Chapter 1
Steady-state stability analysis of power system models

In this Chapter, we conduct stability analysis based on the approximate linearization of power system models. The structure of this Chapter is as follows. First, in Section **??**, we derive a linear approximation model for the power system model described by a system of ordinary differential equations using Kron reduction of the generator buses. Then, in Section **??**, we explain the method for numerically analyzing the stability of the derived linear approximation model. We also confirm through numerical simulation that the stability of the linear approximation model depends not only on the physical constants of the generators, loads, and transmission lines, but also on the selection of the steady-state power flow. Additionally, in Section **??**, we explore advanced topics and demonstrate how the stability of the linear approximation model can be analyzed using the concept of passivity in dynamic systems.

COFFEE BREAK

Derivation of the approximate linear system:
Consider the nonlinear system:

$$\dot{x}(t) = f\big(x(t)\big) + Bu(t)$$

where $f(0) = 0$. The function $f(x)$ can be expressed near the origin by a Taylor expansion as:

$$f(x) = f(0) + \frac{\partial f}{\partial x}(0)x + \text{Second or higher order term}$$

Here, $f(x)$ and x are expressed as $f_i(x)$ and x_i, respectively, and $\frac{\partial f}{\partial x}(x)$ is the *Jacobian matrix* with the (i, j) element given by $\frac{\partial f_i}{\partial x_j}(x)$. By using this Jacobian matrix, we define:

$$A := \frac{\partial f}{\partial x}(0)$$

Then, when the magnitudes of the state $x(t)$ and input $u(t)$ are sufficiently small, the behavior of the nonlinear system can be approximated by the behavior of the linear system obtained by neglecting terms of degree 2 or higher in the function f:

$$\dot{x}^{\mathrm{lin}}(t) = Ax^{\mathrm{lin}}(t) + Bu^{\mathrm{lin}}(t)$$

Note that even if $u(t)$ and $u^{\mathrm{lin}}(t)$ are the same, the state $x(t)$ of the nonlinear system and the state $x^{\mathrm{lin}}(t)$ of the approximate linear system may not be exactly the same.

1 Stability analysis based on linear approximation

1.1 Approximate linearization of the power system model

In this section, we derive an approximate linear model for the power system model where each bus has a generator connected, which is equivalent to the Kron-reduced differential equation system model discussed in Section **??**. We derive the approximate linear model for the steady-state flow state. The differential equation system model is given by:

$$\begin{cases} \dot{\delta}_i = \omega_0 \Delta\omega_i \\ M_i \Delta\dot{\omega}_i = -D_i \Delta\omega_i - f_i(\delta, E) + P_{\mathrm{mech}i} \\ \tau_i \dot{E}_i = -\frac{X_i}{X_i'} E_i + \left(X_i - X_i'\right) g_i(\delta, E) + V_{\mathrm{field}i} \end{cases} \qquad i \in \mathcal{I}_{\mathrm{G}} \tag{1}$$

However, δ and E are vectors obtained by vertically arranging δ_i and E_i, respectively. The nonlinear terms representing the interactions between generators are expressed as follows:

$$\begin{aligned} f_i(\delta, E) &:= -E_i \sum_{j=1}^{N} E_j \left(B_{ij}^{\mathrm{red}} \sin \delta_{ij} - G_{ij}^{\mathrm{red}} \cos \delta_{ij}\right), \\ g_i(\delta, E) &:= -\sum_{j=1}^{N} E_j \left(B_{ij}^{\mathrm{red}} \cos \delta_{ij} + G_{ij}^{\mathrm{red}} \sin \delta_{ij}\right) \end{aligned} \tag{2}$$

In addition, $\delta_{ij} := \delta_i - \delta_j$ is defined. Note that, due to the properties of reduced admittance, the reduced conductance and reduced susceptance satisfy the symmetry condition:

$$G_{ij}^{\mathrm{red}} = G_{ji}^{\mathrm{red}}, \qquad B_{ij}^{\mathrm{red}} = B_{ji}^{\mathrm{red}}, \qquad \forall (i, j) \in \mathcal{I}_{\mathrm{G}} \times \mathcal{I}_{\mathrm{G}}$$

To obtain the partial derivatives of these nonlinear functions with respect to each variable, we define:

$$
\begin{aligned}
k_{ij}(\delta_{ij}) &:= -B_{ij}^{\mathrm{red}} \cos \delta_{ij} - G_{ij}^{\mathrm{red}} \sin \delta_{ij}, \\
h_{ij}(\delta_{ij}) &:= -B_{ij}^{\mathrm{red}} \sin \delta_{ij} + G_{ij}^{\mathrm{red}} \cos \delta_{ij}
\end{aligned}
\tag{3}
$$

Then, for f_i, we obtain:

$$
\begin{aligned}
\frac{\partial f_i}{\partial \delta_i} &= E_i \sum_{j=1, j\neq i}^{N} E_j k_{ij}(\delta_{ij}), \quad & \frac{\partial f_i}{\partial E_i} &= 2E_i h_{ii}(\delta_{ii}) + \sum_{j=1, j\neq i}^{N} E_j h_{ij}(\delta_{ij}), \\
\frac{\partial f_i}{\partial \delta_j} &= -E_i E_j k_{ij}(\delta_{ij}), & \frac{\partial f_i}{\partial E_j} &= E_i h_{ij}(\delta_{ij})
\end{aligned}
\tag{4}
$$

where $j \neq i$.

Similarly, we can obtain the partial derivatives of g_i as follows:

$$
\begin{aligned}
\frac{\partial g_i}{\partial \delta_i} &= -\sum_{j=1, j\neq i}^{N} E_j h_{ij}(\delta_{ij}), \quad & \frac{\partial g_i}{\partial E_i} &= k_{ii}(\delta_{ii}), \\
\frac{\partial g_i}{\partial \delta_j} &= E_j h_{ij}(\delta_{ij}), & \frac{\partial g_i}{\partial E_j} &= k_{ij}(\delta_{ij})
\end{aligned}
\tag{5}
$$

We denote the steady-state values of the internal state of generator i as $(\delta_i^\star, E_i^\star)$ and the steady-state values of external inputs as $(P_{\mathrm{mech}i}^\star, V_{\mathrm{field}i}^\star)$ for the differential equation system in equation **??**. Furthermore, we use symbols without the subscript i to represent the vector of these values for all $i \in \mathcal{I}_{\mathrm{G}}$. For example, $\delta^\star$ denotes the vector $(\delta_i^\star)_{i \in \mathcal{I}_{\mathrm{G}}}$. With these steady-state values, we can write the following system of equations:

$$
\begin{cases}
0 = -f_i\left(\delta^\star, E^\star\right) + P_{\mathrm{mech}i}^\star \\
0 = -\frac{X_i}{X_i'} E_i^\star + \left(X_i - X_i'\right) g_i\left(\delta^\star, E^\star\right) + V_{\mathrm{field}i}^\star
\end{cases}
\qquad i \in \mathcal{I}_{\mathrm{G}}
\tag{6}
$$

Here, note that we assume the steady-state value of the frequency deviation $\Delta\omega_i$ in Eq. **??** is zero for all $i \in \mathcal{I}_{\mathrm{G}}$. The validity of Eq. **??** corresponds to setting the steady-state values of the external input $(P_{\mathrm{mech}}^\star, V_{\mathrm{field}}^\star)$ to appropriate values that achieve supply-demand balance. By linearizing the system around this steady state, the approximate linear model is obtained as:

$$
\begin{bmatrix} \dot{\delta}^{\mathrm{lin}} \\ M\Delta\dot{\omega}^{\mathrm{lin}} \\ \tau \dot{E}^{\mathrm{lin}} \end{bmatrix}
=
\begin{bmatrix} 0 & \omega_0 I & 0 \\ -L & -D & -C \\ B & 0 & A \end{bmatrix}
\begin{bmatrix} \delta^{\mathrm{lin}} \\ \Delta\omega^{\mathrm{lin}} \\ E^{\mathrm{lin}} \end{bmatrix}
+
\begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{bmatrix}
\begin{bmatrix} P_{\mathrm{mech}}^{\mathrm{lin}} \\ V_{\mathrm{field}}^{\mathrm{lin}} \end{bmatrix}
\tag{7}
$$

Note that the state and input variables with the subscript "lin" are vectors consisting of small deviations from the corresponding variables with their steady-state values as the reference. Also,

$$M := \mathsf{diag}\,(M_i)_{i\in\mathcal{I}_\mathrm{G}}\,, \qquad D := \mathsf{diag}(D_i)_{i\in\mathcal{I}_\mathrm{G}}, \qquad \tau := \mathsf{diag}\,(\tau_i)_{i\in\mathcal{I}_\mathrm{G}}$$

are diagonal matrices where $\mathsf{diag}(\cdot)$ is an operator that creates a diagonal matrix from a vector.

Furthermore, for the functions k_{ij} and h_{ij} defined in Equation **??**, the (i, j) element of the matrices $\hat{L}$, $\hat{A}$, $\hat{B}$, and $\hat{C}$, defined as:

$$\hat{L}_{ij} := \begin{cases} E_i^\star \sum_{j=1,j\neq i}^N E_j^\star k_{ij}(\delta_{ij}^\star), & i = j \\ -E_i^\star E_j^\star k_{ij}(\delta_{ij}^\star), & i \neq j \end{cases}$$
$$\hat{A}_{ij} := \begin{cases} k_{ii}(\delta_{ii}^\star) - \frac{X_i}{X_i'(X_i - X_i')}, & i = j \\ k_{ij}(\delta_{ij}^\star), & i \neq j \end{cases}$$
$$\hat{B}_{ij} := \begin{cases} -\sum_{j=1,j\neq i}^N E_j^\star h_{ij}(\delta_{ij}^\star), & i = j \\ E_j^\star h_{ij}(\delta_{ij}^\star), & i \neq j \end{cases}$$
$$\hat{C}_{ij} := \begin{cases} \sum_{j=1,j\neq i}^N E_j^\star h_{ij}(\delta_{ij}^\star), & i = j \\ E_i^\star h_{ij}(\delta_{ij}^\star), & i \neq j \end{cases}$$

The matrices L, A, B, and C are then defined as follows:

$$\begin{aligned} L &:= \hat{L}, \\ A &:= \mathsf{diag}\left(X_i - X_i'\right)_{i\in\mathcal{I}_\mathrm{G}} \hat{A}, \\ B &:= \mathsf{diag}\left(X_i - X_i'\right)_{i\in\mathcal{I}_\mathrm{G}} \hat{B}, \\ C &:= \mathsf{diag}\big(2E_i^\star h_{ii}(\delta_{ii}^\star)\big)_{i\in\mathcal{I}_\mathrm{G}} + \hat{C} \end{aligned} \tag{8}$$

Note that $\delta_{ij}^\star := \delta_i^\star - \delta_j^\star$. It should be noted that the system matrix (L, A, B, C) is a function of the steady-state values $(\delta^\star, E^\star)$. The block diagram of this approximate linear model is shown in Figure **??**. Here, P^{lin} represents the approximately linearized active power supplied by the generators. Note that generally $X_i > X_i'$ for all i.

In power system engineering, the value obtained by differentiating the generator's active power with respect to the rotor angle at the steady-state is called the **synchronizing power coefficient** [**?**, Section 8.4]. That is, the matrix L in the approximate linear model given by equation **??** corresponds to the synchronizing power coefficient. However, in power system engineering, it is common to define the synchronizing power coefficient using the one-machine infinite-bus system model explained in Section **??**, so it is a scalar value rather than a matrix.

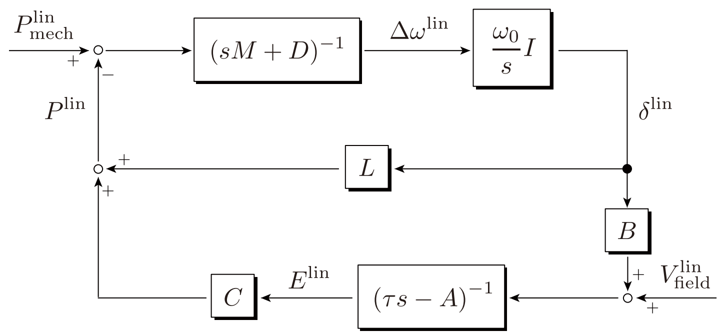


Fig. 1 Block Diagram of Approximate Linear Model

1.2 Stability analysis of approximate linear models

1.2.1 Stability of approximate linear models

In this section, we consider numerically analyzing the stability of the approximate linear model. Whether the approximate linear model of Equation **??** is stable or not is characterized by whether the internal states of the generator groups return to the steady state satisfying the simultaneous equations of Equation **??** in the event of a small disturbance in the power system, such as temporary minor fluctuations in mechanical input or excitation input of the generators, impedance values of the loads, current or voltage values of the transmission lines, etc., from the reference values at the steady state. In power system engineering, stability against such small fluctuations is called **small signal stability**.

It should be noted that the stability of the approximate linear model of Equation **??** depends on the selection of the steady-state values of the internal states of the generator groups $(\delta^\star, E^\star)$ and the steady-state values of external inputs $(P^\star_{\mathrm{mech}}, V^\star_{\mathrm{field}})$. Furthermore, changes in the admittance of the transmission lines or the impedance of the loads alter the reduced conductance $G^{\mathrm{red}}ij$ and reduced susceptance $B^{\mathrm{red}}ij$ of Equation **??**. Therefore, the stability of the approximate linear model varies depending on various model parameters mentioned above. The purpose of this section is to numerically examine the relationship between the changes in these model parameters and the stability of the approximate linear model.

1.2.2 Stability analysis based on eigenvalues of the system matrix

For the approximate linear model in Equation **??**, if we appropriately choose the steady-state values $(\delta^\star, E^\star)$ of the internal states as parameters, then the system matrix (L, A, B, C) in Equation **??** and the steady-state values $(P^\star_{\mathrm{mech}}, V^\star_{\mathrm{field}})$ of the external inputs satisfying Equation **??** are determined dependently. Here, we consider setting

$$P_{\mathrm{mech}i}(t) = P^{\star}_{\mathrm{mech}i}, \qquad V_{\mathrm{field}i}(t) = V^{\star}_{\mathrm{field}i}, \qquad \forall t \geq 0$$

for all $i \in \mathcal{I}$G in the nonlinear differential equation system model in Equation **??**. We then assess the stability of the system using the eigenvalues of the system matrix.

This means that in the approximate linear model of Equation **??** the following values are set:

$$P^{\mathrm{lin}}_{\mathrm{mech}}(t) = 0, \qquad V^{\mathrm{lin}}_{\mathrm{field}}(t) = 0, \qquad \forall t \geq 0$$

In the following, under this assumption, we analyze the stability of an autonomous approximate linear model with input set identically to zero, given by:

$$\begin{bmatrix} \dot{\delta}^{\mathrm{lin}} \\ \Delta\dot{\omega}^{\mathrm{lin}} \\ \dot{E}^{\mathrm{lin}} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \omega_0 I & 0 \\ -M^{-1}L & -M^{-1}D & -M^{-1}C \\ \tau^{-1}B & 0 & \tau^{-1}A \end{bmatrix}}_{\Psi} \begin{bmatrix} \delta^{\mathrm{lin}} \\ \Delta\omega^{\mathrm{lin}} \\ E^{\mathrm{lin}} \end{bmatrix} \tag{9}$$

Specifically, by examining the sign of the real part of the eigenvalues of the matrix Ψ, we can determine the stability of this approximate linear model. However, it should be noted that Ψ generally has at least one zero eigenvalue. In fact, from the structure of the matrices L and B in equation **??**, we have:

$$L\mathbb{1} = 0, \qquad B\mathbb{1} = 0 \tag{10}$$

Therefore, for any model parameters, we have:

$$\Psi v = 0, \qquad v := \begin{bmatrix} \mathbb{1} \\ 0 \\ 0 \end{bmatrix}$$

This means that v is an eigenvector of Ψ corresponding to a zero eigenvalue. If the real parts of all eigenvalues, except for the zero eigenvalue, are negative, then for any initial value, the solution trajectory of Equation **??** satisfies:

$$\lim_{t\to\infty} \delta^{\mathrm{lin}}(t) = c_0\mathbb{1}, \qquad \lim_{t\to\infty} \Delta\omega^{\mathrm{lin}}(t) = 0, \qquad \lim_{t\to\infty} E^{\mathrm{lin}}(t) = 0 \tag{11}$$

Here, c_0 is a constant determined by the initial value. Note that the value of c_0 does not make a significant difference in the analytical results. This is because in the differential equation system model of Equation **??**, the rotor angle δ_i of a generator has meaning only in relation to the difference between the rotor angle δ_j of other generators. Specifically, if $(\delta^\star, E^\star)$ satisfies the system of equations in equation **??** for a certain $(P^\star_{\mathrm{mech}}, V^\star_{\mathrm{field}})$, then $(\delta^\star + c_0\mathbb{1}, E^\star)$ also satisfies the same system of equations. Therefore, $\delta^\star$ and $\delta^\star + c_0\mathbb{1}$ are essentially equivalent steady-state values where all generator rotor angles are rotated by the same amount of c_0. Equation

?? means the asymptotic convergence of solution trajectories to these essentially equivalent steady-state values.

2 stability analysis of approximate linear models using numerical calculations

2.1 Implementation of approximate linearization using a group of partitioned modules

In this section, we explain the implementation method for obtaining an approximate linear model numerically. Specifically, we describe how to add the functionality of linearization to the program that has been segmented into module groups as explained in Sections **??** and **??**.

In the numerical simulation program of the power system created in Section **??**, the following state and output equations are implemented for each device as differential and algebraic equations, respectively:

$$\dot{x}_i = f_i^{(1)}(x_i, \boldsymbol{V}_i, \boldsymbol{I}_i, u_i), \qquad 0 = f_i^{(2)}(x_i, \boldsymbol{V}_i, \boldsymbol{I}_i, u_i)$$

In the following, we derive the approximate linear model in the vicinity of the equilibrium point $(x_i^\star, \boldsymbol{V}_i^\star, \boldsymbol{I}_i^\star, u_i^\star)$ for the device of interest. Specifically, we explain the implementation method of the linear approximation function to the program that has been partitioned into the module group described in Sections **??** and **??**.

For the numerical simulation program of the power system created in Section **??**, differential equations for the state and algebraic equations for the output are implemented for each device as:

$$\dot{x}_i = f_i^{(1)}(x_i, \boldsymbol{V}_i, \boldsymbol{I}_i, u_i), \qquad 0 = f_i^{(2)}(x_i, \boldsymbol{V}_i, \boldsymbol{I}_i, u_i)$$

We consider the linearization of the functions $f_i^{(1)}$ and $f_i^{(2)}$ as follows:

$$\begin{aligned} f_i^{(1)}(x_i, \boldsymbol{V}_i, \boldsymbol{I}_i, u_i) &\approx A_i(x_i - x_i^\star) + B_{u_i} u_i \\ &\quad + B_{\boldsymbol{V}_i} \begin{bmatrix} \mathsf{Re}[\boldsymbol{V}_i - \boldsymbol{V}^\star] \\ \mathrm{i}[\boldsymbol{V}_i - \boldsymbol{V}^\star] \end{bmatrix} + B_{\boldsymbol{I}_i} \begin{bmatrix} \mathsf{Re}[\boldsymbol{I}_i - \boldsymbol{I}_i^\star] \\ \mathrm{i}[\boldsymbol{I}_i - \boldsymbol{I}_i^\star] \end{bmatrix} \end{aligned} \tag{12}$$

$$\begin{aligned} f_i^{(2)}(x_i, \boldsymbol{V}_i, \boldsymbol{I}_i, u_i) &\approx C_i(x_i - x_i^\star) + D_{u_i} u_i \\ &\quad + D_{\boldsymbol{V}_i} \begin{bmatrix} \mathsf{Re}[\boldsymbol{V}_i - \boldsymbol{V}^\star] \\ \mathrm{i}[\boldsymbol{V}_i - \boldsymbol{V}^\star] \end{bmatrix} + D_{\boldsymbol{I}_i} \begin{bmatrix} \mathsf{Re}[\boldsymbol{I}_i - \boldsymbol{I}_i^\star] \\ \mathrm{i}[\boldsymbol{I}_i - \boldsymbol{I}_i^\star] \end{bmatrix} \end{aligned} \tag{13}$$

A system of simultaneous equations for each machine and algebraic equations for the entire power system can be used to obtain an expression using ordinary differential equations for the approximate linear model by eliminating all $\boldsymbol{V}_i - \boldsymbol{V}_i^\star$ and $\boldsymbol{I}_i - \boldsymbol{I}_i^\star$, where $i \in 1, \ldots, N$, as follows:

$$\boldsymbol{I}_i - \boldsymbol{I}_i^\star = \sum_{j=1}^{N} \boldsymbol{Y}_{ij}(\boldsymbol{V}_j - \boldsymbol{V}_j^\star), \qquad i \in \{1, \ldots, N\}$$

Here, $\boldsymbol{Y}_{ij}$ represents the (i, j)th element of the admittance matrix $\boldsymbol{Y}$. Let us check the specific implementation method with the following example.

Example 1.1 (Implementation of Approximate Linear Model)

Equations **??** and **??** depend on the dynamic characteristics of the device, so it is natural to implement the calculation of coefficient matrices such as A_i and B_{u_i} in the classes of devices such as generators and loads in the implementation example of Section **??**. For example, in the generator model:

$$A_i = \begin{bmatrix} 0 & \omega_0 & 0 \\ 0 & -\frac{D_i}{M_i} & 0 \\ -\frac{1}{\tau_i}(\frac{X_i}{X_i'} - 1)|\boldsymbol{V}_i^\star| \sin(\delta_i^\star - \angle \boldsymbol{V}_i^\star) & 0 & -\frac{X_i}{\tau_i X_i'} \end{bmatrix}$$

$$B_{u_i} = \begin{bmatrix} 0 \\ \frac{1}{M_i} \\ 0 \end{bmatrix}, \qquad B_{\boldsymbol{V}_i} = \begin{bmatrix} 0 & 0 \\ -\frac{\mathrm{Re}[\boldsymbol{I}_i^\star]}{M_i} & -\frac{\mathrm{i}[\boldsymbol{I}_i^\star]}{M_i} \\ \frac{1}{\tau_i}(\frac{X_i}{X_i'} - 1)\cos\delta_i^\star & \frac{1}{\tau_i}(\frac{X_i}{X_i'} - 1)\sin\delta_i^\star \end{bmatrix}$$

$$B_{\boldsymbol{I}_i} = \begin{bmatrix} 0 & 0 \\ -\frac{\mathrm{Re}[\boldsymbol{V}_i^\star]}{M_i} & -\frac{\mathrm{i}[\boldsymbol{V}_i^\star]}{M_i} \\ 0 & 0 \end{bmatrix}, \qquad C_i = \begin{bmatrix} E_i^\star \cos\delta_i^\star & 0 & \sin(\delta_i^\star) \\ E_i^\star \sin\delta_i^\star & 0 & -\cos(\delta_i^\star) \end{bmatrix}$$

$$D_{u_i} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad D_{\boldsymbol{V}_i} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad D_{\boldsymbol{I}_i} = \begin{bmatrix} -X_i' & 0 \\ 0 & -X_i' \end{bmatrix}$$

If the calculation of these coefficient matrices is added to the `generator` class as a method named `get_linear_matrix`, the program **??** is obtained.

```
classdef generator < handle

  properties
(Same as lines 4-11 in program 3-23)
    x_equilibrium
    V_equilibrium
    I_equilibrium
  end

  methods
```

```
(Same as lines 7 through 21 in program 3-34)

    function x_equilibrium = set_equilibrium(obj, V, I, P, Q)

(Same as lines 10 through 23 of program 3-28)

      obj.x_equilibrium = x_equilibrium;
      obj.V_equilibrium = V;
      obj.I_equilibrium = I;
    end

    function [A, Bu, BV, BI, C, Du, DV, DI] =...
        get_linear_matrix(obj)

      X = obj.X;
      X_prime = obj.X_prime;
      D = obj.D;
      M = obj.M;
      tau = obj.tau;

      omega0 = obj.omega0;
      delta = obj.x_equilibrium(1);
      E = obj.x_equilibrium(3);
      V = obj.V_equilibrium;
      Vabs = abs(obj.V_equilibrium);
      Vangle = angle(obj.V_equilibrium);
      I = obj.I_equilibrium;
      A = [0, omega0, 0;
        0, -D/M, 0;
        -(X/X_prime-1)*Vabs*sin(delta-Vangle)/tau,...
        0, -X/X_prime/tau];
      Bu = [0; 1/M; 0];
      BV = [0, 0;
        -real(I)/M, -imag(I)/M;
        (X/X_prime-1)*cos(delta)/tau,...
        (X/X_prime-1)*sin(delta)/tau];
      BI = [0, 0;
        -real(V)/M, -imag(V)/M;
        0, 0];
      C = [E*cos(delta), 0, sin(delta);
        E*sin(delta), 0, -cos(delta)];
      Du = [0; 0];
      DV = [0, -1; 1, 0];
      DI = -X_prime*eye(2);
    end

  end

end
```

Program 1.1 generator.m

In Program **??**, lines 18 to 20 in `set_equilibrium` store information about the equilibrium point used in the calculation of the approximate linear model.

If implemented similarly for the constant impedance load model, the program **??** is obtained.

```
classdef load_impedance < handle

  properties
    z
    I_equilibrium
  end

  methods

(Same as lines 7 through 18 in program 3-35)

    function x_equilibrium = set_equilibrium(obj, V, I, P, Q)
      x_equilibrium = [];
      obj.z = -V/I;
      obj.I_equilibrium = I;
    end

    function [A, Bu, BV, BI, C, Du, DV, DI] =...
        get_linear_matrix(obj)

      A = [];
      Bu = zeros(0, 2);
      BV = zeros(0, 2);
      BI = zeros(0, 2);
      C = zeros(2, 0);
      I = obj.I_equilibrium;
      z = obj.z;
      Du = [real(z)*real(I), imag(z)*imag(I);
        real(z)*imag(I), imag(z)*real(I)];
      DV = eye(2);
      DI = [real(z), -imag(z); imag(z), real(z)];
    end

  end

end
```

Program 1.2 load_impedance.m

By using the class of equipment such as modified generators and loads, the function for obtaining an approximate linear model can be described as shown in Program **??**.

```
function sys = get_linear_model(a_component, Y)

  A = cell(numel(a_component), 1);
  Bu = cell(numel(a_component), 1);
  BV = cell(numel(a_component), 1);
  BI = cell(numel(a_component), 1);
```

```
7    C = cell(numel(a_component), 1);
8    Du = cell(numel(a_component), 1);
9    DV = cell(numel(a_component), 1);
10   DI = cell(numel(a_component), 1);
11
12   for k = 1:numel(a_component)
13     component = a_component{k};
14     [A{k}, Bu{k}, BV{k}, BI{k}, C{k}, Du{k}, DV{k}, DI{k}] =...
15       component.get_linear_matrix();
16   end
17
18   A = blkdiag(A{:});
19   Bu = blkdiag(Bu{:});
20   BV = blkdiag(BV{:});
21   BI = blkdiag(BI{:});
22   C = blkdiag(C{:});
23   Du = blkdiag(Du{:});
24   DV = blkdiag(DV{:});
25   DI = blkdiag(DI{:});
26
27   Ymat = zeros(size(Y, 1)*2, size(Y, 2)*2);
28   Ymat(1:2:end, 1:2:end) = real(Y);
29   Ymat(2:2:end, 1:2:end) = imag(Y);
30   Ymat(1:2:end, 2:2:end) = -imag(Y);
31   Ymat(2:2:end, 2:2:end) = real(Y);
32
33   nx = size(A, 1);
34
35   A11 = A;
36   A12 = [BV, BI];
37   A21 = [C; zeros(size(Ymat, 1), nx)];
38   A22 = [DV, DI; Ymat, -eye(size(Ymat))];
39
40   B1 = Bu;
41   B2 = [Du; zeros(size(Ymat, 1), size(Du, 2))];
42
43
44   Aout = A11 - A12/A22*A21;
45   Bout = B1 - A12/A22*B2;
46   Cout = eye(nx);
47   Dout = 0;
48
49   sys = ss(Aout, Bout, Cout, Dout);
50
51 end
```

Program 1.3 get_linear_model.m

In lines 12 to 16 of Program **??**, the coefficient matrix of the approximate linear model is obtained from each equipment. Additionally, by eliminating the voltage and current phases of all buses from lines 27 to 47, an expression for the approximate linear model's system of ordinary differential equations is obtained.

The approximate linear model can be used as follows by using Program **??**.

```
1 (Same as lines 1 through 23 in Program 3-30)
2
3 sys = get_linear_model(a_component, Y);
4
5 sys = sys(2, 1);
6 nyquist(sys)
```

Program 1.4 main_linearization.m

In this example, an approximate linear model is constructed in line 5 with the mechanical input $P_{\mathrm{mech}1}$ of generator 1 as input and the frequency deviation $\Delta\omega_1$ of generator 1 as output. In addition, a Nyquist plot is drawn in line 6.

In the mathematical analysis of Section **??**, an approximate linear model is derived from a nonlinear system of ordinary differential equations where all buses are Kron reduced. On the other hand, in the numerical implementation of this section, the nonlinear differential-algebraic equation system is first linearized, and then Kron reduction is applied to construct the ordinary differential equation system. It should be noted that this is because in the power system model with Kron reduction, expressions generally involve a mixture of information about equipment, buses, and transmission lines.

To increase the readability and expandability of the program, it is important to modularize each element appropriately, as in the implementation of this section.

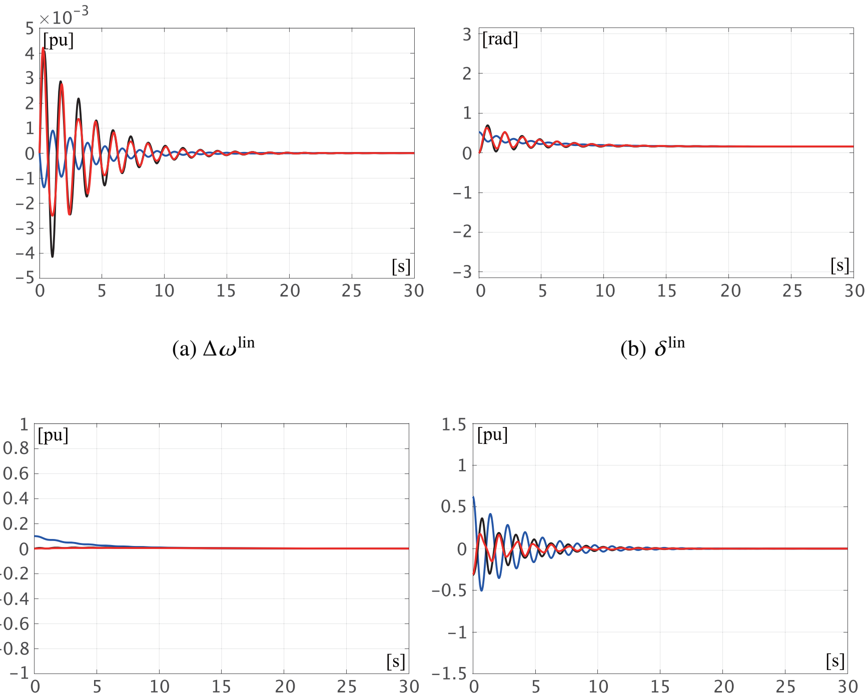
2.2 Numerical analysis of small signal stability

Let us perform a stability analysis based on approximate linearization for an actual electrical power system model consisting of three generators.

Example 1.2 Numerical stability analysis of the linearized model Let us consider an electrical power system model consisting of three generators discussed in the Example **??**. The constant of the generators and transmission lines are set to the same value as in the Example **??**, and a linear approximation model for Equation **??** is derived with the approximate of the steady value shown in **??**. Figure **??** shows the time response When the initial values are set as follows to correspond to Equation **??**:

$$\delta^{\mathrm{lin}}(0) = \begin{bmatrix} \frac{\pi}{6} \\ 0 \\ 0 \end{bmatrix}, \quad \Delta\omega^{\mathrm{lin}}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad E^{\mathrm{lin}}(0) = \begin{bmatrix} 0.1 \\ 0 \\ 0 \end{bmatrix} \tag{14}$$

The blue, black, and red lines represent generators 1, 2, and 3, respectively. From this figure, we can see that the internal state of the generator group converges asymptotically as given in (**??**). Moreover, it approximately reproduces the initial value response of the nonlinear model shown in Fig. **??**.



(a) $\Delta\omega^{\mathrm{lin}}$ (b) δ^{lin}

(c) E^{lin} (d) P^{lin}

Fig. 2 Initial value response of approximate linear model
(Blue: Generator 1, Black: Generator 2, Red: Generator 3)

Next, we parameterize the constants and steady-state values of the generators and transmission lines to analyze the stability of the resulting approximate linear model. For the generator constants, we compare the cases where all the damping coefficients are set to 10 and where they are set to 0.1, i.e., we consider the two cases:

$$(D_1, D_2, D_3) = (10, 10, 10), \qquad (D_1, D_2, D_3) = (0.1, 0.1, 0.1)$$

Other constants are set to the values in Table **??**. In addition, the steady-state values of the rotor angle differences are expressed in terms of a parameter $\theta_1 \in [0, 1]$ as follows:

$$\delta_{12}^{\star} = -\frac{\pi}{2}\theta_1, \qquad \delta_{13}^{\star} = \frac{\pi}{2}\theta_1 \tag{15}$$

Here, θ_1 is a parameter that specifies the magnitude of the rotor angle difference in the steady-state. By varying this value, the system matrix in **??** changes. Note that the steady-state values of the internal voltages are not changed from the values in Table **??**.

The admittance matrix is also modified as follows. Using the admittance values $\boldsymbol{y}_{12}$ and $\boldsymbol{y}_{23}$ in Equation **??**, the admittance matrix of the power system in Equation **??** is constructed. The real part of this admittance matrix, which is the conductance matrix, is denoted as G_0, and the imaginary part, which is the susceptance matrix, is denoted as B_0. Specifically,

$$\begin{aligned} G_0 &= \begin{bmatrix} 1.3652 & -1.3652 & 0 \\ -1.3652 & 3.3074 & -1.9422 \\ 0 & -1.9422 & 1.9422 \end{bmatrix}, \\ B_0 &= \begin{bmatrix} -11.6041 & 11.6041 & 0 \\ 11.6041 & -22.1148 & 10.5107 \\ 0 & 10.5107 & -10.5107 \end{bmatrix} \end{aligned} \tag{16}$$

Using the parameter $\theta_2 \in [0, 5]$, we express the reference admittance matrix as follows:

$$\boldsymbol{Y}_0(\theta_2) := \theta_2 G_0 + \boldsymbol{j} B_0 \tag{17}$$

Here, θ_2 is a parameter that specifies the size of the real part (conductance matrix). For comparison, we consider two parameterized admittance matrices:

$$\boldsymbol{Y} = \boldsymbol{Y}_0(\theta_2), \qquad \boldsymbol{Y} = \frac{\boldsymbol{Y}_0(\theta_2)}{100}$$

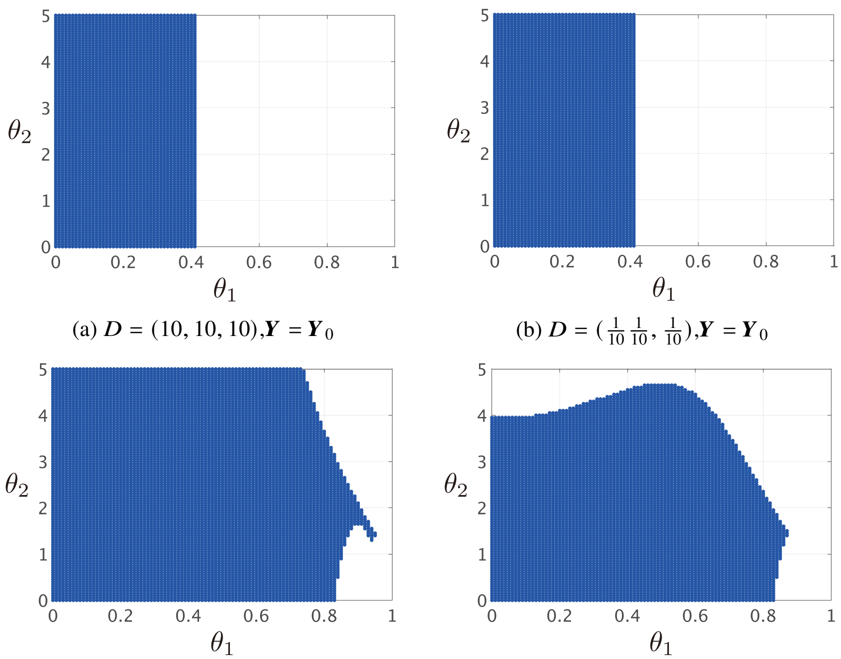
The changes in the admittance matrix are approximately represented in the linearized model by the changes in the values of the reduced conductor $B^{\mathrm{red}}ij$ and the reduced susceptance $G^{\mathrm{red}}ij$ in equation **??**. The parameter settings for the comparison are summarized in Table **??**.

Let us numerically analyze the stability of the approximate linear model by varying the parameters (θ_1, θ_2) for each case (a)-(d) in Table **??**. Specifically, we will check whether the approximate linear model is stable or not by examining the eigenvalues of Ψ in Equation **??** by varying θ_1 and θ_2 on a grid of 100 equidistant points each. The results are shown in Figure **??**. The blue area represents the parameter region where the approximate linear model is stable. First, in the case of (a), we see that the approximate linear model is stable regardless of the size of the conductance matrix specified by θ_2, as long as θ_1 is below approximately 0.4, which corresponds to a rotor angle difference of approximately 36° in the steady state. The same result is obtained for case (b), where the generator's damping coefficient is small at 0.1.

Next, we examine the results for cases (c) and (d), where the admittance matrix is multiplied by $\frac{1}{100}$. In this case, we find that when θ_2 is small and the size of the conductance matrix is around 1, the approximate linear model is stable as long as the rotor angle difference in the steady state is below approximately 76°. We also find that as θ_2 increases to 2 or more, the upper limit of the rotor angle difference for stability of the approximate linear model decreases.

Table 1 Parameter settings to compare

	$D = (10, 10, 10)$	$D = (0.1, 0.1, 0.1)$
$\boldsymbol{Y} = \boldsymbol{Y}_0$	(a)	(b)
$\boldsymbol{Y} = \boldsymbol{Y}_0/100$	(c)	(d)



(a) $D = (10, 10, 10), \boldsymbol{Y} = \boldsymbol{Y}_0$

(b) $D = (\frac{1}{10} \frac{1}{10}, \frac{1}{10}), \boldsymbol{Y} = \boldsymbol{Y}_0$

(c) $D = (10, 10, 10), \boldsymbol{Y} = \frac{\boldsymbol{Y}_0}{100}$

(d) $D = (\frac{1}{10}, \frac{1}{10}, \frac{1}{10}), \boldsymbol{Y} = \frac{\boldsymbol{Y}_0}{100}$

Fig. 3 Area of parameters where the approximate linear model is stable

3 Mathematical stability analysis of the linearized model‡

3.1 Small signal stability of the linearized model‡

In this section, we mathematically analyze the stability of the linearized model given in equation **??**. The stability is characterized by the eigenvalues of the matrix Ψ. However, as discussed in section **??**, Ψ is not regular and the eigenspace for the zero eigenvalue is given by

$$\mathcal{M} = \mathsf{span}\left\{\begin{bmatrix} \mathbb{1} \\ 0 \\ 0 \end{bmatrix}\right\} \tag{18}$$

This eigenspace represents the set of equivalent steady-state values obtained by varying the phase angles of all generators while maintaining their relative values constant. Therefore, it is not a problem which point in the equilibrium set of Equation **??** the state of the approximate linear model converges to. Based on this fact, we give the following definition:

COFFEE BREAK

Eigenspace of a square matrix: For a square matrix A and a given eigenvalue λ, the *eigenspace* $\mathcal{V}_\lambda$ is defined as:

$$\mathcal{V}_\lambda := \mathsf{ker}(\lambda I - A)$$

where ker denotes the kernel of the matrix. If $v_1, \ldots, v_k$ are all linearly independent eigenvectors corresponding to a specific eigenvalue λ, then:

$$\mathcal{V}_\lambda = \mathsf{span}\{v_1, \ldots, v_k\}$$

which means that it is a linear space spanned by the eigenvectors associated with a specific eigenvalue.

Definition 1.1 (Small-signal stability of the linear approximation model)
Consider the linearized model given by Equation **??**. For any initial values, if the internal state converges to one of the equilibrium points in the set $\mathcal{M}$ defined by Equation **??**, the linearized model is said to be **steady-state stable**.

The small-signal stability in Definition **??** means that for any initial condition, Equation **??** holds. Note that the value of c_0 in Equation **??** is arbitrary, so we express its arbitrariness as "converging to one of the equilibrium points in $\mathcal{M}$."

In power system engineering, the term "small-signal stability" is widely used to discuss the stability of a power system against small disturbances using an approximate linear model. However, introducing a mathematical definition like Definition **??** is not common practice.

In the following discussion, we assume that the kernel space of Ψ in Equation **??** is one-dimensional and that Equation **??** holds:

$$\mathsf{ker}\,\Psi = \mathcal{M} \tag{19}$$

It is clear from the structure of the matrix Ψ that $\mathcal{M}$ is a subset of $\mathsf{ker}\,\Psi$, but it should be noted that we are assuming that $\mathsf{ker}\,\Psi$ is one-dimensional and that the equality holds. If the kernel space were two-dimensional or greater, the invariant eigenspace would be larger than $\mathcal{M}$, and the approximate linear model would not be in a steady state stability. Therefore, equation **??** is a necessary condition for the steady-state stability of the linear approximation model. In particular, when A is invertible, and using the definition of $L_0 := L - CA^{-1}B$, the necessary condition can be equivalently expressed as:

$$\mathsf{ker}\,L_0 = \mathsf{span}\,\{\mathbb{1}\} \tag{20}$$

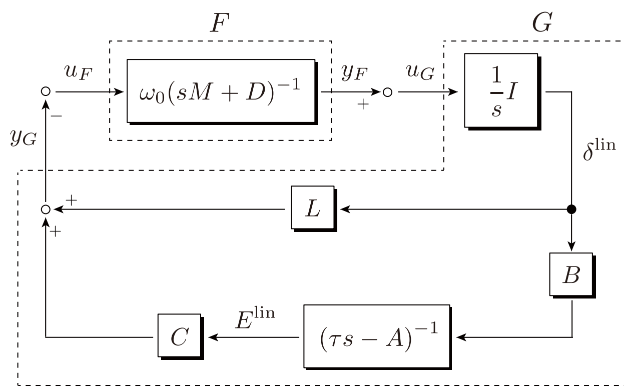


Fig. 4 Feedback system representation of approximate linear models

Note that this matrix L_0 plays an important role in the subsequent analysis.

The relationship between equation **??** and equation **??** can be verified as follows. Because the $(1,2)$ block of the matrix Ψ is invertible, the necessary and sufficient condition for the kernel of Ψ to be equal to $\mathcal{M}$ is:

$$\ker \begin{bmatrix} -L & -C \\ B & A \end{bmatrix} = \operatorname{span} \left\{ \begin{bmatrix} \mathbb{1} \\ 0 \end{bmatrix} \right\}$$

In particular, when A is invertible, we have

$$\begin{bmatrix} -L & -C \\ B & A \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \qquad \Longleftrightarrow \qquad L_0 x = 0, \qquad y = -A^{-1}Bx$$

Thus, the necessary condition is equivalent to equation **??**.

For the following discussion, we introduce the following fundamental terminology.

Definition 1.2 (Stability of square matrix) For a square matrix A, if all the real parts of its eigenvalues are negative, A is called **stable** or **asymptotically stable**.

3.2 Passivity of approximate linear models‡

3.2.1 Representation of Approximate Linear Models with Feedback

Let us consider describing the approximate linear model of Equation **??** as a feedback system of two subsystems (Figure **??**). The first subsystem is described as a system of differential equations for the deviation of the angular frequency:

$$F : \begin{cases} M\Delta\dot{\omega}^{\mathrm{lin}} = -D\Delta\omega^{\mathrm{lin}} + u_F \\ \quad y_F = \omega_0 \Delta\omega^{\mathrm{lin}} \end{cases} \tag{21}$$

In this book, we refer to this subsystem as the **mechanical subsystem**. The mechanical subsystem is determined solely by the physical constants of the generator set $(M_i, D_i)i \in \mathcal{I}\mathrm{G}$ and the reference angular frequency ω_0, and does not depend on the steady-state values of the internal state $(\delta^\star, E^\star)$.

The second subsystem is described as a system of differential equations with respect to the rotor angle and internal voltage as follows:

$$G : \begin{cases} \quad \dot{\delta}^{\mathrm{lin}} = u_G \\ \tau\dot{E}^{\mathrm{lin}} = AE^{\mathrm{lin}} + B\delta^{\mathrm{lin}} \\ \quad y_G = CE^{\mathrm{lin}} + L\delta^{\mathrm{lin}} \end{cases} \tag{22}$$

We call this subsystem the **electrical subsystem** [1]. The electrical subsystem not only depends on the physical constants of the generator group, $(\tau_i)i \in \mathcal{I}\mathrm{G}$, but also on the steady-state values of the internal states, $(\delta^\star, E^\star)$. In fact, the system matrix (L, A, B, C) in equation **??** is a function of $(\delta^\star, E^\star)$.

If the two subsystems' inputs and outputs are coupled through negative feedback as follows:

$$u_F = -y_G, \qquad u_G = y_F \tag{23}$$

the approximate linear model of Equation **??** is represented. The subsequent analysis of steady-state stability is based on the property called the *passivity* of the mechanical and electrical subsystems. It is well-known that a negative feedback system of a passive subsystem is stable.

3.2.2 Passivity of the mechanical subsystem

The mechanical subsystem F in Equation **??** has the following strong passivity:

Definition 1.3 (Passivity of linear systems)
Consider the linear system

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \tag{24}$$

Using the symmetric matrix P, define the function:

$$W(x) := \frac{1}{2}x^{\mathsf{T}}Px \tag{25}$$

---

[1] The terms "mechanical subsystem" and "electrical subsystem" introduced here are unique to this book.

For any u, if there exists a semi-positive definite matrix P that satisfies:

$$\frac{d}{dt}W\big(x(t)\big) \leq u^{\mathsf{T}}(t)y(t), \qquad \forall t \geq 0 \tag{26}$$

Then, we call Σ **passive**.

In particular, when there exists a positive definite number ρ such that in addition to the semi-positive definite matrix, the inequality:

$$\frac{d}{dt}W\big(x(t)\big) \leq u^{\mathsf{T}}(t)y(t) - \rho \left\|y(t)\right\|^2, \qquad \forall t \geq 0 \tag{27}$$

is satisfied, then we call Σ **strictly passive**.

The function $W(x)$ in Definition **??** is generally called the **storage function**. Moreover, the inequality in Equation **??** describes a type of passivity that is more strictly called **output-strict passivity**, where the energy represented by the storage function $W(x)$ dissipates more quickly in proportion to the square of the output compared to the passivity in Equation **??**.

The mechanical subsystem F of Equation**??** having strong passivity can be confirmed as follows. First, the subsystem is written as below:

$$F : \begin{cases} \dot{x}_F = A_F x_F + B_F u_F \\ y_F = C_F x_F \end{cases} \tag{28}$$

where state x_F represents $\Delta\omega^{\mathrm{lin}}$, and the system matrices are:

$$A_F := -M^{-1}D, \qquad B_F := M^{-1}, \qquad C_F := \omega_0 I$$

Additionally, we define the symmetric matrix P_F as:

$$P_F := \omega_0 M$$

and since M is positive definite, P_F is also positive definite. Then, the following inequalities hold:

$$A_F^{\mathsf{T}} P_F + P_F A_F \leq -\frac{2\min\{D_i\}}{\omega_0} C_F^{\mathsf{T}} C_F, \qquad P_F B_F = C_F^{\mathsf{T}}$$

Therefore, when the storage function is defined as:

$$W_F(x_F) := \frac{1}{2} x_F^{\mathsf{T}} P_F x_F \tag{29}$$

the time derivative along the solution trajectory of F is evaluated as:

$$
\begin{aligned}
\frac{d}{dt}W_F\big(x_F(t)\big) &= \nabla W_F^{\mathsf{T}}(x_F)\frac{dx_F}{dt} \\
&= \big(P_F x_F(t)\big)^{\mathsf{T}}\big(A_F x_F(t) + B_F u_F(t)\big) \\
&= y_F^{\mathsf{T}}(t)u_F(t) + \frac{1}{2}x_F^{\mathsf{T}}(t)\left(A_F^{\mathsf{T}}P_F + P_F A_F\right)x_F(t) \\
&\leq y_F^{\mathsf{T}}(t)u_F(t) - \tfrac{\min\{D_i\}}{\omega_0}\|y_F(t)\|^2
\end{aligned}
\tag{30}
$$

where $\nabla W_F(x_F)$ is the gradient function obtained by partially differentiating $W_F(x_F)$ with respect to its elements and arranging them vertically. From this, it can be seen that the machine subsystem F in Equation **??** is strongly passive for any positive definite $(M_i, D_i)_{i\in\mathcal{I}_{\mathrm{G}}}$. Note that the function $W_F(x_F)$ represents the mechanical kinetic energy of the power system.

3.2.3 Passivity of the electrical subsystem

Next, we consider the electrical subsystem G in Equation **??**. Unlike the mechanical subsystem F, the electrical subsystem G only possesses passivity under limited conditions. While this may seem arbitrary, we consider the case where all reduced conductances in Equation **??** are zero, in other words:

$$G_{ij}^{\mathrm{red}} = 0, \qquad \forall(i,j) \in \mathcal{I}_{\mathrm{G}} \times \mathcal{I}_{\mathrm{G}} \tag{31}$$

Excluding special cases, the condition in Equation **??** only holds when the conductance of all transmission lines in the power system is zero, or equivalently, the resistance of all transmission lines is zero. In this case, for the functions $k_{ij}(\delta_{ij})$ and $h_{ij}(\delta_{ij})$ defined in Equation **??**, the following holds:

$$k_{ij}(\delta_{ij}^{\star}) = k_{ji}(\delta_{ji}^{\star}), \qquad h_{ij}(\delta_{ij}^{\star}) = -h_{ji}(\delta_{ji}^{\star}), \qquad h_{ii}(\delta_{ii}^{\star}) = 0$$

Therefore, for the system matrix (L, A, B, C) in Equation **??**, it holds that:

$$L = L^{\mathsf{T}}, \qquad \hat{A} = \hat{A}^{\mathsf{T}}, \qquad C = -\hat{B}^{\mathsf{T}} \tag{32}$$

In the following, we use the symmetric structure of this special system matrix to analyze the passivity of the electrical subsystem.

First, let us express the electrical subsystem G of Equation **??** as follows:

$$G : \begin{cases} \dot{x}_G = A_G x_G + B_G u_G \\ y_G = C_G x_G \end{cases} \tag{33}$$

where the state x_G is a column vector obtained by concatenating δ^{lin} and E^{lin}, and Ω is a positive definite diagonal matrix defined as follows:

$$\Omega := \mathrm{diag}\left(\sqrt{\frac{X_i - X_i'}{\tau_i}}\right)_{i \in \mathcal{I}_G}$$

The system matrices are expressed as:

$$A_G := \begin{bmatrix} 0 & 0 \\ \Omega^2 \hat{B} & \Omega^2 \hat{A} \end{bmatrix}, \qquad B_G := \begin{bmatrix} I \\ 0 \end{bmatrix}, \qquad C_G := \begin{bmatrix} L & -\hat{B}^\mathsf{T} \end{bmatrix}$$

Furthermore, we define the symmetric matrix P_G as follows:

$$P_G := \begin{bmatrix} L & -\hat{B}^\mathsf{T} \\ -\hat{B} & -\hat{A} \end{bmatrix} \tag{34}$$

The following inequalities hold for these matrices:

$$A_G^\mathsf{T} P_G + P_G A_G \le 0, \qquad P_G B_G = C_G^\mathsf{T} \tag{35}$$

If we calculated the left of the inequality, it can be expressed as follows using a symmetric matrix $\hat{A}_\Omega := \Omega \hat{A} \Omega$:

$$\frac{A_G^\mathsf{T} P_G + P_G A_G}{2} = \begin{bmatrix} \Omega \hat{B} & 0 \\ 0 & \Omega^{-1} \end{bmatrix}^\mathsf{T} \underbrace{\begin{bmatrix} -I & -\hat{A}_\Omega \\ -\hat{A}_\Omega & -\hat{A}_\Omega^2 \end{bmatrix}}_{Y} \begin{bmatrix} \Omega \hat{B} & 0 \\ 0 & \Omega^{-1} \end{bmatrix}$$

Here, the top-left block $-I$ of Y is negative definite, and the Schur complement of Y with respect to $-I$ is 0, which implies that Y is negative semi-definite. Therefore, the matrix inequality in Equation **??** holds.

COFFEE BREAK

Schur complement:
Let a symmetric matrix M be partitioned as

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^\mathsf{T} & M_{22} \end{bmatrix}$$

Then, the **Schur complement** of M with respect to M_{22} is defined as:

$$M/M_{22} := M_{11} - M_{12} M_{22}^{-1} M_{12}^\mathsf{T}$$

Similarly, the Schur complement of M with respect to M_{11} is defined as:

$$M/M_{11} := M_{22} - M_{12}^\mathsf{T} M_{11}^{-1} M_{12}$$

If the matrix M_{22} is positive definite, then M is positive semidefinite if and only if M/M_{22} is positive semidefinite. The same fact holds for the Schur complement of M with respect to M_{11} [**?**]. The same fact holds if we replace positive semidefinite with positive definite.

Properties of semidefinite matrices:

For any positive semidefinite (or negative semidefinite) matrix $Y \in \mathbb{R}^{n\times n}$ and any matrix $X \in \mathbb{R}^{n\times m}$, the matrix $X^\mathsf{T} Y X$ is positive semidefinite (or negative semidefinite). This can be shown from the fact that

$$v^\mathsf{T} Y v \geq 0, \quad \forall v \in \mathbb{R}^n \qquad \Longrightarrow \qquad (Xw)^\mathsf{T} Y (Xw) \geq 0, \quad \forall w \in \mathbb{R}^m$$

By using the relationship given by Equation **??**, the time derivative of the storage function $W_G(x_G)$ along the solution trajectory of G can be evaluated, where $W_G(x_G)$ is defined by Equation **??**, similarly to Equation **??**:

$$W_G(x_G) := \frac{1}{2} x_G^\mathsf{T} P_G x_G \tag{36}$$

$$\frac{d}{dt} W_G\big(x_G(t)\big) \leq y_G^\mathsf{T}(t) u_G(t) \tag{37}$$

However, to show the passivity of G, P_G in equation **??** must be positive semi-definite. If the matrix A in equation **??** is stable, then it can be shown that:

$$A = S^2 \hat{A} \qquad \Longleftrightarrow \qquad S^{-1} A S = S \hat{A} S$$

where $S := \mathsf{diag}\left(\sqrt{X_i - X_i'}\right) i \in \mathcal{I}$G. Here, $\hat{A}$ is negative definite. Under this condition, the necessary and sufficient condition for P_G in Equation **??** to be positive semi-definite is that the Schur complement of $-\hat{A}$ is positive semi-definite, in other words:

$$L_0 = L_0^\mathsf{T} \geq 0 \tag{38}$$

where L_0 is defined by Equation **??** and can be expressed as $L_0 = L + \hat{B}^\mathsf{T} \hat{A}^{-1} \hat{B}$ using Equation **??**. To summarize the above discussion, the following definition is introduced.

Definition 1.4 (Passive power transmission condition) For the system matrix (L, A, B, C) of Equation **??**, the following three conditions are together called **passive power transmission conditions**. [2]

(i) Matrix A is stable.
(ii) As in Equation **??**, all reduced conductances are zero.
(iii) For the matrix L_0 of Equation **??**, the matrix inequality of Equation **??** holds.

Each of these conditions may be referred to individually as the passive transmission condition (i), and so on.

Based on the above discussions, we can see that the passive power transmission conditions describe the conditions necessary for the electrical system G of Equation

[2] "Passive power transmission conditions" is a term unique to this book.

?? to be passive. Furthermore, these conditions are necessary for the linear approximation model to be statically stable for the passivity of an electrical subsystem and arbitrary physical constant. The details are discussed in Section **??** and Section **??**. Function $W_G(x_G)$ indicates the electrical potential energy of an electrical power system.

3.3 Analysis of small signal stability based on passivity[‡]

3.3.1 Stability analysis of feedback systems

In the following, under the passive power transmission conditions defined in Definition **??**, the small signal stability of the linear approximation model given in Equation **??** is analyzed for electric subsystems that are passive. The stability of their feedback systems is also analyzed.

Since the inequalities in Equations **??** and **??** hold, their sum is given by:

$$\begin{aligned} &\frac{d}{dt}\left\{W_F\big(x_F(t)\big) + W_G\big(x_G(t)\big)\right\} \\ &\qquad\qquad \leq \underbrace{y_F^{\mathsf{T}}(t)u_F(t) + y_G^{\mathsf{T}}(t)u_G(t)}_{\star} - \tfrac{\min\{D_i\}}{\omega_0}\|y_F(t)\|^2 \end{aligned}$$

By substituting the feedback coupling equation in Equation **??** into this inequality, the term indicated by "$\star$" is cancelled out, and the inequality for the entire feedback system can be expressed as:

$$\frac{d}{dt}\left\{W_F\big(x_F(t)\big) + W_G\big(x_G(t)\big)\right\} \leq -\tfrac{\min\{D_i\}}{\omega_0}\|y_F(t)\|^2 \tag{39}$$

In other words, the sum of the functions $W_F(x_F)$ and $W_G(x_G)$ is monotonically non-increasing with respect to the time evolution along the feedback system trajectory. Furthermore, since the lower bounds of $W_F(x_F)$ and $W_G(x_G)$ are both 0, their sum asymptotically converges to a certain value as time passes sufficiently. This means that the value of the time derivative on the left-hand side of Equation **??** converges to 0. Additionally, since the right-hand side of Equation **??** is negative when $y_F(t) \neq 0$ and is only 0 when $y_F(t) = 0$, the following is obtained:

$$\lim_{t\to\infty} y_F(t) = 0 \tag{40}$$

Furthermore, focusing on the output equation of Equation **??**, since the output y_F is a constant multiple of the internal state $\Delta\omega^{\mathrm{lin}}$, it can be understood that for the mechanical subsystem F:

$$y_F(t) = 0, \quad \forall t \geq 0 \qquad \Longrightarrow \qquad \Delta\omega^{\mathrm{lin}}(t) = 0, \quad \forall t \geq 0 \tag{41}$$

This is a property called **observability** in system control engineering. Therefore, from Equations **??** and **??**, for the approximated linear model of Equation **??**, for any initial value $(\Delta\omega^{\text{lin}}(0), \delta^{\text{lin}}(0), E^{\text{lin}}(0))$, the following holds:

$$\lim_{t\to\infty} \Delta\omega^{\text{lin}}(t) = 0 \tag{42}$$

In other words, the internal state of the mechanical subsystem F in Equation **??** in the feedback system converges to 0 asymptotically.

COFFEE BREAK

Observability: For a linear system Σ described by Equation **??**, if the output $y(t)$ is identically zero, then the internal state $x(t)$ is also identically zero, and Σ is said to be **observable** in this case. The necessary and sufficient condition for Σ to be observable is given by Equation **??**, where n is the dimension of the state.

$$\mathsf{ker} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = \{0\} \tag{43}$$

In some contexts, a pair of matrices that satisfies Equation **??** is referred to as an observable pair (C, A).

Controllability: For a linear system Σ described by Equation **??**, if there exists an input $u(t)$ such that for all initial states $x(0)$, there exists a time $T > 0$ such that $x(T) = 0$, then Σ is said to be **controllable** . The necessary and sufficient condition for Σ to be controllable is given by Equation **??**, where n is the dimension of the state.

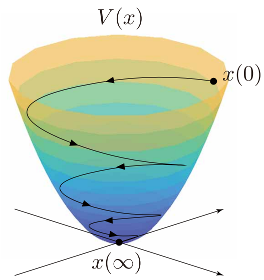
$$\mathsf{im} \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = \mathbb{R}^n \tag{44}$$

Here, im denotes the **image** of the matrix. In some contexts, a pair of matrices that satisfies Equation **??** is referred to as a controllable pair (A, B)

Lyapunov function: Consider the observable linear system Σ described by Equation **??**, where the input $u(t)$ is identically zero. Let $V(x)$ be a positive semi-definite function that satisfies $V(x) \geq 0$ for all x and $V(0) = 0$. If there exists a positive constant ρ such that the derivative of $V(x)$ along the solution trajectory of Σ satisfies

$$\frac{d}{dt}V\big(x(t)\big) = \nabla V^{\mathsf{T}}(x)\frac{dx}{dt}(t) \leq -\rho\|y(t)\|^2, \qquad \forall t \geq 0$$

then the solution trajectory $x(t)$ converges asymptotically to zero for any initial state. Such a function $V(x)$ is called a **Lyapunov function** .



[h]

The fact that the value of the Lyapunov function decreases monotonically along the solution trajectory of the system can be interpreted as a type of energy dissipation over time (Figure **??**). Similar stability analyses based on Lyapunov functions can also be applied to nonlinear systems.

On the one hand, it is not possible to deduce from the above discussion whether the internal state of the electric subsystem G in Equation **??** converges asymptotically to 0. Specifically, from the relation in Equation **??** and the asymptotic convergence in Equation **??**, it can be derived that the input and output of the two subsystems satisfy:

$$\lim_{t\to\infty} u_F(t) = 0, \qquad \lim_{t\to\infty} u_G(t) = 0, \qquad \lim_{t\to\infty} y_G(t) = 0$$

However, since the electric subsystem is not observable, it cannot be concluded that its internal state converges asymptotically to zero. Assuming that the electric subsystem is observable, it can be concluded that for any initial values:

$$\lim_{t\to\infty} \delta^{\mathrm{lin}}(t) = 0, \qquad \lim_{t\to\infty} E^{\mathrm{lin}}(t) = 0$$

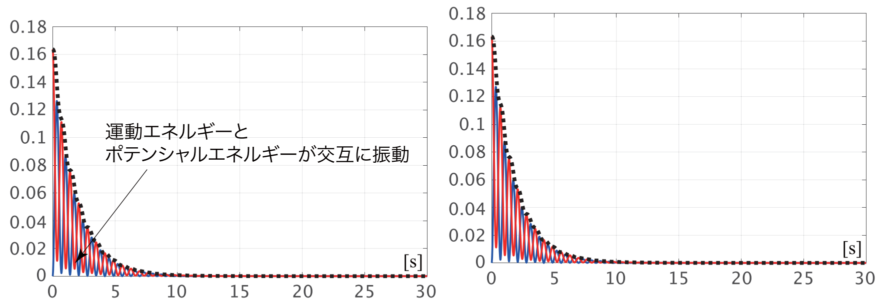
However, this implies that c_0 must always be equal to 0 in Equation **??**. This fact contradicts the instability of Equation **??** due to the zero eigenvalue of Ψ. It should be noted that except for some special cases, the electric subsystem is controllable.

Example 1.3 Time evolution of stored energy Consider the approximated linear model discussed in the first half of Example **??**. First, consider the case where the passive transmission condition (ii) is satisfied, i.e., when the conductance of both transmission lines is 0. Specifically, set the admittance values of the transmission lines to:

$$\boldsymbol{y}_{12} = -\boldsymbol{j}11.6041, \qquad \boldsymbol{y}_{23} = -\boldsymbol{j}10.5107 \tag{45}$$

This corresponds to setting the parameter θ_2 to 0 in Equation **??**. In this case, the matrix A becomes stable. Also, all the eigenvalues of L_0 in Equation **??** become non-negative. Thus, the passive transmission conditions (i) and (iii) hold.

Consider the time response of the initial values in Equation **??** and calculate the time variation of the kinetic energy $W_F(x_F)$ in Equation **??** and the potential



(a) If passive transmission condition (ii) is met

(b) If passive transmission condition (ii) is not met

Fig. 5 Time variation of the accumulation function according to Example ??
(Blue: W_F, Red: W_G, Black:$W_F + W_G$)

energy $W_G(x_G)$ in Equation **??**. The calculation results are shown in Figure **??**(a) where the blue and red solid lines represent $W_F(x_F)$ and $W_G(x_G)$, respectively, and the black dashed line represents their sum, which is the total energy of the system. From this figure, it can be seen that while the kinetic and potential energies increase and decrease alternatively, the total energy of the system, which is the sum of these energies, decreases monotonically. The decrease in total energy over time can be interpreted as energy loss due to friction caused by the damping coefficient.

Next, as a reference, let us show the results when the passive power transmission condition (ii) is not satisfied. Specifically, we set $\boldsymbol{Y}_0(1)$ in Equation **??** to be the admittance matrix $\boldsymbol{Y}$ by setting θ_2 to 1. This is equivalent to calculating the time variation of the kinetic and potential energies for the initial value response in Figure **??**. Note that when the passive power transmission condition (ii) is not satisfied, P_G in Equation **??** does not become a symmetric matrix, but the potential energy $W_G(x_G)$ can still be calculated using the definition in Equation **??**. The calculation result in Figure **??**(b) is almost identical to that in Figure **??**(a). This fact suggests that even when the conductance of the transmission line is not zero, the electrical potential energy can be approximately calculated based on the definition in Equation **??**.

3.3.2 Basis transformation for separating unobservable state variables

Let us consider deriving an observable subsystem from the electrical subsystem G in Equation **??** by removing the common component of unobservable rotor angles. Specifically, we apply a basis transformation to the state δ^{lin} in Equation **??** to derive a set of differential equations describing only the deviations of the rotor angles. Note that the following basis transformation is always applicable regardless of whether the passive power delivery condition holds or not.

COFFEE BREAK

Basis transformation of linear systems:
In the state equation:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

each element $x_i(t)$ of the n-dimensional state vector $x(t)$ can be represented as the component of the expansion with respect to the basis $e_1, \ldots, e_n$, such as:

$$x(t) = e_1 x_1(t) + \cdots + e_n x_n(t)$$

where e_i is an n-dimensional vector with only the i-th element being 1. This basis is called the **standard basis** and represents the time evolution of the "components" of the state vector $x(t)$ in a certain basis. We now consider representing $x(t)$ in another basis $v_1, \ldots, v_n$ such that

$$x(t) = v_1 \xi_1(t) + \cdots + v_n \xi_n(t)$$

where $\xi_i(t)$ is the component of the basis vector v_i. Let us denote the matrix obtained by arranging the vectors v_i horizontally as V and the vector obtained by arranging $\xi_i(t)$ vertically as $\xi(t)$. Then, the linear transformation $x(t) = V\xi(t)$ corresponds to this representation. In this case, the state equation is transformed as:

$$\dot{\xi}(t) = V^{-1}AV\xi(t) + V^{-1}Bu(t)$$

where V_a and V_b are the matrices obtained by arranging the basis vectors of $\mathcal{V}_a$ and $\mathcal{V}_b$ horizontally, respectively, the transformed state equation is obtained as:

$$x(t) = V_a \xi_a(t) + V_b \xi_b(t)$$

where W_a and W_b are matrices that satisfy:

$$\begin{bmatrix} \dot{\xi}_a(t) \\ \dot{\xi}_b(t) \end{bmatrix} = \begin{bmatrix} W_a A V_a & W_a A V_b \\ W_b A V_a & W_b A W_b \end{bmatrix} \begin{bmatrix} \xi_a(t) \\ \xi_b(t) \end{bmatrix}$$

However, V_a and V_b are matrices consisting of the basis vectors of $\mathcal{V}a$ and $\mathcal{V}b$, respectively. Wa and Wb are matrices that satisfy:

$$\begin{bmatrix} W_a \\ W_b \end{bmatrix} = \begin{bmatrix} V_a & V_b \end{bmatrix}^{-1} \qquad \Longleftrightarrow \qquad \begin{bmatrix} V_a & V_b \end{bmatrix} \begin{bmatrix} W_a \\ W_b \end{bmatrix} = I$$

In this representation, $\xi_a(t)$ represents the component of $x(t)$ with respect to the subspace $\mathsf{span}\,\mathcal{V}_a$. Similarly, $\xi_b(t)$ represents the component of $x(t)$ with respect to the subspace $\mathsf{span}\,\mathcal{V}_b$.

The change of basis explained below can be applied regardless of whether passive power transmission conditions hold. δ^{lin} is expanded using a matrix $W \in \mathbb{R}^{N\times(N-1)}$:

$$\delta^{\text{lin}} = W\delta_{\text{e}}^{\text{lin}} + \mathbb{1}\overline{\delta}_{\text{e}}^{\text{lin}} \tag{46}$$

Here, $\mathbb{1}$ is a base vector that expresses the common component of δ^{lin}, while W is a matrix with base vectors that express other deviation components. In other words, the common component of δ^{lin} is $\overline{\delta}_{\text{e}}^{\text{lin}}$, and deviation components are $\delta_{\text{e}}^{\text{lin}}$. The common component $\overline{\delta}_{\text{e}}^{\text{lin}}$ is one-dimensional, while deviation components $\delta_{\text{e}}^{\text{lin}}$ are $(N-1)$-dimensional.

Next, let us consider the inverse transformation of Equation **??**. Specifically, let us consider a matrix $W^{\dagger} \in \mathbb{R}^{(N-1)\times N}$:

$$\delta^{\text{lin}} = \underbrace{\begin{bmatrix} W & \mathbb{1} \end{bmatrix}}_{T} \begin{bmatrix} \delta_{\text{e}}^{\text{lin}} \\ \overline{\delta}_{\text{e}}^{\text{lin}} \end{bmatrix} \quad \Longleftrightarrow \quad \begin{bmatrix} \delta_{\text{e}}^{\text{lin}} \\ \overline{\delta}_{\text{e}}^{\text{lin}} \end{bmatrix} = \underbrace{\begin{bmatrix} W^{\dagger} \\ \frac{1}{N}\mathbb{1}^{\mathsf{T}} \end{bmatrix}}_{T^{-1}} \delta^{\text{lin}}$$

For this inverse transformation to exist, the column vector of W must be orthogonal to $\mathbb{1}$. This can be confirmed as follows. From the relationship of inverse transformation, the following must hold:

$$T^{-1}T = \begin{bmatrix} W^{\dagger}W & W^{\dagger}\mathbb{1} \\ \frac{1}{N}\mathbb{1}^{\mathsf{T}}W & \frac{1}{N}\mathbb{1}^{\mathsf{T}}\mathbb{1} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 1 \end{bmatrix}$$

In other words, W and $W^{\dagger}$ are matrices that satisfy:

$$\mathbb{1}^{\mathsf{T}}W = 0, \qquad W^{\dagger}W = I, \qquad W^{\dagger}\mathbb{1} = 0$$

Therefore, from the first equation, we can see that the column vectors of W must be orthogonal to $\mathbb{1}$. Note that W and $W^{\dagger}$ can be constructed using an appropriate matrix $U \in \mathbb{R}^{N\times(N-1)}$ that satisfies $\mathbb{1}^{\mathsf{T}}U = 0$ and $U^{\mathsf{T}}U$ is invertible, as follows:

$$W = U(U^{\mathsf{T}}U)^{-1}, \qquad W^{\dagger} = U^{\mathsf{T}}$$

In this case, the product $WW^{\dagger}$ can be expressed as the **orthogonal projection matrix** onto the orthogonal complement of $\operatorname{span}\mathbb{1}$:

$$WW^{\dagger} = I - \frac{1}{N}\mathbb{1}\mathbb{1}^{\mathsf{T}} \tag{47}$$

The pseudoinverse of W obtained in this way is called the **Moore-Penrose pseudoinverse** [**?**].

In Figure **??**, the subspace $\operatorname{span}\mathbb{1}$ is shown by a black arrow, and the orthogonal complement space $\operatorname{span}\mathbb{1}^{\perp}$ is shown as a plane perpendicular to it. When a vector v is multiplied by the orthogonal projection matrix $WW^{\dagger}$, the projection of v onto $\operatorname{span}\mathbb{1}^{\perp}$, which is the shadow cast by v in the direction perpendicular to $\operatorname{span}\mathbb{1}$, is obtained as $WW^{\dagger}v$. Furthermore, the complementary relationship, shown

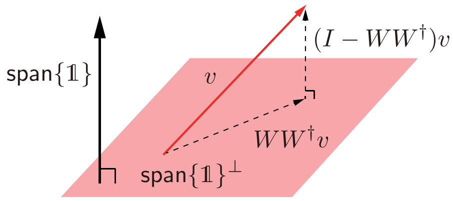


Fig. 6 Conceptual diagram of orthogonal projection

below, indicates that it is the orthogonal projection matrix onto span $\mathbb{1}$'s orthogonal complement, that is, span $\mathbb{1}$.

$$I - WW^\dagger = \tfrac{1}{N}\mathbb{1}\mathbb{1}^\top$$

We apply the above-mentioned change of basis to the electrical subsystem G of Equation **??**. First, if we substitute Equation **??** into a differential equation related to δ^{lin}, the following is obtained:

$$W\dot{\delta}_{\mathrm{e}}^{\mathrm{lin}} + \mathbb{1}\dot{\bar{\delta}}_{\mathrm{e}}^{\mathrm{lin}} = u_G$$

By multiplying the left-hand side of this differential equation by $W^\dagger$ or $\frac{1}{N}\mathbb{1}^\top$, we obtain:

$$\dot{\delta}_{\mathrm{e}}^{\mathrm{lin}} = W^\dagger u_G, \qquad \dot{\bar{\delta}}_{\mathrm{e}}^{\mathrm{lin}} = \frac{1}{N}\mathbb{1}^\top u_G$$

Next, noting that the relationship in Equation **??** holds for matrices L and B, the differential equation and output equation with respect to E^{lin} can be rewritten as:

$$\tau\dot{E}^{\mathrm{lin}} = AE^{\mathrm{lin}} + BW\delta_{\mathrm{e}}^{\mathrm{lin}}, \qquad y_G = CE^{\mathrm{lin}} + LW\delta_{\mathrm{e}}^{\mathrm{lin}}$$

Therefore, the transformed electric subsystem is given by:

$$G : \begin{cases} \dot{\bar{\delta}}_{\mathrm{e}}^{\mathrm{lin}} = \frac{1}{N}\mathbb{1}^\top u_G \\ \dot{\delta}_{\mathrm{e}}^{\mathrm{lin}} = W^\dagger u_G \\ \tau\dot{E}^{\mathrm{lin}} = AE^{\mathrm{lin}} + BW\delta_{\mathrm{e}}^{\mathrm{lin}} \\ y_G = CE^{\mathrm{lin}} + LW\delta_{\mathrm{e}}^{\mathrm{lin}} \end{cases} \tag{48}$$

One notable point about this system representation is that the common component of δ^{lin}, represented by $\bar{\delta}_{\mathrm{e}}^{\mathrm{lin}}$, is affected by the input u_G but has no effect on the output y_G. In other words, $\bar{\delta}_{\mathrm{e}}^{\mathrm{lin}}$ is an unobservable state variable.

By removing the differential equation of $\bar{\delta}_{\mathrm{e}}^{\mathrm{lin}}$ from Equation **??**, $(N-1)$-dimensional controllable and observable subsystem is obtained as:

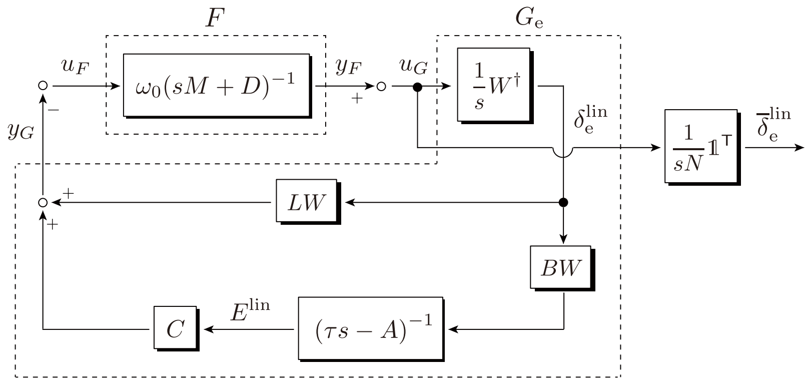


Fig. 7 Basis-transformed approximate linear model

$$
G_e : \begin{cases} \dot{\delta}_e^{lin} = W^\dagger u_G \\ \tau \dot{E}^{lin} = AE^{lin} + BW\delta_e^{lin} \\ y_G = CE^{lin} + LW\delta_e^{lin} \end{cases} \tag{49}
$$

Here, please note that from observability of G_e, the following holds.

$$
y_G(t) = 0, \quad \forall t \geq 0 \qquad \Longrightarrow \qquad \begin{bmatrix} \delta_e^{lin}(t) \\ E^{lin}(t) \end{bmatrix} = 0, \quad \forall t \geq 0 \tag{50}
$$

The fact mentioned above is important for the analysis of the steady-state stability of the approximate linear model in equation **??**. As a reference, the block diagram of the approximate linear model transformed by the change of basis is shown in Figure **??**.

It should be noted that it is a necessary and sufficient condition for G_e to be controllable and observable that the pair $(\tau^{-1}A, \tau^{-1}B)$ is controllable and the pair $(C, \tau^{-1}A)$ is observable. In the following, we assume controllability and observability. Note that an exact proof is not always easy, but assuming that the rank of B and C is $(N-1)$ or higher in most situations, there are no practical obstacles to analysis.

3.3.3 Small signal stability analysis based on passivity

In the following, assuming the passive power transfer condition defined in Definition **??**, we show the passivity of G_e in Equation **??** using the same procedure as that of the electrical subsystem G in Equation **??**. To this end, we express G_e in the form of:

$$
G_e : \begin{cases} \dot{x}_{G_e} = A_{G_e} x_{G_e} + B_{G_e} u_G \\ y_G = C_{G_e} x_{G_e} \end{cases} \tag{51}
$$

where x_{G_e} is a vector composed of $\delta_\mathrm{e}^\mathrm{lin}$ and E^lin, and:

$$A_{G_\mathrm{e}} := \begin{bmatrix} 0 & 0 \\ \Omega^2 \hat{B} W & \Omega^2 \hat{A} \end{bmatrix}, \quad B_{G_\mathrm{e}} := \begin{bmatrix} W^\dagger \\ 0 \end{bmatrix}, \quad C_{G_\mathrm{e}} := \begin{bmatrix} LW & -\hat{B}^\mathsf{T} \end{bmatrix}$$

Additionally, we define the positive semidefinite matrix P_{G_e} as:

$$P_{G_\mathrm{e}} := \begin{bmatrix} W & 0 \\ 0 & I \end{bmatrix}^\mathsf{T} \underbrace{\begin{bmatrix} L & -\hat{B}^\mathsf{T} \\ -\hat{B} & -\hat{A} \end{bmatrix}}_{P_G} \begin{bmatrix} W & 0 \\ 0 & I \end{bmatrix} \tag{52}$$

Note that if P_G in Equation **??** is positive semi-definite, then P_{G_e} is also positive semi-definite. In this case, by noting that $\hat{B} W W^\dagger = \hat{B}$ and $LWW^\dagger = L$ from the relationship in Equation **??**, we can see that the following equation holds:

$$A_{G_\mathrm{e}}^\mathsf{T} P_{G_\mathrm{e}} + P_{G_\mathrm{e}} A_{G_\mathrm{e}} \preceq 0, \qquad P_{G_\mathrm{e}} B_{G_\mathrm{e}} = C_{G_\mathrm{e}}^\mathsf{T} \tag{53}$$

Note that the left matrix inequality is shown from the same equation as **??**:

$$\frac{A_{G_\mathrm{e}}^\mathsf{T} P_{G_\mathrm{e}} + P_{G_\mathrm{e}} A_{G_\mathrm{e}}}{2} = \begin{bmatrix} \Omega \hat{B} W & 0 \\ 0 & \Omega^{-1} \end{bmatrix}^\mathsf{T} \underbrace{\begin{bmatrix} -I & -\hat{A}_\Omega \\ -\hat{A}_\Omega & -\hat{A}_\Omega^2 \end{bmatrix}}_{Y} \begin{bmatrix} \Omega \hat{B} W & 0 \\ 0 & \Omega^{-1} \end{bmatrix}$$

Therefore, the time derivative along the solution trajectory of G_e of the storage function:

$$W_{G_\mathrm{e}}(x_{G_\mathrm{e}}) := \frac{1}{2} x_{G_\mathrm{e}}^\mathsf{T} P_{G_\mathrm{e}} x_{G_\mathrm{e}}$$

can be evaluated as:

$$\frac{d}{dt} W_{G_\mathrm{e}}\big(x_{G_\mathrm{e}}(t)\big) \le y_G^\mathsf{T}(t) u_G(t) \tag{54}$$

Thus, G_e in Equation **??** is passive. Note that this inequality is equivalent to the inequality in Equation **??**, and the values of the two storage functions satisfy:

$$W_G\big(x_G(t)\big) = W_{G_\mathrm{e}}\big(x_{G_\mathrm{e}}(t)\big), \qquad \forall t \ge 0$$

By considering the observability of G_e shown by Equation **??**, the following is true for the arbitrary initial value of the solution trajectory of the linear approximation model of Equation **??**:

$$\lim_{t\to\infty} \Delta\omega^\mathrm{lin}(t) = 0, \qquad \lim_{t\to\infty} \begin{bmatrix} \delta_\mathrm{e}^\mathrm{lin}(t) \\ E^\mathrm{lin}(t) \end{bmatrix} = 0 \tag{55}$$

Therefore, from the relationship of the change of basis of Equation **??**, we can see that Equation **??** holds for the arbitrary initial value. In other words, the linear approximation model of Equation **??** is statically stable. Also:

$$c_0 = \lim_{t\to\infty} \overline{\delta}_{\mathrm{e}}^{\mathrm{lin}}(t)$$

and state variables $\overline{\delta}_{\mathrm{e}}^{\mathrm{lin}}$ follow the differential equation of Equation **??**.

We summarize the previous discussion in the following theorem.

Theorem 1.1 (Small signal stability of the linear approximation model based on passivity) *For any steady-state value $(\delta^\star, E^\star)$ that satisfies the passive power transfer condition defined in Definition **??**, the electrical subsystem G given in equation **??** is passive. Additionally, for any positive constants $(M_i, D_i, \tau_i)i \in \mathcal{I}\mathrm{G}$, the approximate linear model given in equation **??** is steady-state stable.*

As discussed in Theorem **??**, under the passive power transmission conditions, the linear approximation model is statically stable for combinations of all physical constants $(M_i, D_i, \tau_i)_{i\in\mathcal{I}_{\mathrm{G}}}$. Analysis based on passivity allows stability independent of model parameters.

3.4 Necessary conditions for the approxiamte linear model to be passive‡

3.4.1 Passivity and positive realness

It is known that the passivity of a linear system is mathematically equivalent to the property called positive realness of its transfer function. In this section, we consider the necessity of the passive power transmission condition defined in Definition **??** from the viewpoint of the passivity of the electrical subsystem based on this equivalence.

COFFEE BREAK

Transfer function: For a linear system:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

its **transfer function** is defined as:

$$Q(s) := C(sI - A)^{-1}B + D$$

When the Laplace transform of the input $u(t)$ is $U(s)$ and the Laplace transform of the output $y(t)$ is $Y(s)$, the transfer function relates to the system as $Y(s) = Q(s)U(s)$. The input-output behavior of a linear system is characterized by its transfer function.

The transfer function from the input u_G to the output y_G of the electrical subsystem G in Equation **??** is given by:

$$G(s) := -\frac{1}{s}\underbrace{\left\{-C\left(\tau s - A\right)^{-1}B - L\right\}}_{H(s)} \tag{56}$$

Note that since unobservable state variables are not relevant to the input-output characteristics, the transfer function of G_e in Equation **??** is also equal to $G(s)$. Hereafter, we consider the case where the transfer function $H(s)$ in Equation **??** is stable. The stability of the transfer function is defined as follows.

Definition 1.5 (Stability of a transfer function) When the real part of all poles of the transfer function $Q(s)$ is negative, $Q(s)$ is called **stable**.

The poles of a transfer function are the zeros of the denominator polynomial. It is known that $H(s)$ in equation **??** being stable is equivalent to all the real parts of the eigenvalues of the matrix $\tau^{-1}A$ being negative.

Furthermore, the **Positive realness** of a transfer function is defined as follows.

Definition 1.6 (Positive realness of a transfer function) For a square transfer function $Q(s)$, the following is defined:

$$\Omega_0 := \{\omega_0 \in \mathbb{R} : \text{ The pure imaginary number } \boldsymbol{j}\omega_0 \text{ is a pole of } Q(s)\} \tag{57}$$

$Q(s)$ is called **positive real** if the following three conditions are satisfied.

- The real part of all poles of $Q(s)$ is nonpositive.
- For all $\omega \in [0, \infty) \setminus \Omega_0$, $Q(\boldsymbol{j}\omega) + Q^{\mathsf{T}}(-\boldsymbol{j}\omega)$ is positive semi-definite.
- When there are poles of a pure imaginary number, their multiplicity is 1, and the following is true for the remaining number:

$$\lim_{s\to \boldsymbol{j}\omega_0} (s - \boldsymbol{j}\omega_0)Q(s) = \lim_{s\to \boldsymbol{j}\omega_0} \{(s - \boldsymbol{j}\omega_0)Q(s)\}^* \geq 0, \qquad \forall \omega_0 \in \Omega_0$$

In Definition **??**, the two most important conditions are the first and second ones. The first condition expresses the stability of the transfer function. However, it also includes the case where the real part of the pole is 0. The second condition concerns the positive definiteness of the complex symmetric part of the transfer function evaluated on the imaginary axis. In particular, if $Q(s)$ is a scalar, that is, both the input and output are scalars, the second condition expresses that the real part of $Q(\boldsymbol{j}\omega)$ is non-negative for all $\omega \in [0, \infty) \setminus \Omega_0$. However, it should be noted that for matrix-valued $Q(s)$, in general,

$$Q(\boldsymbol{j}\omega) + Q^{\mathsf{T}}(-\boldsymbol{j}\omega) \neq 2\,\mathsf{Re}\,[Q(\boldsymbol{j}\omega)]$$

Also, for $Q(s)$ with real coefficients, $Q^{\mathsf{T}}(-\boldsymbol{j}\omega)$ is equal to $Q(\boldsymbol{j}\omega)^*$. The third condition is exceptional in the case when $Q(s)$ has pure imaginary poles. However, it is necessary to analyze transfer functions with poles at the origin, such as $G(s)$ in Equation **??**.

COFFEE BREAK

Complex Symmetric and Skew Hermitian Parts of a Square Matrix: Any square matrix M can be decomposed as:

$$M = \frac{M + M^*}{2} + \frac{M - M^*}{2}$$

where $\frac{M+M^*}{2}$ is called the **complex symmetric part** (Hermitian part) of M, and $\frac{M-M^*}{2}$ is called the **complex skew-symmetric part** (skew Hermitian part) of M.

In system control engineering, it is known that passivity in Definition **??** and positive realness in Definition **??** are equivalent. If we apply Lemma **??** at the end of this chapter to the discussion in this section, the necessary and sufficient condition for $G(s)$ in Equation **??** to be positive real is the existence of a positive definite matrix P_{G_e} that satisfies Equation **??** for the controllable and observable state space realization G_e in Equation **??**. This is equivalent to the passivity of G_e defined by the inequality in Equation **??**. Furthermore, the positive realness of P_{G_e} in Equation **??** is shown by the fact that both the Schur complements with respect to $-\hat{A}$ and $-\hat{A}$ of the matrix $W^\mathsf{T}\left(L + \hat{B}^\mathsf{T}\hat{A}^{-1}\hat{B}\right)W$ are positive definite, where $W^\mathsf{T}L_0W$ satisfies Equation **??** and the column vectors of W in Equation **??** are orthogonal to $\mathbb{1}$.

3.4.2 Necessary condition for the transfer function of the electrical subsystem to be positive-real

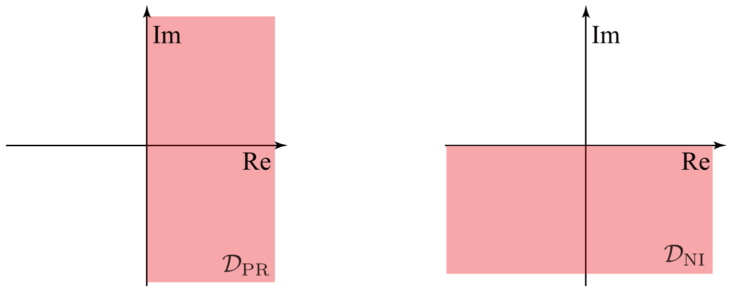
As a mathematical preparation for deriving necessary conditions, we introduce the concept of **negative imaginaryness** of transfer functions, which is similar to positive-realness [**?**, **?**].

Definition 1.7 (Negative imaginariness of a transfer function) For a square transfer function $Q(s)$ without a pole at the origin, we define Ω_0 of Equation **??**. When the following three conditions are satisfied, $Q(s)$ is called **negative imaginary**.

- The real part of all poles of $Q(s)$ is nonpositive.
- For all $\omega \in (0, \infty) \setminus \Omega_0$, $\boldsymbol{j}\left\{Q(\boldsymbol{j}\omega) - Q^\mathsf{T}(-\boldsymbol{j}\omega)\right\}$ is positive semi-definite.
- When there is a pole of a pure imaginary number, their multiplicity is 1, and the following holds for the remaining numbers:

$$\lim_{s\to\boldsymbol{j}\omega_0}(s - \boldsymbol{j}\omega_0)\boldsymbol{j}Q(s) = \lim_{s\to\boldsymbol{j}\omega_0}\{(s - \boldsymbol{j}\omega_0)\boldsymbol{j}Q(s)\}^* \geq 0, \quad \forall\omega_0 \in \Omega_0$$

While Definition **??** defines positive-realness based on the positive-definiteness of the complex symmetric part of a transfer function, Definition **??** defines negative-imaginaryness based on the positive-semidefiniteness of the complex anti-symmetric part of the transfer function multiplied by the imaginary unit. Note that since the eigenvalues of a complex anti-symmetric matrix are purely imaginary, their product with the imaginary unit is real. In particular, when $Q(s)$ is a scalar transfer function,



(a) $\mathcal{D}_{\mathrm{PR}}$ (b) $\mathcal{D}_{\mathrm{NI}}$

Fig. 8 Positive realms and negative imaginary realms

the second condition implies that the imaginary part of $Q(\boldsymbol{j}\omega)$ is non-positive for all $\omega \in (0, \infty) \setminus \Omega_0$. Furthermore, in the following discussion, we focus only on the second condition to consider the negative-imaginaryness of stable transfer functions. Similar to positive-realness, negative-imaginaryness can also be characterized as the solvability of matrix inequalities, as described in Lemma **??** at the end of this chapter.

COFFEE BREAK

Nyquist plot: The plot of the frequency response function $Q(\boldsymbol{j}\omega)$ for $\omega \in \mathbb{R}$ on the complex plane is called the **Nyquist plot**. It is often used in geometric analysis of stability for feedback systems. The analysis method is called the **Nyquist stability criterion**. Note that when $Q(s)$ is a scalar and the coefficients of the numerator and denominator polynomials are real, the plot of $Q(\boldsymbol{j}\omega)$ for negative ω is symmetrical to the plot for positive ω with respect to the real axis.

The relationship of positive realness and negative imaginariness is explained with Figure **??**. If the transfer function $Q(s)$ is scalar, $Q(s)$ being positive real can be understood as the trajectory related to non-negative ω of the frequency response function $Q(\boldsymbol{j}\omega)$ is included in the range $\mathcal{D}_{\mathrm{PR}}$ shown in **??**(a).

$$\mathcal{D}_{\mathrm{PR}} = \boldsymbol{j}\mathcal{D}_{\mathrm{NI}}$$

and $-\frac{1}{\boldsymbol{j}} = \boldsymbol{j}$ for $G(s)$ and $H(s)$ of Equation **??**, the following is derived:

$$G(\boldsymbol{j}\omega) \in \mathcal{D}_{\mathrm{PR}}, \quad \forall \omega > 0 \qquad \Longleftrightarrow \qquad H(\boldsymbol{j}\omega) \in \mathcal{D}_{\mathrm{NI}}, \quad \forall \omega > 0$$

Therefore, a negative imaginariness analysis of $H(s)$ is equivalent to a positive realness analysis of $G(s)$. To be accurate, $G(\boldsymbol{j}\omega)$ and $H(\boldsymbol{j}\omega)$ are complex matrices; thus, $\mathcal{D}_{\mathrm{PR}}$ and $\mathcal{D}_{\mathrm{NI}}$ should be redefined with a set of positive semi-definite matrices.

Therefore, a negative imaginariness analysis of $H(s)$ is equivalent to a positive realness analysis of $G(s)$. Based on this fact, the passive power transmission conditions (ii) and (iii) are necessary conditions for $G(s)$ to be positive real.

Theorem 1.2 (Positive realness of the transfer function of an electrical subsystem) *Translated with DeepL For an arbitrary $(\delta^\star, E^\star)$ where the transfer function $H(s)$ of Equation* **??** *becomes stable, a condition necessary for $H(s)$ to be negative imaginary is that the passive power transmission condition (ii) of definition* **??** *holds. In addition, a condition necessary for the transfer function $G(s)$ of Equation* **??** *to be positive real is that the passive power transmission conditions (ii) and (iii) hold.*

Proof First, we show that for any $(\delta^\star, E^\star)$ for which $H(s)$ is stable, if $H(s)$ is negative imaginary, then the passive transmission condition (ii), i.e., the Equation **??**, holds. Now

$$\lim_{\omega\to\infty} \boldsymbol{j}\left\{H(\boldsymbol{j}\omega) - H^\mathsf{T}(-\boldsymbol{j}\omega)\right\} = \boldsymbol{j}\left(-L + L^\mathsf{T}\right) \geq 0$$

Therefore, L must be symmetric for $H(s)$ to be negative vacuity. Thus, we have $K_{IJ}(\delta^\star_{IJ}) = K_{JI}(\delta^\star_{JI})$. In other words

$$G^{\mathrm{red}}_{ij} \sin \delta^\star_{ij} = 0, \qquad \forall (i, j) \in \mathcal{I}_{\mathrm{G}} \times \mathcal{I}_{\mathrm{G}}$$

This implies $\delta^\star_i \neq \delta^\star_j$ for (i, j) where $G^{\mathrm{red}}_{ij} = 0$. Also, when $\delta^\star_i = \delta^\star_j$, continuity regarding parameter variation of eigenvalues for matrices with parameters implies that $\tau^{-1}A$ for $\delta^\star + \gamma e_i$ is There exists a sufficiently small $\gamma > 0$ such that it is stable. However, e_i denotes a vector where only the ith i-element is 1 and the rest are 0. Therefore, there exists a vector

$$G^{\mathrm{red}}_{ij} = 0, \qquad \forall i \neq j \tag{58}$$

Furthermore, if $H(s)$ is negative imaginary, then

$$\lim_{\omega\to 0} \boldsymbol{j}\left\{H(\boldsymbol{j}\omega) - H^\mathsf{T}(-\boldsymbol{j}\omega)\right\} = \boldsymbol{j}\left(-L_0 + L_0^\mathsf{T}\right) \geq 0$$

Therefore, L_0 in equation**??** must also be symmetric. When the Equation**??** holds.

$$C = \mathsf{diag}\left(2E^\star_i G^{\mathrm{red}}_{ii}\right) - \hat{B}^\mathsf{T}$$

Note that L_0 is given by:

$$L_0 = \underbrace{L + \hat{B}^\mathsf{T}\hat{A}^{-1}\hat{B}}_{L_1} - \underbrace{\mathsf{diag}(2E^\star_i G^{\mathrm{red}}_{ii})\hat{A}^{-1}\hat{B}}_{L_2}$$

However, $\hat{A}$ is a symmetric matrix defined by the Equation **??**. On the other hand, for L_2 to be symmetric for any $E^\star$, it must have $G^{\mathrm{red}}_{ii} = 0$ for all i. From this, for any $(\delta^\star, E^\star)$ for which $H(s)$ is stable, if $H(s)$ is negative imaginary, then the expression **??** holds.

Next, we show that $H(s)$ is negative imaginary for any $(\delta^\star, E^\star)$ for which $H(s)$ is stable if the expression**??** holds. This requires that L is symmetric and:

$$\tilde{A}^\mathsf{T} P + P\tilde{A} \preceq 0, \qquad P\tilde{A}^{-1}\tilde{B} = C^\mathsf{T} \tag{59}$$

It is enough to show that there exists a positive definite matrix P satisfying. However, we need to show that there exists a positive definite matrix P such that

$$\tilde{A} := \tau^{-1}A, \qquad \tilde{B} := \tau^{-1}B$$

,**??**

$$k_{ij}(\delta^\star_{ij}) = k_{ji}(\delta^\star_{ji}), \qquad h_{ij}(\delta^\star_{ij}) = -h_{ji}(\delta^\star_{ji}), \qquad h_{ii}(\delta^\star_{ii}) = 0$$

Since L is symmetric, it follows that L is symmetric. Also, since $H(s)$ is stable

$$\tilde{A} = \mathsf{diag}\left(\frac{X_i - X_i'}{\tau_i}\right)\hat{A}$$

Since $X_i > X_i'$, $\hat{A}$ in Equation **??** is negative definite. Therefore, we can choose $-\hat{A}$ as the positive definite matrix P satisfying Equation **??**, which shows that $H(s)$ is negative-definite.

Next, we show the equivalence for $G(s)$. Since $H(s)$ is stable, the only pole on the imaginary axis of $G(s)$ is the origin, and its degree of overlap is 1. Therefore, $G(s)$ is The necessary and sufficient condition for being positively real is:

$$G(\boldsymbol{j}\omega) + G^\mathsf{T}(-\boldsymbol{j}\omega) \succeq 0, \qquad \forall\omega \in \mathbb{R} \setminus \{0\} \tag{60}$$

is valid, and the following can be established:

$$\lim_{s\to 0} sG(s) = \lim_{s\to 0}\{sG(s)\}^\mathsf{T} \succeq 0 \tag{61}$$

When the Equation **??** holds, then the Equation **??** holds.

$$G(\boldsymbol{j}\omega) + G^\mathsf{T}(-\boldsymbol{j}\omega) = \frac{\boldsymbol{j}}{\omega}\left\{H(\boldsymbol{j}\omega) - H^\mathsf{T}(-\boldsymbol{j}\omega)\right\}, \qquad \forall\omega \in \mathbb{R} \setminus \{0\} \tag{62}$$

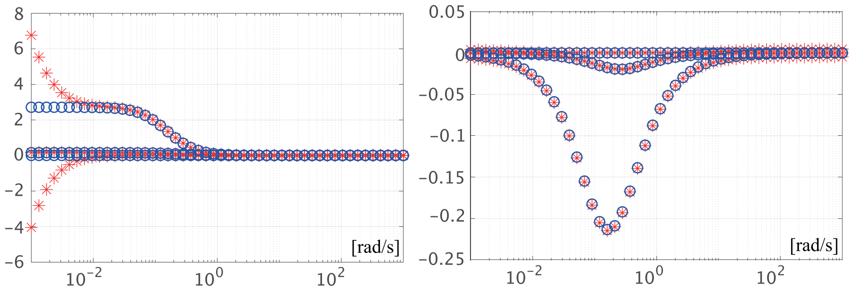
It is shown from $H(s)$ that $H(s)$ is negative imaginary. And,

$$\lim_{s\to 0} sG(s) = L - C\tilde{A}^{-1}\tilde{B} = L - CA^{-1}B$$

Therefore, the semi-positive definiteness of equation **??** is equivalent to the passive transmission condition (iii), i.e., the condition of equation **??**. Note that when the passive transmission condition (ii) holds, L is symmetric and:

$$C\tilde{A}^{-1}\tilde{B} = CP^{-1}C^\mathsf{T}$$

is also symmetric, which also shows the symmetry of the expression **??**.



(a) $\frac{G(\boldsymbol{j}\omega)+G^{\mathsf{T}}(-\boldsymbol{j}\omega)}{2}$ (b) $\frac{H(\boldsymbol{j}\omega)-H^{\mathsf{T}}(-\boldsymbol{j}\omega)}{2\boldsymbol{j}}$

Fig. 9 $\boldsymbol{G(s)H(s)}$ (Blue: passive transmission condition (ii) is satisfied, Red: not satisfied)

Conversely, if the passive power transmission conditions (ii) or (iii) do not hold, then $G(s)$ is not positively real. The latter is evident from the fact that the condition in equation **??** is equal to the condition in equation **??**. Also, when the passive power transmission condition (ii) does not hold, since $H(s)$ is not negative vacuous, there exists a point $\omega_0 \geq 0$ and a sufficiently small $\epsilon > 0$ such that:

$$\lambda_{\min}\left[\boldsymbol{j}\left\{H(\boldsymbol{j}(\omega_0+\alpha)) - H^{\mathsf{T}}(-\boldsymbol{j}(\omega_0+\alpha))\right\}\right] < 0, \qquad \forall\alpha \in (0, \epsilon]$$

where $\lambda_{\min}$ denotes the smallest eigenvalue. Thus, for all $\omega \in (\omega_0, \omega_0+\epsilon]$ that are not 0, the complex symmetric part of $G(\boldsymbol{j}\omega)$ is not semidefinite. □

Let us confirm the result of Theorem **??** with an example.

Example 1.4 (Transmission loss and positive realness of the transfer function of an electrical subsystemTransmission loss and positive realness of the transfer function of an electrical subsystem) For the electrical power system model consisting of three generators discussed in the Example **??**, let us examine the positive realness of $G(s)$ and negative imaginariness of $H(s)$. For comparison, we will calculate a case where the passive power transmission condition (ii) is satisfied, and when it is not satisfied. Specifically, similar to the Example **??**, we set $\boldsymbol{Y}_0(0)$ and $\boldsymbol{Y}_0(1)$ as the admittance matrix $\boldsymbol{Y}$ of the power grid. With the horizontal axis as frequency ω, the eigenvalue of the Hermitian part of $G(\boldsymbol{j}\omega)$ is shown in Figure **??**(a), while the imaginary part of the eigenvalue of the skew Hermitian part of $H(\boldsymbol{j}\omega)$ is shown in Figure **??**(b). The blue circle indicates when the passive power transmission condition (ii) is satisfied, while red indicates when it is not satisfied. With this Figure, we can see that when the conductance of a transmission line is not 0, the Hermitian part of $G(\boldsymbol{j}\omega)$ is positive semi-definite in a low frequency band.

The significance of the passive power transmission condition (iii), which appeared as a condition for P_G of Equation **??** to be positive semi-definite, can be explained as follows. In the electrical subsystem G of Equation **??**, let us look at the equation of state for the internal voltage:

$$\tau \dot{E}^{\text{lin}} = AE^{\text{lin}} + B\delta^{\text{lin}}$$

With this differential equation, let us consider a limit at which the time constant $(\tau_i)_{i\in\mathcal{I}_{\text{G}}}$ asymptotically approaches 0. This is equivalent to "a limit for which the time it takes for the internal voltage to reach a steady state is sufficiently shorter than the fluctuations in δ^{lin}". At this time, the following approximation holds:

$$E^{\text{lin}}(t) \simeq -A^{-1}B\delta^{\text{lin}}(t), \qquad \forall t \geq 0 \tag{63}$$

If A is not stable; in other words, if the passive power transmission condition (i) does not hold, state E^{lin} dissipates. The method to approximate a differential equation with an algebraic equation using such a difference in the timescale of state variables is called **singular perturbation approximation** in control systems engineering. Actually, the dynamic characteristics of the internal voltage often have smaller time constants compared to the dynamic characteristics of mechanical turbines.

If we assume Equation **??** establishes an equation and substitutes as an output equation of Equation **??**, the following low-dimensional approximation of the electrical system is obtained:

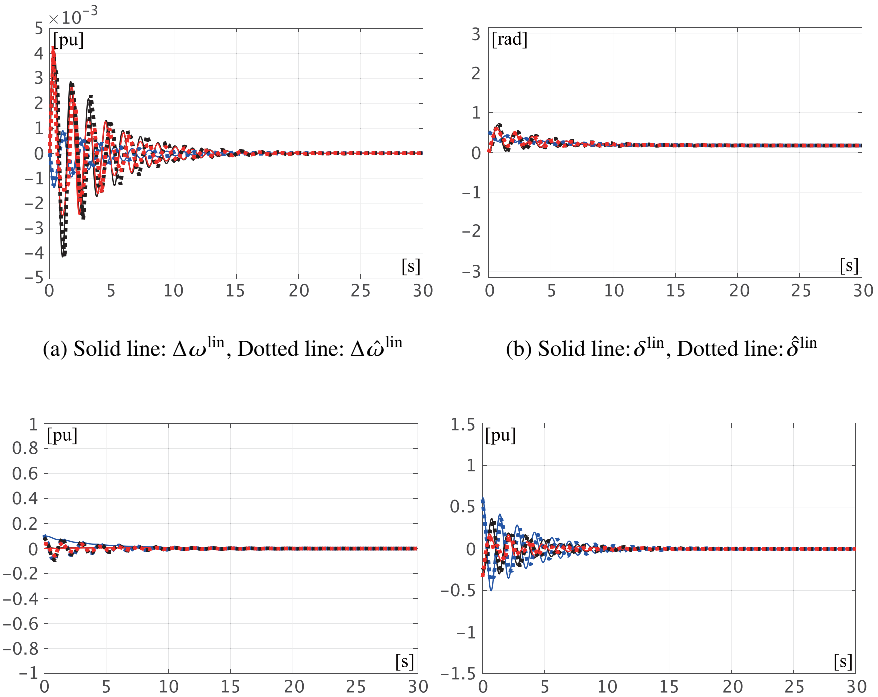
$$\hat{G} : \begin{cases} \dot{\hat{\delta}}^{\text{lin}} = u_G \\ y_G = L_0\hat{\delta}^{\text{lin}} \end{cases} \tag{64}$$

To show that it is an approximation, we classified state variables as $\hat{\delta}^{\text{lin}}$. The entire linear approximation model of Equation **??** is approximated as a differential equation system where the second-order oscillator is combined by this singular perturbation approximation:

$$M\ddot{\hat{\delta}}^{\text{lin}} + D\dot{\hat{\delta}}^{\text{lin}} + \omega_0 L_0\hat{\delta}^{\text{lin}} = 0 \tag{65}$$

This result shows that the passive power transmission condition (iii) shows the "positive semi-definite nature of a spring constant matrix" when the time constant is small. It can be interpreted as equivalent to a dynamic spring constant of the electrical subsystem G of Equation **??**.

Example 1.5 (Singular perturbation approximation of a linear approximation model) As a reference, Figure **??** shows the time response of a second-order oscillator system of Equation **??** to the linear approximation model discussed in the Example **??**. The solid line is the response of the original linear approximation model, while the dashed line is the response of a second-order oscillator system after applying the singular perturbation approximation. Also:



(a) Solid line: $\Delta\omega^{\mathrm{lin}}$, Dotted line: $\Delta\hat{\omega}^{\mathrm{lin}}$

(b) Solid line: δ^{lin}, Dotted line: $\hat{\delta}^{\mathrm{lin}}$

(c) Solid line: E^{lin}, Dotted line: $\hat{E}^{\mathrm{lin}}$

(d) Solid line: P^{lin}, Dotted line: $\hat{P}^{\mathrm{lin}}$

Fig. 10 Time response when low-dimensional approximation is applied
(Blue: Generator 1, Black: Generator 2, Red: Generator 3)

$$\Delta\hat{\omega}^{\mathrm{lin}} := \omega_0^{-1}\dot{\hat{\delta}}^{\mathrm{lin}}, \qquad \hat{E}^{\mathrm{lin}} := -A^{-1}B\hat{\delta}^{\mathrm{lin}}, \qquad \hat{P}^{\mathrm{lin}} := L\hat{\delta}^{\mathrm{lin}} + C\hat{E}^{\mathrm{lin}}$$

The initial value of the linear approximation model is given as follows in response to Equation **??**:

$$\hat{\delta}^{\mathrm{lin}}(0) = \begin{bmatrix} \frac{\pi}{6} \\ 0 \\ 0 \end{bmatrix}, \qquad \Delta\hat{\omega}^{\mathrm{lin}}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Figure **??** shows that the time response of both is largely consistent with the peak value of the oscillations and attenuation rate.

3.5 Necessary conditions for the linear approximation model to be statically stable‡

Below, we discuss the necessity of the passive power transmission condition (iii) from the viewpoint of small signal stability of the linear approximation model of Equation **??**. Specifically, it shows that the passive power transmission condition (iii) is a necessary condition for the linear approximation model to be statically stable regardless of the physical constants of generators. As shown in the discussions of Section **??**, the passive power transmission condition (i) is a necessary condition for the linear approximation model to not be unstable for the time constant $(\tau_i)_{i\in\mathcal{I}_\mathrm{G}}$ at a sufficiently small limit. When the matrix A is not stable, this can be confirmed from the dissipation of the internal voltage since the singular perturbation approximation of Equation **??** cannot be applied.

When the passive power transmission condition (ii) does not hold, since L_0 is usually not symmetrical, we consider generalization of the passive power transmission condition (iii) so that it can be applied to unsymmetrical L_0:

$$\mathbf{\Lambda}(L_0) \subseteq [0, \infty) \tag{66}$$

However, $\mathbf{\Lambda}(L_0)$ shows a set of eigenvalues of L_0. The conditions of Equation **??** show that all eigenvalues of L_0 are "non-negative real numbers". Below, we call this generalized condition the passive power transmission condition (iii) $'$ of definition **??**. When L_0 is symmetrical, the passive power transmission conditions (iii) and (iii)$'$ are equivalent. The following lemma is presented.

Lemma 1.1 (Necessary condition for the small signal stability of a second-order oscillator system) *Let us consider a second-order oscillator system of Equation* **??**. *For an arbitrary initial value and arbitrary positive definite number* $(M_i, D_i)_{i\in\mathcal{I}_\mathrm{G}}$, *a condition necessary for a certain constant* c_0 *to exist and the following to hold:*

$$\lim_{t\to\infty} \hat{\delta}^{\mathrm{lin}}(t) = c_0\mathbb{1} \tag{67}$$

is that the passive power transmission condition (iii)$'$ holds.

Proof Translated with DeepL If the passive transmission condition (iii)$'$ does not hold, then there exists a certain positive constant $(M_i, D_i)_{i\in\mathcal{I}_\mathrm{G}}$ such that the Equation **??**. For this purpose, the following two cases will be discussed.

(a) Among the eigenvalues of L_0, there exist eigenvalues whose real part is negative or pure imaginary.
(b) There exist eigenvalues of L_0 whose real part is positive and whose imaginary part is nonzero.

First, let's consider the case (a). In the following, we choose constant matrices as $M = \omega_0 I$ and $D = \omega_0 dI$. In this case, the eigen equation of the equation **??** are

$$\begin{bmatrix} 0 & I \\ -L_0 & -dI \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = \lambda \begin{bmatrix} v \\ w \end{bmatrix}$$

If w is eliminated from this equation by substitution, the following is obtained:

$$\left(\lambda^2 I + d\lambda I + L_0\right) v = 0$$

This eigenvalue is an eigenvector with v of L_0 and for that eigenvalue κ, and therefore the following is true.

$$\lambda^2 + d\lambda + \kappa = 0 \qquad \Longleftrightarrow \qquad \lambda = \frac{-d \pm \sqrt{d^2 - 4\kappa}}{2} \tag{68}$$

Therefore, in the case of (a), it is sufficient to show that the real part of $\sqrt{d^2 - -4\kappa}$ is larger than d. In general, for any complex number z

$$\mathsf{Re}[z] = \sqrt{\mathsf{Re}[z^2] + (\mathrm{i}[z])^2}$$

Since it can be expressed as $z = \sqrt{d^2 - -4\kappa}$, the following result can be obtained:

$$\mathsf{Re}\left[\sqrt{d^2 - 4\kappa}\right] = \sqrt{d^2 - 4\,\mathsf{Re}[\kappa] + (\mathrm{i}[z])^2}$$

This value is in case (a), that is, when the real part of κ is negative, or the real part of κ is 0 and the imaginary part of κ is non-zero.Must be greater than d. Therefore, the secondary oscillator system of the equation **??** is unstable.

Next, consider the case of (b). In the following, it is shown that there exists a positive constant d such that the eigenvalue λ of the expression **??** is a pure imaginary number. When L_0 in the execution column has a complex eigenvalue, there is always something with a negative imaginary part. The eigenvalue is expressed as $\kappa = \alpha + \beta \boldsymbol{j}$ using $\alpha > 0$ and $\beta < 0$. For this κ, there exists $\omega \neq 0$ and $d > 0$ that satisfy:

$$-d + \sqrt{d^2 - 4\kappa} = \omega \boldsymbol{j}$$

If you transfer $-d$ on the left side and square both sides:

$$-4(\alpha + \beta \boldsymbol{j}) = 2d\omega \boldsymbol{j} - \omega^2$$

This equation is satisfied by choosing $\omega = 2\sqrt{\alpha}$, $d = -\frac{\beta}{\sqrt{\alpha}}$. Therefore, since the secondary oscillator system has a steady-state vibration solution, the equation **??** does not hold. □

Lemma **??** shows that, for a limit where the time constant of the internal voltage is sufficiently small, the passive power transmission condition (iii)′ is a necessary condition for the linear approximation model to be statically stable against the arbitrary physical constants of generators. Furthermore, with Theorem **??**, when the passive power transmission conditions (i)–(iii) hold, the linear approximation model is stable against arbitrary physical constants. Based on these facts, the conclusion of this Section is summarized in the following Theorem.

Theorem 1.3 (Small signal stability of the linear approximation model) *For an arbitrary positive definite number $(M_i, D_i, \tau_i)_{i \in \mathcal{I}_G}$, a necessary condition for the linear approximation model of Equation* **??** *to be statically stable is that the passive power transmission conditions (i) and (iii)′ of definition* **??** *hold. Specifically, when the passive power transmission condition (ii) holds, the above-mentioned necessary condition for the small signal stability is that the passive power transmission conditions (i) and (iii)′ hold.*

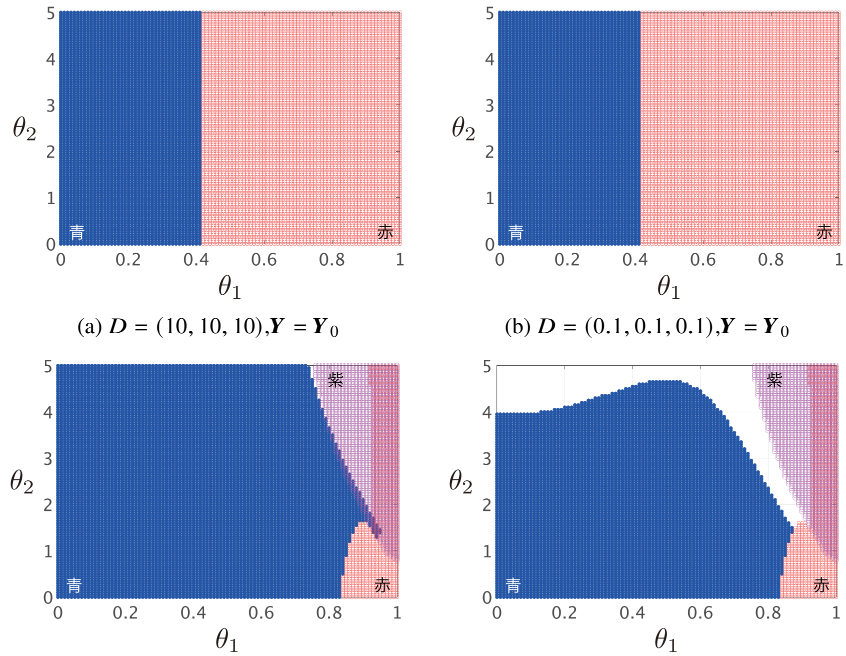
We present an analytical example of the small signal stability of the linear approximate model using Theorem **??**.

Example 1.6 (Small signal stability analysis based on the passive power transmission conditions) Using Theorem **??**, let us analyze the small signal stability of the linear approximation model consisting of three generators discussed in the Example **??**. The physical constants of generators are set to the same value as Example **??**. Since the passive power transmission condition (i) was satisfied for all parameters, we also plotted the range or parameters where the passive power transmission condition (iii) ′ is not satisfied in Figure **??**. When the eigenvalue of L_0 of Equation **??** includes those where the real part is negative, it is shown in red. When the eigenvalue of complex numbers is included, it is shown with purple. This shows when the passive power transmission condition (ii) holds for the range on the horizontal axis where θ_2 is 0.

Theorem **??** shows that the ranges shown with red and purple are “dangerous parameter ranges where the linear approximation model is always unstable with some physical constant settings”. When the passive power transmission condition (ii) holds; in other words, for parameters on the horizontal axis where θ_2 is 0, as long as θ_1 is set to a value that is not red, the linear approximation model has a small signal stability regardless of the value of these constants.

What we need to pay attention to in the result of Figure **??** is that the parameter range shown in red accurately captures some or all of the boundaries with the blue parameter range, where static stability is achieved with the above-described physical constants. The necessity of the passive power transmission condition (iii)′ shown with Theorem **??** means that there is at least one setting of physical constants for generators where the linear approximation model is unstable. Therefore, for the specific constants set in the Example **??**, the parameter ranges for static stability cannot always be captured accurately. On the other hand, the fact that the some boundaries of the blue range are accurately captured indicates that when the real part of L_0 has negative eigenvalues, in many cases, the linear approximation model is unstable.

In the cases of (a) and (b), there is no blue range. In other words, for the searched parameters, the eigenvalue of L_0 is a real number. Generally, as long as θ_2 is not 0, L_0 is a non-symmetrical matrix; thus, it is unclear whether only L_0 has a real eigenvalue. On the other hand, in (c) and (d) where the admittance matrix is multiplied by $\frac{1}{100}$, if θ_1 and θ_2 are relatively large, L_0 has a complex eigenvalue. However, in a realistic setting, it has been confirmed that L_0 often only has real eigenvalues.



(a) $\boldsymbol{D} = (10, 10, 10), \boldsymbol{Y} = \boldsymbol{Y}_0$ (b) $\boldsymbol{D} = (0.1, 0.1, 0.1), \boldsymbol{Y} = \boldsymbol{Y}_0$

(c) $\boldsymbol{D} = (10, 10, 10), \boldsymbol{Y} = \frac{\boldsymbol{Y}_0}{100}$ (d) $\boldsymbol{D} = (0.1, 0.1, 0.1), \boldsymbol{Y} = \frac{\boldsymbol{Y}_0}{100}$

Fig. 11 Regions of parameters for which the approximate linear model is stable

Mathematical Supplement

Lemma 1.2 *Stable and square transfer function*

$$Q(s) = C(sI - A)^{-1}B + D$$

$,(A, B),(C, A)$ $,Q(s),$ P

$$\begin{bmatrix} A^\top P + PA & PB - C^\top \\ B^\top P - C & -(D + D^\top) \end{bmatrix} \preceq 0$$

[**?**, Theorem 5.31] [**?**, Theorem 3] , [**?**],

$$Q(s) = C(sI - A)^{-1}B + D$$

$,(A, B),(C, A)$ $,Q(s),D,,$ P

$$A^\top P + PA \preceq 0, \qquad -PA^{-1}B = C^\top$$

[**?**, Lemma 7]