1; -0.1];`:

```
 Iteration  Func-count     f(x)
     0           3         2.8125
     1           6        1.89068

    29          62           1.75



x_sol =

    0.0000
   -1.2247
```

As you can see with this Example, based on the initial value, even equations with solutions might not yield correct solutions. For practice, it is important to provide initial values that are close to the desired solution.

6.2 Simple implementation of the power flow calculation

Next, let us introduce a simple method to implement a program that performs power flow calculations.

[

Implementation method of power flow calculations] Let us think about performing power flow calculations for the power system in Figure **??**. Power flow calculation is a type of calculation to solve algebraic equations. Thus, as in the Example **??**, $f(x)$ of $f(x) = 0$ must be implemented.

Let us think about obtaining the steady power flow distribution that is consistent with the datasheet of **??**(a) when setting the admittance of two transmission lines to the value of Equation **??**.

$$\left(|\boldsymbol{I}_1^\star|, \angle\boldsymbol{I}_1^\star, |\boldsymbol{V}_1^\star|, \angle\boldsymbol{V}_1^\star, |\boldsymbol{I}_2^\star|, \angle\boldsymbol{I}_2^\star, |\boldsymbol{V}_2^\star|, \angle\boldsymbol{V}_2^\star, |\boldsymbol{I}_3^\star|, \angle\boldsymbol{I}_3^\star, |\boldsymbol{V}_3^\star|, \angle\boldsymbol{V}_3^\star\right)$$

When the admittance matrix $\boldsymbol{Y}$ is provided, current is determined uniquely dependent on voltage; thus, power flow calculation becomes a practical step by which to determine a set of voltages. As such, variables x in algebraic equations just need to be the voltage phasor of bus bars under a steady state. Here, we express the real and imaginary parts of the voltage phasor of all bus bars in a line as x.

[2] As discussed above, the voltage phasor and current phasor of each bus bar can be expressed as a function of x. Similarly, the active power and reactive power of each bus bar also become a function of x. If these are expressed as $\hat{\boldsymbol{V}}_i(x)$, $\hat{\boldsymbol{I}}_i(x)$, $\hat{P}_i(x)$, and $\hat{Q}_i(x)$, algebraic equations to be solved are:

$$\begin{aligned}
|\boldsymbol{V}_1^\star| - |\hat{\boldsymbol{V}}_1(x)| &= 0 \\
\angle\boldsymbol{V}_1^\star - \angle\hat{\boldsymbol{V}}_1(x) &= 0 \\
P_2^\star - \hat{P}_2(x) &= 0 \\
Q_2^\star - \hat{Q}_2(x) &= 0 \\
P_3^\star - \hat{P}_3(x) &= 0 \\
|\boldsymbol{V}_3^\star| - |\hat{\boldsymbol{V}}_3(x)| &= 0
\end{aligned}$$

If we implement the functions on the left side, it becomes a program **??**.

Though this function has five output arguments, with `fsolve`, only the first output argument is used. The remaining arguments were added to confirm the result.

[count, title=func_ex2.m]

```
function [out, Vhat, Ihat, Phat, Qhat] = func_ex2(x)
% x: [Real(V1), Imag(V1),
%     Real(V2), Imag(V2),
%     Real(V3), Imag(V3)]';

y12 = 1.3652 - 11.6040j;
y23 = -10.5107j;
Y = [y12, -y12, 0;
    -y12, y12+y23, -y23;
    0, -y23, y23];

V1abs = 2;
V1angle = 0;
```

[2] It is the same when expressing the absolute value and argument of the voltage phasor as x

```
P2 = -3;
Q2 = 0;

P3 = 0.5;
V3abs = 2;

V1hat = x(1) + 1j*x(2);
V2hat = x(3) + 1j*x(4);
V3hat = x(5) + 1j*x(6);

Vhat = [V1hat; V2hat; V3hat];

Ihat = Y*Vhat;
PQhat = Vhat.*conj(Ihat);
Phat = real(PQhat);
Qhat = imag(PQhat);

out = [V1abs-abs(V1hat); V1angle-angle(V1hat);
    P2-Phat(2); Q2-Qhat(2);
    P3-Phat(3); V3abs-abs(V3hat)];
end
```

As discussed in the Example **??**, if searching the solution of algebraic equations with convergence calculation, setting of the initial value is important. An initial value often used by the power flow calculation is **flat start**. With flat start, for all bus bars, the initial value of x is set as:

$$|\hat{\boldsymbol{V}}_i(x)| = 1, \qquad \angle \boldsymbol{V}_i(x) = 0$$

This corresponds to setting the real and imaginary parts of $\boldsymbol{V}_i$ of 1 and 0, respectively. The program is as follows:

[count,title=main_ex2.m]

```
x0 = [1; 0; 1; 0; 1; 0];
options = optimoptions('fsolve', 'Display', 'iter');
x_sol = fsolve(@func_ex2, x0, options);

[~, V, I, P, Q] = func_ex2(x_sol);
Vabs = abs(V);
Vangle = angle(V);
display('Vabs:'), display(Vabs')
display('Vangle:'), display(Vangle')
display('P:'), display(P')
display('Q:'), display(Q')
```

The initial value is set in the first row. The result of executing the program **??** is shown. This result is consistent with the values shown in **??**(b).

```
 Iteration  Func-count     f(x)
     0          7           11.25
     1         14          3.28327

     5         42      4.19428e-28



Vabs:
    2.0000    1.9918    2.0000

Vangle:
    0.0000   -0.0538   -0.0419

P:
    2.5158   -3.0000    0.5000

Q:
   -0.0347    0.0000    0.1759
```

6.3 Implementation method of the power flow calculation that considers increased scale

In the previous section, we discussed the implementation method of the power flow calculation. With this implementation, each time the admittance matrix is changed, the function related to the program **??** must be rewritten. Furthermore, when the number or type of bus bars are changed, the entire program must be rewritten. In this Section, we consider dividing the program into modules so that power flow calculations can be performed for a large-scale electrical power system and the program can be shared and implemented by multiple people. In this manner, the program becomes structured and clear. First, let us think about separating the implementation of the admittance matrix of the program **??**.

First, let us think about separating the implementation of the admittance matrix of the program **??**.

[

Separation of the implementation of the admittance matrix] In the program **??**, let us separate the part that specifies the admittance matrix. To do this, the function of variables Y is used as the input argument. For example, the following correction is possible.

[count, title=func_ex3.m]

```
function [out, Vhat, Ihat, Phat, Qhat] = func_ex3(x, Y)
(Same as lines 12 through 34 of program 3-3)
end
```

However, the specification of `fsolve` demands a function that uses the optimizing parameter as the only argument; thus, the program **??** cannot be used directly. This problem can be solved by using a technology called **currying** for the function. Specifically, it is corrected as follows:

[count,title=main_ex3.m]

```
x0 = [1; 0; 1; 0; 1; 0];
options = optimoptions('fsolve', 'Display', 'iter');

y12 = 1.3652 - 11.6040j;
y23 = -10.5107j;
Y = [y12, -y12, 0;
    -y12, y12+y23, -y23;
    0, -y23, y23];

func_curried = @(x) func_ex3(x, Y);

x_sol = fsolve(func_curried, x0, options);
[~, V, I, P, Q] = func_curried(x_sol);
3-46
```

Translated with www.DeepL.com/Translator (free version) The "`@(x) expression of x`" used creates "a function that returns an expression of `x` for the argument `x`". In this way, a function that is generated without writing the function name is called **anonymous function**. In this case, the variable Y, which is not included in the argument of the anonymous function, is always a constant that exists in the workspace. This allows us to generate a new function that retains only the first argument,`x`, of the two input arguments,`x` and Y, of the function `func_ex3(x, Y)`.

Translated with www.DeepL.com/Translator (free version) Thus, culling is the creation of a new function in which the variables given to some of the multiple arguments are fixed and only the remaining variables are used as arguments. Using this technique, a function with an arbitrary number of arguments can be used as a function with the required number of arguments. This allows separation of functions without the use of global variables.

Translated with DeepL Next, consider the part of the program**??** that calculates the constraint conditions at each bus bar. In the program**??**, the constraint conditions for each bus bar are written down directly, so the program needs to be rewritten each time the number of bus bars is changed. In addition, the case classification by the type of bus bar is implicitly included, which means that the case classification process needs to be modified when the number of bus bars is increased.

In order to write a program with good visibility without using case separation, it is recommended to use the object-oriented concept of **polymorphism**. In the following examples, the idea of polymorphism will be explained through implementation examples.

[

Separation of bus bar implementation] In the power flow calculation, different constraints are set based on different types of bus bars. At this time, the bus bar of each type has a commonality of "having constraints". If we implement this common property as a method with a common name, we can create a program that is easy to handle. Specifically, the class of bus bars is defined as shown in Program **??** to Program **??**. If we use the instance of these classes, the power flow calculation can be executed as below.

[count,title=bus_slack.m]

```
classdef bus_slack

  properties
    Vabs
    Vangle
  end

  methods
    function obj = bus_slack(Vabs, Vangle)
      obj.Vabs = Vabs;
      obj.Vangle = Vangle;
    end

    function out = get_constraint(obj, Vr, Vi, P, Q)
      Vabs = norm([Vr; Vi]);
      Vangle = atan2(Vi, Vr);
      out = [Vabs-obj.Vabs; Vangle-obj.Vangle];
    end
  end
end
```

[count,title=bus_genertor.m]

```
classdef bus_generator

  properties
    P
    Vabs
```

```
    end

    methods
      function obj = bus_generator(P, Vabs)
        obj.P = P;
        obj.Vabs = Vabs;
      end

      function out = get_constraint(obj, Vr, Vi, P, Q)
        Vabs = norm([Vr; Vi]);
        out = [obj.P-P; Vabs-obj.Vabs];
      end
    end
end
```

[count,title=bus_load.m]

```
classdef bus_load

    properties
      P
      Q
    end

    methods
      function obj = bus_load(P, Q)
        obj.P = P;
        obj.Q = Q;
      end

      function out = get_constraint(obj, Vr, Vi, P, Q)
        out = [P-obj.P; Q-obj.Q];
      end
    end
end
```

Translated with DeepL All of the classes defined here, `bus_slack`, `bus_generator`, and `bus_load`, have a method named `get_constraint` with common inputs and outputs. In this case, programs using these classes can calculate constraints appropriately by calling `get_constraint` without being aware of which type of bus bar they are. If we rewrite the Program **??** using this concept, we obtain the following.

[count, title=func_power_flow.m]

```
function [out, V, I, P, Q] = func_power_flow(x, Y, a_bus)

V = x(1:2:end) + 1j*x(2:2:end);
I = Y*V;
```

```
PQhat = V.*conj(I);
P = real(PQhat);
Q = imag(PQhat);

out_cell = cell(numel(a_bus), 1);

for i = 1:numel(a_bus)
  bus = a_bus{i};
  out_cell{i} = bus.get_constraint(...
    real(V(i)), imag(V(i)), P(i), Q(i));
end
out = vertcat(out_cell{:});

end
```

Translated with DeepL The program **??** assumes that the cell array, which is the set of mother lines, is input as `a_bus`. An example of how to use this function is shown in the program **??**.

[count,title=main_ex4.m]

```
(Same as lines 1-8 in program 3-6)

a_bus = cell(3, 1);
a_bus{1} = bus_slack(2, 0);
a_bus{2} = bus_load(-3, 0);
a_bus{3} = bus_generator(0.5, 2);

func_curried = @(x) func_power_flow(x, Y, a_bus);

x_sol = fsolve(func_curried, x0, options);

(Same as Program 3-4, line 6 and following)
```

Translated with DeepL until end of chapter In the program **??**, the slack bus bar, load bus bar, and generator bus bar are defined respectively and assigned to the cell array `a_bus`. At this time, in the program**??**, `get_constraint` is executed for each bus bar assigned to `a_bus`. Since `get_constraint` is implemented for each bus line type, the appropriate processing is performed without having to describe the case. Such different processing for similar program calls is called polymorphism or polymorphism, and is one of the important concepts in object-oriented programming.

In the implementation of the program **??**, the constraint is computed by calling a method named `get_constraint` method, which has a common name without being aware of the type of bus bar. (called in programs 1-10, not 3-11?). Also, is the backslash before the underscore unnecessary?. The description is simplified. This property of being able to use the same method name regardless of the type of instance is called **polymorphism** or **polymorphism**, and is one of the important concepts in object-oriented programming. Using polymorphism not only simplifies not only simplifies the description, but also

An implementation using polymorphism such as Program **??** has the advantage that it is easy to deal with changes in the number or configuration of bus bars. That is, the Program **??** can be used without modification simply by changing the number and definition of bus bars in lines 3 to 6 of the Program **??**.

Furthermore, it is easy to add new types of bus bars. The program **??** assumes only that there exists a method named `get_constraint` with appropriate inputs and outputs for the bus bar. In other words, if it has a `get_constraint`, it is defined to be a bus bar. This is a concept called duck typing, derived from "if it walks like a duck and quacks like a duck, then it is a duck". This allows the module designer to define a new class of bus bar and implement only `get_constraint` without paying attention to the internal processing of the program **??**. In other words, this clarifies the scope of the implementation to be done by the module designer.

The above has allowed us to prospectively implement a program for calculating the tidal currents given an admittance matrix. Furthermore, let us implement the computation of the admittance matrix by dividing it into a group of modules.

[

Object-oriented computation of admittance matrices] An admittance matrix is generally defined by several transmission lines. Therefore, we consider creating a class that represents a transmission line and using a set of instances of the class. Since a transmission line connects two bus bars, the numbers of those bus bars need to be specified. In addition, in general, regardless of the type of transmission line model

$$\begin{bmatrix} \boldsymbol{I}_{\text{from}} \\ \boldsymbol{I}_{\text{to}} \end{bmatrix} = \boldsymbol{Y}_{\text{branch}} \begin{bmatrix} \boldsymbol{V}_{\text{from}} \\ \boldsymbol{V}_{\text{to}} \end{bmatrix} \tag{51}$$

The admittance matrix $\boldsymbol{Y}_{\text{branch}}$ is determined satisfying where $\boldsymbol{V}_{\text{from}}$ and $\boldsymbol{V}_{\text{to}}$ are the voltages at the bus bars at both ends of the transmission line and $\boldsymbol{I}_{\text{from}}$ and $\boldsymbol{I}_{\text{to}}$ are the currents entering the transmission line from the bus bars at both ends. For example, for the simple transmission line treated in the example**??**

$$\boldsymbol{Y}_{\text{branch}} = \begin{bmatrix} \boldsymbol{y} & -\boldsymbol{y} \\ -\boldsymbol{y} & \boldsymbol{y} \end{bmatrix}$$

In summary, if the information on the bus bar at both ends and the admittance matrix Y_{branch} are returned, it is safe to define it as a transmission line. An example implementation of a transmission line based on this concept is shown below.

[count,title=branch.m]

```
classdef branch

  properties
```

```
    y
    from
    to
  end

  methods
    function obj = branch(from, to, y)
      obj.from = from;
      obj.to = to;
      obj.y = y;
    end

    function Y = get_admittance_matrix(obj)
      y = obj.y;
      Y = [y, -y;
        -y,  y];
    end
  end
end
```

The function to compute the admittance matrix using this transmission line class is implemented as follows [count,title=get_admittance_matrix.m]

```
function Y = get_admittance_matrix(n_bus, a_branch)

Y = zeros(n_bus, n_bus);

for i = 1:numel(a_branch)
  br = a_branch{i};
  Y_branch = br.get_admittance_matrix();
  Y([br.from, br.to], [br.from, br.to]) =...
    Y([br.from, br.to], [br.from, br.to]) + Y_branch;
end

end
```

The input argument `n_bus` is the number of bus lines. Also, `a_branch` is a cell array containing the transmission lines. This program can be used as follows

[count,title=main_admittance_matrix.m]

```
a_branch = cell(2, 1);
a_branch{1} = branch(1, 2, 1.3652-11.6040j);
a_branch{2} = branch(2, 3, -10.5107j);

Y = get_admittance_matrix(3, a_branch)
```

In the program**??**, we only assume that the transmission line has the data `from` and `to` and that it has the method `get_admittance_matrix` and the method `get_admittance_matrix`. Therefore, when considering other transmission line models such as π-type models When considering other transmission line models, such as π-type models, the variables `to` and `from` and the method `get_admittance_matrix` are also assumed to be used. The program program **??** can be used without any modification by simply implementing a class with variables named `to`, `from` and a method named `get_admittance_matrix`.

From the above, the program to calculate tidal currents can be summarized as follows.

[

Tidal current calculation implementation results] Translated with DeepL The program implemented in the above example is summarized and organized as a function that performs a tidal flow calculation Program**??**.

[count,title=calculate_power_flow.m]

```
function [V, I, P, Q] = calculate_power_flow(a_bus, a_branch)

n_bus = numel(a_bus);
Y = get_admittance_matrix(n_bus, a_branch);

func_curried = @(x) func_power_flow(x, Y, a_bus);

x0 = kron(ones(n_bus, 1), [1; 0]);
options = optimoptions('fsolve', 'Display', 'iter');
x_sol = fsolve(func_curried, x0, options);

[~, V, I, P, Q] = func_curried(x_sol);

end
```

Translated with DeepL This program receives a set of bus lines and a set of transmission lines and performs a tidal current calculation It can be used as Program **??**.

[count,title=main_power_flow.m]

```
a_bus = cell(3, 1);
a_bus{1} = bus_slack(2, 0);
a_bus{2} = bus_load(-3, 0);
a_bus{3} = bus_generator(0.5, 2);

a_branch = cell(2, 1);
a_branch{1} = branch(1, 2, 1.3652-11.6040j);
a_branch{2} = branch(2, 3, -10.5107j);
```

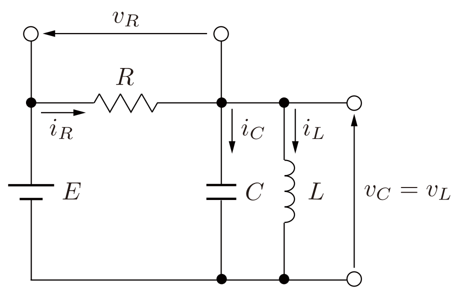


Fig. 11 Differential Algebraic Equations Example: LC Parallel Circuit

```
[V, I, P, Q] = calculate_power_flow(a_bus, a_branch);

display('Vabs:'), display(abs(V)')
display('Vangle:'), display(angle(V)')
display('P:'), display(P')
display('Q:'), display(Q')
```

Translated with DeepL If it is necessary to perform tidal flow calculations for different power grids, only the definitions of bus bar and transmission line in lines 1 to 8 of the Program **??** can be changed, and the other programs can be used without any modification. When a new type of bus bar or transmission line is needed, a new class that implements the data and methods with the defined names can be implemented. These features make the program more readable and extensible than a simple implementation such as **??**.

---

7 How to implement time response calculations for power system models

Translated with DeepL This section describes how to numerically calculate the time response of a power system model using MATLAB . The computation of the time response of a power system model requires solving a system of differential equations that expresses the dynamic characteristics of the equipment and a system of differential algebraic equations that The time response of the power system model requires solving a system of differential algebraic equations, which is a system of differential equations representing the dynamic characteristics of the equipment and a system of algebraic equations for the power flow. First, let's take a simple system of differential-algebraic equations as an example and see how to solve it.

```
[
```

Numerical solution of simple systems of differential-algebraic equations] Translated with DeepL Numerical simulation is performed for the simple electric circuit shown in Figure **??**, where R, L, C, and E are all 1. The dynamic elements of this circuit are the coil L and the capacitor C, whose differential equations are given by:

$$L\dot{i}_L = v_L \tag{52a}$$
$$C\dot{v}_C = i_C \tag{52b}$$

However, the initial values are $i_L(0) = 0$ and $v_C(0) = 0$. Also, from Ohm's law and Kirchhoff's law, we get the following algebraic equations.

$$v_R = Ri_R \tag{53a}$$
$$i_R = i_L + i_C \tag{53b}$$
$$v_L = v_C \tag{53c}$$
$$E = v_C + v_R \tag{53d}$$

Using the algebraic equations in equation (**??**) and eliminating redundant variables such as v_R and i_R, we obtain an equivalent system of ordinary differential equations can be obtained. This operation corresponds to Kron reduction, and the resulting system of ordinary differential equations is

$$\dot{i}_L = \frac{1}{L}v_C \tag{54a}$$
$$\dot{v}_C = \frac{1}{RC}(E - v_C) - \frac{1}{C}i_L \tag{54b}$$

First, let's write a program to solve this ordinary differential equation system. The solver of MATLAB for the ODE system is generally `ode45`. In `ode45`, the solution of the ordinary differential equation system can be obtained by implementing the function $f(t, x)$ of the ordinary differential equation $\dot{x} = f(t, x)$. Implementing the right-hand side of an expression (**??**) results in the Program **??**.

[count,title=func_RLC_ode.m]

```
function dx = func_RLC_ode(x, R, C, L, E)

iL = x(1);
vC = x(2);

diL = vC/L;
dvC = (E-vC)/R/C - iL/C;

dx = [diL; dvC];

end
```

As a result, when ode45 is executed and the ordinary differential equation system is solved, the program **??** becomes:

[count,title=main_RLC_ode.m]

```
R = 1;
L = 1;
C = 1;
E = 1;

func = @(t, x) func_RLC_ode(x, R, C, L, E);
x0 = [0; 0];
tspan = [0 30];

[t, x] = ode45(func, tspan, x0);

plot(t, x)
```

The output variable x in the program **??** is a matrix in which the time series of i_L and v_C are arranged vertically. Therefore, note that the time series of other variables such as i_C and i_R need to be additionally calculated using the algebraic equation of the equation (**??**).

Next, consider directly finding the solution of the system of differential algebra equations without going through Kron reduction. In this case, since the physical algebraic equation can be written down as it is, there is an advantage that the description is easy even if the system becomes complicated. One of the commands that can solve the system of differential algebra with MATLAB is ode15s. The target is a system of differential algebraic equations described in the following format.

$$M\dot{x} = f(t, x) \tag{55}$$

Applying the expression (**??**) and the expression (**??**):

$$\underbrace{\begin{bmatrix} 0 & L & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & C \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} \dot{i}_R \\ \dot{i}_L \\ \dot{i}_C \\ \dot{v}_R \\ \dot{v}_L \\ \dot{v}_C \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} v_L \\ i_C \\ v_R - Ri_R \\ i_R - i_L - i_C \\ v_L - v_C \\ E - v_C - v_R \end{bmatrix}}_{f(t,x)}$$

Note that the order in which the elements on the 3rd to 6th lines on the right side are placed is arbitrary. Implementing this right-hand side results in a program **??**.

[count,title=func_RLC_dae.m]

```
function dx = func_RLC_dae(x, R, C, L, E)
```

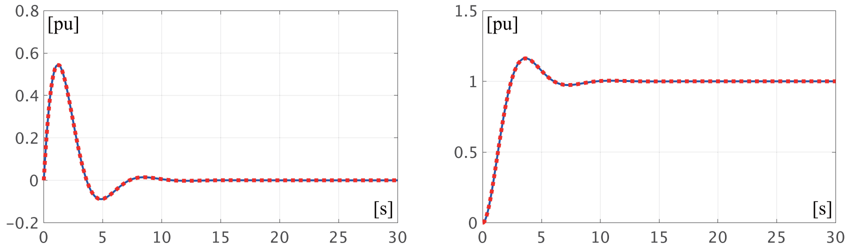
```
iR = x(1);
iL = x(2);
iC = x(3);
vR = x(4);
vL = x(5);
vC = x(6);

diL = vL;
dvC = iC;

con1 = vC-vL;
con2 = E-vC-vR;
con3 = iR-(iC+iL);
con4 = vR-iR*R;

dx = [diL; dvC; con1; con2; con3; con4];
end
```

This function can be used to solve a system of differential algebraic equations such as in Program **??**.

[count,title=main_RLC_dae.m]

```
R = 1;
C = 1;
L = 1;
E = 1;

M = zeros(6, 6);
M(1, 2) = L;
M(2, 6) = C;

x0 = zeros(6, 1);
tspan = [0 30];

options = odeset('Mass', M);
[t, x] = ode15s(@(t, x) func_RLC_dae(x, R, C, L, E),...
  tspan, x0, options);

plot(t, x(:, [2, 6]))
```

In this program, M of the expression **??** is set in the option on the 13th line. Also, the initial value of the state is set in the 10th line, but only the state of the differential equation, that is, the 2nd and 6th elements, has meaning. The state of the algebraic equation does not need to be calculated and set by the user himself because the value that satisfies the equation is automatically searched by `ode15s`. In fact, the algebraic equation is not satisfied by the initial value with all the elements as 0, but the solution of the system of differential algebraic equations is calculated without



(a) Time response of i_L. (b) Time response of v_C.

Fig. 12 Time response of LC parallel circuits (Blue solid line: ode45, red dashed line: ode15s)

any problem. The solution `x` obtained in the 15th line is a matrix in which the time series of $v_R, v_L, v_C, i_R, i_L, i_C$ are arranged, and all of them are different from the case of equivalent conversion to the ordinary differential equation system. The time series of the variables of is calculated at once.

Figure **??** shows the time response of i_L and v_C when converted to an ordinary differential equation system and when the solution of the differential algebraic equation system is directly calculated. Two lines are displayed in each figure: the solution by verb — ode45 — (solid blue line) and the solution by verb — ode15s — (red dashed line). Obviously, the two solutions are equal.

7.1 Simple implementation of time response calculations for power system models

Translated with DeepL Describe how to implement a numerical simulation of a power system model.

[

Simple implementation of power system simulation] Let us implement a program to numerically compute the time response of the power system model treated in the example **??**. In order to describe this system, we need the differential equations in eq:gendynVIst for bus bars 1 and 3, to which the generators are connected. Also, the algebraic equation of eq:phVI holds. A constant-impedance load model is connected to bus bar 2, and the algebraic equation of eq:cimp is satisfied. Furthermore, the algebraic system of equations of Kirchhoff's law, eq:ohmY2, holds for the entire system at any given time.

In the following, let x be a vector of δ_1, δ_1, δ_omega_1, E_1, δ_3, δ_3, $\boldsymbol{V}_1$, $\boldsymbol{V}_2$, $\boldsymbol{V}_3$, $\boldsymbol{I}_1$, $\boldsymbol{I}_2$ and $\boldsymbol{I}_3$ arranged vertically

$$M\dot{x} = f(t,x)$$

Consider a description in the form of $f(t,x)$. An example implementation of the function $f(t,x)$ on the right-hand side would be the Program **??**.

[count,title=func_simulation_3bus.m]

```
function dx = func_simulation_3bus(x, Y, parameter)

delta1 = x(1);
omega1 = x(2);
E1 = x(3);
delta3 = x(4);
omega3 = x(5);
E3 = x(6);
V1 = x(7)  + 1j*x(8);
V2 = x(9)  + 1j*x(10);
V3 = x(11) + 1j*x(12);
I1 = x(13) + 1j*x(14);
I2 = x(15) + 1j*x(16);
I3 = x(17) + 1j*x(18);

omega0 = parameter.omega0;

X1 = parameter.X1;
X1_prime = parameter.X1_prime;
M1 = parameter.M1;
D1 = parameter.D1;
tau1 = parameter.tau1;
Pmech1 = parameter.Pmech1;
Vfield1 = parameter.Vfield1;

z2 = parameter.z2;

X3 = parameter.X3;
X3_prime = parameter.X3_prime;
M3 = parameter.M3;
D3 = parameter.D3;
tau3 = parameter.tau3;
Pmech3 = parameter.Pmech3;
Vfield3 = parameter.Vfield3;

P1 = real(V1*conj(I1));
P3 = real(V3*conj(I3));

dx1 = [omega0 * omega1;
    (-D1*omega1-P1+Pmech1)/M1;
```

```
    (-X1/X1_prime*E1+...
    (X1/X1_prime-1)*abs(V1)*cos(delta1-angle(V1))+Vfield1)/tau1];

dx3 = [omega0 * omega3;
    (-D3*omega3-P3+Pmech3)/M3;
    (-X3/X3_prime*E3+...
    (X3/X3_prime-1)*abs(V3)*cos(delta3-angle(V3))+Vfield3)/tau3];

con1 = I1-(E1*exp(1j*delta1)-V1)/(1j*X1_prime);
con2 = V2+z2*I2;
con3 = I3-(E3*exp(1j*delta3)-V3)/(1j*X3_prime);

con_network = [I1; I2; I3] - Y*[V1; V2; V3];

dx = [dx1; dx3;real(con1); imag(con1); real(con2);
    imag(con2); real(con3); imag(con3);
    real(con_network); imag(con_network)];

end
```

In this program, the voltage and current phasors in `x` are represented as real and imaginary parts side by side. In addition, in order to specify variables collectively, we use the structure `parameter`. Solving a system of differential-algebraic equations using the program **??** results in the program **??**.

[count,title=main_simulation_simple.m]

```
a_branch = cell(2, 1);
a_branch{1} = branch(1, 2, 1.3652-11.6040j);
a_branch{2} = branch(2, 3, -10.5107j);
Y = get_admittance_matrix(3, a_branch);

parameter = struct();

parameter.M1 = 100;
parameter.D1 = 10;
parameter.tau1 = 5.14;
parameter.X1 = 1.569;
parameter.X1_prime = 0.936;
parameter.Pmech1 = 2.5158;
parameter.Vfield1 = 2.7038;

parameter.z2 = 1.3224;

parameter.M3 = 12;
parameter.D3 = 10;
parameter.tau3 = 8.97;
```

```
parameter.X3 = 1.220;
parameter.X3_prime = 0.667;
parameter.Pmech3 = 0.5000;
parameter.Vfield3 = 2.1250;

parameter.omega0 = 60*2*pi;

x0 = [0.5357 + pi/6; 0; 2.3069 + 0.1;...
    0.0390; 0; 2.0654; zeros(12, 1)];
M = blkdiag(eye(6), zeros(12, 12));

tspan = [0 50];

options = odeset('Mass', M);
func = @(t, x) func_simulation_3bus(x, Y, parameter);
[t, x] = ode15s(func, tspan, x0, options);

plot(t, x(:, [2, 5]))
```

In the Program**??**, the initial value response of the power system model is calculated and the angular frequency deviations of the two generators are drawn.

7.2 Implementation method for time response calculation using a group of partitioned modules

Translated with DeepL This section describes how to separate the program described in the previous section by function and change it to a highly extensible program.

[

Modularization of generators and loads] The program **??** consists of two steps: dividing the input `x` into the state variables, voltage and current of each device, and computing differential and algebraic equations. Let's consider how to write a program that executes these two steps in a clear view.

First, in order to properly partition the variable `x`, we need to know the number of states of each instrument. Also, for all instruments, the time derivative of the state $\frac{dx}{dt}$ in the differential equation and $f(x)$ in the algebraic equation $f(x) = 0$ are computed. Using the concept of duck typing, if a device has functions to return the number of states, perform time differentiation, and compute algebraic equations, it can be defined as a device. If we implement a generator and a load as a device with these functions, then we have a Program **??** and a Program **??**.

[count,title=generator.m]

```
classdef generator < handle

properties
  omega0
  X
  X_prime
  M
  D
  tau
  Pmech
  Vfield
end

methods
  function obj = generator(omega0, M, D, tau,...
      X, X_prime, Pmech, Vfield)

    obj.omega0 = omega0;
    obj.X = X;
    obj.X_prime = X_prime;
    obj.M = M;
    obj.D = D;
    obj.tau = tau;
    obj.Pmech = Pmech;
    obj.Vfield = Vfield;
  end

  function nx = get_nx(obj)
    nx = 3;
  end

  function [dx, con] = get_dx_constraint(obj, x, V, I)
    delta = x(1);
    omega = x(2);
    E = x(3);
    P = real(V*conj(I));

    Pmech = obj.Pmech;
    Vfield = obj.Vfield;

    X = obj.X;
    X_prime = obj.X_prime;
    D = obj.D;
    M = obj.M;
    tau = obj.tau;
```

```
      omega0 = obj.omega0;

      dE = (-X/X_prime*E+...
        (X/X_prime-1)*abs(V)*cos(delta-angle(V))...
        +Vfield)/tau;
      dx = [omega0 * omega;
        (-D*omega-P+Pmech)/M;
        dE];
      con = I-(E*exp(1j*delta)-V)/(1j*X_prime);
      con = [real(con); imag(con)];
    end
  end

  end
```

[count,title=load_impedance.m]

```
classdef load_impedance < handle

properties
  z
end

methods
  function obj = load_impedance(z)
    obj.z = z;
  end

  function nx = get_nx(obj)
    nx = 0;
  end

  function [dx, con] = get_dx_constraint(obj, x, V, I)
    dx = [];
    z = obj.z;
    con = V+z*I;
    con = [real(con); imag(con)];
  end
end

end
```

In these programs, the method `get_nx` returns the number of states and `get_dx_constraint` returns the time derivative of the state and the constraint conditions. With the equipment defined in this way, the Program**??** can be rewritten as Program**??**.

[count,title=func_simulation.m]

```
function out = func_simulation(t, x, Y, a_component)

n_component = numel(a_component);
x_split = cell(n_component, 1);
V = zeros(n_component, 1);
I = zeros(n_component, 1);

idx = 0;
for k = 1:n_component
  nx = a_component{k}.get_nx();
  x_split{k} = x(idx+(1:nx));
  idx = idx + nx;
end

for k = 1:n_component
  V(k) = x(idx+1) + x(idx+2)*1j;
  idx = idx + 2;
end

for k = 1:n_component
  I(k) = x(idx+1) + x(idx+2)*1j;
  idx = idx + 2;
end

dx = cell(n_component, 1);
con = cell(n_component, 1);

for k = 1:n_component
  component = a_component{k};
  xk = x_split{k};
  Vk = V(k);
  Ik = I(k);
  [dx{k}, con{k}] = component.get_dx_constraint(xk, Vk, Ik);
end

con_network = I - Y*V;

out = vertcat(dx{:});
out = [out; vertcat(con{:})];
out = [out; real(con_network); imag(con_network)];

end
```

In the Program??, it is assumed that the cell array of the device is input toa_component. In addition, the variable x is divided in lines 9 through 23. At this time, by obtaining the number of device states in line 10, the division can be performed appropriately. In lines 28 to 34, the time derivative of the state and the constraints are computed. In line 33 of the program, we call get_dx_constraint, which is implemented using polymorphism. In line 38 of the Program,vertcat(dx{:}) vertically combines all the elements of the cell array dx. That is, [dx{1}; dx{2}; ... ; dx{end}] is equal to

Let us consider running a numerical simulation using the Program??. Specifically, the numerical simulation part of the Program?? can be summarized as follows:

[count,title=simulate_power_system.m]

```
function [t, x, V, I] = ...
  simulate_power_system(a_component, Y, x0, tspan)

n_component = numel(a_component);
a_nx = zeros(n_component, 1);
for k = 1:n_component
  component = a_component{k};
  a_nx(k) = component.get_nx();
end
nx = sum(a_nx);

M = blkdiag(eye(nx), zeros(n_component*4));
options = odeset('Mass', M, 'RelTol', 1e-6);

y0 = [x0(:); zeros(4*n_component, 1)];

[t, y] = ode15s(@(t, x) func_simulation(t, x, Y, a_component),...
  tspan, y0, options);


x = cell(n_component, 1);
V = zeros(numel(t), n_component);
I = zeros(numel(t), n_component);

idx = 0;

for k = 1:n_component
  x{k} = y(:, idx+(1:a_nx(k)));
  Vk = y(:, nx+2*(k-1)+(1:2));
  V(:, k) = Vk(:, 1) + 1j*Vk(:, 2);
  Ik = y(:, nx+2*n_component+2*(k-1)+(1:2));
  I(:, k) = Ik(:, 1) + 1j*Ik(:, 2);
  idx = idx + a_nx(k);
```

```
end

end
```

In the Program**??**, lines 5 to 9 define the matrixM— using the number of states obtained from each device. In line 17, the differential algebraic equations are solved using the program **??**. In lines 27 to 34, the result is divided into the state variables of each device, the voltage phasors and current phasors of each bus bar, and returned.

Furthermore, rewriting the Program**??** results in the Program**??**.

[count,title=main_simulation_3bus.m]

```
a_branch = cell(2, 1);
a_branch{1} = branch(1, 2, 1.3652-11.6040j);
a_branch{2} = branch(2, 3, -10.5107j);

Y = get_admittance_matrix(3, a_branch);

gen1 = generator(60*2*pi, 100, 10, 5.14,...
  1.569, 0.936, 2.5158, 2.7038);

load2 = load_impedance(1.3224);

gen3 = generator(60*2*pi, 12, 10, 8.97,...
  1.220, 0.667, 0.5000, 2.1250);

a_component = {gen1; load2; gen3};

x0 = [0.5357 + pi/6; 0; 2.3069 + 0.1; 0.0390; 0; 2.0654];
tspan = [0 50];

[t, x, V, I] = simulate_power_system(a_component, Y, x0, tspan);

plot(t, [x{1}(:, 2), x{3}(:, 2)])
```

In the Program**??**,an array `a_component` of devices representing generators and loads is created and the function `simulate_power_system` defined in the Program **??** is executed. The function `simulate_power_system` defined in the program**??** is executed. In the numerical calculation of the time response, when changing the structure of the power grid or the devices connected to it, it is only necessary to change the program `a_branch` and `a_component` in the Program **??**, and not to change the Program **??** from **??**. In the case of using equipment different from the generator or load in the Program **??** or Program **??**, you only need to implement and use a class with `get_nx` and `get_dx_constraint`. These modularizations make the program more readable and extensible compared to the implementation of the Example **??**.

In the example program **??**, the modularization of generators and loads allows for a prospective implementation of the time response calculation of the power system model. However, in the program **??**, the values of the external input of the generator and the impedance of the load need to be pre-computed to match the desired equilibrium point. Since these values depend on the dynamic characteristics of each device, there remains room for improvement in terms of separation of functions. To solve this problem, let us consider an extension of the program described in the example **??** to make it functionally more separated.

[

Add methods to calculate steady state conditions for generators and loads] In the numerical computation of the time response of the power system model, we consider achieving a steady tidal state that achieves the desired power supply. In this case, the Equations (**??**) and **??** must be satisfied for the generator and load, respectively. Since these relations are related to the state equations of each device, it is appropriate to implement them in the classes `generator` and `load_impedance`. From this point of view, adding a method to calculate the steady state that achieves the desired power supply and demand to the program**??** and the program**??** will result in the Program **??** and Program **??**.

[count,title=generator.m]

```
classdef generator < handle

(Same as lines 3-12 of program 3-23)

methods

(Same as lines 15-57 in program 3-23)

  function x_equilibrium = set_equilibrium(obj, V, I, P, Q)
    Vabs = abs(V);
    Vangle = angle(V);

    X = obj.X;
    X_prime = obj.X_prime;

    delta = Vangle + atan(P/(Q+Vabs^2/X_prime));
    E = X_prime/Vabs*sqrt((Q+Vabs^2/X_prime)^2+P^2);

    x_equilibrium = [delta; 0; E];

    obj.Pmech = P;
    obj.Vfield = X*E/X_prime ...
      - (X/X_prime-1)*Vabs*cos(delta-Vangle);
  end
```

```
end

end
```

[count,title=load_impedance.m]

```
classdef load_impedance < handle

(Same as lines 3-5 in program 3-24)

methods

(Same as lines 8 through 21 in Program 3-24)

  function x_equilibrium = set_equilibrium(obj, V, I, P, Q)
    x_equilibrium = [];
    obj.z = -V/I;
  end

end

end
```

In these programs, `set_equilibrium` takes the bus bar voltage phasor `V`, current phasor`I`, active power`P`, reactive power`Q` at the desired steady state and returns a steady state value`x_ equilibrium`. Theoretically, it is not necessary to give `I`, only `V` and `P` are sufficient, but for convenience, we also give `I`. A program that performs a simulation similar to the one shown in the example **??** using the parameters obtained from the tidal current calculation can be written as follows.

[count,title=main_simulation_3bus_equilibrium.m]

```
a_bus = cell(3, 1);
a_bus{1} = bus_slack(2, 0);
a_bus{2} = bus_load(-3, 0);
a_bus{3} = bus_generator(0.5, 2);

a_branch = cell(2, 1);
a_branch{1} = branch(1, 2, 1.3652-11.6040j);
a_branch{2} = branch(2, 3, -10.5107j);

gen1 = generator(60*2*pi, 100, 10, 5.14, 1.569, 0.936, [], []);
load2 = load_impedance([]);
gen3 = generator(60*2*pi, 12, 10, 8.97, 1.220, 0.667, [], []);

a_component = {gen1; load2; gen3};
```

```
[V, I, P, Q] = calculate_power_flow(a_bus, a_branch);
Y = get_admittance_matrix(3, a_branch);

x_equilibrium = cell(numel(a_component), 1);
for k=1:numel(a_component)
   x_equilibrium{k} =...
     a_component{k}.set_equilibrium(V(k), I(k), P(k), Q(k));
end

x0 = vertcat(x_equilibrium{:});
x0(1) = x0(1) + pi/6;
x0(3) = x0(3) + 0.1;
tspan = [0 50];

[t, x, V, I] = simulate_power_system(a_component, Y, x0, tspan);

plot(t, [x{1}(:, 2), x{3}(:, 2)])
```

The program **??** clarifies the process of defining a power system model using classes of bus bar, transmission line, generator, and load, calculating the tidal current, and computing the time response. The user only needs to specify the physical constants of each device and can run the numerical simulation without paying attention to the internal dynamic characteristics. In this example, it is assumed that the equipment model has a method called `set_equilibrium`. This is equivalent to changing the definition of the device in duck typing.

The above examples have dealt with the numerical computation of the initial value response of power system models. Next, we will discuss how to implement the time response calculation for ground faults.

[

Numerical calculation of time response to ground fault] In order to calculate the response to a ground fault, the voltage at the bus bar where the fault occurred should be fixed at 0 for a certain time, as described in section **??**. This can be implemented as follows by modifying some constraints in line 36 of the Program **??**.

[count,title=func_simulation.m]

```
function out = func_simulation(x, Y, a_component, bus_fault)

(Same as lines 3-34 of program 3-25)

con_network = I - Y*V;
con_network(bus_fault) = V(bus_fault);
```

```
(Same as lines 38 through 40 in program 3-25)

end
```

In this program, an input argument has been added, and it is assumed that the number of the bus bar where the ground fault occurs is assigned to `bus_fault`. The Program **??** is changed to correspond to this change, resulting in Program **??**.

[count,title=simulate_power_system.m]

```
function [t, x, V, I] = simulate_power_system(a_component, ...
  Y, x0, tspan,bus_fault, tspan_fault)

if nargin < 5
  bus_fault = [];
end

if nargin < 6
  tspan_fault = [0, 0];
end

(Same as lines 4-15 of program 3-26)

if isempty(bus_fault)
  [t, y] = ode15s(...
    @(t, x) func_simulation(t, x, Y, a_component, []),...
    tspan, y0, options);
else
  [t1, y1] = ode15s(...
    @(t, x) func_simulation(t, x, Y, a_component, bus_fault),...
  tspan_fault, y0, options);

  [t2, y2] = ode15s(...
    @(t, x) func_simulation(t, x, Y, a_component, []),...
  [tspan_fault(2), tspan(2)], y1(end, :), options);

  t = [t1; t2];
  y = [y1; y2];
end

(Same as lines 21 through 34 of program 3-26)

end
```

In this program, the bus bar and time interval where the ground fault occurs are added to the input arguments. However, lines 4 through 10 set default values if these values are not entered, so the program can be used without specifying a ground fault as in the Program **??**. Such a property that allows past programs to be used in newer versions is called **backward compatibility**.

In the Program **??**, an earth fault is specified in lines 19 to 21, and the numerical simulation after the earth fault is resolved is performed in lines 23 to 25. Note that the initial value of the state in the numerical simulation after the ground fault is cleared is the final value of the time response calculation during the ground fault. If a program that numerically simulates the time response to a ground fault is written using this program, it will be Program**??**.

[count,title=main_simulation_3bus_fault.m]

```
(Same as lines 1 through 23 of program 3-30)

x0 = vertcat(x_equilibrium{:});
tspan = [0 50];

fault_bus = 1;
fault_tspan = [0, 50e-3];

[t, x, V, I] = simulate_power_system(a_component, Y, x0, tspan,...
  fault_bus, fault_tspan);

plot(t, [x{1}(:, 2), x{3}(:, 2)])
```

Execution of this program yields results equivalent to **??**.

Finally, the calculation of the time response to the input signal is described.

[

Numerical computation of time response to input signals] In the examples we have dealt with so far, we will consider numerical simulations in which the mechanical input P_{mech} of the generator is varied and the magnitude of the load is varied, as in the example**??**. For this purpose, we modify the program so that it can take into account external inputs.

In order to take external inputs into account, the number of inputs that each device receives must be explicitly stated. From this perspective, we add to the definition of an instrument "a method that returns the number of inputs". Furthermore, if we modify the Programs **??** and **??** to reflect the external inputs in the time differentiation of the state variables and the computation of the constraint conditions, the Programs **??** and **??**.

[count,title=generator.m]

```
classdef generator < handle

(Same as lines 3-12 of program 3-23)
```

```
methods

(Same as lines 15-57 in program 3-23)

  function nu = get_nu(obj)
    nx = 1;
  end

  function [dx, con] = get_dx_constraint(obj, x, V, I, u)

(Same as lines 33 through 36 of program 3-23)

    Pmech = obj.Pmech + u;

(Same as lines 39-56 in program 3-23)

  end

(Same as lines 9 through 24 of program 3-28)

end
end
```

[count,title=load_impedance.m]

```
classdef load_impedance < handle

(Same as lines 3-5 in program 3-24)

methods

(Same as lines 8 through 21 in Program 3-24)

  function nu = get_nu(obj)
    nu = 2;
  end

  function [dx, con] = get_dx_constraint(obj, x, V, I, u)
    dx = [];
    z = real(obj.z)*(1+u(1)) + 1j*imag(obj.z)*(1+u(2));
    con = V+z*I;
    con = [real(con); imag(con)];
  end

(Same as lines 9 through 12 in program 3-29)
```

```
end

end
```

These programs return the number of inputs that `get_nu` can receive. They are also modified so that `get_dx_constraint` receives the input `u` and processes it appropriately. The input to the generator represents the increment of P_{mech}, and the input to the load represents the ratio of the real to the imaginary part of the impedance change. Modifying the Programs **??** and **??** so that the input response can be calculated using these devices will result in the Programs **??** and **??**.

[count,title=func_simulation.m]

```
function out = func_simulation(t, x, Y, a_component,...
  bus_fault, U, bus_U)

(Same as lines 3-26 of program 3-25)

for k = 1:n_component

(Same as lines 29-32 of program 3-25)

  if ismember(k, bus_U)
    uk = U{bus_U==k}(t);
  else
    uk = zeros(component.get_nu(), 1);
  end

  [dx{k}, con{k}] = component.get_dx_constraint(xk, Vk, Ik, uk);

end

(Same as lines 5-6 in program 3-31)

(Same as lines 38 through 40 in program 3-25)

end
```

[count,title=simulate_power_system.m]

```
function [t, x, V, I] = simulate_power_system(a_component, ...
  Y, x0, tspan,bus_fault, tspan_fault, U, bus_U)

(Same as lines 4 through 10 in program 3-32)

if nargin < 7
    U = {};
```

```
    bus_U = [];
end

(Same as lines 4-15 of program 3-26)

if isempty(bus_fault)
  func = @(t, x) func_simulation(t, x, Y, a_component,...
    [], U, bus_U);
  [t, y] = ode15s(func, tspan, y0, options);
else
  func = @(t, x) func_simulation(t, x, Y, a_component,...
    bus_fault, U, bus_U);
  [t1, y1] = ode15s(func, tspan_fault, y0, options);

  func = @(t, x) func_simulation(t, x, Y, a_component,...
    [], U, bus_U);
  [t2, y2] = ode15s(func, ...
    [tspan_fault(2), tspan(2)], y1(end, :), options);

  t = [t1; t2];
  y = [y1; y2];
end
(Same as lines 21 through 34 of program 3-26)

end
```

In these programs, U and bus_U are added to the input arguments. Here, it is assumed that bus_U is assigned the bus line number that specifies the input signal. In addition, U is a cell array with the specified number of bus lines, and its elements are functions that receive the time t and return the input signal.

An example of a program that uses the modified function to perform the simulation is the Program **??**.

[count,title=main_simulation_3bus_input.m]

```
  (Same as lines 1 through 25 of program 3-30)

tspan = [0 50];

bus_U = [1; 2];
U = {@(t) 0; @(t) [0.05*t/50; 0.05*t/50]};

[t, x, V, I] = simulate_power_system(a_component, Y, x0,...
    tspan, [], [], U, bus_U);

plot(t, [x{1}(:, 2), x{3}(:, 2)])
```

In the Program **??**, the input signal is defined in lines 6 and 7. In this program, the load impedance is increased by 5% in 50 seconds. The input signal on bus bar 1 has no meaning because it always returns 0, but it is added to illustrate how to specify it. Also, no input is specified for bus bar 3, but in this case, 0 is automatically input. With the above program modifications, it is possible to perform time response calculations for external inputs. Since this modification takes care to maintain backward compatibility, the initial value response to various power systems and the time response to ground faults and external inputs can be calculated by appropriately rewriting program**??**. At this time, the other program groups can be used without any modification except for the Program **??**. This is the advantage of the implementation divided into modules.

Mathematical Supplement

Translated with DeepL

Lemma 1.3 *For real constants r_i, ω_i, and phi_i.*

$$\boldsymbol{C}_n(t) := \sum_{i=1}^{n} r_i e^{\boldsymbol{j}(\omega_i t + \phi_i)}$$

However, $r_i > 0$ and $\phi_i \in [0, 2\pi)$. In this case, the necessary and sufficient condition for $\boldsymbol{C}_1$ to be a constant independent of t is $\omega_1 = 0$. Also, the necessary and sufficient condition for $\boldsymbol{C}_2$ to be a constant independent of t is $\omega_1 = \omega_2 = 0$. Besides that,

$$\omega_1 = \omega_2, \qquad r_1 = r_2, \qquad |\phi_2 - \phi_1| = \pi$$

Furthermore, when ω_1, ω_2, and ω_3 are all non-zero, the necessary and sufficient condition for $\boldsymbol{C}_3$ to be a constant independent of t is

$$\omega_1 = \omega_2 = \omega_3, \qquad \sum_{i=1}^{3} r_i e^{\boldsymbol{j}\phi_i} = 0$$

Proof Translated with DeepL First, by setting the derivative of $\boldsymbol{C}_1$ with respect to t to 0:

$$r_1 \omega_1 e^{\boldsymbol{j}(\omega_1 t + \phi_1)} = 0$$

Therefore, the necessary and sufficient condition for $\boldsymbol{C}_1$ to be a constant independent of t is that $\omega_1 = 0$. Next, by setting the derivative of $\boldsymbol{C}_2$ with respect to t to 0:

$$r_1\omega_1 e^{\boldsymbol{j}(\omega_1 t+\phi_1)} + r_2\omega_2 e^{\boldsymbol{j}(\omega_2 t+\phi_2)} = 0 \tag{56}$$

Multiply both sides by $e^{-\boldsymbol{j}(\omega_1 t+\phi_1)}$ and further differentiate by t.

$$r_2\omega_2(\omega_2-\omega_1)e^{\boldsymbol{j}\{((\omega_2-\omega_1)t+\phi_2-\phi_1)\}} = 0$$

This is equivalent to $\omega_2(\omega_2-\omega_1)=0$. Thus,
$omega_1 = \omega_2$ is obtained. In particular, if
$omega_1 = \omega_2 = 0$, then the equation**??** is satisfied for any r_1, r_2, ϕ_1, ϕ_1. Also, when ω_1 and ω_2 are non-zero:

$$r_1 e^{\boldsymbol{j}\phi_1} + r_2 e^{\boldsymbol{j}\phi_2} = 0$$

This is equivalent to $r_1 = r_2$ and $|\phi_2-\phi_1| = \pi$.

Finally, consider $\boldsymbol{C}_3$. As before, by setting the derivative of $\boldsymbol{C}_2$ with respect to t to 0

$$r_1\omega_1 e^{\boldsymbol{j}(\omega_1 t+\phi_1)} + r_2\omega_2 e^{\boldsymbol{j}(\omega_2 t+\phi_2)} + r_3\omega_3 e^{\boldsymbol{j}(\omega_3 t+\phi_3)} = 0 \tag{57}$$

Multiply both sides by $e^{-\boldsymbol{j}(\omega_1 t+\phi_1)}$ and further differentiate by t.

$$r_2\omega_2(\omega_2-\omega_1)e^{\boldsymbol{j}\{(\omega_2-\omega_1)t+\phi_2-\phi_1)\}} + r_3\omega_3(\omega_3-\omega_1)e^{\boldsymbol{j}\{(\omega_3-\omega_1)t+\phi_3-\phi_1)\}} = 0$$

Similarly, multiplying both sides by $e^{-\boldsymbol{j}\{(\omega_2-\omega_1)t+\phi_2-\phi_1\}}$ and further differentiating by t yields

$$r_3\omega_3(\omega_3-\omega_1)(\omega_3-\omega_2)e^{\boldsymbol{j}\{(\omega_3-\omega_2)t+\phi_3-\phi_2)\}} = 0$$

This is because $\omega_3 \neq 0$.

$$(\omega_3-\omega_1)(\omega_3-\omega_2) = 0 \tag{58a}$$

By a similar procedure, from Equation **??**

$$(\omega_2-\omega_1)(\omega_3-\omega_2) = 0, \qquad (\omega_3-\omega_1)(\omega_2-\omega_1) = 0 \tag{58b}$$

Equation**??** is equivalent to $\omega_1 = \omega_2 = \omega_3$. In this case, since
$omega_1$,ω_2,ω_3 are non-zero, the Equation **??** becomes:

$$\sum_{i=1}^{n} r_i e^{\boldsymbol{j}\phi_i} = 0$$