Chapter 1
Steady-state stability analysis of power system models

In this Chapter, we conduct stability analysis based on the approximate linearization of power system models. The structure of this Chapter is as follows. First, in Section 1, we derive a linear approximation model for the power system model described by a system of ordinary differential equations using Kron reduction of the generator buses. Then, in Section 2, we explain the method for numerically analyzing the stability of the derived linear approximation model. We also confirm through numerical simulation that the stability of the linear approximation model depends not only on the physical constants of the generators, loads, and transmission lines, but also on the selection of the steady-state power flow. Additionally, in Section 3, we explore advanced topics and demonstrate how the stability of the linear approximation model can be analyzed using the concept of passivity in dynamic systems.

COFFEE BREAK

Derivation of the approximate linear system:
Consider the nonlinear system:

$$\dot{x}(t) = f\big(x(t)\big) + Bu(t)$$

where $f(0) = 0$. The function $f(x)$ can be expressed near the origin by a Taylor expansion as:

$$f(x) = f(0) + \frac{\partial f}{\partial x}(0)x + \text{Second or higher order term}$$

Here, $f(x)$ and x are expressed as $f_i(x)$ and x_i, respectively, and $\frac{\partial f}{\partial x}(x)$ is the *Jacobian matrix* with the (i, j) element given by $\frac{\partial f_i}{\partial x_j}(x)$. By using this Jacobian matrix, we define:

$$A := \frac{\partial f}{\partial x}(0)$$

Then, when the magnitudes of the state $x(t)$ and input $u(t)$ are sufficiently small, the behavior of the nonlinear system can be approximated by the behavior of the linear system obtained by neglecting terms of degree 2 or higher in the function f:

$$\dot{x}^{\mathrm{lin}}(t) = Ax^{\mathrm{lin}}(t) + Bu^{\mathrm{lin}}(t)$$

Note that even if $u(t)$ and $u^{\mathrm{lin}}(t)$ are the same, the state $x(t)$ of the nonlinear system and the state $x^{\mathrm{lin}}(t)$ of the approximate linear system may not be exactly the same.

1 Stability analysis based on linear approximation

1.1 Approximate linearization of the power system model

In this section, we derive an approximate linear model for the power system model where each bus has a generator connected, which is equivalent to the Kron-reduced differential equation system model discussed in Section **??**. We derive the approximate linear model for the steady-state flow state. The differential equation system model is given by:

$$\begin{aligned} \dot{\delta}_i &= \omega_0 \Delta\omega_i \\ M_i \Delta\dot{\omega}_i &= -D_i \Delta\omega_i - f_i(\delta, E) + P_{\mathrm{mech}i} \qquad\qquad i \in \mathcal{I}_{\mathrm{G}} \\ \tau_i \dot{E}_i &= -\tfrac{X_i}{X_i'} E_i + \left(X_i - X_i'\right) g_i(\delta, E) + V_{\mathrm{field}i} \end{aligned} \tag{1}$$

However, δ and E are vectors obtained by vertically arranging δ_i and E_i, respectively. The nonlinear terms representing the interactions between generators are expressed as follows:

$$\begin{aligned} f_i(\delta, E) &:= -E_i \sum_{j=1}^{N} E_j \left(B_{ij}^{\mathrm{red}} \sin\delta_{ij} - G_{ij}^{\mathrm{red}} \cos\delta_{ij}\right), \\ g_i(\delta, E) &:= -\sum_{j=1}^{N} E_j \left(B_{ij}^{\mathrm{red}} \cos\delta_{ij} + G_{ij}^{\mathrm{red}} \sin\delta_{ij}\right) \end{aligned} \tag{2}$$

In addition, $\delta_{ij} := \delta_i - \delta_j$ is defined. Note that, due to the properties of reduced admittance, the reduced conductance and reduced susceptance satisfy the symmetry condition:

$$G_{ij}^{\mathrm{red}} = G_{ji}^{\mathrm{red}}, \qquad B_{ij}^{\mathrm{red}} = B_{ji}^{\mathrm{red}}, \qquad \forall (i, j) \in \mathcal{I}_{\mathrm{G}} \times \mathcal{I}_{\mathrm{G}}$$

To obtain the partial derivatives of these nonlinear functions with respect to each variable, we define:

$$
\begin{aligned}
k_{ij}(\delta_{ij}) &:= -B_{ij}^{\mathrm{red}} \cos\delta_{ij} - G_{ij}^{\mathrm{red}} \sin\delta_{ij}, \\
h_{ij}(\delta_{ij}) &:= -B_{ij}^{\mathrm{red}} \sin\delta_{ij} + G_{ij}^{\mathrm{red}} \cos\delta_{ij}
\end{aligned}
\tag{3}
$$

Then, for f_i, we obtain:

$$
\begin{aligned}
\frac{\partial f_i}{\partial \delta_i} &= E_i \sum_{j=1, j\neq i}^{N} E_j k_{ij}(\delta_{ij}), \quad & \frac{\partial f_i}{\partial E_i} &= 2E_i h_{ii}(\delta_{ii}) + \sum_{j=1, j\neq i}^{N} E_j h_{ij}(\delta_{ij}), \\
\frac{\partial f_i}{\partial \delta_j} &= -E_i E_j k_{ij}(\delta_{ij}), & \frac{\partial f_i}{\partial E_j} &= E_i h_{ij}(\delta_{ij})
\end{aligned}
\tag{4}
$$

where $j \neq i$.

Similarly, we can obtain the partial derivatives of g_i as follows:

$$
\begin{aligned}
\frac{\partial g_i}{\partial \delta_i} &= -\sum_{j=1, j\neq i}^{N} E_j h_{ij}(\delta_{ij}), \quad & \frac{\partial g_i}{\partial E_i} &= k_{ii}(\delta_{ii}), \\
\frac{\partial g_i}{\partial \delta_j} &= E_j h_{ij}(\delta_{ij}), & \frac{\partial g_i}{\partial E_j} &= k_{ij}(\delta_{ij})
\end{aligned}
\tag{5}
$$

We denote the steady-state values of the internal state of generator i as $(\delta_i^\star, E_i^\star)$ and the steady-state values of external inputs as $(P_{\mathrm{mech}i}^\star, V_{\mathrm{field}i}^\star)$ for the differential equation system in equation 1. Furthermore, we use symbols without the subscript i to represent the vector of these values for all $i \in \mathcal{I}_{\mathrm{G}}$. For example, $\delta^\star$ denotes the vector $(\delta_i^\star)_{i\in\mathcal{I}_{\mathrm{G}}}$. With these steady-state values, we can write the following system of equations:

$$
\left.
\begin{aligned}
0 &= -f_i\left(\delta^\star, E^\star\right) + P_{\mathrm{mech}i}^\star \\
0 &= -\tfrac{X_i}{X_i'} E_i^\star + \left(X_i - X_i'\right) g_i\left(\delta^\star, E^\star\right) + V_{\mathrm{field}i}^\star
\end{aligned}
\right. \qquad i \in \mathcal{I}_{\mathrm{G}}
\tag{6}
$$

Here, note that we assume the steady-state value of the frequency deviation $\Delta\omega_i$ in Eq. 1 is zero for all $i \in \mathcal{I}_{\mathrm{G}}$. The validity of Eq. 6 corresponds to setting the steady-state values of the external input $(P_{\mathrm{mech}}^\star, V_{\mathrm{field}}^\star)$ to appropriate values that achieve supply-demand balance. By linearizing the system around this steady state, the approximate linear model is obtained as:

$$
\begin{bmatrix} \dot{\delta}^{\mathrm{lin}} \\ M\Delta\dot{\omega}^{\mathrm{lin}} \\ \tau\dot{E}^{\mathrm{lin}} \end{bmatrix}
=
\begin{bmatrix} 0 & \omega_0 I & 0 \\ -L & -D & -C \\ B & 0 & A \end{bmatrix}
\begin{bmatrix} \delta^{\mathrm{lin}} \\ \Delta\omega^{\mathrm{lin}} \\ E^{\mathrm{lin}} \end{bmatrix}
+
\begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{bmatrix}
\begin{bmatrix} P_{\mathrm{mech}}^{\mathrm{lin}} \\ V_{\mathrm{field}}^{\mathrm{lin}} \end{bmatrix}
\tag{7}
$$

Note that the state and input variables with the subscript "lin" are vectors consisting of small deviations from the corresponding variables with their steady-state values as the reference. Also,

$$M := \mathsf{diag}\,(M_i)_{i\in\mathcal{I}_\mathrm{G}}\,, \qquad D := \mathsf{diag}(D_i)_{i\in\mathcal{I}_\mathrm{G}}, \qquad \tau := \mathsf{diag}\,(\tau_i)_{i\in\mathcal{I}_\mathrm{G}}$$

are diagonal matrices where $\mathsf{diag}(\cdot)$ is an operator that creates a diagonal matrix from a vector.

Furthermore, for the functions k_{ij} and h_{ij} defined in Equation 3, the (i, j) element of the matrices $\hat{L}$, $\hat{A}$, $\hat{B}$, and $\hat{C}$, defined as:

$$\hat{L}_{ij} := \begin{cases} E_i^\star \sum_{j=1, j\neq i}^N E_j^\star k_{ij}(\delta_{ij}^\star), & i = j \\ -E_i^\star E_j^\star k_{ij}(\delta_{ij}^\star), & i \neq j \end{cases}$$
$$\hat{A}_{ij} := \begin{cases} k_{ii}(\delta_{ii}^\star) - \frac{X_i}{X_i'(X_i - X_i')}, & i = j \\ k_{ij}(\delta_{ij}^\star), & i \neq j \end{cases}$$
$$\hat{B}_{ij} := \begin{cases} -\sum_{j=1, j\neq i}^N E_j^\star h_{ij}(\delta_{ij}^\star), & i = j \\ E_j^\star h_{ij}(\delta_{ij}^\star), & i \neq j \end{cases}$$
$$\hat{C}_{ij} := \begin{cases} \sum_{j=1, j\neq i}^N E_j^\star h_{ij}(\delta_{ij}^\star), & i = j \\ E_i^\star h_{ij}(\delta_{ij}^\star), & i \neq j \end{cases}$$

The matrices L, A, B, and C are then defined as follows:

$$\begin{aligned} L &:= \hat{L}, \\ A &:= \mathsf{diag}\left(X_i - X_i'\right)_{i\in\mathcal{I}_\mathrm{G}} \hat{A}, \\ B &:= \mathsf{diag}\left(X_i - X_i'\right)_{i\in\mathcal{I}_\mathrm{G}} \hat{B}, \\ C &:= \mathsf{diag}\big(2E_i^\star h_{ii}(\delta_{ii}^\star)\big)_{i\in\mathcal{I}_\mathrm{G}} + \hat{C} \end{aligned} \tag{8}$$

Note that $\delta_{ij}^\star := \delta_i^\star - \delta_j^\star$. It should be noted that the system matrix (L, A, B, C) is a function of the steady-state values $(\delta^\star, E^\star)$. The block diagram of this approximate linear model is shown in Figure 1. Here, P^{lin} represents the approximately linearized active power supplied by the generators. Note that generally $X_i > X_i'$ for all i.

In power system engineering, the value obtained by differentiating the generator's active power with respect to the rotor angle at the steady-state is called the **synchronizing power coefficient** [**?**, Section 8.4]. That is, the matrix L in the approximate linear model given by equation 7 corresponds to the synchronizing power coefficient. However, in power system engineering, it is common to define the synchronizing power coefficient using the one-machine infinite-bus system model explained in Section **??**, so it is a scalar value rather than a matrix.

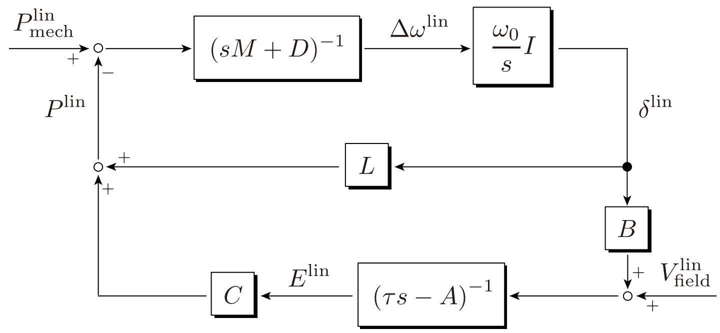


Fig. 1 Block Diagram of Approximate Linear Model

1.2 Stability analysis of approximate linear models

1.2.1 Stability of approximate linear models

In this section, we consider numerically analyzing the stability of the approximate linear model. Whether the approximate linear model of Equation 7 is stable or not is characterized by whether the internal states of the generator groups return to the steady state satisfying the simultaneous equations of Equation 6 in the event of a small disturbance in the power system, such as temporary minor fluctuations in mechanical input or excitation input of the generators, impedance values of the loads, current or voltage values of the transmission lines, etc., from the reference values at the steady state. In power system engineering, stability against such small fluctuations is called **small signal stability**.

It should be noted that the stability of the approximate linear model of Equation 7 depends on the selection of the steady-state values of the internal states of the generator groups $(\delta^\star, E^\star)$ and the steady-state values of external inputs $(P^\star_{\mathrm{mech}}, V^\star_{\mathrm{field}})$. Furthermore, changes in the admittance of the transmission lines or the impedance of the loads alter the reduced conductance $G^{\mathrm{red}}ij$ and reduced susceptance $B^{\mathrm{red}}ij$ of Equation 3. Therefore, the stability of the approximate linear model varies depending on various model parameters mentioned above. The purpose of this section is to numerically examine the relationship between the changes in these model parameters and the stability of the approximate linear model.

1.2.2 Stability analysis based on eigenvalues of the system matrix

For the approximate linear model in Equation 7, if we appropriately choose the steady-state values $(\delta^\star, E^\star)$ of the internal states as parameters, then the system matrix (L, A, B, C) in Equation 8 and the steady-state values $(P^\star_{\mathrm{mech}}, V^\star_{\mathrm{field}})$ of the external inputs satisfying Equation 6 are determined dependently. Here, we consider setting

$$P_{\mathrm{mech}i}(t) = P^{\star}_{\mathrm{mech}i}, \qquad V_{\mathrm{field}i}(t) = V^{\star}_{\mathrm{field}i}, \qquad \forall t \geq 0$$

for all $i \in \mathcal{I}\mathrm{G}$ in the nonlinear differential equation system model in Equation **??**. We then assess the stability of the system using the eigenvalues of the system matrix.

This means that in the approximate linear model of Equation 7 the following values are set:

$$P^{\mathrm{lin}}_{\mathrm{mech}}(t) = 0, \qquad V^{\mathrm{lin}}_{\mathrm{field}}(t) = 0, \qquad \forall t \geq 0$$

In the following, under this assumption, we analyze the stability of an autonomous approximate linear model with input set identically to zero, given by:

$$\begin{bmatrix} \dot{\delta}^{\mathrm{lin}} \\ \Delta\dot{\omega}^{\mathrm{lin}} \\ \dot{E}^{\mathrm{lin}} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \omega_0 I & 0 \\ -M^{-1}L & -M^{-1}D & -M^{-1}C \\ \tau^{-1}B & 0 & \tau^{-1}A \end{bmatrix}}_{\Psi} \begin{bmatrix} \delta^{\mathrm{lin}} \\ \Delta\omega^{\mathrm{lin}} \\ E^{\mathrm{lin}} \end{bmatrix} \tag{9}$$

Specifically, by examining the sign of the real part of the eigenvalues of the matrix Ψ, we can determine the stability of this approximate linear model. However, it should be noted that Ψ generally has at least one zero eigenvalue. In fact, from the structure of the matrices L and B in equation 8, we have:

$$L\mathbb{1} = 0, \qquad B\mathbb{1} = 0 \tag{10}$$

Therefore, for any model parameters, we have:

$$\Psi v = 0, \qquad v := \begin{bmatrix} \mathbb{1} \\ 0 \\ 0 \end{bmatrix}$$

This means that v is an eigenvector of Ψ corresponding to a zero eigenvalue. If the real parts of all eigenvalues, except for the zero eigenvalue, are negative, then for any initial value, the solution trajectory of Equation 9 satisfies:

$$\lim_{t\to\infty} \delta^{\mathrm{lin}}(t) = c_0\mathbb{1}, \qquad \lim_{t\to\infty} \Delta\omega^{\mathrm{lin}}(t) = 0, \qquad \lim_{t\to\infty} E^{\mathrm{lin}}(t) = 0 \tag{11}$$

Here, c_0 is a constant determined by the initial value. Note that the value of c_0 does not make a significant difference in the analytical results. This is because in the differential equation system model of Equation 1, the rotor angle δ_i of a generator has meaning only in relation to the difference between the rotor angle δ_j of other generators. Specifically, if $(\delta^\star, E^\star)$ satisfies the system of equations in equation 6 for a certain $(P^\star_{\mathrm{mech}}, V^\star_{\mathrm{field}})$, then $(\delta^\star + c_0\mathbb{1}, E^\star)$ also satisfies the same system of equations. Therefore, $\delta^\star$ and $\delta^\star + c_0\mathbb{1}$ are essentially equivalent steady-state values where all generator rotor angles are rotated by the same amount of c_0. Equation

11 means the asymptotic convergence of solution trajectories to these essentially equivalent steady-state values.

2 stability analysis of approximate linear models using numerical calculations

2.1 Implementation of approximate linearization using a group of partitioned modules

In this section, we explain the implementation method for obtaining an approximate linear model numerically. Specifically, we describe how to add the functionality of linearization to the program that has been segmented into module groups as explained in Sections **??** and **??**.

In the numerical simulation program of the power system created in Section **??**, the following state and output equations are implemented for each device as differential and algebraic equations, respectively:

$$\dot{x}_i = f_i^{(1)}(x_i, \boldsymbol{V}_i, \boldsymbol{I}_i, u_i), \qquad 0 = f_i^{(2)}(x_i, \boldsymbol{V}_i, \boldsymbol{I}_i, u_i)$$

In the following, we derive the approximate linear model in the vicinity of the equilibrium point $(x_i^\star, \boldsymbol{V}_i^\star, \boldsymbol{I}_i^\star, u_i^\star)$ for the device of interest. Specifically, we explain the implementation method of the linear approximation function to the program that has been partitioned into the module group described in Sections **??** and **??**.

For the numerical simulation program of the power system created in Section **??**, differential equations for the state and algebraic equations for the output are implemented for each device as:

$$\dot{x}_i = f_i^{(1)}(x_i, \boldsymbol{V}_i, \boldsymbol{I}_i, u_i), \qquad 0 = f_i^{(2)}(x_i, \boldsymbol{V}_i, \boldsymbol{I}_i, u_i)$$

We consider the linearization of the functions $f_i^{(1)}$ and $f_i^{(2)}$ as follows:

$$\begin{aligned} f_i^{(1)}(x_i, \boldsymbol{V}_i, \boldsymbol{I}_i, u_i) &\approx A_i(x_i - x_i^\star) + B_{u_i} u_i \\ &\quad + B_{\boldsymbol{V}_i} \begin{bmatrix} \mathrm{Re}[\boldsymbol{V}_i - \boldsymbol{V}^\star] \\ \mathrm{i}[\boldsymbol{V}_i - \boldsymbol{V}^\star] \end{bmatrix} + B_{\boldsymbol{I}_i} \begin{bmatrix} \mathrm{Re}[\boldsymbol{I}_i - \boldsymbol{I}_i^\star] \\ \mathrm{i}[\boldsymbol{I}_i - \boldsymbol{I}_i^\star] \end{bmatrix} \end{aligned} \tag{12}$$

$$\begin{aligned} f_i^{(2)}(x_i, \boldsymbol{V}_i, \boldsymbol{I}_i, u_i) &\approx C_i(x_i - x_i^\star) + D_{u_i} u_i \\ &\quad + D_{\boldsymbol{V}_i} \begin{bmatrix} \mathrm{Re}[\boldsymbol{V}_i - \boldsymbol{V}^\star] \\ \mathrm{i}[\boldsymbol{V}_i - \boldsymbol{V}^\star] \end{bmatrix} + D_{\boldsymbol{I}_i} \begin{bmatrix} \mathrm{Re}[\boldsymbol{I}_i - \boldsymbol{I}_i^\star] \\ \mathrm{i}[\boldsymbol{I}_i - \boldsymbol{I}_i^\star] \end{bmatrix} \end{aligned} \tag{13}$$

A system of simultaneous equations for each machine and algebraic equations for the entire power system can be used to obtain an expression using ordinary differential equations for the approximate linear model by eliminating all $\boldsymbol{V}_i - \boldsymbol{V}_i^\star$ and $\boldsymbol{I}_i - \boldsymbol{I}_i^\star$, where $i \in 1, \ldots, N$, as follows:

$$\boldsymbol{I}_i - \boldsymbol{I}_i^\star = \sum_{j=1}^{N} \boldsymbol{Y}_{ij}(\boldsymbol{V}_j - \boldsymbol{V}_j^\star), \qquad i \in \{1, \ldots, N\}$$

Here, $\boldsymbol{Y}_{ij}$ represents the (i, j)th element of the admittance matrix $\boldsymbol{Y}$. Let us check the specific implementation method with the following example.

Example 1.1 (Implementation of Approximate Linear Model)

Equations 12 and 13 depend on the dynamic characteristics of the device, so it is natural to implement the calculation of coefficient matrices such as A_i and B_{u_i} in the classes of devices such as generators and loads in the implementation example of Section **??**. For example, in the generator model:

$$A_i = \begin{bmatrix} 0 & \omega_0 & 0 \\ 0 & -\frac{D_i}{M_i} & 0 \\ -\frac{1}{\tau_i}(\frac{X_i}{X_i'} - 1)|\boldsymbol{V}_i^\star| \sin(\delta_i^\star - \angle \boldsymbol{V}_i^\star) & 0 & -\frac{X_i}{\tau_i X_i'} \end{bmatrix}$$

$$B_{u_i} = \begin{bmatrix} 0 \\ \frac{1}{M_i} \\ 0 \end{bmatrix}, \qquad B_{\boldsymbol{V}_i} = \begin{bmatrix} 0 & 0 \\ -\frac{\mathrm{Re}[\boldsymbol{I}_i^\star]}{M_i} & -\frac{\mathrm{i}[\boldsymbol{I}_i^\star]}{M_i} \\ \frac{1}{\tau_i}(\frac{X_i}{X_i'} - 1)\cos\delta_i^\star & \frac{1}{\tau_i}(\frac{X_i}{X_i'} - 1)\sin\delta_i^\star \end{bmatrix}$$

$$B_{\boldsymbol{I}_i} = \begin{bmatrix} 0 & 0 \\ -\frac{\mathrm{Re}[\boldsymbol{V}_i^\star]}{M_i} & -\frac{\mathrm{i}[\boldsymbol{V}_i^\star]}{M_i} \\ 0 & 0 \end{bmatrix}, \qquad C_i = \begin{bmatrix} E_i^\star \cos\delta_i^\star & 0 & \sin(\delta_i^\star) \\ E_i^\star \sin\delta_i^\star & 0 & -\cos(\delta_i^\star) \end{bmatrix}$$

$$D_{u_i} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad D_{\boldsymbol{V}_i} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad D_{\boldsymbol{I}_i} = \begin{bmatrix} -X_i' & 0 \\ 0 & -X_i' \end{bmatrix}$$

If the calculation of these coefficient matrices is added to the `generator` class as a method named `get_linear_matrix`, the program 1.1 is obtained.

```
classdef generator < handle

  properties
(Same as lines 4-11 in program 3-23)
    x_equilibrium
    V_equilibrium
    I_equilibrium
  end

  methods
```

```

(Same as lines 7 through 21 in program 3-34)

    function x_equilibrium = set_equilibrium(obj, V, I, P, Q)

(Same as lines 10 through 23 of program 3-28)

      obj.x_equilibrium = x_equilibrium;
      obj.V_equilibrium = V;
      obj.I_equilibrium = I;
    end

    function [A, Bu, BV, BI, C, Du, DV, DI] =...
        get_linear_matrix(obj)

      X = obj.X;
      X_prime = obj.X_prime;
      D = obj.D;
      M = obj.M;
      tau = obj.tau;

      omega0 = obj.omega0;
      delta = obj.x_equilibrium(1);
      E = obj.x_equilibrium(3);
      V = obj.V_equilibrium;
      Vabs = abs(obj.V_equilibrium);
      Vangle = angle(obj.V_equilibrium);
      I = obj.I_equilibrium;
      A = [0, omega0, 0;
        0, -D/M, 0;
        -(X/X_prime-1)*Vabs*sin(delta-Vangle)/tau,...
        0, -X/X_prime/tau];
      Bu = [0; 1/M; 0];
      BV = [0, 0;
        -real(I)/M, -imag(I)/M;
        (X/X_prime-1)*cos(delta)/tau,...
        (X/X_prime-1)*sin(delta)/tau];
      BI = [0, 0;
        -real(V)/M, -imag(V)/M;
        0, 0];
      C = [E*cos(delta), 0, sin(delta);
        E*sin(delta), 0, -cos(delta)];
      Du = [0; 0];
      DV = [0, -1; 1, 0];
      DI = -X_prime*eye(2);
    end

  end

end
```

Program 1.1 generator.m

In Program 1.1, lines 18 to 20 in `set_equilibrium` store information about the equilibrium point used in the calculation of the approximate linear model.

If implemented similarly for the constant impedance load model, the program 1.2 is obtained.

```
classdef load_impedance < handle

  properties
    z
    I_equilibrium
  end

  methods

(Same as lines 7 through 18 in program 3-35)

    function x_equilibrium = set_equilibrium(obj, V, I, P, Q)
      x_equilibrium = [];
      obj.z = -V/I;
      obj.I_equilibrium = I;
    end

    function [A, Bu, BV, BI, C, Du, DV, DI] =...
        get_linear_matrix(obj)

      A = [];
      Bu = zeros(0, 2);
      BV = zeros(0, 2);
      BI = zeros(0, 2);
      C = zeros(2, 0);
      I = obj.I_equilibrium;
      z = obj.z;
      Du = [real(z)*real(I), imag(z)*imag(I);
        real(z)*imag(I), imag(z)*real(I)];
      DV = eye(2);
      DI = [real(z), -imag(z); imag(z), real(z)];
    end

  end

end
```

Program 1.2 load_impedance.m

By using the class of equipment such as modified generators and loads, the function for obtaining an approximate linear model can be described as shown in Program 1.3.

```
function sys = get_linear_model(a_component, Y)

  A = cell(numel(a_component), 1);
  Bu = cell(numel(a_component), 1);
  BV = cell(numel(a_component), 1);
  BI = cell(numel(a_component), 1);
```

```
7    C = cell(numel(a_component), 1);
8    Du = cell(numel(a_component), 1);
9    DV = cell(numel(a_component), 1);
10   DI = cell(numel(a_component), 1);
11
12   for k = 1:numel(a_component)
13     component = a_component{k};
14     [A{k}, Bu{k}, BV{k}, BI{k}, C{k}, Du{k}, DV{k}, DI{k}] =...
15       component.get_linear_matrix();
16   end
17
18   A = blkdiag(A{:});
19   Bu = blkdiag(Bu{:});
20   BV = blkdiag(BV{:});
21   BI = blkdiag(BI{:});
22   C = blkdiag(C{:});
23   Du = blkdiag(Du{:});
24   DV = blkdiag(DV{:});
25   DI = blkdiag(DI{:});
26
27   Ymat = zeros(size(Y, 1)*2, size(Y, 2)*2);
28   Ymat(1:2:end, 1:2:end) = real(Y);
29   Ymat(2:2:end, 1:2:end) = imag(Y);
30   Ymat(1:2:end, 2:2:end) = -imag(Y);
31   Ymat(2:2:end, 2:2:end) = real(Y);
32
33   nx = size(A, 1);
34
35   A11 = A;
36   A12 = [BV, BI];
37   A21 = [C; zeros(size(Ymat, 1), nx)];
38   A22 = [DV, DI; Ymat, -eye(size(Ymat))];
39
40   B1 = Bu;
41   B2 = [Du; zeros(size(Ymat, 1), size(Du, 2))];
42
43
44   Aout = A11 - A12/A22*A21;
45   Bout = B1 - A12/A22*B2;
46   Cout = eye(nx);
47   Dout = 0;
48
49   sys = ss(Aout, Bout, Cout, Dout);
50
51 end
```

Program 1.3 get_linear_model.m

In lines 12 to 16 of Program 1.3, the coefficient matrix of the approximate linear model is obtained from each equipment. Additionally, by eliminating the voltage and current phases of all buses from lines 27 to 47, an expression for the approximate linear model's system of ordinary differential equations is obtained.

The approximate linear model can be used as follows by using Program 1.3.

```
1 (Same as lines 1 through 23 in Program 3-30)
2
3 sys = get_linear_model(a_component, Y);
4
5 sys = sys(2, 1);
6 nyquist(sys)
```

Program 1.4 main_linearization.m

In this example, an approximate linear model is constructed in line 5 with the mechanical input $P_{\mathrm{mech}1}$ of generator 1 as input and the frequency deviation $\Delta\omega_1$ of generator 1 as output. In addition, a Nyquist plot is drawn in line 6.

In the mathematical analysis of Section 1.1, an approximate linear model is derived from a nonlinear system of ordinary differential equations where all buses are Kron reduced. On the other hand, in the numerical implementation of this section, the nonlinear differential-algebraic equation system is first linearized, and then Kron reduction is applied to construct the ordinary differential equation system. It should be noted that this is because in the power system model with Kron reduction, expressions generally involve a mixture of information about equipment, buses, and transmission lines.

To increase the readability and expandability of the program, it is important to modularize each element appropriately, as in the implementation of this section.

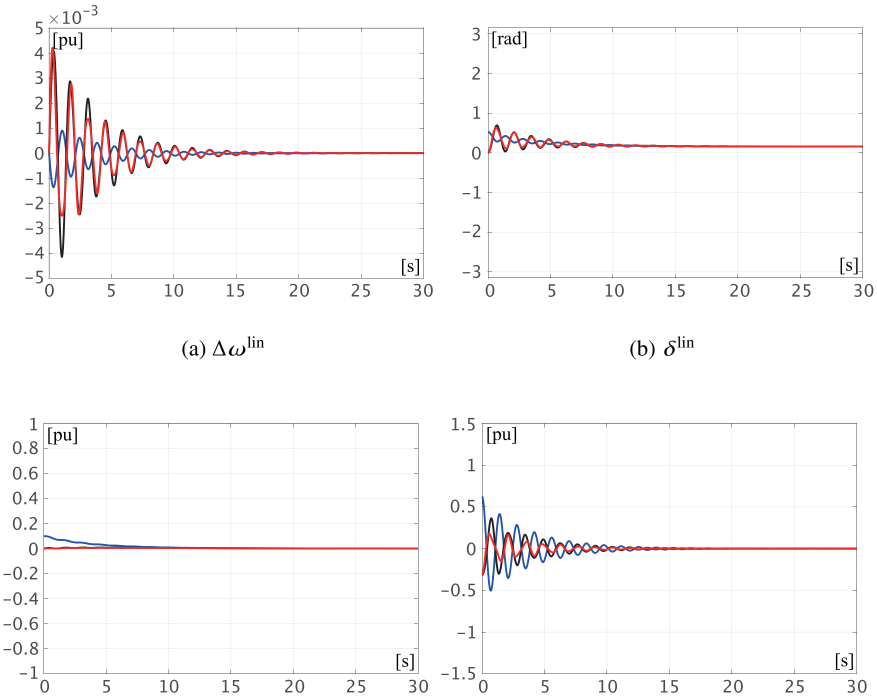
2.2 Numerical analysis of small signal stability

Let us perform a stability analysis based on approximate linearization for an actual electrical power system model consisting of three generators.

Example 1.2 Numerical stability analysis of the linearized model Let us consider an electrical power system model consisting of three generators discussed in the Example **??**. The constant of the generators and transmission lines are set to the same value as in the Example **??**, and a linear approximation model for Equation 9 is derived with the approximate of the steady value shown in **??**. Figure 2 shows the time response When the initial values are set as follows to correspond to Equation **??**:

$$\delta^{\mathrm{lin}}(0) = \begin{bmatrix} \frac{\pi}{6} \\ 0 \\ 0 \end{bmatrix}, \quad \Delta\omega^{\mathrm{lin}}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad E^{\mathrm{lin}}(0) = \begin{bmatrix} 0.1 \\ 0 \\ 0 \end{bmatrix} \tag{14}$$

The blue, black, and red lines represent generators 1, 2, and 3, respectively. From this figure, we can see that the internal state of the generator group converges asymptotically as given in (11). Moreover, it approximately reproduces the initial value response of the nonlinear model shown in Fig. **??**.



(a) $\Delta\omega^{\rm lin}$ (b) $\delta^{\rm lin}$

(c) $E^{\rm lin}$ (d) $P^{\rm lin}$

Fig. 2 Initial value response of approximate linear model
(Blue: Generator 1, Black: Generator 2, Red: Generator 3)

Next, we parameterize the constants and steady-state values of the generators and transmission lines to analyze the stability of the resulting approximate linear model. For the generator constants, we compare the cases where all the damping coefficients are set to 10 and where they are set to 0.1, i.e., we consider the two cases:

$$(D_1, D_2, D_3) = (10, 10, 10), \qquad (D_1, D_2, D_3) = (0.1, 0.1, 0.1)$$

Other constants are set to the values in Table **??**. In addition, the steady-state values of the rotor angle differences are expressed in terms of a parameter $\theta_1 \in [0, 1]$ as follows:

$$\delta_{12}^\star = -\frac{\pi}{2}\theta_1, \qquad \delta_{13}^\star = \frac{\pi}{2}\theta_1 \tag{15}$$

Here, θ_1 is a parameter that specifies the magnitude of the rotor angle difference in the steady-state. By varying this value, the system matrix in 8 changes. Note that the steady-state values of the internal voltages are not changed from the values in Table **??**.

The admittance matrix is also modified as follows. Using the admittance values $\boldsymbol{y}_{12}$ and $\boldsymbol{y}_{23}$ in Equation **??**, the admittance matrix of the power system in Equation **??** is constructed. The real part of this admittance matrix, which is the conductance matrix, is denoted as G_0, and the imaginary part, which is the susceptance matrix, is denoted as B_0. Specifically,

$$
\begin{aligned}
G_0 &= \begin{bmatrix} 1.3652 & -1.3652 & 0 \\ -1.3652 & 3.3074 & -1.9422 \\ 0 & -1.9422 & 1.9422 \end{bmatrix}, \\
B_0 &= \begin{bmatrix} -11.6041 & 11.6041 & 0 \\ 11.6041 & -22.1148 & 10.5107 \\ 0 & 10.5107 & -10.5107 \end{bmatrix}
\end{aligned}
\tag{16}
$$

Using the parameter $\theta_2 \in [0, 5]$, we express the reference admittance matrix as follows:

$$
\boldsymbol{Y}_0(\theta_2) := \theta_2 G_0 + \boldsymbol{j} B_0 \tag{17}
$$

Here, θ_2 is a parameter that specifies the size of the real part (conductance matrix). For comparison, we consider two parameterized admittance matrices:

$$
\boldsymbol{Y} = \boldsymbol{Y}_0(\theta_2), \qquad \boldsymbol{Y} = \tfrac{\boldsymbol{Y}_0(\theta_2)}{100}
$$

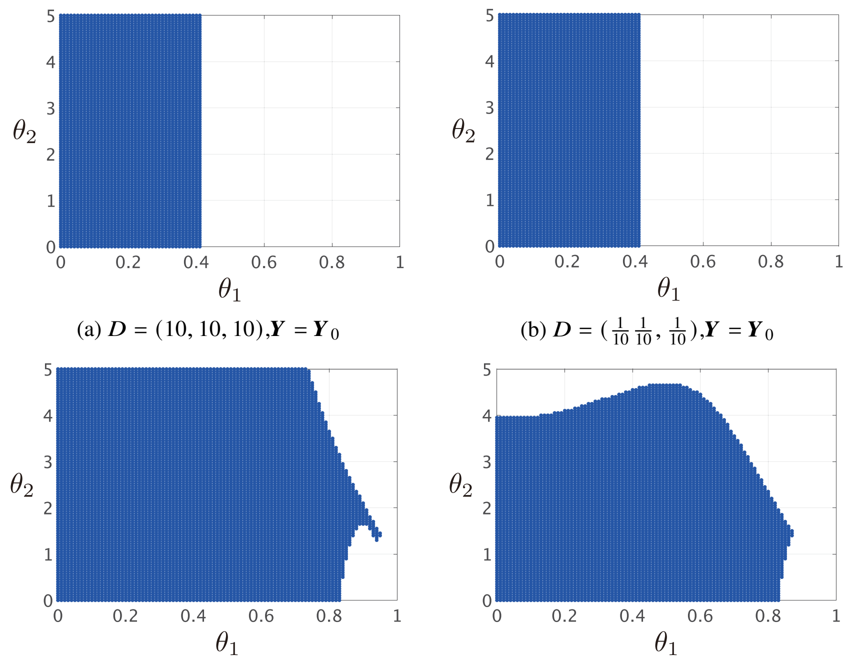
The changes in the admittance matrix are approximately represented in the linearized model by the changes in the values of the reduced conductor $B^{\mathrm{red}}ij$ and the reduced susceptance $G^{\mathrm{red}}ij$ in equation 3. The parameter settings for the comparison are summarized in Table 1.

Let us numerically analyze the stability of the approximate linear model by varying the parameters (θ_1, θ_2) for each case (a)-(d) in Table 1. Specifically, we will check whether the approximate linear model is stable or not by examining the eigenvalues of Ψ in Equation 9 by varying θ_1 and θ_2 on a grid of 100 equidistant points each. The results are shown in Figure 3. The blue area represents the parameter region where the approximate linear model is stable. First, in the case of (a), we see that the approximate linear model is stable regardless of the size of the conductance matrix specified by θ_2, as long as θ_1 is below approximately 0.4, which corresponds to a rotor angle difference of approximately 36° in the steady state. The same result is obtained for case (b), where the generator's damping coefficient is small at 0.1.

Next, we examine the results for cases (c) and (d), where the admittance matrix is multiplied by $\frac{1}{100}$. In this case, we find that when θ_2 is small and the size of the conductance matrix is around 1, the approximate linear model is stable as long as the rotor angle difference in the steady state is below approximately 76°. We also find that as θ_2 increases to 2 or more, the upper limit of the rotor angle difference for stability of the approximate linear model decreases.

Table 1 Parameter settings to compare

	$D = (10, 10, 10)$	$D = (0.1, 0.1, 0.1)$
$\boldsymbol{Y} = \boldsymbol{Y}_0$	(a)	(b)
$\boldsymbol{Y} = \boldsymbol{Y}_0/100$	(c)	(d)



(a) $\boldsymbol{D} = (10, 10, 10), \boldsymbol{Y} = \boldsymbol{Y}_0$

(b) $\boldsymbol{D} = (\frac{1}{10}, \frac{1}{10}, \frac{1}{10}), \boldsymbol{Y} = \boldsymbol{Y}_0$

(c) $\boldsymbol{D} = (10, 10, 10), \boldsymbol{Y} = \frac{\boldsymbol{Y}_0}{100}$

(d) $\boldsymbol{D} = (\frac{1}{10}, \frac{1}{10}, \frac{1}{10}), \boldsymbol{Y} = \frac{\boldsymbol{Y}_0}{100}$

Fig. 3 Area of parameters where the approximate linear model is stable

3 Mathematical stability analysis of the linearized model

3.1 Small signal stability of the linearized model

In this section, we mathematically analyze the stability of the linearized model given in Equation 9. The stability is characterized by the eigenvalues of the matrix Ψ. However, as discussed in section 2, Ψ is not regular and the eigenspace for the zero eigenvalue is given by

$$\mathcal{M} = \mathrm{span}\left\{ \begin{bmatrix} \mathbb{1} \\ 0 \\ 0 \end{bmatrix} \right\} \tag{18}$$

This eigenspace represents the set of equivalent steady-state values obtained by varying the phase angles of all generators while maintaining their relative values constant. Therefore, it is not a problem which point in the equilibrium set of Equation 18 the state of the approximate linear model converges to. Based on this fact, we give the following definition:

COFFEE BREAK

Eigenspace of a square matrix: For a square matrix A and a given eigenvalue λ, the *eigenspace* $\mathcal{V}_\lambda$ is defined as:

$$\mathcal{V}_\lambda := \mathsf{ker}(\lambda I - A)$$

where ker denotes the kernel of the matrix. If $v_1, \ldots, v_k$ are all linearly independent eigenvectors corresponding to a specific eigenvalue λ, then:

$$\mathcal{V}_\lambda = \mathsf{span}\{v_1, \ldots, v_k\}$$

which means that it is a linear space spanned by the eigenvectors associated with a specific eigenvalue.

Definition 1.1 (Small-signal stability of the linear approximation model)
Consider the linearized model given by Equation 9. For any initial values, if the internal state converges to one of the equilibrium points in the set $\mathcal{M}$ defined by Equation 18, the linearized model is said to be **steady-state stable**.

The small-signal stability in Definition 1.1 means that for any initial condition, Equation 11 holds. Note that the value of c_0 in Equation 11 is arbitrary, so we express its arbitrariness as "converging to one of the equilibrium points in $\mathcal{M}$."

In power system engineering, the term "small-signal stability" is widely used to discuss the stability of a power system against small disturbances using an approximate linear model. However, introducing a mathematical definition like Definition 1.1 is not common practice.

In the following discussion, we assume that the kernel space of Ψ in Equation 9 is one-dimensional and that Equation 19 holds:

$$\mathsf{ker}\,\Psi = \mathcal{M} \tag{19}$$

It is clear from the structure of the matrix Ψ that $\mathcal{M}$ is a subset of $\mathsf{ker}\,\Psi$, but it should be noted that we are assuming that $\mathsf{ker}\,\Psi$ is one-dimensional and that the equality holds. If the kernel space were two-dimensional or greater, the invariant eigenspace would be larger than $\mathcal{M}$, and the approximate linear model would not be in a steady state stability. Therefore, equation 19 is a necessary condition for the steady-state stability of the linear approximation model. In particular, when A is invertible, and using the definition of $L_0 := L - CA^{-1}B$, the necessary condition can be equivalently expressed as:

$$\mathsf{ker}\,L_0 = \mathsf{span}\,\{\mathbb{1}\} \tag{20}$$

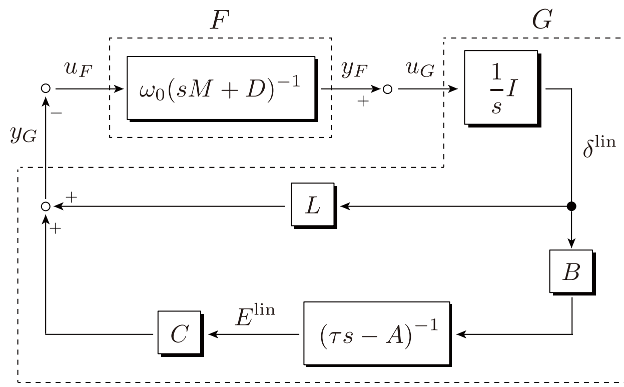


Fig. 4 Feedback system representation of approximate linear models

Note that this matrix L_0 plays an important role in the subsequent analysis.

The relationship between equation **??** and equation 20 can be verified as follows. Because the $(1, 2)$ block of the matrix Ψ is invertible, the necessary and sufficient condition for the kernel of Ψ to be equal to $\mathcal{M}$ is:

$$\ker \begin{bmatrix} -L & -C \\ B & A \end{bmatrix} = \mathrm{span} \left\{ \begin{bmatrix} \mathbb{1} \\ 0 \end{bmatrix} \right\}$$

In particular, when A is invertible, we have

$$\begin{bmatrix} -L & -C \\ B & A \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \qquad \Longleftrightarrow \qquad L_0 x = 0, \qquad y = -A^{-1} B x$$

Thus, the necessary condition is equivalent to equation 20.

For the following discussion, we introduce the following fundamental terminology.

Definition 1.2 (Stability of square matrix) For a square matrix A, if all the real parts of its eigenvalues are negative, A is called **stable** or **asymptotically stable**.

3.2 Passivity of approximate linear models

3.2.1 Representation of Approximate Linear Models with Feedback

Let us consider describing the approximate linear model of Equation 9 as a feedback system of two subsystems (Figure 4). The first subsystem is described as a system of differential equations for the deviation of the angular frequency:

$$F : \begin{cases} M\Delta\dot{\omega}^{\mathrm{lin}} = -D\Delta\omega^{\mathrm{lin}} + u_F \\ y_F = \omega_0\Delta\omega^{\mathrm{lin}} \end{cases} \tag{21}$$

In this book, we refer to this subsystem as the **mechanical subsystem**. The mechanical subsystem is determined solely by the physical constants of the generator set $(M_i, D_i)i \in \mathcal{I}\mathrm{G}$ and the reference angular frequency ω_0, and does not depend on the steady-state values of the internal state $(\delta^\star, E^\star)$.

The second subsystem is described as a system of differential equations with respect to the rotor angle and internal voltage as follows:

$$G : \begin{cases} \dot{\delta}^{\mathrm{lin}} = u_G \\ \tau\dot{E}^{\mathrm{lin}} = AE^{\mathrm{lin}} + B\delta^{\mathrm{lin}} \\ y_G = CE^{\mathrm{lin}} + L\delta^{\mathrm{lin}} \end{cases} \tag{22}$$

We call this subsystem the **electrical subsystem** [1]. The electrical subsystem not only depends on the physical constants of the generator group, $(\tau_i)i \in \mathcal{I}\mathrm{G}$, but also on the steady-state values of the internal states, $(\delta^\star, E^\star)$. In fact, the system matrix (L, A, B, C) in equation 8 is a function of $(\delta^\star, E^\star)$.

If the two subsystems' inputs and outputs are coupled through negative feedback as follows:

$$u_F = -y_G, \qquad u_G = y_F \tag{23}$$

the approximate linear model of Equation 9 is represented. The subsequent analysis of steady-state stability is based on the property called the *passivity* of the mechanical and electrical subsystems. It is well-known that a negative feedback system of a passive subsystem is stable.

3.2.2 Passivity of the mechanical subsystem

The mechanical subsystem F in Equation 21 has the following strong passivity:

Definition 1.3 (Passivity of linear systems)
Consider the linear system

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \tag{24}$$

Using the symmetric matrix P, define the function:

$$W(x) := \frac{1}{2}x^\top Px \tag{25}$$

---

[1] The terms ”mechanical subsystem” and ”electrical subsystem” introduced here are unique to this book.

For any u, if there exists a semi-positive definite matrix P that satisfies:

$$\frac{d}{dt}W\big(x(t)\big) \leq u^{\mathsf{T}}(t)y(t), \qquad \forall t \geq 0 \tag{26}$$

Then, we call Σ **passive**.

In particular, when there exists a positive definite number ρ such that in addition to the semi-positive definite matrix, the inequality:

$$\frac{d}{dt}W\big(x(t)\big) \leq u^{\mathsf{T}}(t)y(t) - \rho\,\|y(t)\|^2, \qquad \forall t \geq 0 \tag{27}$$

is satisfied, then we call Σ **strictly passive**.

The function $W(x)$ in Definition 1.3 is generally called the **storage function**. Moreover, the inequality in Equation 27 describes a type of passivity that is more strictly called **output-strict passivity**, where the energy represented by the storage function $W(x)$ dissipates more quickly in proportion to the square of the output compared to the passivity in Equation 26.

The mechanical subsystem F of Equation21 having strong passivity can be confirmed as follows. First, the subsystem is written as below:

$$F : \begin{cases} \dot{x}_F = A_F x_F + B_F u_F \\ y_F = C_F x_F \end{cases} \tag{28}$$

where state x_F represents $\Delta\omega^{\mathrm{lin}}$, and the system matrices are:

$$A_F := -M^{-1}D, \qquad B_F := M^{-1}, \qquad C_F := \omega_0 I$$

Additionally, we define the symmetric matrix P_F as:

$$P_F := \omega_0 M$$

and since M is positive definite, P_F is also positive definite. Then, the following inequalities hold:

$$A_F^{\mathsf{T}}P_F + P_F A_F \leq -\frac{2\min\{D_i\}}{\omega_0}C_F^{\mathsf{T}}C_F, \qquad P_F B_F = C_F^{\mathsf{T}}$$

Therefore, when the storage function is defined as:

$$W_F(x_F) := \frac{1}{2}x_F^{\mathsf{T}}P_F x_F \tag{29}$$

the time derivative along the solution trajectory of F is evaluated as:

$$
\begin{aligned}
\frac{d}{dt}W_F\big(x_F(t)\big) &= \nabla W_F^{\mathsf{T}}(x_F)\frac{dx_F}{dt} \\
&= \big(P_F x_F(t)\big)^{\mathsf{T}}\big(A_F x_F(t) + B_F u_F(t)\big) \\
&= y_F^{\mathsf{T}}(t)u_F(t) + \frac{1}{2}x_F^{\mathsf{T}}(t)\left(A_F^{\mathsf{T}}P_F + P_F A_F\right)x_F(t) \\
&\leq y_F^{\mathsf{T}}(t)u_F(t) - \tfrac{\min\{D_i\}}{\omega_0}\|y_F(t)\|^2
\end{aligned}
\tag{30}
$$

where $\nabla W_F(x_F)$ is the gradient function obtained by partially differentiating $W_F(x_F)$ with respect to its elements and arranging them vertically. From this, it can be seen that the machine subsystem F in Equation 21 is strongly passive for any positive definite $(M_i, D_i)_{i\in\mathcal{I}_G}$. Note that the function $W_F(x_F)$ represents the mechanical kinetic energy of the power system.

3.2.3 Passivity of the electrical subsystem

Next, we consider the electrical subsystem G in Equation 22. Unlike the mechanical subsystem F, the electrical subsystem G only possesses passivity under limited conditions. While this may seem arbitrary, we consider the case where all reduced conductances in Equation 2 are zero, in other words:

$$
G_{ij}^{\mathrm{red}} = 0, \qquad \forall (i,j) \in \mathcal{I}_{\mathrm{G}} \times \mathcal{I}_{\mathrm{G}}
\tag{31}
$$

Excluding special cases, the condition in Equation 31 only holds when the conductance of all transmission lines in the power system is zero, or equivalently, the resistance of all transmission lines is zero. In this case, for the functions $k_{ij}(\delta_{ij})$ and $h_{ij}(\delta_{ij})$ defined in Equation 3, the following holds:

$$
k_{ij}(\delta_{ij}^\star) = k_{ji}(\delta_{ji}^\star), \qquad h_{ij}(\delta_{ij}^\star) = -h_{ji}(\delta_{ji}^\star), \qquad h_{ii}(\delta_{ii}^\star) = 0
$$

Therefore, for the system matrix (L, A, B, C) in Equation 8, it holds that:

$$
L = L^{\mathsf{T}}, \qquad \hat{A} = \hat{A}^{\mathsf{T}}, \qquad C = -\hat{B}^{\mathsf{T}}
\tag{32}
$$

In the following, we use the symmetric structure of this special system matrix to analyze the passivity of the electrical subsystem.

First, let us express the electrical subsystem G of Equation 22 as follows:

$$
G : \begin{cases} \dot{x}_G = A_G x_G + B_G u_G \\ y_G = C_G x_G \end{cases}
\tag{33}
$$

where the state x_G is a column vector obtained by concatenating δ^{lin} and E^{lin}, and Ω is a positive definite diagonal matrix defined as follows:

$$\Omega := \mathrm{diag}\left(\sqrt{\frac{X_i - X_i'}{\tau_i}}\right)_{i\in\mathcal{I}_G}$$

The system matrices are expressed as:

$$A_G := \begin{bmatrix} 0 & 0 \\ \Omega^2\hat{B} & \Omega^2\hat{A} \end{bmatrix}, \qquad B_G := \begin{bmatrix} I \\ 0 \end{bmatrix}, \qquad C_G := \begin{bmatrix} L & -\hat{B}^\top \end{bmatrix}$$

Furthermore, we define the symmetric matrix P_G as follows:

$$P_G := \begin{bmatrix} L & -\hat{B}^\top \\ -\hat{B} & -\hat{A} \end{bmatrix} \tag{34}$$

The following inequalities hold for these matrices:

$$A_G^\top P_G + P_G A_G \leq 0, \qquad P_G B_G = C_G^\top \tag{35}$$

If we calculated the left of the inequality, it can be expressed as follows using a symmetric matrix $\hat{A}_\Omega := \Omega\hat{A}\Omega$:

$$\frac{A_G^\top P_G + P_G A_G}{2} = \begin{bmatrix} \Omega\hat{B} & 0 \\ 0 & \Omega^{-1} \end{bmatrix}^\top \underbrace{\begin{bmatrix} -I & -\hat{A}_\Omega \\ -\hat{A}_\Omega & -\hat{A}_\Omega^2 \end{bmatrix}}_{Y} \begin{bmatrix} \Omega\hat{B} & 0 \\ 0 & \Omega^{-1} \end{bmatrix}$$

Here, the top-left block $-I$ of Y is negative definite, and the Schur complement of Y with respect to $-I$ is 0, which implies that Y is negative semi-definite. Therefore, the matrix inequality in Equation 35 holds.

COFFEE BREAK

Schur complement:
Let a symmetric matrix M be partitioned as

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^\top & M_{22} \end{bmatrix}$$

Then, the **Schur complement** of M with respect to M_{22} is defined as:

$$M/M_{22} := M_{11} - M_{12}M_{22}^{-1}M_{12}^\top$$

Similarly, the Schur complement of M with respect to M_{11} is defined as:

$$M/M_{11} := M_{22} - M_{12}^\top M_{11}^{-1}M_{12}$$

If the matrix M_{22} is positive definite, then M is positive semidefinite if and only if M/M_{22} is positive semidefinite. The same fact holds for the Schur complement of M with respect to M_{11} [**?**]. The same fact holds if we replace positive semidefinite with positive definite.

Properties of semidefinite matrices:
For any positive semidefinite (or negative semidefinite) matrix $Y \in \mathbb{R}^{n \times n}$ and any matrix $X \in \mathbb{R}^{n \times m}$, the matrix $X^{\mathsf{T}} Y X$ is positive semidefinite (or negative semidefinite). This can be shown from the fact that

$$v^{\mathsf{T}} Y v \geq 0, \quad \forall v \in \mathbb{R}^{n} \qquad \Longrightarrow \qquad (Xw)^{\mathsf{T}} Y (Xw) \geq 0, \quad \forall w \in \mathbb{R}^{m}$$

By using the relationship given by Equation 35, the time derivative of the storage function $W_G(x_G)$ along the solution trajectory of G can be evaluated, where $W_G(x_G)$ is defined by Equation 36, similarly to Equation 30:

$$W_G(x_G) := \frac{1}{2} x_G^{\mathsf{T}} P_G x_G \tag{36}$$

$$\frac{d}{dt} W_G\big(x_G(t)\big) \leq y_G^{\mathsf{T}}(t) u_G(t) \tag{37}$$

However, to show the passivity of G, P_G in equation 34 must be positive semi-definite. If the matrix A in equation 8 is stable, then it can be shown that:

$$A = S^2 \hat{A} \qquad \Longleftrightarrow \qquad S^{-1} A S = S \hat{A} S$$

where $S := \mathsf{diag}\left(\sqrt{X_i - X_i'}\right) i \in \mathcal{I}$G. Here, $\hat{A}$ is negative definite. Under this condition, the necessary and sufficient condition for P_G in Equation 34 to be positive semi-definite is that the Schur complement of $-\hat{A}$ is positive semi-definite, in other words:

$$L_0 = L_0^{\mathsf{T}} \geq 0 \tag{38}$$

where L_0 is defined by Equation **??** and can be expressed as $L_0 = L + \hat{B}^{\mathsf{T}} \hat{A}^{-1} \hat{B}$ using Equation 32. To summarize the above discussion, the following definition is introduced.

Definition 1.4 (Passive power transmission condition) For the system matrix (L, A, B, C) of Equation 8, the following three conditions are together called **passive power transmission conditions**. [2]

(i) Matrix A is stable.
(ii) As in Equation 31, all reduced conductances are zero.
(iii) For the matrix L_0 of Equation **??**, the matrix inequality of Equation 38 holds.

Each of these conditions may be referred to individually as the passive transmission condition (i), and so on.

[2] "Passive power transmission conditions" is a term unique to this book.

Based on the above discussions, we can see that the passive power transmission conditions describe the conditions necessary for the electrical system G of Equation 22 to be passive. Furthermore, these conditions are necessary for the linear approximation model to be statically stable for the passivity of an electrical subsystem and arbitrary physical constant. The details are discussed in Section 3.4 and Section 3.5. Function $W_G(x_G)$ indicates the electrical potential energy of an electrical power system.

3.3 Analysis of small signal stability based on passivity

3.3.1 Stability analysis of feedback systems

In the following, under the passive power transmission conditions defined in Definition 1.4, the small signal stability of the linear approximation model given in Equation 9 is analyzed for electric subsystems that are passive. The stability of their feedback systems is also analyzed.

Since the inequalities in Equations 30 and 37 hold, their sum is given by:

$$\begin{aligned}&\frac{d}{dt}\left\{W_F\big(x_F(t)\big) + W_G\big(x_G(t)\big)\right\}\\ &\qquad \le \underbrace{y_F^{\mathsf{T}}(t)u_F(t) + y_G^{\mathsf{T}}(t)u_G(t)}_{\star} - \tfrac{\min\{D_i\}}{\omega_0}\|y_F(t)\|^2\end{aligned}$$

By substituting the feedback coupling equation in Equation 23 into this inequality, the term indicated by "$\star$" is cancelled out, and the inequality for the entire feedback system can be expressed as:

$$\frac{d}{dt}\left\{W_F\big(x_F(t)\big) + W_G\big(x_G(t)\big)\right\} \le -\tfrac{\min\{D_i\}}{\omega_0}\|y_F(t)\|^2 \tag{39}$$

In other words, the sum of the functions $W_F(x_F)$ and $W_G(x_G)$ is monotonically non-increasing with respect to the time evolution along the feedback system trajectory. Furthermore, since the lower bounds of $W_F(x_F)$ and $W_G(x_G)$ are both 0, their sum asymptotically converges to a certain value as time passes sufficiently. This means that the value of the time derivative on the left-hand side of Equation 39 converges to 0. Additionally, since the right-hand side of Equation 39 is negative when $y_F(t) \neq 0$ and is only 0 when $y_F(t) = 0$, the following is obtained:

$$\lim_{t\to\infty} y_F(t) = 0 \tag{40}$$

Furthermore, focusing on the output equation of Equation 21, since the output y_F is a constant multiple of the internal state $\Delta\omega^{\mathrm{lin}}$, it can be understood that for the mechanical subsystem F:

$$y_F(t) = 0, \quad \forall t \geq 0 \qquad \Longrightarrow \qquad \Delta\omega^{\mathrm{lin}}(t) = 0, \quad \forall t \geq 0 \tag{41}$$

This is a property called **observability** in system control engineering. Therefore, from Equations 40 and 41, for the approximated linear model of Equation 9, for any initial value $(\Delta\omega^{\mathrm{lin}}(0), \delta^{\mathrm{lin}}(0), E^{\mathrm{lin}}(0))$, the following holds:

$$\lim_{t\to\infty} \Delta\omega^{\mathrm{lin}}(t) = 0 \tag{42}$$

In other words, the internal state of the mechanical subsystem F in Equation 21 in the feedback system converges to 0 asymptotically.

COFFEE BREAK

Observability: For a linear system Σ described by Equation 24, if the output $y(t)$ is identically zero, then the internal state $x(t)$ is also identically zero, and Σ is said to be **observable** in this case. The necessary and sufficient condition for Σ to be observable is given by Equation 43, where n is the dimension of the state.

$$\mathsf{ker} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = \{0\} \tag{43}$$

In some contexts, a pair of matrices that satisfies Equation 43 is referred to as an observable pair (C, A).

Controllability: For a linear system Σ described by Equation 24, if there exists an input $u(t)$ such that for all initial states $x(0)$, there exists a time $T > 0$ such that $x(T) = 0$, then Σ is said to be **controllable** . The necessary and sufficient condition for Σ to be controllable is given by Equation 44, where n is the dimension of the state.

$$\mathsf{im} \left[\, B \; AB \; \cdots \; A^{n-1}B \,\right] = \mathbb{R}^n \tag{44}$$

Here, im denotes the **image** of the matrix. In some contexts, a pair of matrices that satisfies Equation 44 is referred to as a controllable pair (A, B)

Lyapunov function: Consider the observable linear system Σ described by Equation 24, where the input $u(t)$ is identically zero. Let $V(x)$ be a positive semi-definite function that satisfies $V(x) \geq 0$ for all x and $V(0) = 0$. If there exists a positive constant ρ such that the derivative of $V(x)$ along the solution trajectory of Σ satisfies

$$\frac{d}{dt}V\big(x(t)\big) = \nabla V^{\mathsf{T}}(x)\frac{dx}{dt}(t) \leq -\rho\|y(t)\|^2, \qquad \forall t \geq 0$$

then the solution trajectory $x(t)$ converges asymptotically to zero for any initial state. Such a function $V(x)$ is called a **Lyapunov function** .
The fact that the value of the Lyapunov function decreases monotonically along the solution trajectory of the system can be interpreted as a type of energy dissipation over time (Figure 5). Similar stability analyses based on Lyapunov functions can also be applied to nonlinear systems.

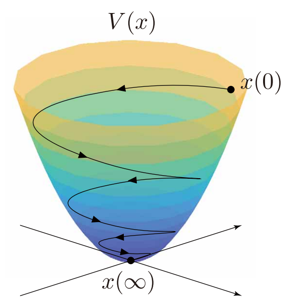


Fig. 5 Monotonically decreasing values along the solution trajectory of the Lyapunov function

On the one hand, it is not possible to deduce from the above discussion whether the internal state of the electric subsystem G in Equation 22 converges asymptotically to 0. Specifically, from the relation in Equation 23 and the asymptotic convergence in Equation 42, it can be derived that the input and output of the two subsystems satisfy:

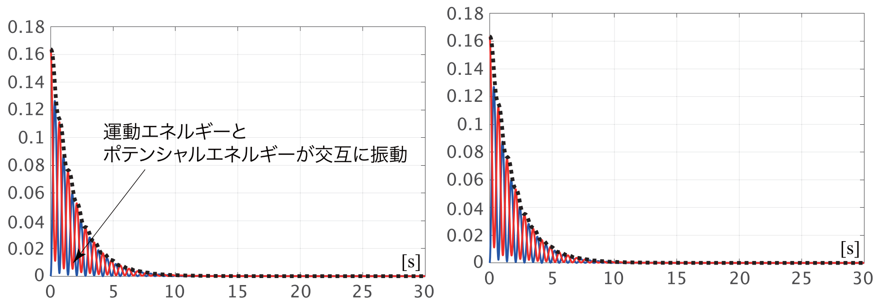
$$\lim_{t\to\infty} u_F(t) = 0, \qquad \lim_{t\to\infty} u_G(t) = 0, \qquad \lim_{t\to\infty} y_G(t) = 0$$

However, since the electric subsystem is not observable, it cannot be concluded that its internal state converges asymptotically to zero. Assuming that the electric subsystem is observable, it can be concluded that for any initial values:

$$\lim_{t\to\infty} \delta^{\text{lin}}(t) = 0, \qquad \lim_{t\to\infty} E^{\text{lin}}(t) = 0$$

However, this implies that c_0 must always be equal to 0 in Equation 11. This fact contradicts the instability of Equation 9 due to the zero eigenvalue of Ψ. It should be noted that except for some special cases, the electric subsystem is controllable.

Example 1.3 Time evolution of stored energy Consider the approximated linear model discussed in the first half of Example 1.2. First, consider the case where the passive transmission condition (ii) is satisfied, i.e., when the conductance of both transmission lines is 0. Specifically, set the admittance values of the transmission lines to:



(a) If passive transmission condition (ii) is met

(b) If passive transmission condition (ii) is not met

Fig. 6 Time variation of the accumulation function according to Example 1.2
(Blue: W_F, Red: W_G, Black:$W_F + W_G$)

$$\boldsymbol{y}_{12} = -\boldsymbol{j}11.6041, \qquad \boldsymbol{y}_{23} = -\boldsymbol{j}10.5107 \tag{45}$$

This corresponds to setting the parameter θ_2 to 0 in Equation 17. In this case, the matrix A becomes stable. Also, all the eigenvalues of L_0 in Equation **??** become non-negative. Thus, the passive transmission conditions (i) and (iii) hold.

Consider the time response of the initial values in Equation 14 and calculate the time variation of the kinetic energy $W_F(x_F)$ in Equation 29 and the potential energy $W_G(x_G)$ in Equation 36. The calculation results are shown in Figure 6(a) where the blue and red solid lines represent $W_F(x_F)$ and $W_G(x_G)$, respectively, and the black dashed line represents their sum, which is the total energy of the system. From this figure, it can be seen that while the kinetic and potential energies increase and decrease alternatively, the total energy of the system, which is the sum of these energies, decreases monotonically. The decrease in total energy over time can be interpreted as energy loss due to friction caused by the damping coefficient.

Next, as a reference, let us show the results when the passive power transmission condition (ii) is not satisfied. Specifically, we set $\boldsymbol{Y}_0(1)$ in Equation 17 to be the admittance matrix $\boldsymbol{Y}$ by setting θ_2 to 1. This is equivalent to calculating the time variation of the kinetic and potential energies for the initial value response in Figure 2. Note that when the passive power transmission condition (ii) is not satisfied, P_G in Equation 34 does not become a symmetric matrix, but the potential energy $W_G(x_G)$ can still be calculated using the definition in Equation 36. The calculation result in Figure 6(b) is almost identical to that in Figure 6(a). This fact suggests that even when the conductance of the transmission line is not zero, the electrical potential energy can be approximately calculated based on the definition in Equation 36.

3.3.2 Basis transformation for separating unobservable state variables

Let us consider deriving an observable subsystem from the electrical subsystem G in Equation 22 by removing the common component of unobservable rotor angles. Specifically, we apply a basis transformation to the state δ^{lin} in Equation 7 to derive a set of differential equations describing only the deviations of the rotor angles. Note that the following basis transformation is always applicable regardless of whether the passive power delivery condition holds or not.

COFFEE BREAK

Basis transformation of linear systems:
In the state equation:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

each element $x_i(t)$ of the n-dimensional state vector $x(t)$ can be represented as the component of the expansion with respect to the basis $e_1, \ldots, e_n$, such as:

$$x(t) = e_1 x_1(t) + \cdots + e_n x_n(t)$$

where e_i is an n-dimensional vector with only the i-th element being 1. This basis is called the **standard basis** and represents the time evolution of the "components" of the state vector $x(t)$ in a certain basis. We now consider representing $x(t)$ in another basis $v_1, \ldots, v_n$ such that

$$x(t) = v_1 \xi_1(t) + \cdots + v_n \xi_n(t)$$

where $\xi_i(t)$ is the component of the basis vector v_i. Let us denote the matrix obtained by arranging the vectors v_i horizontally as V and the vector obtained by arranging $\xi_i(t)$ vertically as $\xi(t)$. Then, the linear transformation $x(t) = V\xi(t)$ corresponds to this representation. In this case, the state equation is transformed as:

$$\dot{\xi}(t) = V^{-1}AV\xi(t) + V^{-1}Bu(t)$$

where V_a and V_b are the matrices obtained by arranging the basis vectors of $\mathcal{V}_a$ and $\mathcal{V}_b$ horizontally, respectively, the transformed state equation is obtained as:

$$x(t) = V_a \xi_a(t) + V_b \xi_b(t)$$

where W_a and W_b are matrices that satisfy:

$$\begin{bmatrix} \dot{\xi}_a(t) \\ \dot{\xi}_b(t) \end{bmatrix} = \begin{bmatrix} W_a A V_a & W_a A V_b \\ W_b A V_a & W_b A W_b \end{bmatrix} \begin{bmatrix} \xi_a(t) \\ \xi_b(t) \end{bmatrix}$$

However, V_a and V_b are matrices consisting of the basis vectors of $\mathcal{V}a$ and $\mathcal{V}b$, respectively. Wa and Wb are matrices that satisfy:

$$\begin{bmatrix} W_a \\ W_b \end{bmatrix} = \begin{bmatrix} V_a \ V_b \end{bmatrix}^{-1} \qquad \Longleftrightarrow \qquad \begin{bmatrix} V_a \ V_b \end{bmatrix} \begin{bmatrix} W_a \\ W_b \end{bmatrix} = I$$

In this representation, $\xi_a(t)$ represents the component of $x(t)$ with respect to the subspace $\mathsf{span}\,\mathcal{V}_a$. Similarly, $\xi_b(t)$ represents the component of $x(t)$ with respect to the subspace $\mathsf{span}\,\mathcal{V}_b$.

The change of basis explained below can be applied regardless of whether passive power transmission conditions hold. δ^{lin} is expanded using a matrix $W \in \mathbb{R}^{N\times(N-1)}$:

$$\delta^{\text{lin}} = W\delta_{\text{e}}^{\text{lin}} + \mathbb{1}\overline{\delta}_{\text{e}}^{\text{lin}} \tag{46}$$

Here, $\mathbb{1}$ is a base vector that expresses the common component of δ^{lin}, while W is a matrix with base vectors that express other deviation components. In other words, the common component of δ^{lin} is $\overline{\delta}_{\text{e}}^{\text{lin}}$, and deviation components are $\delta_{\text{e}}^{\text{lin}}$. The common component $\overline{\delta}_{\text{e}}^{\text{lin}}$ is one-dimensional, while deviation components $\delta_{\text{e}}^{\text{lin}}$ are $(N-1)$-dimensional.

Next, let us consider the inverse transformation of Equation 46. Specifically, let us consider a matrix $W^{\dagger} \in \mathbb{R}^{(N-1)\times N}$:

$$\delta^{\text{lin}} = \underbrace{\begin{bmatrix} W \ \mathbb{1} \end{bmatrix}}_{T} \begin{bmatrix} \delta_{\text{e}}^{\text{lin}} \\ \overline{\delta}_{\text{e}}^{\text{lin}} \end{bmatrix} \quad \Longleftrightarrow \quad \begin{bmatrix} \delta_{\text{e}}^{\text{lin}} \\ \overline{\delta}_{\text{e}}^{\text{lin}} \end{bmatrix} = \underbrace{\begin{bmatrix} W^{\dagger} \\ \frac{1}{N}\mathbb{1}^{\mathsf{T}} \end{bmatrix}}_{T^{-1}} \delta^{\text{lin}}$$

For this inverse transformation to exist, the column vector of W must be orthogonal to $\mathbb{1}$. This can be confirmed as follows. From the relationship of inverse transformation, the following must hold:

$$T^{-1}T = \begin{bmatrix} W^{\dagger}W & W^{\dagger}\mathbb{1} \\ \frac{1}{N}\mathbb{1}^{\mathsf{T}}W & \frac{1}{N}\mathbb{1}^{\mathsf{T}}\mathbb{1} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 1 \end{bmatrix}$$

In other words, W and $W^{\dagger}$ are matrices that satisfy:

$$\mathbb{1}^{\mathsf{T}}W = 0, \qquad W^{\dagger}W = I, \qquad W^{\dagger}\mathbb{1} = 0$$

Therefore, from the first equation, we can see that the column vectors of W must be orthogonal to $\mathbb{1}$. Note that W and $W^{\dagger}$ can be constructed using an appropriate matrix $U \in \mathbb{R}^{N\times(N-1)}$ that satisfies $\mathbb{1}^{\mathsf{T}}U = 0$ and $U^{\mathsf{T}}U$ is invertible, as follows:

$$W = U(U^{\mathsf{T}}U)^{-1}, \qquad W^{\dagger} = U^{\mathsf{T}}$$

In this case, the product $WW^{\dagger}$ can be expressed as the **orthogonal projection matrix** onto the orthogonal complement of $\mathsf{span}\,\mathbb{1}$:

$$WW^{\dagger} = I - \frac{1}{N}\mathbb{1}\mathbb{1}^{\mathsf{T}} \tag{47}$$

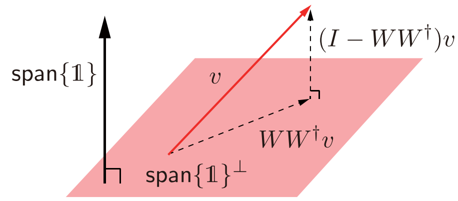


Fig. 7 Conceptual diagram of orthogonal projection

The pseudoinverse of W obtained in this way is called the **Moore-Penrose pseudoinverse** [**?**].

In Figure 7, the subspace $\mathsf{span}\,\mathbb{1}$ is shown by a black arrow, and the orthogonal complement space $\mathsf{span}\,\mathbb{1}^\perp$ is shown as a plane perpendicular to it. When a vector v is multiplied by the orthogonal projection matrix $WW^\dagger$, the projection of v onto $\mathsf{span}\,\mathbb{1}^\perp$, which is the shadow cast by v in the direction perpendicular to $\mathsf{span}\,\mathbb{1}$, is obtained as $WW^\dagger v$. Furthermore, the complementary relationship, shown below, indicates that it is the orthogonal projection matrix onto $\mathsf{span}\,\mathbb{1}$'s orthogonal complement, that is, $\mathsf{span}\,\mathbb{1}$.

$$I - WW^\dagger = \tfrac{1}{N}\mathbb{1}\mathbb{1}^\top$$

We apply the above-mentioned change of basis to the electrical subsystem G of Equation 22. First, if we substitute Equation 46 into a differential equation related to δ^{lin}, the following is obtained:

$$W\dot{\delta}_{\mathrm{e}}^{\mathrm{lin}} + \mathbb{1}\dot{\bar{\delta}}_{\mathrm{e}}^{\mathrm{lin}} = u_G$$

By multiplying the left-hand side of this differential equation by $W^\dagger$ or $\frac{1}{N}\mathbb{1}^\top$, we obtain:

$$\dot{\delta}_{\mathrm{e}}^{\mathrm{lin}} = W^\dagger u_G, \qquad \dot{\bar{\delta}}_{\mathrm{e}}^{\mathrm{lin}} = \frac{1}{N}\mathbb{1}^\top u_G$$

Next, noting that the relationship in Equation 10 holds for matrices L and B, the differential equation and output equation with respect to E^{lin} can be rewritten as:

$$\tau \dot{E}^{\mathrm{lin}} = AE^{\mathrm{lin}} + BW\delta_{\mathrm{e}}^{\mathrm{lin}}, \qquad y_G = CE^{\mathrm{lin}} + LW\delta_{\mathrm{e}}^{\mathrm{lin}}$$

Therefore, the transformed electric subsystem is given by:

$$G : \begin{cases} \dot{\bar{\delta}}_{\mathrm{e}}^{\mathrm{lin}} = \frac{1}{N}\mathbb{1}^\top u_G \\ \dot{\delta}_{\mathrm{e}}^{\mathrm{lin}} = W^\dagger u_G \\ \tau \dot{E}^{\mathrm{lin}} = AE^{\mathrm{lin}} + BW\delta_{\mathrm{e}}^{\mathrm{lin}} \\ y_G = CE^{\mathrm{lin}} + LW\delta_{\mathrm{e}}^{\mathrm{lin}} \end{cases} \tag{48}$$

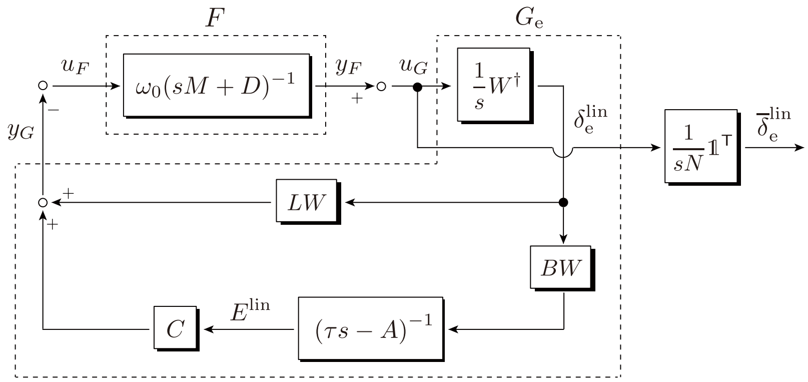


Fig. 8 Basis-transformed approximate linear model

One notable point about this system representation is that the common component of δ^lin, represented by $\overline{\delta}_\mathrm{e}^\mathrm{lin}$, is affected by the input u_G but has no effect on the output y_G. In other words, $\overline{\delta}_\mathrm{e}^\mathrm{lin}$ is an unobservable state variable.

By removing the differential equation of $\overline{\delta}_\mathrm{e}^\mathrm{lin}$ from Equation 48, $(N-1)$-dimensional controllable and observable subsystem is obtained as:

$$G_\mathrm{e} : \begin{cases} \dot{\delta}_\mathrm{e}^\mathrm{lin} = W^\dagger u_G \\ \tau \dot{E}^\mathrm{lin} = AE^\mathrm{lin} + BW\delta_\mathrm{e}^\mathrm{lin} \\ y_G = CE^\mathrm{lin} + LW\delta_\mathrm{e}^\mathrm{lin} \end{cases} \tag{49}$$

Here, please note that from observability of G_e, the following holds.

$$y_G(t) = 0, \quad \forall t \geq 0 \qquad \Longrightarrow \qquad \begin{bmatrix} \delta_\mathrm{e}^\mathrm{lin}(t) \\ E^\mathrm{lin}(t) \end{bmatrix} = 0, \quad \forall t \geq 0 \tag{50}$$

The fact mentioned above is important for the analysis of the steady-state stability of the approximate linear model in equation 9. As a reference, the block diagram of the approximate linear model transformed by the change of basis is shown in Figure 8.

It should be noted that it is a necessary and sufficient condition for G_e to be controllable and observable that the pair $(\tau^{-1}A, \tau^{-1}B)$ is controllable and the pair $(C, \tau^{-1}A)$ is observable. In the following, we assume controllability and observability. Note that an exact proof is not always easy, but assuming that the rank of B and C is $(N-1)$ or higher in most situations, there are no practical obstacles to analysis.

3.3.3 Small signal stability analysis based on passivity

In the following, assuming the passive power transfer condition defined in Definition 1.4, we show the passivity of G_{e} in Equation 49 using the same procedure as that of the electrical subsystem G in Equation 22. To this end, we express G_{e} in the form of:

$$G_{\mathrm{e}}: \begin{cases} \dot{x}_{G_{\mathrm{e}}} = A_{G_{\mathrm{e}}} x_{G_{\mathrm{e}}} + B_{G_{\mathrm{e}}} u_G \\ y_G = C_{G_{\mathrm{e}}} x_{G_{\mathrm{e}}} \end{cases} \tag{51}$$

where $x_{G_{\mathrm{e}}}$ is a vector composed of $\delta_{\mathrm{e}}^{\mathrm{lin}}$ and E^{lin}, and:

$$A_{G_{\mathrm{e}}} := \begin{bmatrix} 0 & 0 \\ \Omega^2 \hat{B} W & \Omega^2 \hat{A} \end{bmatrix}, \quad B_{G_{\mathrm{e}}} := \begin{bmatrix} W^{\dagger} \\ 0 \end{bmatrix}, \quad C_{G_{\mathrm{e}}} := \begin{bmatrix} LW & -\hat{B}^{\mathsf{T}} \end{bmatrix}$$

Additionally, we define the positive semidefinite matrix $P_{G_{\mathrm{e}}}$ as:

$$P_{G_{\mathrm{e}}} := \begin{bmatrix} W & 0 \\ 0 & I \end{bmatrix}^{\mathsf{T}} \underbrace{\begin{bmatrix} L & -\hat{B}^{\mathsf{T}} \\ -\hat{B} & -\hat{A} \end{bmatrix}}_{P_G} \begin{bmatrix} W & 0 \\ 0 & I \end{bmatrix} \tag{52}$$

Note that if P_G in Equation 34 is positive semi-definite, then $P_{G_{\mathrm{e}}}$ is also positive semi-definite. In this case, by noting that $\hat{B}WW^{\dagger} = \hat{B}$ and $LWW^{\dagger} = L$ from the relationship in Equation 47, we can see that the following equation holds:

$$A_{G_{\mathrm{e}}}^{\mathsf{T}} P_{G_{\mathrm{e}}} + P_{G_{\mathrm{e}}} A_{G_{\mathrm{e}}} \leq 0, \qquad P_{G_{\mathrm{e}}} B_{G_{\mathrm{e}}} = C_{G_{\mathrm{e}}}^{\mathsf{T}} \tag{53}$$

Note that the left matrix inequality is shown from the same equation as 35:

$$\frac{A_{G_{\mathrm{e}}}^{\mathsf{T}} P_{G_{\mathrm{e}}} + P_{G_{\mathrm{e}}} A_{G_{\mathrm{e}}}}{2} = \begin{bmatrix} \Omega \hat{B} W & 0 \\ 0 & \Omega^{-1} \end{bmatrix}^{\mathsf{T}} \underbrace{\begin{bmatrix} -I & -\hat{A}_{\Omega} \\ -\hat{A}_{\Omega} & -\hat{A}_{\Omega}^2 \end{bmatrix}}_{Y} \begin{bmatrix} \Omega \hat{B} W & 0 \\ 0 & \Omega^{-1} \end{bmatrix}$$

Therefore, the time derivative along the solution trajectory of G_{e} of the storage function:

$$W_{G_{\mathrm{e}}}(x_{G_{\mathrm{e}}}) := \frac{1}{2} x_{G_{\mathrm{e}}}^{\mathsf{T}} P_{G_{\mathrm{e}}} x_{G_{\mathrm{e}}}$$

can be evaluated as:

$$\frac{d}{dt} W_{G_{\mathrm{e}}}\big(x_{G_{\mathrm{e}}}(t)\big) \leq y_G^{\mathsf{T}}(t) u_G(t) \tag{54}$$

Thus, G_{e} in Equation 49 is passive. Note that this inequality is equivalent to the inequality in Equation 37, and the values of the two storage functions satisfy:

$$W_G\big(x_G(t)\big) = W_{G_{\mathrm{e}}}\big(x_{G_{\mathrm{e}}}(t)\big), \qquad \forall t \geq 0$$

By considering the observability of G_e shown by Equation 50, the following is true for the arbitrary initial value of the solution trajectory of the linear approximation model of Equation 9:

$$\lim_{t\to\infty} \Delta\omega^{\mathrm{lin}}(t) = 0, \qquad \lim_{t\to\infty} \begin{bmatrix} \delta_{\mathrm{e}}^{\mathrm{lin}}(t) \\ E^{\mathrm{lin}}(t) \end{bmatrix} = 0 \tag{55}$$

Therefore, from the relationship of the change of basis of Equation 46, we can see that Equation 11 holds for the arbitrary initial value. In other words, the linear approximation model of Equation 9 is statically stable. Also:

$$c_0 = \lim_{t\to\infty} \overline{\delta}_{\mathrm{e}}^{\mathrm{lin}}(t)$$

and state variables $\overline{\delta}_{\mathrm{e}}^{\mathrm{lin}}$ follow the differential equation of Equation 48.

We summarize the previous discussion in the following theorem.

Theorem 1.1 (Small signal stability of the linear approximation model based on passivity) *For any steady-state value* $(\delta^\star, E^\star)$ *that satisfies the passive power transfer condition defined in Definition 1.4, the electrical subsystem G given in equation 22 is passive. Additionally, for any positive constants* $(M_i, D_i, \tau_i) i \in \mathcal{I}\mathrm{G}$, *the approximate linear model given in equation 9 is steady-state stable.*

As discussed in Theorem 1.1, under the passive power transmission conditions, the linear approximation model is statically stable for combinations of all physical constants $(M_i, D_i, \tau_i)_{i\in\mathcal{I}_{\mathrm{G}}}$. Analysis based on passivity allows stability independent of model parameters.

3.4 Necessary conditions for the approxiamte linear model to be passive

3.4.1 Passivity and positive realness

It is known that the passivity of a linear system is mathematically equivalent to the property called positive realness of its transfer function. In this section, we consider the necessity of the passive power transmission condition defined in Definition 1.4 from the viewpoint of the passivity of the electrical subsystem based on this equivalence.

COFFEE BREAK

Transfer function: For a linear system:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

its **transfer function** is defined as:

$$Q(s) := C(sI - A)^{-1}B + D$$

When the Laplace transform of the input $u(t)$ is $U(s)$ and the Laplace transform of the output $y(t)$ is $Y(s)$, the transfer function relates to the system as $Y(s) = Q(s)U(s)$. The input-output behavior of a linear system is characterized by its transfer function.

The transfer function from the input u_G to the output y_G of the electrical subsystem G in Equation 22 is given by:

$$G(s) := -\frac{1}{s}\underbrace{\left\{-C\left(\tau s - A\right)^{-1}B - L\right\}}_{H(s)} \tag{56}$$

Note that since unobservable state variables are not relevant to the input-output characteristics, the transfer function of G_e in Equation 49 is also equal to $G(s)$. Hereafter, we consider the case where the transfer function $H(s)$ in Equation 56 is stable. The stability of the transfer function is defined as follows.

Definition 1.5 (Stability of a transfer function) When the real part of all poles of the transfer function $Q(s)$ is negative, $Q(s)$ is called **stable**.

The poles of a transfer function are the zeros of the denominator polynomial. It is known that $H(s)$ in equation 56 being stable is equivalent to all the real parts of the eigenvalues of the matrix $\tau^{-1}A$ being negative.

Furthermore, the **Positive realness** of a transfer function is defined as follows.

Definition 1.6 (Positive realness of a transfer function) For a square transfer function $Q(s)$, the following is defined:

$$\Omega_0 := \{\omega_0 \in \mathbb{R} : \text{ The pure imaginary number } \boldsymbol{j}\omega_0 \text{ is a pole of } Q(s)\} \tag{57}$$

$Q(s)$ is called **positive real** if the following three conditions are satisfied.

- The real part of all poles of $Q(s)$ is nonpositive.
- For all $\omega \in [0, \infty) \setminus \Omega_0$, $Q(\boldsymbol{j}\omega) + Q^{\mathsf{T}}(-\boldsymbol{j}\omega)$ is positive semi-definite.
- When there are poles of a pure imaginary number, their multiplicity is 1, and the following is true for the remaining number:

$$\lim_{s\to\boldsymbol{j}\omega_0}(s - \boldsymbol{j}\omega_0)Q(s) = \lim_{s\to\boldsymbol{j}\omega_0}\{(s - \boldsymbol{j}\omega_0)Q(s)\}^* \geq 0, \qquad \forall\omega_0 \in \Omega_0$$

In Definition 1.6, the two most important conditions are the first and second ones. The first condition expresses the stability of the transfer function. However, it also includes the case where the real part of the pole is 0. The second condition concerns the positive definiteness of the complex symmetric part of the transfer function evaluated on the imaginary axis. In particular, if $Q(s)$ is a scalar, that is, both the input and output are scalars, the second condition expresses that the real part of $Q(\boldsymbol{j}\omega)$ is non-negative for all $\omega \in [0, \infty) \setminus \Omega_0$. However, it should be noted that for matrix-valued $Q(s)$, in general,

$$Q(\boldsymbol{j}\omega) + Q^{\mathsf{T}}(-\boldsymbol{j}\omega) \neq 2\,\mathsf{Re}\,[Q(\boldsymbol{j}\omega)]$$

Also, for $Q(s)$ with real coefficients, $Q^{\mathsf{T}}(-\boldsymbol{j}\omega)$ is equal to $Q(\boldsymbol{j}\omega)^*$. The third condition is exceptional in the case when $Q(s)$ has pure imaginary poles. However, it is necessary to analyze transfer functions with poles at the origin, such as $G(s)$ in Equation 56.

COFFEE BREAK

Complex Symmetric and Skew Hermitian Parts of a Square Matrix: Any square matrix M can be decomposed as:

$$M = \frac{M + M^*}{2} + \frac{M - M^*}{2}$$

where $\frac{M+M^*}{2}$ is called the **complex symmetric part** (Hermitian part) of M, and $\frac{M-M^*}{2}$ is called the **complex skew-symmetric part** (skew Hermitian part) of M.

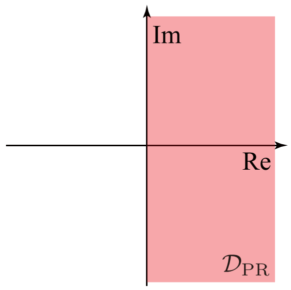
In system control engineering, it is known that passivity in Definition 1.3 and positive realness in Definition 1.6 are equivalent. If we apply Lemma 1.2 at the end of this chapter to the discussion in this section, the necessary and sufficient condition for $G(s)$ in Equation 56 to be positive real is the existence of a positive definite matrix P_{G_e} that satisfies Equation 55 for the controllable and observable state space realization G_e in Equation 51. This is equivalent to the passivity of G_e defined by the inequality in Equation 54. Furthermore, the positive realness of P_{G_e} in Equation 52 is shown by the fact that both the Schur complements with respect to $-\hat{A}$ and $-\hat{A}$ of the matrix $W^{\mathsf{T}}\left(L + \hat{B}^{\mathsf{T}}\hat{A}^{-1}\hat{B}\right)W$ are positive definite, where $W^{\mathsf{T}}L_0W$ satisfies Equation 20 and the column vectors of W in Equation 46 are orthogonal to $\mathbb{1}$.

3.4.2 Necessary condition for the transfer function of the electrical subsystem to be positive-real

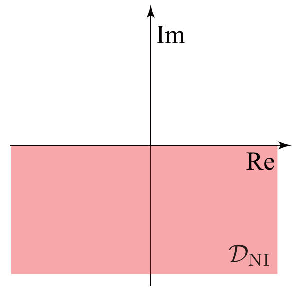
As a mathematical preparation for deriving necessary conditions, we introduce the concept of **negative imaginaryness** of transfer functions, which is similar to positive-realness [**?**, **?**].

Definition 1.7 (Negative imaginariness of a transfer function) For a square transfer function $Q(s)$ without a pole at the origin, we define Ω_0 of Equation 57. When the following three conditions are satisfied, $Q(s)$ is called **negative imaginary**.

- The real part of all poles of $Q(s)$ is nonpositive.
- For all $\omega \in (0, \infty) \setminus \Omega_0$, $\boldsymbol{j}\left\{Q(\boldsymbol{j}\omega) - Q^{\mathsf{T}}(-\boldsymbol{j}\omega)\right\}$ is positive semi-definite.



(a) $\mathcal{D}_{\mathrm{PR}}$



(b) $\mathcal{D}_{\mathrm{NI}}$

Fig. 9 Positive realms and negative imaginary realms

- When there is a pole of a pure imaginary number, their multiplicity is 1, and the following holds for the remaining numbers:

$$\lim_{s \to \boldsymbol{j}\omega_0} (s - \boldsymbol{j}\omega_0)\boldsymbol{j}Q(s) = \lim_{s \to \boldsymbol{j}\omega_0} \{(s - \boldsymbol{j}\omega_0)\boldsymbol{j}Q(s)\}^* \geq 0, \quad \forall \omega_0 \in \Omega_0$$

While Definition 1.6 defines positive-realness based on the positive-definiteness of the complex symmetric part of a transfer function, Definition 1.7 defines negative-imaginaryness based on the positive-semidefiniteness of the complex anti-symmetric part of the transfer function multiplied by the imaginary unit. Note that since the eigenvalues of a complex anti-symmetric matrix are purely imaginary, their product with the imaginary unit is real. In particular, when $Q(s)$ is a scalar transfer function, the second condition implies that the imaginary part of $Q(\boldsymbol{j}\omega)$ is non-positive for all $\omega \in (0, \infty) \setminus \Omega_0$. Furthermore, in the following discussion, we focus only on the second condition to consider the negative-imaginaryness of stable transfer functions. Similar to positive-realness, negative-imaginaryness can also be characterized as the solvability of matrix inequalities, as described in Lemma 1.3 at the end of this chapter.

COFFEE BREAK

Nyquist plot: The plot of the frequency response function $Q(\boldsymbol{j}\omega)$ for $\omega \in \mathbb{R}$ on the complex plane is called the **Nyquist plot**. It is often used in geometric analysis of stability for feedback systems. The analysis method is called the **Nyquist stability criterion**. Note that when $Q(s)$ is a scalar and the coefficients of the numerator and denominator polynomials are real, the plot of $Q(\boldsymbol{j}\omega)$ for negative ω is symmetrical to the plot for positive ω with respect to the real axis.

The relationship of positive realness and negative imaginariness is explained with Figure 9. If the transfer function $Q(s)$ is scalar, $Q(s)$ being positive real can be understood as the trajectory related to non-negative ω of the frequency response function $Q(\boldsymbol{j}\omega)$ is included in the range $\mathcal{D}_{\mathrm{PR}}$ shown in 9(a).

$$\mathcal{D}_{\mathrm{PR}} = \boldsymbol{j}\mathcal{D}_{\mathrm{NI}}$$

and $-\frac{1}{\boldsymbol{j}} = \boldsymbol{j}$ for $G(s)$ and $H(s)$ of Equation 56, the following is derived:

$$G(\boldsymbol{j}\omega) \in \mathcal{D}_{\mathrm{PR}}, \quad \forall\omega > 0 \qquad \Longleftrightarrow \qquad H(\boldsymbol{j}\omega) \in \mathcal{D}_{\mathrm{NI}}, \quad \forall\omega > 0$$

Therefore, a negative imaginariness analysis of $H(s)$ is equivalent to a positive realness analysis of $G(s)$. To be accurate, $G(\boldsymbol{j}\omega)$ and $H(\boldsymbol{j}\omega)$ are complex matrices; thus, $\mathcal{D}_{\mathrm{PR}}$ and $\mathcal{D}_{\mathrm{NI}}$ should be redefined with a set of positive semi-definite matrices.

Therefore, a negative imaginariness analysis of $H(s)$ is equivalent to a positive realness analysis of $G(s)$. Based on this fact, the passive power transmission conditions (ii) and (iii) are necessary conditions for $G(s)$ to be positive real.

Theorem 1.2 *Positive realness of electrical subsystem transfer function For any $(\delta^\star, E^\star)$ where the transfer function $H(s)$ given by Equation 56 is stable, a necessary and sufficient condition for $H(s)$ to be negative imaginary is that the passive power transmission condition (ii) in Definition 1.4 holds. Furthermore, a necessary and sufficient condition for the transfer function $G(s)$ given by Equation 56 to be positive real is that both passive power transmission conditions (ii) and (iii) in Definition 1.4 hold.*

Proof First, we show that if $H(s)$ is negative imaginary for any $(\delta^\star, E^\star)$ where $H(s)$ is stable, then the passive power transmission condition (ii), which is expressed in equation 31, holds true. Now, since:

$$\lim_{\omega\to\infty} \boldsymbol{j}\left\{H(\boldsymbol{j}\omega) - H^{\mathsf{T}}(-\boldsymbol{j}\omega)\right\} = \boldsymbol{j}\left(-L + L^{\mathsf{T}}\right) \geq 0$$

L must be symmetric for $H(s)$ to be negative imaginary. Therefore, we obtain $k_{ij}(\delta^\star_{ij}) = k_{ji}(\delta^\star_{ji})$.

Therefore, L must be symmetric for $H(s)$ to be negative vacuity. Thus, we have $K_{IJ}(\delta^\star_{IJ}) = K_{JI}(\delta^\star_{JI})$.

In other words:

$$G^{\mathrm{red}}_{ij} \sin\delta^\star_{ij} = 0, \qquad \forall(i,j) \in \mathcal{I}_{\mathrm{G}} \times \mathcal{I}_{\mathrm{G}}$$

This implies $\delta^\star_i \neq \delta^\star_j$ for (i,j) where $G^{\mathrm{red}}_{ij} = 0$. Also, even in the case where $\delta^\star_i = \delta^\star_j$, there exists a sufficiently small $\gamma > 0$ such that $\tau^{-1}A$ is stable for $\delta^\star + \gamma e_i$ due to the continuity of the eigenvalue parameter variation for matrices with parameters. Here, e_i represents a vector with only the i-th element being 1 and the others being 0. Therefore, we obtain:

$$G^{\mathrm{red}}_{ij} = 0, \qquad \forall i \neq j \tag{58}$$

Furthermore, if $H(s)$ is negative imaginary, then L_0 in equation **??** must also be symmetric, because we have:

$$\lim_{\omega\to 0} \boldsymbol{j}\left\{H(\boldsymbol{j}\omega) - H^{\mathsf{T}}(-\boldsymbol{j}\omega)\right\} = \boldsymbol{j}\left(-L_0 + L_0^{\mathsf{T}}\right) \geq 0$$

When equation 58 holds, it should be noted that L_0 can be expressed as follows:

$$C = \mathsf{diag}\left(2E_i^{\star} G_{ii}^{\mathrm{red}}\right) - \hat{B}^{\mathsf{T}}$$

Note that L_0 is given by:

$$L_0 = \underbrace{L + \hat{B}^{\mathsf{T}}\hat{A}^{-1}\hat{B}}_{L_1} - \underbrace{\mathsf{diag}(2E_i^{\star} G_{ii}^{\mathrm{red}})\hat{A}^{-1}\hat{B}}_{L_2}$$

where $\hat{A}$ is a symmetric matrix defined in equation 8. Therefore, L_1 is symmetric. On the other hand, for L_2 to be symmetric for any $E^{\star}$, it must be the case that $G_{ii}^{\mathrm{red}} \neq 0$ for all i. From this, it follows that if $H(s)$ is negative imaginary for any $(\delta^{\star}, E^{\star})$ such that $H(s)$ is stable, then equation 31 holds.

Next, we will show that if Equation 31 holds, then $H(s)$ is negative imaginary for any $(\delta^{\star}, E^{\star})$ that makes $H(s)$ stable. For this purpose, it is enough to show that there exists a positive definite matrix P that satisfies L is symmetric and

$$\tilde{A}^{\mathsf{T}}P + P\tilde{A} \preceq 0, \qquad P\tilde{A}^{-1}\tilde{B} = C^{\mathsf{T}} \tag{59}$$

$$\tilde{A} := \tau^{-1}A, \qquad \tilde{B} := \tau^{-1}B$$

Note that if equation 31 holds, then

$$k_{ij}(\delta_{ij}^{\star}) = k_{ji}(\delta_{ji}^{\star}), \qquad h_{ij}(\delta_{ij}^{\star}) = -h_{ji}(\delta_{ji}^{\star}), \qquad h_{ii}(\delta_{ii}^{\star}) = 0$$

which implies that L is symmetric. Moreover, since $H(s)$ is stable, we have

$$\tilde{A} = \mathsf{diag}\left(\frac{X_i - X_i'}{\tau_i}\right)\hat{A}$$

and since $X_i > X_i'$, the matrix $\hat{A}$ in Equation 8 is negative definite. Therefore, we can choose $P = -\hat{A}$ as a positive definite matrix that satisfies Equation 59, which implies that $H(s)$ is negative imaginary.

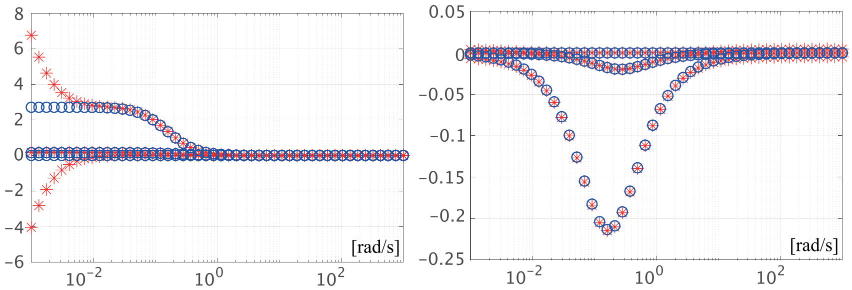
Next, we show the equivalence of $G(s)$. Since $H(s)$ is stable, the poles of $G(s)$ on the imaginary axis are only at the origin and have a multiplicity of 1. Therefore, the necessary and sufficient condition for $G(s)$ to be positive real is:

$$G(\boldsymbol{j}\omega) + G^{\mathsf{T}}(-\boldsymbol{j}\omega) \succeq 0, \qquad \forall\omega \in \mathbb{R} \setminus \{0\} \tag{60}$$

Then the following can be established:

$$\lim_{s\to 0} sG(s) = \lim_{s\to 0}\{sG(s)\}^{\mathsf{T}} \succeq 0 \tag{61}$$

When the Equation 31 holds, then the Equation 60 holds.



(a) Eigenvalue of $\frac{G(\boldsymbol{j}\omega)+G^{\mathsf{T}}(-\boldsymbol{j}\omega)}{2}$ (b) Eigenvalue of $\frac{H(\boldsymbol{j}\omega)-H^{\mathsf{T}}(-\boldsymbol{j}\omega)}{2\boldsymbol{j}}$

Fig. 10 Positive realness of $G(s)$ and negative imaginariness of$H(s)$
(Blue: passive transmission condition (ii) is satisfied, Red: not satisfied)

$$G(\boldsymbol{j}\omega) + G^{\mathsf{T}}(-\boldsymbol{j}\omega) = \frac{\boldsymbol{j}}{\omega}\left\{H(\boldsymbol{j}\omega) - H^{\mathsf{T}}(-\boldsymbol{j}\omega)\right\}, \qquad \forall\omega \in \mathbb{R}\setminus\{0\} \tag{62}$$

It is shown from $H(s)$ that $H(s)$ is negative imaginary. Moreover:

$$\lim_{s\to 0} sG(s) = L - C\tilde{A}^{-1}\tilde{B} = L - CA^{-1}B$$

Therefore, the positive semi-dfiniteness of equation 61 is equivalent to the passive transmission condition (iii), i.e., the condition of Equation 38. Note that when the passive transmission condition (ii) holds, L is symmetric and:

$$C\tilde{A}^{-1}\tilde{B} = CP^{-1}C^{\mathsf{T}}$$

is also symmetric, which also shows the symmetry of the expression 61.

On the contrary, if either the passive power transmission condition (ii) or (iii) is not satisfied, it indicates that $G(s)$ is not positive real. As for the latter, it is evident from the fact that the condition in Equation 38 is equivalent to the condition in Equation 61. Additionally, when the passive power transmission condition (ii) is not satisfied, $H(s)$ is not purely imaginary, and there exist some point $\omega_0 \geq 0$ and sufficiently small $\epsilon > 0$ such that:

$$\lambda_{\min}\left[\boldsymbol{j}\left\{H(\boldsymbol{j}(\omega_0+\alpha)) - H^{\mathsf{T}}(-\boldsymbol{j}(\omega_0+\alpha))\right\}\right] < 0, \qquad \forall\alpha \in (0,\epsilon]$$

where $\lambda_{\min}$ denotes the minimum eigenvalue. Therefore, for all non-zero $\omega \in (\omega_0, \omega_0+\epsilon]$, the complex symmetric part of $G(\boldsymbol{j}\omega)$ is not semi-positive definite. □

Let us confirm the result of Theorem 1.2 with the following example.

Example 1.4 Transmission loss and positive realness of transfer function of electrical subsystem

Let's examine the positive realness of $G(s)$ and the negative imaginary property of $H(s)$ for the power system model composed of three generators that we dealt with in Example 1.2. We will calculate these properties in two cases: when the passive power transmission condition (ii) is satisfied and when it is not, for comparison purposes. Specifically, as in Example 1.3, we set two types of $\boldsymbol{Y}_0(0)$ and $\boldsymbol{Y}_0(1)$ as the admittance matrix $\boldsymbol{Y}$ of the transmission network. The eigenvalues of the complex symmetric part of $G(\boldsymbol{j}\omega)$ and the imaginary parts of the eigenvalues of the complex skew-symmetric part of $H(\boldsymbol{j}\omega)$ are plotted against the frequency ω on the horizontal axis in Figure 10(a) and Figure 10(b), respectively. The blue circles indicate when the passive power transmission condition (ii) is satisfied, and the red asterisks indicate when it is not. From this figure, we can see that if the conductance of the transmission lines is non-zero, the complex symmetric part of $G(\boldsymbol{j}\omega)$ is not semi-positive definite in the low-frequency band.

The meaning of the passive power transmission condition (iii), which appeared as a condition for P_G in Equation 34 to be semi-positive definite, can also be explained as follows. Consider the state equation of the internal voltage in the electrical subsystem G given in Equation 22:

$$\tau \dot{E}^{\mathrm{lin}} = AE^{\mathrm{lin}} + B\delta^{\mathrm{lin}}$$

Let us focus on the limit where the time constant $(\tau_i)_{i\in\mathcal{I}_{\mathrm{G}}}$ tends to 0. This corresponds to considering the limit where "the time it takes for the internal voltage to reach a steady state is much shorter than the variation of δ^{lin}." In this case, the following approximation holds:

$$E^{\mathrm{lin}}(t) \simeq -A^{-1}B\delta^{\mathrm{lin}}(t), \qquad \forall t \geq 0 \tag{63}$$

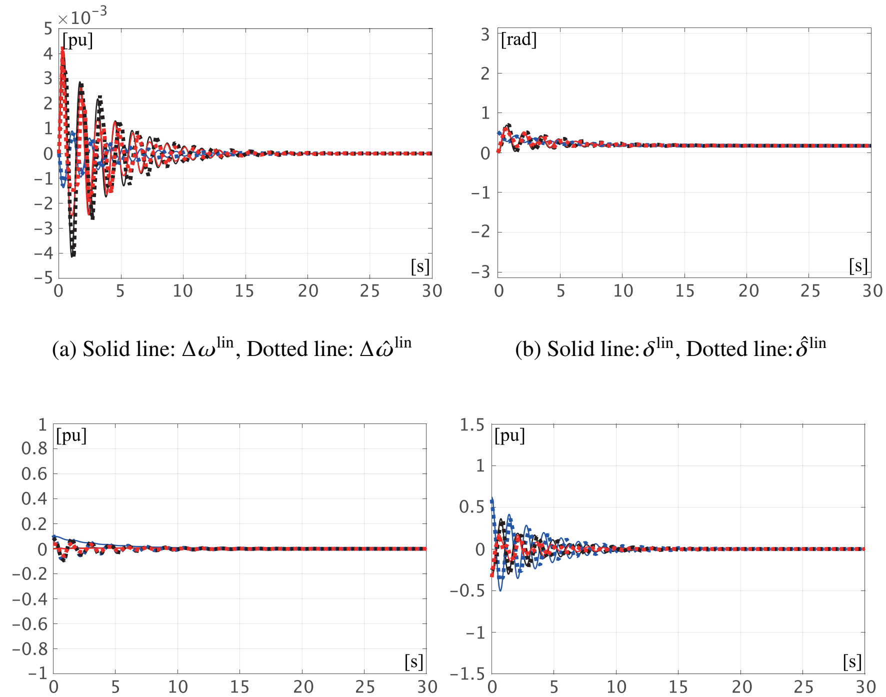
However, if A is unstable, i.e., if the passive power transmission condition (i) does not hold, the state E^{lin} diverges. Systems with state variables that have different time scales are called **singularly perturbed systems** [**?**] in the field of control engineering. A differential equation system with sufficiently small time constants can be approximated by an algebraic equation system. In fact, the dynamic characteristics of internal voltage often have smaller time constants than the mechanical turbine dynamics.

Assuming that equation 63 holds as an equality, substituting it into the output equation of equation 22 yields a low-dimensional approximation of the electrical subsystem as follows:

$$\hat{G} : \begin{cases} \dot{\hat{\delta}}^{\mathrm{lin}} = u_G \\ y_G = L_0 \hat{\delta}^{\mathrm{lin}} \end{cases} \tag{64}$$

However, to indicate that this is an approximation, the state variable is distinguished as $\hat{\delta}^{\mathrm{lin}}$.

Using the low-dimensional approximation of this singular perturbation system, the entire approximate linear model of equation 9 is approximated as a system of coupled differential equations with two second-order oscillators:



(a) Solid line: $\Delta\omega^{\mathrm{lin}}$, Dotted line: $\Delta\hat{\omega}^{\mathrm{lin}}$

(b) Solid line: δ^{lin}, Dotted line: $\hat{\delta}^{\mathrm{lin}}$

(c) Solid line: E^{lin}, Dotted line: $\hat{E}^{\mathrm{lin}}$

(d) Solid line: P^{lin}, Dotted line: $\hat{P}^{\mathrm{lin}}$

Fig. 11 Time response when low-dimensional approximation is applied
(Blue: Generator 1, Black: Generator 2, Red: Generator 3)

$$M\ddot{\hat{\delta}}^{\mathrm{lin}} + D\dot{\hat{\delta}}^{\mathrm{lin}} + \omega_0 L_0 \hat{\delta}^{\mathrm{lin}} = 0 \tag{65}$$

From this result, it can be understood that the passive power transmission condition (iii) represents the ”semi-positive definiteness of the spring constant matrix” in the case of small time constants. Furthermore, the electrical subsystem G of Equation 22 can be interpreted as corresponding to a dynamic spring constant.

Example 1.5 Low-dimensional approximation for an approximate linear model As a reference, the time response of the second-order oscillator system of Equation 65 for the approximate linear model discussed in Example 1.2 is shown in Figure 11. The solid lines represent the response of the original approximate linear model, while the dashed lines represent the response of the second-order oscillator system obtained by applying the low-dimensional approximation of the singular perturbation system. In addition,

$$\Delta\hat{\omega}^{\text{lin}} := \omega_0^{-1}\dot{\hat{\delta}}^{\text{lin}}, \qquad \hat{E}^{\text{lin}} := -A^{-1}B\hat{\delta}^{\text{lin}}, \qquad \hat{P}^{\text{lin}} := L\hat{\delta}^{\text{lin}} + C\hat{E}^{\text{lin}}$$

The initial values for the approximate linear model are given by:

$$\hat{\delta}^{\text{lin}}(0) = \begin{bmatrix} \frac{\pi}{6} \\ 0 \\ 0 \end{bmatrix}, \qquad \Delta\hat{\omega}^{\text{lin}}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From Figure 11, it can be seen that the time response of both systems matches well in terms of the peak values and decay rates of the oscillations.

3.5 Necessary condition for the steady-state stability of the approximate linear model

In the following, we discuss the necessity of the passive power transmission condition (iii) for the steady-state stability of the approximate linear model in equation 9. Specifically, we show that the passive power transmission condition (iii) is a necessary condition for the steady-state stability of the approximate linear model regardless of the physical constants of the generator group.

Note that as shown in section 3.4, the passive power transmission condition (i) is a necessary condition for the stability of the approximate linear model even in the limit where the time constants $(\tau_i)i \in \mathcal{I}\text{G}$ are sufficiently small. This can be verified by the fact that if the matrix A is unstable, the low-dimensional approximation of the singular perturbation system in equation 63 cannot be applied, and the internal voltage diverges.

When the passive power transmission condition (ii) does not hold, L_0 is generally not symmetric, so a generalized version of the passive power transmission condition (iii) was introduced to apply it to non-symmetric L_0 as well:

$$\mathbf{\Lambda}(L_0) \subseteq [0, \infty) \tag{66}$$

Here, $\mathbf{\Lambda}(L_0)$ denotes the set of eigenvalues of L_0. The condition in Equation 66 represents that all eigenvalues of L_0 are "non-negative real numbers". In the following, we refer to this generalized condition as the passive electrical condition (iii)′, defined in Definition 1.4. Note that when L_0 is symmetric, the passive electrical conditions (iii) and (iii)′ are equivalent. The following lemma is proven.

Lemma 1.1 (Necessary condition for the small signal stability of a second-order oscillator system) *Consider the second-order oscillator system given by Equation 65. For any initial conditions and any positive definite $(M_i, D_i)i \in \mathcal{I}\text{G}$, the necessary condition for the existence of a constant c_0 such that Equation 67 holds as $t \to \infty$ is that the passive power condition (iii)′ holds.*

$$\lim_{t\to\infty} \hat{\delta}^{\text{lin}}(t) = c_0\mathbb{1} \tag{67}$$

Proof If the passive transmission condition (iii)′ does not hold, we show that there exist positive constants $(M_i, D_i)_{i\in\mathcal{I}_G}$ such that Equation 67 does not hold. For this purpose, we discuss the following two cases:

(a) There exist eigenvalues of L_0 with negative real part or purely imaginary part.
(b) There exist eigenvalues of L_0 with positive real part and nonzero imaginary part.

First, let us consider case (a). In what follows, we choose the constant matrices $M = \omega_0 I$ and $D = \omega_0 dI$. Then, the eigenvalue equation for Equation 65 is given by:

$$\begin{bmatrix} 0 & I \\ -L_0 & -dI \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = \lambda \begin{bmatrix} v \\ w \end{bmatrix}$$

Eliminating w by substitution from this equation, we obtain:

$$\left(\lambda^2 I + d\lambda I + L_0\right) v = 0$$

This eigenvalue equation implies that v is an eigenvector of L_0, and for its eigenvalue κ, we have:

$$\lambda^2 + d\lambda + \kappa = 0 \qquad \Longleftrightarrow \qquad \lambda = \frac{-d \pm \sqrt{d^2 - 4\kappa}}{2} \tag{68}$$

Therefore, to show that the real part of $\sqrt{d^2 - 4\kappa}$ is greater than d in case (a), it suffices to prove that $\sqrt{d^2 - 4\kappa}$ has a larger real part than d in the case where the real part of κ is negative or the real part of κ is zero and the imaginary part of κ is nonzero. In general, for any complex number z, we have:

$$\mathsf{Re}[z] = \sqrt{\mathsf{Re}[z^2] + (\mathrm{i}[z])^2}$$

Using this formula with $z = \sqrt{d^2 - -4\kappa}$, we obtain:

$$\mathsf{Re}\left[\sqrt{d^2 - 4\kappa}\right] = \sqrt{d^2 - 4\,\mathsf{Re}[\kappa] + (\mathrm{i}[z])^2}$$

This value is always greater than d in case (a), where the real part of κ is negative or the real part of κ is zero and the imaginary part of κ is nonzero. Therefore, the 2-degree-of-freedom oscillator system in equation 65 is unstable in case (a).

Next, consider the case (b). Below, it is shown that there exists a positive definite number d such that the eigenvalue λ of Equation 68 becomes purely imaginary. If the real matrix L_0 has complex eigenvalues, there must be at least one with a negative imaginary part. Denote this eigenvalue by $\kappa = \alpha + \beta \boldsymbol{j}$, where $\alpha > 0$ and $\beta < 0$. It is shown that there exist some $\omega \neq 0$ and $d > 0$ that satisfy

$$-d + \sqrt{d^2 - 4\kappa} = \omega \boldsymbol{j}$$

Moving $-d$ to the left-hand side and squaring both sides gives:

$$-4(\alpha + \beta \boldsymbol{j}) = 2d\omega \boldsymbol{j} - \omega^2$$

This equation is satisfied if we choose $\omega = 2\sqrt{\alpha}$ and $d = -\frac{\beta}{\sqrt{\alpha}}$. Therefore, the second-order oscillator system has a steady-state oscillatory solution, and Equation 67 does not hold. □

Lemma 1.1 shows that the passive power transfer condition (iii)′ is a necessary condition for the steady-state stability of an approximate linear model for any generator group's physical parameters in the limit of small internal voltage time constants. Furthermore, Theorem 1.1 demonstrates that when the passive power transfer conditions (i)–(iii) hold, the approximate linear model is stable for any physical parameters. Based on these facts, the conclusion of this section is summarized in the following theorem.

Theorem 1.3 (Small signal stability of linearized models) *For any positive parameters* $(M_i, D_i, \tau_i)i \in \mathcal{I}\mathrm{G}$*, the necessary condition for the steady-state stability of the linearized model in Equation 9 is that the passive power transmission conditions (i) and (iii)′ in Definition 1.4 hold.*

In particular, if the passive power transmission condition (ii) holds, then the necessary and sufficient condition for the above steady-state stability is that the passive power transmission conditions (i) and (iii)′ hold.

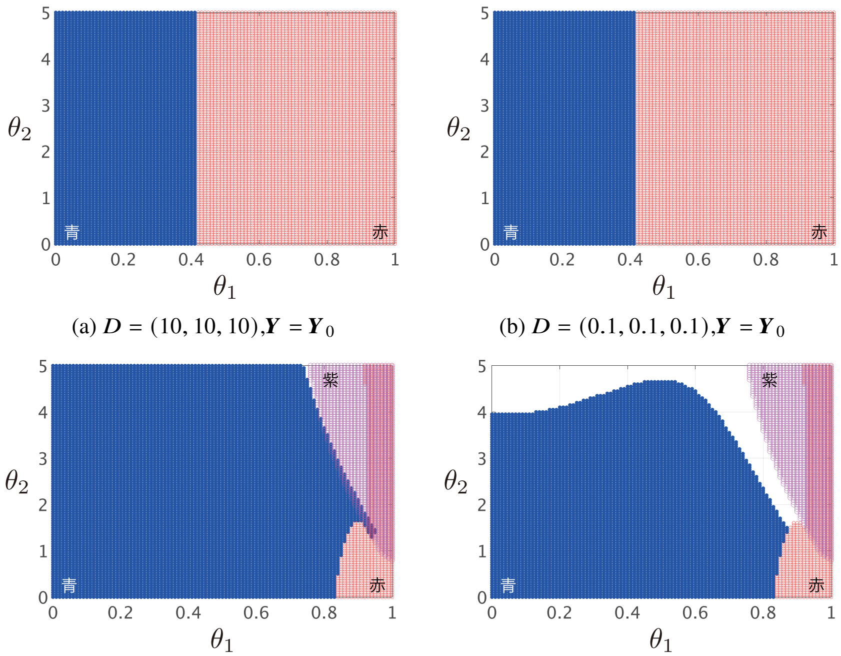
We present an analytical example of the small signal stability of the linear approximate model using Theorem 1.3.

Example 1.6 Small signal stability analysis based on the passive power transmission conditions

Using Theorem 1.3, let's analyze the small signal stability of the approximate linear model consisting of three generators discussed in Example 1.2. The physical constants of the generators are set to the same values as in Example 1.2. Since the passive power transmission condition (i) is satisfied for all parameters, the region of parameters where passive power transmission condition (iii)′ is not satisfied is overlaid on Figure 12. However, the red region shows eigenvalues of L_0 in which the real part is negative, while the purple region shows eigenvalues of L_0 that are complex. The region on the horizontal axis where θ_2 is zero represents the case where the passive power transmission condition (ii) holds.

Theorem 1.3 shows that the red and purple regions are "dangerous parameter regions where the approximate linear model will always become unstable for a certain setting of physical constants." Additionally, when the passive power transmission condition (ii) holds, i.e., for parameters on the horizontal axis where θ_2 is zero, it is shown that the approximate linear model will always be steady-state stable regardless of the values of these constants as long as θ_1 is set to a non-red value.

The noteworthy point from the result in Figure 12 is that the red parameter region accurately captures part or all of the boundary of the blue parameter region, where the model is steady-state stable under the physical constants mentioned above. The necessity of the passive power transmission condition (iii)′ shown in Theorem 1.3 implies that there is at least one setting of physical constants for which the approximate linear model becomes unstable. Therefore, it is not always possible to



(a) $\boldsymbol{D} = (10, 10, 10), \boldsymbol{Y} = \boldsymbol{Y}_0$ (b) $\boldsymbol{D} = (0.1, 0.1, 0.1), \boldsymbol{Y} = \boldsymbol{Y}_0$

(c) $\boldsymbol{D} = (10, 10, 10), \boldsymbol{Y} = \frac{\boldsymbol{Y}_0}{100}$ (d) $\boldsymbol{D} = (0.1, 0.1, 0.1), \boldsymbol{Y} = \frac{\boldsymbol{Y}_0}{100}$

Fig. 12 Regions of parameters for which the approximate linear model is stable

accurately capture the parameter region that ensures steady-state stability for specific constants set in Example 1.2. On the other hand, the fact that part of the boundary of the blue region is accurately captured suggests that the approximate linear model is often unstable when L_0 has eigenvalues with negative real parts.

Also, in cases (a) and (b), it can be seen that there is no purple region. That is, for all the explored parameters, the eigenvalues of L_0 are real numbers. Generally, unless θ_2 is zero, L_0 is an asymmetric matrix, so it is not obvious that L_0 only has real eigenvalues. On the other hand, in cases (c) and (d) where the admittance matrix is multiplied by $\frac{1}{100}$, it is also known that L_0 has complex eigenvalues when θ_1 and θ_2 are relatively large. However, in realistic settings, it has been confirmed that L_0 mostly has real eigenvalues.

Mathematical Appendix

Lemma 1.2 *Consider a stable and square transfer function:*

$$Q(s) = C(sI - A)^{-1}B + D$$

where (A, B) is controllable and (C, A) is observable. The necessary and sufficient condition for $Q(s)$ to be positive real is the existence of a positive definite symmetric matrix P such that:

$$\begin{bmatrix} A^\top P + PA & PB - C^\top \\ B^\top P - C & -(D + D^\top) \end{bmatrix} \leq 0$$

The proof can be found in [**?**, Theorem 5.31] or [**?**, Theorem 3], among others. Also, [**?**] provides a detailed overview of related results.

Lemma 1.3 *Consider a stable and square transfer function*

$$Q(s) = C(sI - A)^{-1}B + D$$

where (A, B) is controllable and (C, A) is observable. The necessary and sufficient condition for $Q(s)$ to be negative imaginary is that D is symmetric and there exists a positive definite symmetric matrix P such that:

$$A^\top P + PA \leq 0, \qquad -PA^{-1}B = C^\top$$

The proof can be found in [**?**, Lemma 7], among others.