Chapter 1
Mathematical model of electrical power systems

In this Chapter, we explain the mathematical model of electrical power systems. In summary, we show that the dynamics of synchronous generators can be described by differential equations, and loads can be described by algebraic equations. Therefore, by combining both the equations of loads and synchronous generators, the entire electrical power system can be expressed as nonlinear differential-algebraic equations.

The current Chapter is structured as follows. First, in Section 1, we introduced to the basic concept of impedance and admittance of circuit elements and phasor representation of current and voltage in AC circuits. Next, in Section 2, we introduce the concept of nodal admittance matrix, which is a mathematical model of power grids. In Sections 3 and 4, mathematical models for generators and loads are explained. Specifically, in Section 3, we show that the model of a power grid composed only by generators can be expressed as a nonlinear ordinary differential equation through Kron reduction of the generator buses. The behavior of such an ordinary differential equation system is shown through a numerical simulation.

1 Foundation of AC circuit theory

1.1 Circuit elements

The basic circuit elements used in mathematical modeling of electrical power systems include resistors, inductors and capacitors. The relationship between terminal voltage and terminal current of each element is presented as follows:

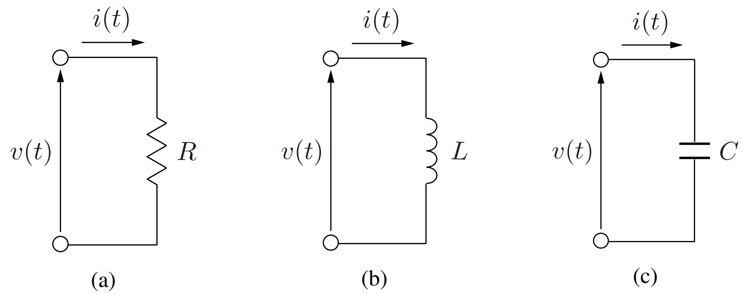


Fig. 1 Basic circuit of resistors, inductors, and capacitors.

(a) **Resistor:** For the resistor with resistance of R [Ω] shown in 1(a), the following relationship holds between the terminal voltage v [V] and terminal current i [A]:

$$v(t) = Ri(t) \tag{1}$$

where $R \geq 0$.

(b) **Inductor:** For the inductor with inductance L [H] shown in 1(b), the following relationship holds between the terminal voltage and terminal current:

$$v(t) = L\frac{di}{dt}(t) \tag{2}$$

where $L \geq 0$.

(c) **Capacitor:** For the capacitor with capacitance C [F] shown in 1(c), the following relationship holds between the terminal voltage and terminal current:

$$i(t) = C\frac{dv}{dt}(t) \tag{3}$$

where $C \geq 0$.

1.2 Instantaneous value and effective value

(a) **Instantaneous value:** The instantaneous value of an AC quantity is an expression of this quantity in function of time t. For example, for a sinusoidal alternating voltage, its instantaneous value can be expressed by:

$$v(t) = V_{\mathrm{m}} \sin(\omega t + \phi) \tag{4}$$

where V_m [V] is the voltage amplitude, ω [rad/s] is the angular frequency, and ϕ [rad] is the phase. The sine wave period T [s] is expressed as follows using ω:

$$T := \frac{2\pi}{\omega} \tag{5}$$

Frequency f [Hz] is expressed as its reciprocal $f := \frac{1}{T}$. Due to the characteristics of the elements presented in Section 1, when the instantaneous value of voltage is a sine wave, the instantaneous value of the current also becomes a sine wave.

(b) **Effective value:** The effective value of an AC quantity corresponds to the square root of the average of the square of the values over a period of time T. Because of this definition, the effective value is also called **RMS value** (root mean square value). For example, for the resistor in 1(a), the average electric power consumed in one period can be calculated as follows:

$$\frac{1}{T}\int_{t}^{t+T} v(\tau)i(\tau)d\tau = \frac{1}{R}\underbrace{\left(\frac{V_\mathrm{m}}{\sqrt{2}}\right)}_{V_\mathrm{e}}^{2} \tag{6}$$

where V_e is the **effective value** of voltage. The effective value of the current is defined in the same manner. Since the average electric power can be described simply by using the effective value of voltage and current, the effective value is often used to perform calculations for AC circuit. The effective value is also used for the phasor representation of voltage and current introduced below.

1.3 Phasor representation

The AC voltage waveform of Equation 4 can be represented in the complex plane as in Figure 2. In this case, $v(t)$ is expressed by the following equation:

$$v(t) = \mathrm{i}\left[V_\mathrm{m}e^{\boldsymbol{j}(\omega t+\phi)}\right] \tag{7}$$

In an electrical power system, the angular frequency ω can be considered constant and equal to the reference angular speed. Under this assumption, the voltage $v(t)$ of Equation 7 can be uniquely expressed by the phase ϕ, which can be derived from ωt and the amplitude V_m. Then, by using the effective value as an expression of the amplitude, we can obtain:

$$\boldsymbol{V} := V_\mathrm{e}e^{\boldsymbol{j}\phi} \tag{8}$$

This is called the **phasor representation** of voltage. When an electrical power system is in a steady state, the phasor $\boldsymbol{V}$ is constant. In other words, the absolute value $|\boldsymbol{V}| = V_\mathrm{e}$ and phase $\angle\boldsymbol{V} = \phi$ are constant. On the other hand, when an electrical

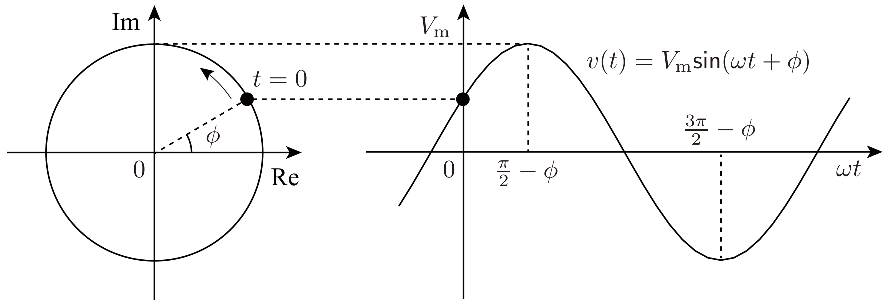


Fig. 2 Complex plane representation of AC voltage

power system is in a transient state, the temporal changes of $|\boldsymbol{V}|$ and $\angle\boldsymbol{V}$ have to be analyzed. The definition for the current phasor $\boldsymbol{I}$ is the same.

1.4 Impedance and admittance

The concept of impedance $\boldsymbol{Z}$ [Ω] arises when expressing the relationship between voltage and curring using the phasor representation explained in Section 1.3. It is equivalent to the resistance in DC circuits, and corresponds to an opposition to alternating current. For the typical circuit elements presented in Section 1, the phasor representations of their terminal voltage and current $\boldsymbol{V}, \boldsymbol{I}$ respect the following relationship.

$$\boldsymbol{V} = \boldsymbol{Z}\boldsymbol{I} \tag{9}$$

The impedance of resistors, inductors and capacitors are, respectively:

$$\boldsymbol{Z}_R := R, \qquad \boldsymbol{Z}_L := \boldsymbol{j}\omega L, \qquad \boldsymbol{Z}_C := \frac{1}{\boldsymbol{j}\omega C}$$

Please note that the characteristics of components such as synchronous generators and power converters may not be expressed using only constant impedances.

The real part of impedance is called **resistance** and the imaginary part is called **reactance**. As standard symbols, R [Ω] is used for resistance and X [Ω] is used for reactance. In other words:

$$\boldsymbol{Z} = R + \boldsymbol{j}X$$

The reciprocal $\boldsymbol{Z}^{-1}$ of impedance is called **admittance**. As a standard symbol, $\boldsymbol{Y}$ [S] is used. In addition, the real part of admittance is called **conductance** and the imaginary part is called **susceptance**. As a standard symbol, G [S] is used for conductance and B [S] is used for susceptance. In other words:

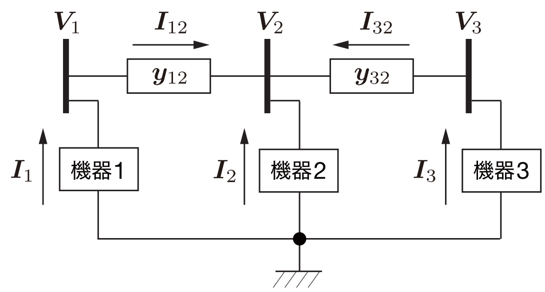


Fig. 3 Power system model composed of three bus bars

$$Y = G + jB$$

The reading of each physical unit presented so far is, V: volt, A: ampere, Ω: ohm, H: henry, F: farad, rad: radian, s: second, Hz: hertz, and S: siemens.

2 Admittance matrix: representation of interaction between connected devices

2.1 Fundamentals of modeling of power grids

In this section we derive the **admittance matrix** of a power grid, which expresses the interaction between the devices connected to an electrical power system

We derive the **admittance matrix** of a power grid that shows the interaction of equipment connected to an electrical power system using a basic transmission line model. The admittance matrix is derived from Ohm's law and Kirchhoff's laws for each bus bar and the transmission lines that connect them. Depending on the literature, the bus bar may also be called a **node** or **bus**. In this book, the bus bar is shown with a thick line in the diagram of an electrical power system. The bus bar is a conductor where the end of the transmission line has been collected. The thin line that connects the bus bar represents the transmission line.

Example 1.1 Admittance matrix of a power grid

Let us consider a simple electrical power system consisting of three bus bars as shown in Fig. 3. Assume that to each bus there is a device connected. In this book, the word "device" refers to synchronous generators and loads. [1] In addition, in an electrical power system model circuit such as that shown in 3, the connection to ground is often omitted for simplification.

[1] In this book, we analyze only load and generators, however, when considering solar generators, wind generators and batteries, these are also classified as "devices".

Below, the voltage phasor of bus bar i with respect to the ground is expressed as $\boldsymbol{V}_i \in \mathbb{C}$, and the current phasor flowing from the device to the bus bar i is expressed as $\boldsymbol{I}_i \in \mathbb{C}$. The voltage and current phasors are unknown variables, therefore it is necessary to find an equation governed by the power grid that establishes a relationship between the current phasor $(\boldsymbol{I}_1, \boldsymbol{I}_2, \boldsymbol{I}_3)$ and the current phasor $(\boldsymbol{V}_1, \boldsymbol{V}_2, \boldsymbol{V}_3)$ of the bus bars. For this purpose, we define the admittance matrix of the power grid.

The admittance of a transmission line that connects bus bars i and j is expressed as $\boldsymbol{y}_{ij} \in \mathbb{C}$, with y_{ij} being known variables for every pair of bus bars ij. Additionally, the current phasor that flows in each transmission line is expressed as $\boldsymbol{I}_{ij} \in \mathbb{C}$, where $\boldsymbol{y}_{ij}$ and $\boldsymbol{y}_{ji}$ are equal. In addition, the sign for $\boldsymbol{I}_{ij}$ is positive for an arbitrarily determined direction, and $\boldsymbol{I}_{ji} = -\boldsymbol{I}_{ij}$ are equal. The current phasor of this transmission line is an intermediate variable that describes the physical relationship of the current phasor and voltage phasor of the bus bars. Specifically, if the sign of the current phasor is defined positive for the direction indicated by the arrows in Fig. 3, the following relationship can be obtained by applying the Ohm's law:

$$\boldsymbol{I}_{12} = \boldsymbol{y}_{12}(\boldsymbol{V}_1 - \boldsymbol{V}_2), \qquad \boldsymbol{I}_{32} = \boldsymbol{y}_{32}(\boldsymbol{V}_3 - \boldsymbol{V}_2)$$

According to Kirchhoff's first law (current law), since the sum of all current on each bus bar is 0, the following relationship is obtained for bus bar 1 to bus bar 3.

$$\boldsymbol{I}_1 - \boldsymbol{I}_{12} = 0, \qquad \boldsymbol{I}_2 + \boldsymbol{I}_{12} + \boldsymbol{I}_{32} = 0, \qquad \boldsymbol{I}_3 - \boldsymbol{I}_{32} = 0$$

Note that Kirchhoff's first law states that the sum of inflow currents and the sum of outflow currents are equal at the point where an electric circuit branches. By replacing the variables $\boldsymbol{I}_{ij}$ by the previously calculated relationship $\boldsymbol{I}_i = \boldsymbol{y}_{ij}(\boldsymbol{V}_1 - \boldsymbol{V}_2)$, we can find the following vectorized version of the Ohm's law:

$$\begin{bmatrix} \boldsymbol{I}_1 \\ \boldsymbol{I}_2 \\ \boldsymbol{I}_3 \end{bmatrix} = \begin{bmatrix} \boldsymbol{y}_{12} & -\boldsymbol{y}_{12} & 0 \\ -\boldsymbol{y}_{12} & \boldsymbol{y}_{12} + \boldsymbol{y}_{32} & -\boldsymbol{y}_{32} \\ 0 & -\boldsymbol{y}_{32} & \boldsymbol{y}_{32} \end{bmatrix} \begin{bmatrix} \boldsymbol{V}_1 \\ \boldsymbol{V}_2 \\ \boldsymbol{V}_3 \end{bmatrix} \tag{10}$$

The complex matrix obtained in this manner is the admittance matrix of the power grid. Since each transmission line is usually expressed as a circuit with an equivalent resistance and inductance, the real part (conductance) of the admittance of the transmission line $\boldsymbol{y}_{ij}$ is non-negative, and the imaginary part (susceptance) is non-positive. Specifically, the imaginary part is usually negative (non-zero).

Below, we consider an electrical power system connected with N bus bars. Then, the admittance matrix $\boldsymbol{Y} \in \mathbb{C}^{N \times N}$ of the power grid gives the following relationship to the current phasor $(\boldsymbol{I}_1, \ldots, \boldsymbol{I}_N)$ and voltage phasor $(\boldsymbol{V}_1, \ldots, \boldsymbol{V}_N)$ of the bus bars.

$$\begin{bmatrix} \boldsymbol{I}_1 \\ \vdots \\ \boldsymbol{I}_N \end{bmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{Y}_{11} & \cdots & \boldsymbol{Y}_{1N} \\ \vdots & \ddots & \vdots \\ \boldsymbol{Y}_{N1} & \cdots & \boldsymbol{Y}_{NN} \end{bmatrix}}_{\boldsymbol{Y}} \begin{bmatrix} \boldsymbol{V}_1 \\ \vdots \\ \boldsymbol{V}_N \end{bmatrix} \tag{11}$$

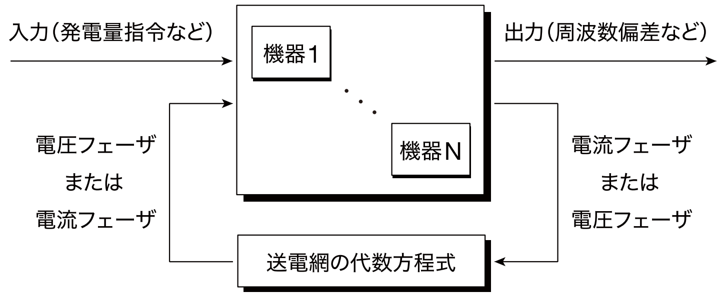


Fig. 4 Schematic diagram of a power system model

Equation 11 can be considered a mathematical model of the power grid that expresses interactions between inputs and outputs of devices connected to a bus bar. Specifically, if we consider the voltage phasor $\boldsymbol{V}_i$ as the output from the device i to the electrical power system, and current phasor $\boldsymbol{I}_i$ as the input from the electrical power system to the device i, $\boldsymbol{I}_i$ can be expressed as a linear combination of output from the other devices:

$$\boldsymbol{I}_i = \boldsymbol{Y}_{i1}\boldsymbol{V}_1 + \cdots + \boldsymbol{Y}_{iN}\boldsymbol{V}_N$$

The real part and imaginary parts of the admittance matrix are called the **conductance matrix** and **susceptance matrix**, respectively.

The simultaneous equations 11 provide partial information about the current and voltage phasors of the bus bars, such that the current and voltage phasors of each bus bar, $(\boldsymbol{I}_1, \ldots, \boldsymbol{I}_N)$ and $(\boldsymbol{V}_1, \ldots, \boldsymbol{V}_N)$, cannot be uniquely determined. To uniquely determine the steady and transient behaviors of the current and voltage phasors of every bus bar, the local relationship between $\boldsymbol{I}_i$ and $\boldsymbol{V}_i$ of each bus bar must be separately determined.

This localized relationship expresses the characteristics of the devices connected to the bus bars and can be considered mathematical models that express the input-output relationship of each device. Specific mathematical models of synchronous generators and loads will be described in detail in Section 3 and beyond.

$$\begin{bmatrix} \boldsymbol{V}_1 \\ \vdots \\ \boldsymbol{V}_N \end{bmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{Z}_{11} & \cdots & \boldsymbol{Z}_{1N} \\ \vdots & \ddots & \vdots \\ \boldsymbol{Z}_{N1} & \cdots & \boldsymbol{Z}_{NN} \end{bmatrix}}_{\boldsymbol{Z}} \begin{bmatrix} \boldsymbol{I}_1 \\ \vdots \\ \boldsymbol{I}_N \end{bmatrix} \tag{12}$$

If the admittance matrix $\boldsymbol{Y}$ in Equation 11 is nonsingular, $\boldsymbol{Z} = \boldsymbol{Y}^{-1}$, however $\boldsymbol{Y}$ is not always nonsingular. For example, if the admittance matrix of Equation 10 is not nonsingular, the following holds:

$$\boldsymbol{V}_1 = \boldsymbol{V}_2 = \boldsymbol{V}_3 \qquad \Longrightarrow \qquad \boldsymbol{I}_1 = \boldsymbol{I}_2 = \boldsymbol{I}_3 = 0$$

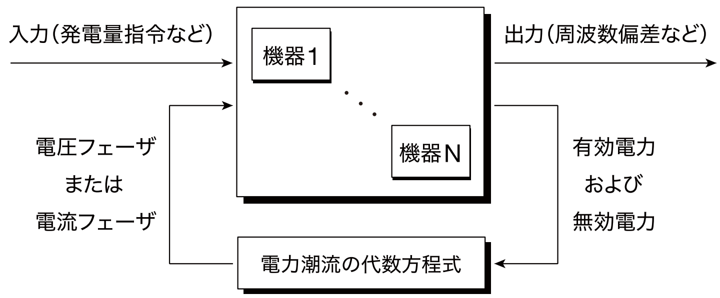


Fig. 5 Schematic diagram of a power system model

The singularity of this admittance matrix indicates that when all current phasors of each bus bar are zero, all voltage phasors are the same; however, this value cannot be uniquely determined. Nevertheless, there is rarely any need to pay attention to the singularity of the admittance matrix within the scope of general analysis.

In addition, active power $P_i \in \mathbb{R}$ and reactive power $Q_i \in \mathbb{R}$ provided from device i to bus bar i are defined by:

$$P_i := \mathrm{Re}\left[\boldsymbol{V}_i \overline{\boldsymbol{I}}_i\right], \qquad Q_i := \mathrm{Im}\left[\boldsymbol{V}_i \overline{\boldsymbol{I}}_i\right]$$

In other words, the following relationship holds between active power, reactive power, bus bar voltage phasor, and bus bar current phasor.

$$P_i + \boldsymbol{j} Q_i = \boldsymbol{V}_i \overline{\boldsymbol{I}}_i \tag{13}$$

Then, by rearranging Equation 11, it is possible to obtain a power system model in which the active and reactive power supplied to the bus bar is the output of the device.

$$\left.\begin{aligned} P_i &= \sum_{j=1}^{N} \mathrm{Re}\left[\overline{\boldsymbol{Y}}_{ij} \boldsymbol{V}_i \overline{\boldsymbol{V}}_j\right] \\ Q_i &= \sum_{j=1}^{N} \mathrm{Im}\left[\overline{\boldsymbol{Y}}_{ij} \boldsymbol{V}_i \overline{\boldsymbol{V}}_j\right] \end{aligned}\right. \qquad i \in \{1, \ldots, N\} \tag{14}$$

Equation 14 is a simultaneous equation that expresses the electric power flow in each bus bar. Figures 4 and 5 are equivalent electrical power system models, but they can be used alternatively according to the analysis puropse.

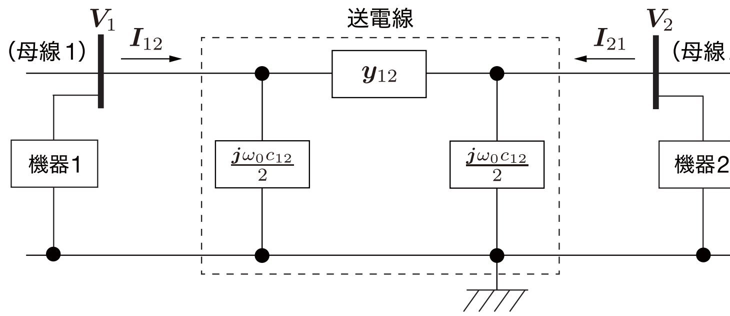


Fig. 6 π-type circuit model of a transmission line with ground capacitance
(Transmission lines with end points at bus bars 1 and 2)

2.2 Power grid model with capacitance to ground

In short transmission lines, the model from the example 1.1 can be used; however, in medium transmission lines that exceed 50 km, the capacitance to ground (capacitance component) created between the transmission lines and the ground cannot be ignored.

When the capacitance to ground, a transmission line is often represented as a π-type equivalent circuit as shown in Figure 6. In Figure 6, a transmission line connecting the bus bars 1 and 2 is shown, where ω_0 is the system frequency and c_{12} is the capacitance to ground. Using this representation of transmission line, the admittance of the power grid is obtained as follows.

Example 1.2 Admittance matrix for the transmission line in a π-type circuit

For an electrical power system similar to Example 1.1, let's derive the admittance matrix when the transmission line is expressed by a π-type equivalent circuit. First, let's consider the relationship of the current phasors $\boldsymbol{I}_{12}$, $\boldsymbol{I}_{21}$, flowing from the bus bars, and the voltage phasors $\boldsymbol{V}_1$, $\boldsymbol{V}_2$ of the bus bars of a transmission line with bus bars 1 and 2 as end points, as illustrated in Figure 6. Then, please note that, unlike in Example 1.1, $\boldsymbol{I}_{21}$ is different from $-\boldsymbol{I}_{12}$. If the current flowing through the transmission line with admittance $\boldsymbol{y}_{12}$ from left to right is expressed as $\boldsymbol{I}'_{12}$, the following equations are obtained from the Kirchhoff's laws:

$$\boldsymbol{I}_{12} = \frac{j\omega_0 c_{12}}{2}\boldsymbol{V}_1 + \boldsymbol{I}'_{12}, \qquad \frac{j\omega_0 c_{12}}{2}\boldsymbol{V}_2 = \boldsymbol{I}_{21} + \boldsymbol{I}'_{12}$$

Moreover, by using the Ohm's law:

$$\boldsymbol{I}'_{12} = \boldsymbol{y}_{12}(\boldsymbol{V}_1 - \boldsymbol{V}_2)$$

By replacing $\boldsymbol{I}'_{12}$ and rearranging the equations, we obtain:

$$\begin{bmatrix} \boldsymbol{I}_{12} \\ \boldsymbol{I}_{21} \end{bmatrix} = \begin{bmatrix} \boldsymbol{y}_{12} + \frac{j\omega_0 c_{12}}{2} & -\boldsymbol{y}_{12} \\ -\boldsymbol{y}_{12} & \boldsymbol{y}_{12} + \frac{j\omega_0 c_{12}}{2} \end{bmatrix} \begin{bmatrix} \boldsymbol{V}_1 \\ \boldsymbol{V}_2 \end{bmatrix}$$

The following is true for the transmission line that uses bus bar 2 and bus bar 3 as end points:

$$\begin{bmatrix} \boldsymbol{I}_{32} \\ \boldsymbol{I}_{23} \end{bmatrix} = \begin{bmatrix} \boldsymbol{y}_{32} + \frac{\boldsymbol{j}\omega_0 c_{32}}{2} & -\boldsymbol{y}_{32} \\ -\boldsymbol{y}_{32} & \boldsymbol{y}_{32} + \frac{\boldsymbol{j}\omega_0 c_{32}}{2} \end{bmatrix} \begin{bmatrix} \boldsymbol{V}_3 \\ \boldsymbol{V}_2 \end{bmatrix}$$

Therefore, by using:

$$\boldsymbol{I}_1 - \boldsymbol{I}_{12} = 0, \qquad \boldsymbol{I}_2 - \boldsymbol{I}_{21} - \boldsymbol{I}_{23} = 0, \qquad \boldsymbol{I}_3 - \boldsymbol{I}_{32} = 0$$

The admittance matrix of the power grid can be obtained as:

$$\begin{bmatrix} \boldsymbol{I}_1 \\ \boldsymbol{I}_2 \\ \boldsymbol{I}_3 \end{bmatrix} = \begin{bmatrix} \boldsymbol{y}_{12} + \frac{\boldsymbol{j}\omega_0 c_{12}}{2} & -\boldsymbol{y}_{12} & 0 \\ -\boldsymbol{y}_{12} & \boldsymbol{y}_{12} + \boldsymbol{y}_{32} + \frac{\boldsymbol{j}\omega_0 (c_{12}+c_{32})}{2} & -\boldsymbol{y}_{32} \\ 0 & -\boldsymbol{y}_{32} & \boldsymbol{y}_{32} + \frac{\boldsymbol{j}\omega_0 c_{32}}{2} \end{bmatrix} \begin{bmatrix} \boldsymbol{V}_1 \\ \boldsymbol{V}_2 \\ \boldsymbol{V}_3 \end{bmatrix}$$

This is consistent with Equation 10 when c_{12} and c_{32} are zero.

2.3 Mathematical properties of the admittance matrix

In this book, we assume power grids that are connected; in other words, there is at least one route connecting two arbitrarily chosen bus bar. For unconnected power grids, the connected parts can be independently discussed. In Figure 7a, the nodes (circles) correspond to the bus bars and the edges (black lines) correspond to the transmission lines.

In the power grid model of Example 1.1 where the capacitance to ground is ignored, the admittance matrix is expressed as $\boldsymbol{Y}_0 \in \mathbb{C}^{N\times N}$. The real and imaginary parts of $\boldsymbol{Y}_0$; in other words, the conductance and susceptance matrices, are expressed as:

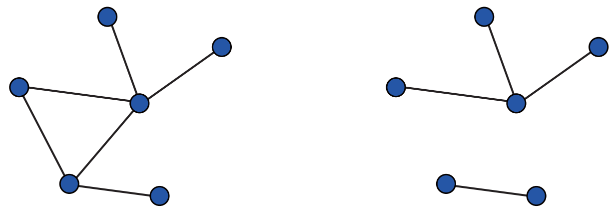
$$G_0 := \mathrm{Re}[\boldsymbol{Y}_0], \qquad B_0 := \mathrm{Im}[\boldsymbol{Y}_0]$$

These matrices have the following properties. First, the sum of elements in all row vectors of these matrices is equal to zero. This is expressed by the following equations, where $\mathbb{1} \in \mathbb{R}^N$ is a vector which all elements are equal to one:

$$\boldsymbol{Y}_0\mathbb{1} = 0 \qquad \Longleftrightarrow \qquad G_0\mathbb{1} = 0, \qquad B_0\mathbb{1} = 0 \tag{15a}$$

Furthermore, since the conductance is non-negative and the susceptance is negative in each transmission line, the conductance matrix is positive semi-definite and the susceptance matrix is negative semi-definite; in other words:

$$G_0 = G_0^\top \succeq 0, \qquad B_0 = B_0^\top \preceq 0 \tag{15b}$$



(a) Connected graph (b) Unconnected graphs

Fig. 7 Connected and disconnected graphs (Connected graph and unconnected graph)

Specifically, based on the fact that the connectivity of the power grid and the susceptance of the transmission line are non-zero, the multiplicity of the zero eigenvalue of B_0 is derived to be 1. This can also be expressed as:

$$\ker B_0 = \mathrm{span}\{\mathbb{1}\} \tag{15c}$$

Under the graph theory perspective, $-B_0$ is called a **graph Laplacian** of a strongly connected weighted undirected graph. The multiplicity of the zero eigenvalue in a graph Laplacian being 1 is a requirement for the corresponding undirected graph to be strongly connected [**?**].

Furthermore, when considering a π-type equivalent circuit for the transmission lines like in Example 1.2, a non-negative value is added to the diagonal element of the susceptance matrix B_0. In other words, the admittance matrix for the power grid model in Examples 1.1 and 1.2 is expressed as:

$$\boldsymbol{Y} = G_0 + \boldsymbol{j}\left(B_0 + \mathrm{diag}(b_i)_{i \in \{1,\ldots,N\}}\right) \tag{16}$$

where b_i is a non-negative constant equivalent to capacitance to ground, and the conductance G_0 and the susceptance B_0 matrices satisfy Equation (15).

3 Mathematical model of synchronous generators

3.1 Classification of generator models based on the level of detail

In electrical power system engineering, different synchronous generator models with different levels of detail, such as the consideration of the field and damper windings, have been used to analyze the stability of electrical power systems [4, 5, 8, 9]. Here, we present four types of models.

(a) Park model The Park model, also called **complete model**, is a model with a high level of details that considers the change in magnetic flux in the stator and field and damper windings. It consists of a two-dimensional linear differential equation (swing equation) that expresses the mechanical dynamic characteristics of the rotor of the generator, and a five to seven-dimensional nonlinear differential equation that expresses the magnetic flux changes in the stator, field winding, and damper winding. The dimension of the latter differential equation that expresses changes in magnetic flux varies based on the number of windings to consider and the setting of variables.

The active and reactive power, which represent the electrical output of a generator, are nonlinear functions of the internal state of the generator. The models used in Japanese works consider a type of damper winding on the d-axis and two types of damper windings on the q-axis in addition to field winding on the d-axis [8]. When there is only one type of damper winding on the q-axis, if the constant is set appropriately, there are no issues in terms of practical use.

(b) **Two-axis model** It is a model that approximates the differential equations to the algebraic equation, assuming that the time constants of the dynamic response of the magnetic flux change in the stator and damper winding are sufficiently small [9. Section 5.4]. It consists of a two-dimensional swing equation and a two-dimensional nonlinear differential equation expressing the magnetic flux change in the field and damper windings. Generally, the dynamic response of the magnetic flux change in the stator and damper winding are sufficiently fast; thus, the behavior of the Park model is generally well simplified.

(c) **One-axis model** In contrast to the two-axis model, the one-axis model, also called **transient model**, is obtained under the assumption that the time constant of the magnetic flux change in the damper winding is sufficiently small [10-12]. It consists of a two-dimensional swing equation and a one-dimensional nonlinear differential equation that expresses the magnetic flux change in the field winding. In this book, we use a one-axis model to perform analysis.The process of deriving a one-axis model from the Park model is explained in [**?**, Section 5] and [**?**, Section 4.15].

(d) **Classical model** It is amodel that ignores the magnetic flux changes in the field and damper windings. It consists of a two-dimensional linear swing equation and the active and reactive power are nonlinear functions of the internal states of the generator. This model is currently widely used to analyze the oscillatory and synchronization phenomena of electrical power systems [10-14].

3.2 Mathematical expression of the one-axis model

(1) **Expressing the relationship of current and voltage with the internal state of the generator as an intermediate variable**

If the interval voltage of a generator i connected to a bus bar i is E_i, and the rotor angle relative to a coordinate system that rotates with angular speed ω_0 is

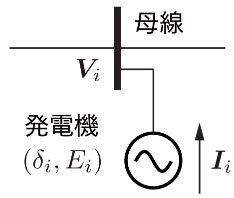


Fig. 8 Generator connected to bus bar

δ_i, the following relationship holds for the voltage phasor $\boldsymbol{V}_i$ of the bus bar i and for the current phasor $\boldsymbol{I}_i$ flowing from the generator to the bus bar i:

$$\boldsymbol{I}_i = \frac{1}{\boldsymbol{j} X_i'} \left(E_i e^{\boldsymbol{j}\delta_i} - \boldsymbol{V}_i \right) \tag{17a}$$

where X_i' is the transient reactance of the generator. Figure 8 illustrates the bus bar and current and voltage phasors. The connection to ground is omitted from the figure.

δ_i and E_i in Equation 17a are intermediate variables representing the internal state of the generator i, and they provide with a dynamic relationship between $\boldsymbol{I}_i$ and $\boldsymbol{V}_i$. In other words, from the perspective of control systems engineering, Equation 17a uses δ_i and E_i as internal states, and can be interpreted as an "output equation" of a generator i, when $\boldsymbol{I}_i$ is the output from the generator to the electrical power system and $\boldsymbol{V}_i$ is the input from the electrical power system to the generator. Alternatively, we can consider $\boldsymbol{V}_i$ as the output and $\boldsymbol{I}_i$ as the input. Details will be discussed later. By multiplying both sides of the Equation 17a with $e^{-\boldsymbol{j}\delta}$ and evaluating its real and imaginary parts separately, we find:

$$\begin{aligned} |\boldsymbol{V}_i| \sin(\delta_i - \angle \boldsymbol{V}_i) &= X_i' |\boldsymbol{I}_i| \cos(\delta_i - \angle \boldsymbol{I}_i), \\ |\boldsymbol{V}_i| \cos(\delta_i - \angle \boldsymbol{V}_i) &= E_i - X_i' |\boldsymbol{I}_i| \sin(\delta_i - \angle \boldsymbol{I}_i) \end{aligned} \tag{17b}$$

Active power P_i provided from the generator i to the bus bar i and can be expressed as:

$$\begin{aligned} P_i &= \mathsf{Re}\left[\boldsymbol{V}_i \bar{\boldsymbol{I}}_i \right] \\ &= \mathsf{Re}\left[|\boldsymbol{V}_i| e^{-\boldsymbol{j}(\delta_i - \angle \boldsymbol{V}_i)} \overline{|\boldsymbol{I}_i| e^{-\boldsymbol{j}(\delta_i - \angle \boldsymbol{I}_i)}} \right] \\ &= |\boldsymbol{V}_i||\boldsymbol{I}_i| \cos(\delta_i - \angle \boldsymbol{V}_i) \cos(\delta_i - \angle \boldsymbol{I}_i) \\ &\qquad\qquad + |\boldsymbol{V}_i||\boldsymbol{I}_i| \sin(\delta_i - \angle \boldsymbol{V}_i) \sin(\delta_i - \angle \boldsymbol{I}_i) \end{aligned}$$

Similarly, reactive power Q_i can be expressed as:

$$
\begin{aligned}
Q_i &= \mathrm{i}\left[\boldsymbol{V}_i \overline{\boldsymbol{I}}_i\right] \\
&= |\boldsymbol{V}_i||\boldsymbol{I}_i| \cos(\delta_i - \angle \boldsymbol{V}_i) \sin(\delta_i - \angle \boldsymbol{I}_i) \\
&\qquad\qquad\qquad - |\boldsymbol{V}_i||\boldsymbol{I}_i| \sin(\delta_i - \angle \boldsymbol{V}_i) \cos(\delta_i - \angle \boldsymbol{I}_i)
\end{aligned}
$$

Thus, by replacing the current phasor using the Equation 17b, the following can be obtained:

$$
\begin{aligned}
P_i &= \frac{E_i |\boldsymbol{V}_i|}{X_i'} \sin(\delta_i - \angle \boldsymbol{V}_i), \\
Q_i &= \frac{E_i |\boldsymbol{V}_i|}{X_i'} \cos(\delta_i - \angle \boldsymbol{V}_i) - \frac{|\boldsymbol{V}_i|^2}{X_i'}
\end{aligned}
\tag{18}
$$

These expressions indicate that the active and reactive power are function of the difference between the rotor angle δ_i and the voltage angle $\angle \boldsymbol{V}_i$ of the bus bar. In typical electrical power system operation, the difference between δ_i and $\angle \boldsymbol{V}_i$ is small; thus, the following approximation holds:

$$
P_i \simeq \frac{E_i |\boldsymbol{V}_i|}{X_i'} (\delta_i - \angle \boldsymbol{V}_i), \qquad Q_i \simeq \frac{|\boldsymbol{V}_i|}{X_i'} (E_i - |\boldsymbol{V}_i|)
$$

The above equations indicate that a difference between δ_i and $\angle \boldsymbol{V}_i$ mainly contributes to the active power, while a difference between E_i and $|\boldsymbol{V}_i|$ contributes to the reactive power.

When the voltage phasor is the input, Equations (17) and 18 can be interpreted as an equivalent deformation of the output equation of the generator under the definition of active power and reactive power in Equation 13. Similarly, if the voltage phasor is cancelled, an output equation, when current phasor is the input, is obtained as:

$$
\begin{aligned}
P_i &= E_i |\boldsymbol{I}_i| \cos(\delta_i - \angle \boldsymbol{I}_i), \\
Q_i &= E_i |\boldsymbol{I}_i| \sin(\delta_i - \angle \boldsymbol{I}_i) - X_i' |\boldsymbol{I}_i|^2
\end{aligned}
\tag{19}
$$

(2) Relational expression of current and voltage in dynamic characteristics of a generator

The swing equation that describes the mechanical dynamics of a synchronous generator is given as follows:

$$
\begin{cases}
\dot{\delta}_i &= \omega_0 \Delta\omega_i \\
M_i \Delta\dot{\omega}_i &= -D_i \Delta\omega_i - P_i + P_{\mathrm{mech}i}
\end{cases}
\tag{20a}
$$

where, $\Delta\omega_i$ is the frequency deviation from the system frequency, ω_0, M_i is the inertia coefficient, D_i is the damping factor, and $P_{\mathrm{mech}i}$ is the mechanical torque. In addition, as a differential equation that expresses the attenuation of magnetic flux, the electromagnetic dynamics of the synchronous generator are given as follows:

$$\tau_i \dot{E}_i = -\frac{X_i}{X_i'} E_i + \left(\frac{X_i}{X_i'} - 1\right) |\boldsymbol{V}_i| \cos(\delta_i - \angle \boldsymbol{V}_i) + V_{\mathrm{field}i} \tag{20b}$$

where, τ_i is the time constant of the field circuit, X_i is the synchronous reactance, and $V_{\mathrm{field}i}$ is the field voltage. From the viewpoint of system control, mechanical torque $P_{\mathrm{mech}i}$ and field voltage $V_{\mathrm{field}i}$ become external inputs. Generally, $X_i > X_i'$ holds.

Summarizing the above, if the voltage magnitude and angle $(|\boldsymbol{V}_i|, \angle \boldsymbol{V}_i)$ are considered inputs from the bus bar i to the generator i, the following equations become the state-space equation that expresses the dynamic characteristics of the generator:

$$\begin{cases} \dot{\delta}_i & = \omega_0 \Delta\omega_i \\ M_i \Delta\dot{\omega}_i & = -D_i \Delta\omega_i - P_i + P_{\mathrm{mech}i} \\ \tau_i \dot{E}_i & = -\frac{X_i}{X_i'} E_i + \left(\frac{X_i}{X_i'} - 1\right) |\boldsymbol{V}_i| \cos(\delta_i - \angle \boldsymbol{V}_i) + V_{\mathrm{field}i} \end{cases} \tag{21a}$$

and the active power P_i is given by Equation 18. The magnitude and angle of the current phasor can be derived from Equation 17b as follows:

$$\begin{aligned} |\boldsymbol{I}_i| &= \sqrt{\left\{\frac{|\boldsymbol{V}_i|}{X_i'} \sin(\delta_i - \angle \boldsymbol{V}_i)\right\}^2 + \left\{\frac{E_i}{X_i'} - \frac{|\boldsymbol{V}_i|}{X_i'} \cos(\delta_i - \angle \boldsymbol{V}_i)\right\}^2}, \\ \angle \boldsymbol{I}_i &= \delta_i - \arctan\left(\frac{E_i - |\boldsymbol{V}_i| \cos(\delta_i - \angle \boldsymbol{V}_i)}{|\boldsymbol{V}_i| \sin(\delta_i - \angle \boldsymbol{V}_i)}\right) \end{aligned} \tag{21b}$$

At this time, the current phasor is considered the output from generator i to the bus bar i. As discussed above, based on the definition of Equation 13, a set of active power and reactive power of Equation 18, (P_i, Q_i), can be considered as an output that is mathematically equivalent to $(|\boldsymbol{I}_i|, \angle \boldsymbol{I}_i)$.

Similarly, the magnitude and angle of the current phasor can be considered inputs from the bus bar i to the generator i. In this case, the following equation becomes the state-space equation that expresses the dynamic characteristics of the generator.

$$\begin{cases} \dot{\delta}_i & = \omega_0 \Delta\omega_i \\ M_i \Delta\dot{\omega}_i & = -D_i \Delta\omega_i - P_i + P_{\mathrm{mech}i} \\ \tau_i \dot{E}_i & = -E_i - (X_i - X_i') |\boldsymbol{I}_i| \sin(\delta_i - \angle \boldsymbol{I}_i) + V_{\mathrm{field}i} \end{cases} \tag{22a}$$

where the active power P_i is expressed as in Equation 19. In addition, the magnitude and angle of the voltage phasor can be derived from Equation 17b as follows:

$$
\begin{aligned}
|\boldsymbol{V}_i| &= \sqrt{\left\{X_i'|\boldsymbol{I}_i|\cos(\delta_i - \angle \boldsymbol{I}_i)\right\}^2 + \left\{E_i - X_i'|\boldsymbol{I}_i|\sin(\delta_i - \angle \boldsymbol{I}_i)\right\}^2}, \\
\angle \boldsymbol{V}_i &= \delta_i - \arctan\left(\frac{X_i'|\boldsymbol{I}_i|\cos(\delta_i - \angle \boldsymbol{I}_i)}{E_i - X_i'|\boldsymbol{I}_i|\sin(\delta_i - \angle \boldsymbol{I}_i)}\right)
\end{aligned} \tag{22b}
$$

In this context, the voltage phasor is regarded as an output from the generator to the bus bar. The set of active power and reactive power of Equation 19, (P_i, Q_i), can also be considered as an output that is mathematically equivalent to $(|\boldsymbol{V}_i|, \angle \boldsymbol{V}_i)$.
The unit of each variable is as follows: the unit for rotor angle δ_i, voltage phasor angle $\angle \boldsymbol{V}_i$, and current phasor angle $\angle \boldsymbol{I}_i$ of the bus bar is [rad]. Frequency deviation $\Delta\omega_i$, internal voltage E_i, absolute value of voltage phasor $|\boldsymbol{V}_i|$, absolute value of current phasor $|\boldsymbol{I}_i|$, active power P_i, active power Q_i, mechanical torque $P_{\mathrm{mech}i}$, and field voltage $V_{\mathrm{field}i}$ are divided by their reference value and, thus are dimensionless values divided by their reference value. Their unit is [pu], which means "per unit". If the system frequency is 50 [Hz], ω_0 is set as 100π. Therefore, the unit of $\omega_0\Delta\omega_i$ is [rad/s].

(3) Relationship with the classical model
With the above one-axis model, the dynamic characteristics of the internal voltage of the generator E_i are taken into consideration. However, in the classical model, this internal voltage is assumed to be constant. Specifically, the following is assumed for the differential equation of internal voltage E_i in Equations (20) and (21):

- Synchronous reactance X_i and transient reactance X_i' are equal, and
- Field voltage $V_{\mathrm{field}i}$ is constant.

Then, if we denote the constant value of the field voltage as $V_i^\star$, the stationary solution of the differential equation in Equation 20b is $E_i(t) = V_i^\star$. In other words, the internal voltage E_i becomes $V_i^\star$, which is constant. For the deeper understanding of the classical model in electrical power system analysis, please refer to [**?**, Section 2.11].

3.3 Kron reduction of a generator bus

In this Section, we analyze an electrical power system model, where the synchronous generator as equipment is connected to each bus bar. Then, if a set of generator buses is expressed as $\mathcal{I}_{\mathrm{G}}$, the number of generator buses $|\mathcal{I}_{\mathrm{G}}|$ is equal to the total number of bus bars N. The model for the entire electrical power system is described as a "differential algebraic equation system", where generators described by Equation 21a are combined with Equation 21b following the input-output relationship of the algebraic equation of Equation 11.

COFFEE BREAK

Differential algebraic equation system: it is a system that described by differential algebraic equations and algebraic equations as follows:

$$\begin{cases} \dot{x}_1 &= f_1(x_1, x_2) \\ 0 &= f_2(x_1, x_2) \end{cases}$$

where x_1 is the state of the differential equation and x_2 is the state of the algebraic equation. For example, in a system where generator models of Equation (21) are combined with the algebraic equation of the power grid of Equation 12, the vector with the internal state of all generators, δ_i $\Delta\omega_i$, E_i, is x_1, while x_2 is the vector which contains all bus voltage phasor variables, $|\boldsymbol{V}_i|$ and $\angle\boldsymbol{V}_i$. When the algebraic equation has a solution for x_2, the solution is expressed as $x_2 = h(x_1)$. By substituting $x_2 = h(x_1)$ into the differential equation we get the following ordinary differential equation system that describes the behavior of x_1:

$$\dot{x}_1 = f_1\big(x_1, h(x_1)\big)$$

The goal of this section is to equivalently transform the system of differential algebraic equations via current and voltage phasors of the generator bus bar $(|\boldsymbol{V}_i|, \angle\boldsymbol{V}_i)_{i\in\mathcal{I}_\mathrm{G}}$ into a system of ordinary differential equations described only by the state variables of the generators, by expressing all voltage phasors as functions of the generator state variables $(\delta_i, E_i)_{i\in\mathcal{I}_\mathrm{G}}$. This transformation is called **Kron reduction** of the generator bus and the steps for performing it are as follows:

(a) Replace the current phasor in the Equation 17a using the algebraic equation representing the power grid in Equation 11. Then, express the voltage phasor $(\boldsymbol{V}_i)_{i\in\mathcal{I}_\mathrm{G}}$ as a function of the generator state variables $(\delta_i, E_i)_{i\in\mathcal{I}_\mathrm{G}}$.

(b) Rewrite the phasors using the Euler's identity:

$$e^{\boldsymbol{j}\,\delta_i}\overline{\boldsymbol{V}}_i = |\boldsymbol{V}_i|\cos(\delta_i - \angle\boldsymbol{V}_i) + \boldsymbol{j}|\boldsymbol{V}_i|\sin(\delta_i - \angle\boldsymbol{V}_i)$$

and rewrite the term of trigonometric function related to $\boldsymbol{V}_i$ included in the model by the state variables $(\delta_i, E_i)_{i\in\mathcal{I}_\mathrm{G}}$.

First, let us consider step (a). If the output equation of Equation 17a is expressed as a vector:

$$\begin{bmatrix} \boldsymbol{I}_1 \\ \vdots \\ \boldsymbol{I}_N \end{bmatrix} = \mathsf{diag}\left(\frac{1}{\boldsymbol{j}X_i'}\right)\left(\mathsf{diag}\left(e^{\boldsymbol{j}\,\delta_i}\right)\begin{bmatrix} E_1 \\ \vdots \\ E_N \end{bmatrix} - \begin{bmatrix} \boldsymbol{V}_1 \\ \vdots \\ \boldsymbol{V}_N \end{bmatrix}\right)$$

Using the algebraic equation of the power grid expressed by Equation 11, and solving the resulting equation for the voltage phasor, the following is obtained:

$$\begin{bmatrix} \boldsymbol{V}_1 \\ \vdots \\ \boldsymbol{V}_N \end{bmatrix} = \left(\mathsf{diag}\left(\frac{1}{\boldsymbol{j}X'_i}\right) + \boldsymbol{Y}\right)^{-1} \mathsf{diag}\left(\frac{e^{\boldsymbol{j}\delta_i}}{\boldsymbol{j}X'_i}\right) \begin{bmatrix} E_1 \\ \vdots \\ E_N \end{bmatrix} \tag{23}$$

In this manner, the voltage phasor $(\boldsymbol{V}_i)_{i\in\mathcal{I}_G}$ of the bus bars is equivalently expressed by the state variables of generators $(\delta_i, E_i)_{i\in\mathcal{I}_G}$. Next, let us consider step (b). If the voltage phasor of the bus bar is expressed in the polar form:

$$\boldsymbol{V}_i = |\boldsymbol{V}_i| e^{\boldsymbol{j}\angle\boldsymbol{V}_i}$$

Then, by multiplying both sides of Equation 23 by $\mathsf{diag}(\frac{e^{-\boldsymbol{j}\delta_i}}{X'_i})$ and taking the complex conjugate, we get:

$$\begin{bmatrix} \frac{|\boldsymbol{V}_1|}{X'_1} e^{\boldsymbol{j}(\delta_1 - \angle\boldsymbol{V}_1)} \\ \vdots \\ \frac{|\boldsymbol{V}_N|}{X'_N} e^{\boldsymbol{j}(\delta_N - \angle\boldsymbol{V}_N)} \end{bmatrix} = \mathsf{diag}\left(e^{\boldsymbol{j}\delta_i}\right) \boldsymbol{\Gamma}^{-1} \mathsf{diag}\left(e^{-\boldsymbol{j}\delta_i}\right) \begin{bmatrix} E_1 \\ \vdots \\ E_N \end{bmatrix} \tag{24}$$

where $\boldsymbol{\Gamma} \in \mathbb{C}^{N\times N}$ is a complex square matrix defined by:

$$\boldsymbol{\Gamma} := \mathsf{diag}\left(X'_i\right) - \boldsymbol{j}\,\mathsf{diag}\left(X'_i\right) \overline{\boldsymbol{Y}}\,\mathsf{diag}\left(X'_i\right) \tag{25}$$

In Equation 24, if the (i, j)th element of $\boldsymbol{\Gamma}^{-1}$ is expressed as $\boldsymbol{\gamma}_{ij}^{-1}$, then:

$$\mathsf{diag}\left(e^{\boldsymbol{j}\delta_i}\right) \boldsymbol{\Gamma}^{-1} \mathsf{diag}\left(e^{-\boldsymbol{j}\delta_i}\right) = \begin{bmatrix} \boldsymbol{\gamma}_{11}^{-1} e^{\boldsymbol{j}(\delta_1-\delta_1)} & \cdots & \boldsymbol{\gamma}_{1N}^{-1} e^{\boldsymbol{j}(\delta_1-\delta_N)} \\ \vdots & \ddots & \vdots \\ \boldsymbol{\gamma}_{N1}^{-1} e^{\boldsymbol{j}(\delta_N-\delta_1)} & \cdots & \boldsymbol{\gamma}_{NN}^{-1} e^{\boldsymbol{j}(\delta_N-\delta_N)} \end{bmatrix}$$

Thus, the real part and imaginary part of the (i, j)th element can be written as:

$$\begin{aligned} \mathrm{Re}\left[\boldsymbol{\gamma}_{ij}^{-1} e^{\boldsymbol{j}(\delta_i-\delta_j)}\right] &= -B_{ij}^{\mathrm{red}} \cos(\delta_i - \delta_j) - G_{ij}^{\mathrm{red}} \sin(\delta_i - \delta_j), \\ \mathrm{Im}\left[\boldsymbol{\gamma}_{ij}^{-1} e^{\boldsymbol{j}(\delta_i-\delta_j)}\right] &= -B_{ij}^{\mathrm{red}} \sin(\delta_i - \delta_j) + G_{ij}^{\mathrm{red}} \cos(\delta_i - \delta_j) \end{aligned}$$

where the reduced conductance and susceptance matrices are defined as:

$$G_{ij}^{\mathrm{red}} := \mathrm{Im}\left[\boldsymbol{\gamma}_{ij}^{-1}\right], \qquad B_{ij}^{\mathrm{red}} := -\,\mathrm{Re}\left[\boldsymbol{\gamma}_{ij}^{-1}\right] \tag{26}$$

In addition, the reduced admittance matrix $\boldsymbol{Y}^{\mathrm{red}}$ is defined as:

$$\boldsymbol{Y}_{ij}^{\mathrm{red}} := G_{ij}^{\mathrm{red}} + \boldsymbol{j} B_{ij}^{\mathrm{red}}$$

This is equal to defining as follows using the above complex matrix $\boldsymbol{\Gamma}^{-1}$:

$$\boldsymbol{Y}^{\mathrm{red}} := -\boldsymbol{j}\boldsymbol{\Gamma}^{-1} \tag{27}$$

Then, Equation 23 can be rewritten as:

$$\begin{aligned}\frac{|\boldsymbol{V}_i|}{X_i'}\cos(\delta_i - \angle\boldsymbol{V}_i) &= -\sum_{j=1}^{N} E_j\left\{B_{ij}^{\mathrm{red}}\cos(\delta_i-\delta_j) + G_{ij}^{\mathrm{red}}\sin(\delta_i-\delta_j)\right\},\\ \frac{|\boldsymbol{V}_i|}{X_i'}\sin(\delta_i - \angle\boldsymbol{V}_i) &= -\sum_{j=1}^{N} E_j\left\{B_{ij}^{\mathrm{red}}\sin(\delta_i-\delta_j) - G_{ij}^{\mathrm{red}}\cos(\delta_i-\delta_j)\right\}\end{aligned} \tag{28}$$

Please note that this equation associates the voltage phasor with the variation of the rotor angle $\delta_i - \delta_j$ and internal voltage E_i. Therefore, a differential algebraic equation model of an electrical power system, in which the generator model of Equation (21) is combined by the simultaneous equation of Equation 11, can be equivalently expressed as a simultaneous ordinary differential equation system related to all $i \in \mathcal{I}_{\mathrm{G}}$.

$$\begin{cases}\dot{\delta}_i = \omega_0\Delta\omega_i\\ M_i\Delta\dot{\omega}_i = -D_i\Delta\omega_i - f_i\left(\delta, E\right) + P_{\mathrm{mech}i}\\ \tau_i\dot{E}_i = -\frac{X_i}{X_i'}E_i + \left(X_i - X_i'\right)g_i\left(\delta, E\right) + V_{\mathrm{field}i}\end{cases} \qquad i \in \mathcal{I}_{\mathrm{G}} \tag{29}$$

However, δ and E are column vectors corresponding to nonlinear interactions between generators. Defining $\delta_{ij} := \delta_i - \delta_j$:

$$\begin{aligned}f_i\left(\delta, E\right) &:= -E_i\sum_{j=1}^{N} E_j\left(B_{ij}^{\mathrm{red}}\sin\delta_{ij} - G_{ij}^{\mathrm{red}}\cos\delta_{ij}\right)\\ g_i\left(\delta, E\right) &:= -\sum_{j=1}^{N} E_j\left(B_{ij}^{\mathrm{red}}\cos\delta_{ij} + G_{ij}^{\mathrm{red}}\sin\delta_{ij}\right)\end{aligned} \tag{30}$$

Based on Equation 29, the behavior of a generator i is only impacted by the relative difference of rotor argument of the remaining generators $(\delta_j)_{j\in\mathcal{I}_{\mathrm{G}}\setminus\{i\}}$. Furthermore, by comparing with Equation 22a, we can see that $f_i(\delta, E)$ of Equation 29 corresponds to the active power P_i output by the generators. If the real part of the admittance matrix $\boldsymbol{Y}$ is zero, in other words, if the conductance of all transmission lines is zero (equivalent resistance equal to zero), the reduced conductance reduced conductance G_{ij}^{red} also becomes 0 for all (i, j). As it will be discussed in Section **??**, this is equivalent to a lossless transmission of active power in the power grid.

In addition, although the admittance matrix $\boldsymbol{Y}$ is a sparse matrix that reflects the graph structure of the power grid, the reduced admittance matrix $\boldsymbol{Y}^{\mathrm{red}}$ in Equation 29 is usually not a sparse matrix. Therefore, note that the system of ordinary differential equations in Equation 29 has a coupled structure in which the internal states of all

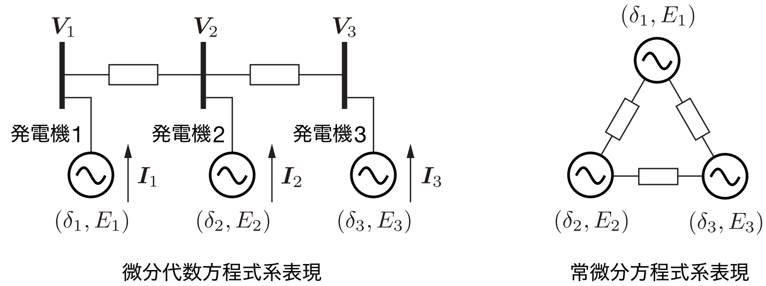


Fig. 9 Changes in bond structure due to Kron reduction

generators interact closely. Let's confirm this fact with numerical examples along with the behavior of the electrical power system model.

COFFEE BREAK

Sparse and dense matrices: A matrix with many 0 elements is called a **sparse matrix**. In contrast, a matrix with almost no 0 elements is called a **dense matrix**. However, how many 0 elements are necessary to call a matrix sparse matrix depends on the context. The inverse matrix of a sparse matrix that is not diagonal is usually a dense matrix.

Table 1 Physical constants of the generator model

i	M_i [s]	D_i [pu]	τ_i [s]	X_i [pu]	X_i' [pu]
1	100	10	5.14	1.569	0.936
2	18	10	5.90	1.651	0.911
3	12	10	8.97	1.220	0.667

Table 2 Steady-state values of external inputs and internal states of the generator

i	$P^{\star}_{\mathrm{mech}i}$ [pu]	$V^{\star}_{\mathrm{filed}i}$ [pu]	$\delta^{\star}_i$ [rad]	$\Delta\omega^{\star}_i$ [pu]	$E^{\star}_i$ [pu]
1	−0.5623	1.5132	0.4656	0	1.4363
2	0.8832	2.2216	1.0903	0	1.8095
3	−0.3160	0.9198	0.6067	0	1.1030

Example 1.3 Behavior of an electrical power system model with a reduced generator bus

Let us consider an electrical power system model consisting of three bus bars as discussed in Example 1.1. Moreover, a synchronous generator is connected to each bus bar and modeled as the one-axis generator model discussed in Section 3.2. Then, the electrical power system model can be draw as the left picture in Figure 9. The values of the physical constants of the generators are chosen as in 1. Since the system frequency is set to 60 [Hz], the value of ω_0 is 120π.

The admittance of the two transmission lines is set as:

$$\boldsymbol{y}_{12} = 1.3652 - \boldsymbol{j}11.6041, \qquad \boldsymbol{y}_{23} = 1.9422 - \boldsymbol{j}10.5107 \tag{31}$$

In this manner, the admittance matrix of the power grid in Equation 10 can be obtained. Then, the real and imaginary parts of the reduced admittance matrix $\boldsymbol{Y}^{\text{red}}$ can be calculated according to Equation 27.

$$G^{\text{red}} = \begin{bmatrix} 0.0073 & 0.0005 & -0.0079 \\ 0.0005 & 0.0041 & -0.0046 \\ -0.0079 & -0.0046 & 0.0125 \end{bmatrix},$$
$$B^{\text{red}} = \begin{bmatrix} -0.3716 & -0.3167 & -0.3800 \\ -0.3167 & -0.3550 & -0.4260 \\ -0.3800 & -0.4260 & -0.6933 \end{bmatrix}$$

Therefore, both reduced conductance and susceptance matrices are dense. This dense structure corresponds to the right image of 9.

Next, let us calculate the time response of the ordinary differential equation system model of Equation 29. In the following, the initial value response is obtained when the mechanical power and the magnetic field, which are external inputs, are fixed to constants. In this example, we assume that the steady-state values of the external inputs and internal states of the power system model are obtained in advance using the steady-state calculation method that will be described in Section **??**. Specifically, the input mechanical power and magnetic field are set to the constant values shown in the first and second columns of Table 2. In this case, the steady-state values of the internal states of the power system are given in columns 3 to 5 of Table 2.

These steady values are one of the solutions that satisfy the following simultaneous equations:

$$\begin{cases} 0 = -f_i\left(\delta^\star, E^\star\right) + P^\star_{\text{mech}i} \\ 0 = -\frac{X_i}{X_i'}E_i^\star + \left(X_i - X_i'\right) g_i\left(\delta^\star, E^\star\right) + V^\star_{\text{field}i} \end{cases} \qquad i \in \{1, 2, 3\}$$

where $\delta^\star$ and $E^\star$ are vectors composed by $\delta_i^\star$ and $E_i^\star$ and the functions f_i and g_i are defined by Equation 30.

First, let us consider a situation in which the steady-state value is perturbed and set as initial value. In other words:

$$\begin{bmatrix} \delta_1(0) \\ \delta_2(0) \\ \delta_3(0) \end{bmatrix} = \begin{bmatrix} \delta_1^\star + \frac{\pi}{6} \\ \delta_2^\star \\ \delta_3^\star \end{bmatrix}, \qquad \begin{bmatrix} E_1(0) \\ E_2(0) \\ E_3(0) \end{bmatrix} = \begin{bmatrix} E_1^\star + 0.1 \\ E_2^\star \\ E_3^\star \end{bmatrix} \tag{32}$$

where the initial value of the frequency deviation is 0. Then, the time response of the electrical power system model is shown in 10.

From this figure, it can be seen that after the angular frequency deviation and rotor deflection angle oscillate for about 15 seconds, they asymptotically converge to the original steady state. However, since the behavior of the rotor angle is only affected by the relative difference among generators, the convergence value of the rotor angle is shifted from the original steady state value by a constant. In other words, for a certain constant c_0:

$$\lim_{t\to\infty} \delta_i(t) = \delta_i^\star + c_0, \qquad \forall i \in \{1, 2, 3\}$$

For any value of c_0, the steady value is essentially equivalent. The voltage phasor of the bus bar shown in Figure 10 can be calculated independently of the internal state of the generators by using the Equation 23. Similarly, the active and reactive power can be calculated independently using Equation 19. Furthermore, since the frequency deviation is equal to 5×10^{-3} [pu] times the system frequency, it is equal to 0.3 [Hz].

Next, let us consider a case where the value of an external input is perturbed. Specifically, let us consider a perturbation in the mechanical power of generator 1.

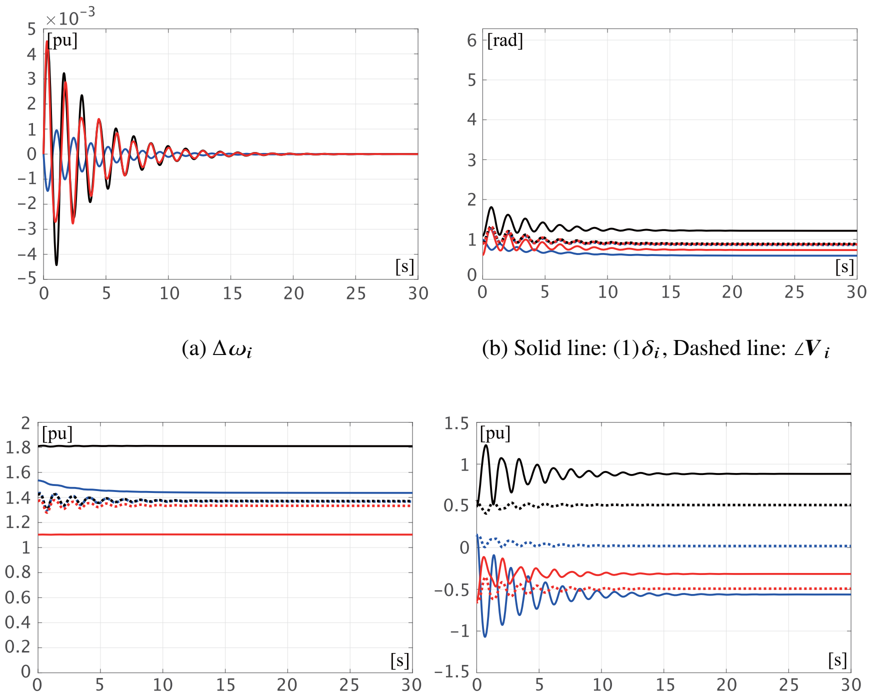
$$\begin{bmatrix} P_{\text{mech}1}(t) \\ P_{\text{mech}2}(t) \\ P_{\text{mech}3}(t) \end{bmatrix} = \begin{bmatrix} P_{\text{mech}1}^\star + 0.05 \\ P_{\text{mech}2}^\star \\ P_{\text{mech}3}^\star \end{bmatrix}$$

Figure 11 shows the time response of the electrical power system model under this condition. The initial value is set to be the same as in Equation 32. In addition, the phase of the rotor angle and the bus voltage phasor are the remainder of the division by π. In this case, it can be seen that the angular frequency deviation does not become 0 in the steady state, and the integral of the rotor angle changes constantly. Figure 12 shows the time response of the electrical power system model when field voltage of generator 1 is similarly perturbed:

$$\begin{bmatrix} V_{\text{filed}1}(t) \\ V_{\text{filed}2}(t) \\ V_{\text{filed}3}(t) \end{bmatrix} = \begin{bmatrix} V_{\text{filed}1}^\star + 0.5 \\ V_{\text{filed}2}^\star \\ V_{\text{filed}3}^\star \end{bmatrix}$$

Similarly, in this situation the frequency deviation in a steady state does not become 0. Therefore, to calculate the time response of an electrical power system model, not only the initial values, but also the external inputs, such as mechanical power and field voltage, must be set to appropriate values.

Since Equation 29 is an ordinary differential equation system model, it is possible to numerically simulate its behavior by using differential equation solvers in MATLAB. However, as shown in Example 1.3, unless the value of the mechanical power and the field voltage are set appropriately, the frequency deviation of each generator does not converge to 0 in the steady state, and the rotor argument constantly deviates from the reference. Thus, to perform a numerical simulation that is



(a) $\Delta\omega_i$ (b) Solid line: (1)δ_i, Dashed line: $\angle \boldsymbol{V}_i$

(c) Solid line: E_i, Dashed line: $|\boldsymbol{V}_i|$ (d) Solid line: P_i, Dashed line: Q_i

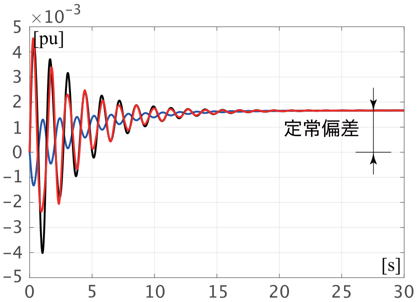
Fig. 10 System time response for perturbated initial value
((Blue: Bus 1, Black: Bus 2, Red: Bus 3))

realistically meaningful, a method to calculate valid equilibrium points and initial values is necessary. These details will be explained in Chapter **??**.

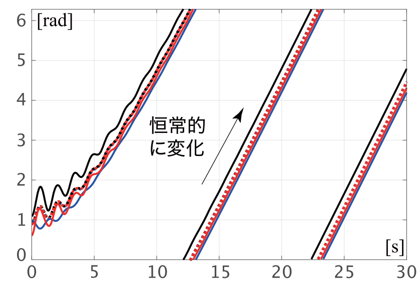
3.4 Derivation of the Kuramoto-type oscillator model

By applying Kron reduction of generator bus to the classical model explained in Section 3.2, a Kuramoto-type oscillator model can be derived.

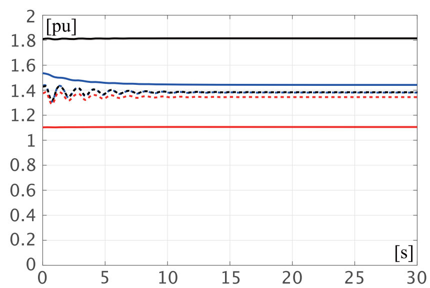
Specifically, if we assume that the value of synchronous and transient reactances, X_i and X_i', are equal, and that the field voltage $V_{\mathrm{field}i}$ is constant and equal to $V_i^\star$, the internal voltage E_i is constant and equal to $V_i^\star$. Therefore, the model of an electrical power system with synchronous generators connected to each bus bar is expressed by the following ordinary differential equation.



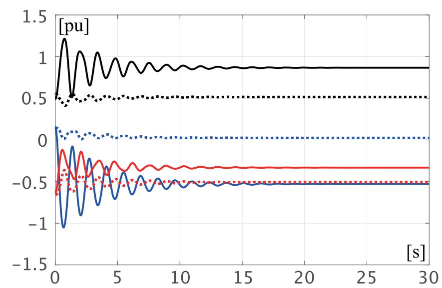
(a) $\Delta\omega_i$



(b) Solid line: δ_i, Dashed line: $\angle \boldsymbol{V}_i$



(c) Solid line: E_i, Dashed line: $|\boldsymbol{V}_i|$



(d) Solid line: P_i, Dashed line: Q_i

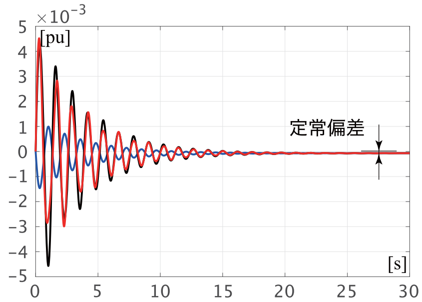
Fig. 11 System time response when perturbation is applied to the mechanical power
(Blue: Bus 1, Black: Bus 2, Red: Bus 3)

$$\begin{cases} \dot{\delta}_i = \omega_0 \Delta\omega_i \\ M_i \Delta\dot{\omega}_i = -D_i \Delta\omega_i - \hat{f}_i(\delta) + P_{\mathrm{mech}i} \end{cases} \qquad i \in \mathcal{I}_{\mathrm{G}} \tag{33}$$

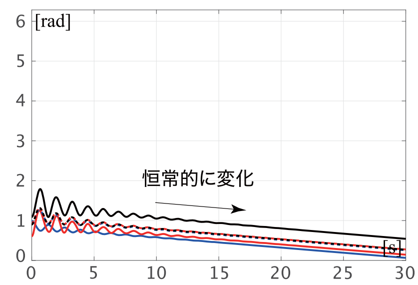
However, the nonlinear term is given by:

$$\hat{f}_i(\delta) := -V_i^\star \sum_{j=1}^{N} V_j^\star \left(B_{ij}^{\mathrm{red}} \sin \delta_{ij} - G_{ij}^{\mathrm{red}} \cos \delta_{ij} \right)$$

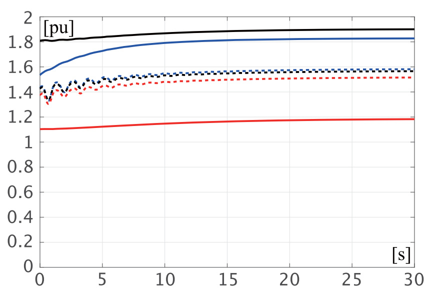
The function $\hat{f}_i(\delta)$ expresses the active power of a generator i. An electrical power system mode whose transmission lines have zero conductance, that is $G_{ij}^{\mathrm{red}} = 0$, is occasionally used.



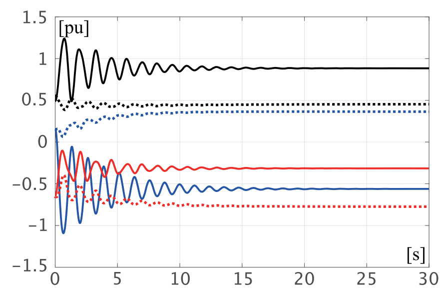
(a) $\Delta\omega_i$



(b) Solid line: δ_i, Dashed line:$\angle \boldsymbol{V}_i$



(c) Solid line: E_i, Dashed line:$|\boldsymbol{V}_i|$



(d) Solid line: P_i, Dashed line:Q_i

Fig. 12 System time response when perturbation is applied to the field voltage
(Blue: Bus 1, Black: Bus 2, Red: Bus 3)

COFFEE BREAK

Kuramoto model:
The following differential equation system comprising of N oscillators moving on the circumference is called **Kuramoto model**.

$$\dot{\theta}_i = \omega_i - \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_i - \theta_j), \qquad i = 1, \dots, N$$

In this context, ω_i is a constant representing the intrinsic angular velocity of the oscillator i, and K is a constant representing the coupling strength. In general, when the coupling strength K is sufficiently large compared to the magnitude of the inhomogeneity of the intrinsic angular speeds $\omega_1, \dots, \omega_N$, the angular speeds of the oscillator, $\dot{\theta}_1, \dots, \dot{\theta}_N$, are asymptotically synchronized.
The Kuramoto model has been analyzed mainly in the field of physics as a mathematical model describing the synchronization phenomena of nonlinear

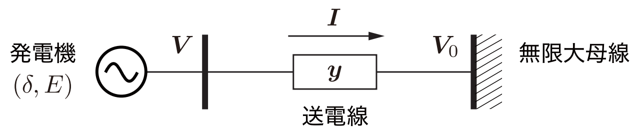


Fig. 13 Single machine infinity bus system model

oscillators, and is known to have a wide range of applications [**?**, **?**]. The nonlinear second-order oscillator model with inertial is also applied to the analysis of synchronization in power systems [**?**, **?**, **?**, **?**].

3.5 Single machine infinite bus system model

The **single machine infinite bus system model** is a simplified electrical power system model that is often used in basic mathematical analysis of electrical power systems, such as in [**?**, Section 1.3] and [**?**, Section 6.3, Section 8.3]. It is an electrical power system model consisting of a generator, a transmission line, and an infinite bus bar as shown in Figure 13. The infinite bus bar is interpreted as a rough approximation of the entire electrical power system, except for the generator of interest, as a ”fixed voltage source”.

Specifically, we assume that voltage phasor of the infinite bus bar $\boldsymbol{V}_0$ is maintained as constant regardless the internal state of the generator. In other words, considering a transmission loss of electric power, the active and reactive power generated by the generator are assumed to be consumed each moment, without excess or deficiency, by the infinite bus bar.

In the following, we only focus on one generator, so we omit the subscript i and denote the variables of the generator and the generator bus. For the dynamic characteristics of the generator expressed in Equation (21), the voltage phasor $\boldsymbol{V}$ and current phasor $\boldsymbol{I}$ of a generator bus have the following relationship:

$$\boldsymbol{I} = \boldsymbol{y}(\boldsymbol{V} - \boldsymbol{V}_0)$$

This is an algebraic equation that determines the input-output relationship of the generator and the electrical power system. For reference, let us derive an ordinary differential equation model from the Kron reduction of the single machinen infinite bus bar system when the resistance of the transmission line is 0 and the reactance x. In other words, the admittance of the transmission line is:

$$\boldsymbol{y} = \frac{1}{\boldsymbol{j}x}$$

Specifically, following the same procedure as of the Kron reduction of a generator bus in Section 3.3, the following two equations are used to replace the current and voltage phasors of the generator bus.

$$\boldsymbol{I} = \frac{1}{\boldsymbol{j}x}(\boldsymbol{V} - \boldsymbol{V}_0), \qquad \boldsymbol{I} = \frac{1}{\boldsymbol{j}X'}(Ee^{\boldsymbol{j}\delta} - \boldsymbol{V})$$

By substituting the voltage phasor and reorganizing the equation, the following is obtained:

$$|\boldsymbol{I}|e^{\boldsymbol{j}(\delta - \angle \boldsymbol{I})} = -\frac{E - |\boldsymbol{V}_0|e^{\boldsymbol{j}\delta}}{\boldsymbol{j}(X' + x)}$$

However, without loss of generality, $\angle \boldsymbol{V} = 0$ with respect to the phase of the infinite bus voltage phasor. Therefore, by replacing the current phasor into Equation 22a, the expression of the ordinary differential equation system is obtained as follows:

$$\begin{cases} \dot{\delta} = \omega_0 \Delta\omega \\ M\Delta\dot{\omega} = -D\Delta\omega - \frac{E|\boldsymbol{V}_0|}{X'+x}\sin\delta + P_{\mathrm{mech}} \\ \tau\dot{E} = -\frac{X+x}{X'+x}E + \frac{X-X'}{X'+x}|\boldsymbol{V}_0|\cos\delta + V_{\mathrm{field}} \end{cases}$$

Similarly, from Equation 19 the active and reactive power supplied from the generator to the bus bar can be calculated as follows:

$$P = \frac{E|\boldsymbol{V}_0|\sin\delta}{X' + x}, \qquad Q = \frac{xE^2 + (X' - x)E|\boldsymbol{V}_0|\cos\delta - X'|\boldsymbol{V}_0|^2}{(X' + x)^2}$$

In practice, since electrical power systems consist of multiple generators, the single machine infinite bus system model is not usually used in this book. The introduction of this model only serves as a reference.

3.6 Mathematical properties of the admittance matrix with reduced generator bus

Below, we mathematically discuss the existence and definiteness of the reduced admittance matrix $\boldsymbol{Y}^{\mathrm{red}}$ of Equation 27. Let us remember that when the capacitance to ground can be ignored, the conductance matrix G_0 is positive semi-definite and the susceptance matrix B_0 is negative definite. Moreover, the reduced conductance and susceptance matrices, which correspond to the real and imaginary parts of $\boldsymbol{Y}^{\mathrm{red}}$ are expressed as follows:

$$G^{\mathrm{red}} := \mathrm{Re}\big[\boldsymbol{Y}^{\mathrm{red}}\big], \qquad B^{\mathrm{red}} := \mathrm{Im}\big[\boldsymbol{Y}^{\mathrm{red}}\big] \tag{34}$$

Then, the following facts hold.

Theorem 1.1 (Existence and definiteness of the reduced admittance matrix)

If the following is true for the admittance matrix $\boldsymbol{Y} \in \mathbb{C}^{N\times N}$ of Equation 16:

$$b_i X_i' \leq 1, \qquad \forall i \in \mathcal{I}_G \tag{35}$$

and at the same time for at least one generator bus the inequality in Equation 35 is strict, then $\boldsymbol{\Gamma}$ of Equation 25 is nonsingular. Moreover, for the admittance matrix $\boldsymbol{Y}^{\mathrm{red}}$ of Equation 27, the reduced conductance matrix G^{red} is positive semi-definite, and the reduced susceptance matrix B^{red} is negative definite.

Proof By using the Mathematical Supplement 1.1 at the end of this chapter, we prove the singularity of $\boldsymbol{\Gamma}$ of Equation 25. From the definition, if M and N are the real and imaginary parts of $\boldsymbol{\Gamma}$, respectively, then:

$$\begin{aligned} M &:= \mathsf{diag}\big(X_i'(1 - b_i X_i')\big) - \mathsf{diag}\left(X_i'\right) B_0 \, \mathsf{diag}\left(X_i'\right), \\ N &:= -\,\mathsf{diag}\left(X_i'\right) G_0 \, \mathsf{diag}\left(X_i'\right) \end{aligned}$$

Here, since B_0 in Equation 16 is negative semi-definite, M is at least positive semi-definite if Equation 35 holds. If $b_i X_i' < 1$ for at least one $i \in \mathcal{I}_G$, then

$$\underbrace{\mathsf{ker}\,\mathsf{diag}\left(X_i'\right) B_0 \, \mathsf{diag}\left(X_i'\right)}_{\mathsf{span}\left\{\mathsf{diag}\left(1/X_i'\right)\mathbb{1}\right\}} \not\subseteq \mathsf{ker}\,\mathsf{diag}\big(X_i'(1 - b_i X_i')\big)$$

and M is positive definite.

Therefore, since N is symmetric, $M + NM^{-1}N$ is positive definite. This implies that $M + NM^{-1}N$ is non-singular. Also, since N is negative semi-definite from the relation in Equation 15b, from the Mathematical Supplement 1.2 at the end of the chapter, it can be shown that the real part of $\boldsymbol{\Gamma}^{-1}$ is positive definite and the imaginary part is negative semi-definite. Therefore, the real part G^{red} of $\boldsymbol{Y}^{\mathrm{red}}$ is positive semi-definite and the imaginary part B^{red} is negative definite. □

Inequality in Equation 35 is a sufficient condition for $\boldsymbol{\Gamma}$ to be nonsingular. However, if $b_i X_i' = 1$ for all generator buses, $\boldsymbol{\Gamma}$ is no longer nonsingular. If all b_i are sufficiently small; in other words, if the capacitance to ground of each transmission line can be ignored as in the Example 1.1, Equation 35 holds. Then, the reduced conductance matrix G^{red}, the real part of $\boldsymbol{Y}^{\mathrm{red}}$, is positive semi-definite. The imaginary part, the reduced susceptance matrix B^{red}, is negative definite. For this reason, the definiteness of the admittance matrix is invariable to the Kron reduction.

3.7 Mathematical model of salient pole synchronous generators

Let us consider a salient pole synchronous generator model that incorporates the difference in reactance between the d and q axes [**?**, **?**, **?**, **?**]. Specifically, let us consider a situation where Equation 17b is as follows:

$$
\begin{aligned}
|\boldsymbol{V}_i| \sin(\delta_i - \angle \boldsymbol{V}_i) &= X_{\mathrm{q}i} |\boldsymbol{I}_i| \cos(\delta_i - \angle \boldsymbol{I}_i), \\
|\boldsymbol{V}_i| \cos(\delta_i - \angle \boldsymbol{V}_i) &= E_i - X'_{\mathrm{d}i} |\boldsymbol{I}_i| \sin(\delta_i - \angle \boldsymbol{I}_i)
\end{aligned}
\tag{36}
$$

where $X'_{\mathrm{d}i}$ is the transient reactance of the d-axis. If $X'_{\mathrm{d}i}$ and $X_{\mathrm{q}i}$ are equal X'_i, Equation 36 is consistent with Equation 17b.

When cancelling the current phasor using Equation 36, the active and reactive power are expressed as:

$$
\begin{aligned}
P_i = {} & \frac{|\boldsymbol{V}_i| E_i}{X'_{\mathrm{d}i}} \sin(\delta_i - \angle \boldsymbol{V}_i) \\
& - \left(\frac{1}{X'_{\mathrm{d}i}} - \frac{1}{X_{\mathrm{q}i}} \right) |\boldsymbol{V}_i|^2 \sin(\delta_i - \angle \boldsymbol{V}_i) \cos(\delta_i - \angle \boldsymbol{V}_i), \\
Q_i = {} & \frac{|\boldsymbol{V}_i| E_i}{X'_{\mathrm{d}i}} \cos(\delta_i - \angle \boldsymbol{V}_i) \\
& - |\boldsymbol{V}_i|^2 \left(\frac{\cos^2(\delta_i - \angle \boldsymbol{V}_i)}{X'_{\mathrm{d}i}} + \frac{\sin^2(\delta_i - \angle \boldsymbol{V}_i)}{X_{\mathrm{q}i}} \right)
\end{aligned}
\tag{37}
$$

Similarly, if the voltage phasor is cancelled, the following is obtained:

$$
\begin{aligned}
P_i = {} & E_i |\boldsymbol{I}| \cos(\delta_i - \angle \boldsymbol{I}_i) \\
& - (X'_{\mathrm{d}i} - X_{\mathrm{q}i}) |\boldsymbol{I}_i|^2 \sin(\delta_i - \angle \boldsymbol{I}_i) \cos(\delta_i - \angle \boldsymbol{I}_i), \\
Q_i = {} & E_i |\boldsymbol{I}_i| \sin(\delta_i - \angle \boldsymbol{I}_i) \\
& - |\boldsymbol{I}_i|^2 \left\{ X'_{\mathrm{d}i} \sin^2(\delta_i - \angle \boldsymbol{I}_i) + X_{\mathrm{q}i} \cos^2(\delta_i - \angle \boldsymbol{I}_i) \right\}
\end{aligned}
\tag{38}
$$

Similar to Equation 21, by combining the swing equation with the electromagnetic dynamics, the following can be obtained.

$$
\begin{cases}
\dot{\delta}_i = \omega_0 \Delta\omega_i \\
M_i \Delta\dot{\omega}_i = -D_i \Delta\omega_i - P_i + P_{\mathrm{mech}i} \\
\tau_i \dot{E}_i = -\frac{X_{\mathrm{d}i}}{X'_{\mathrm{d}i}} E_i + \left(\frac{X_{\mathrm{d}i}}{X'_{\mathrm{d}i}} - 1 \right) |\boldsymbol{V}_i| \cos(\delta_i - \angle \boldsymbol{V}_i) + V_{\mathrm{field}i}
\end{cases}
\tag{39a}
$$

However, for active power P_i, the expression of Equation 37 is used. Here, the voltage phasor is regarded as an input from the electrical power system to the generator i. In addition, from Equation 36, the current phasor can be regarded as an output from the generator to the electrical power system:

$$
\begin{aligned}
|\boldsymbol{I}_i| &= \sqrt{ \left\{ \frac{|\boldsymbol{V}_i|}{X_{\mathrm{q}i}} \sin(\delta_i - \angle \boldsymbol{V}_i) \right\}^2 + \left\{ \frac{E_i}{X'_{\mathrm{d}i}} - \frac{|\boldsymbol{V}_i|}{X'_{\mathrm{d}i}} \cos(\delta_i - \angle \boldsymbol{V}_i) \right\}^2 }, \\
\angle \boldsymbol{I}_i &= \delta_i - \arctan \left(\frac{ \frac{E_i}{X'_{\mathrm{d}i}} - \frac{|\boldsymbol{V}_i|}{X'_{\mathrm{d}i}} \cos(\delta_i - \angle \boldsymbol{V}_i) }{ \frac{|\boldsymbol{V}_i|}{X_{\mathrm{q}i}} \sin(\delta_i - \angle \boldsymbol{V}_i) } \right)
\end{aligned}
\tag{39b}
$$

Similarly, if the current phasor is regarded as an input from the electrical power system to the generator i, the state space representation of the dynamic characteristics of the generator becomes the following:

$$\begin{cases} \dot{\delta}_i = \omega_0 \Delta\omega_i \\ M_i \Delta\dot{\omega}_i = -D_i \Delta\omega_i - P_i + P_{\mathrm{mech}i} \\ \tau_i \dot{E}_i = -E_i - (X_{\mathrm{d}i} - X'_{\mathrm{d}i})|\boldsymbol{I}_i| \sin(\delta_i - \angle \boldsymbol{I}_i) + V_{\mathrm{field}i} \end{cases} \tag{40a}$$

where P_i is calculated through Equation 38.

Moreover, the voltage phasor obtained from Equation 36, is an output from the generator to the electrical power system:

$$\begin{aligned} |\boldsymbol{V}_i| &= \sqrt{\left\{X_{\mathrm{q}i}|\boldsymbol{I}_i|\cos(\delta_i - \angle\boldsymbol{I}_i)\right\}^2 + \left\{E_i - X'_{\mathrm{d}i}|\boldsymbol{I}_i|\sin(\delta_i - \angle\boldsymbol{I}_i)\right\}^2}, \\ \angle\boldsymbol{V}_i &= \delta_i - \arctan\left(\frac{X_{\mathrm{q}i}|\boldsymbol{I}_i|\cos(\delta_i - \angle\boldsymbol{I}_i)}{E_i - X'_{\mathrm{d}i}|\boldsymbol{I}_i|\sin(\delta_i - \angle\boldsymbol{I}_i)}\right) \end{aligned} \tag{40b}$$

This salient pole generator model is consistent with the generator model discussed in Section 3.2 by assuming that $X'_{\mathrm{d}i}$ and $X_{\mathrm{q}i}$ are equal to X'_i and replacing $X_{\mathrm{d}i}$ with X_i. The relationship, $X_{\mathrm{d}i} > X_{\mathrm{q}i} > X'_{\mathrm{d}i}$ usually holds between these reactances.

COFFEE BREAK

Relationship between generator models: The relationship between the 2-axis, 1-axis and classical models is related to the magnitude of the time constant of the electromagnetic dynamics representing the flux variation. The state-space equation of the 2-axis model when the voltage phasor of the bus is regarded as the input is:

$$\begin{cases} \dot{\delta}_i & = \omega_0 \Delta\omega_i \\ M_i \Delta\dot{\omega}_i & = -D_i \Delta\omega_i - P_i + P_{\mathrm{mech}i} \\ \tau_{\mathrm{d}i} \dot{E}_{\mathrm{q}i} & = -\frac{X_{\mathrm{d}i}}{X'_{\mathrm{d}i}} E_{\mathrm{q}i} + \left(\frac{X_{\mathrm{d}i}}{X'_{\mathrm{d}i}} - 1\right) \boldsymbol{V}_{\mathrm{q}i} + V_{\mathrm{field}i} \\ \tau_{\mathrm{q}i} \dot{E}_{\mathrm{d}i} & = -\frac{X_{\mathrm{q}i}}{X'_{\mathrm{q}i}} E_{\mathrm{d}i} + \left(\frac{X_{\mathrm{q}i}}{X'_{\mathrm{q}i}} - 1\right) \boldsymbol{V}_{\mathrm{d}i} \end{cases}$$

where $X'_{\mathrm{q}i}$ is the transient reactance of the q axis.

$$\boldsymbol{V}_{\mathrm{d}i} := |\boldsymbol{V}_i|\sin(\delta_i - \angle\boldsymbol{V}_i), \qquad \boldsymbol{V}_{\mathrm{q}i} := |\boldsymbol{V}_i|\cos(\delta_i - \angle\boldsymbol{V}_i)$$

Additionally, the output active and reactive power are expressed by the following equations:

$$\begin{cases} P_i & = \frac{E_{\mathrm{q}i}}{X'_{\mathrm{d}i}} \boldsymbol{V}_{\mathrm{d}i} - \frac{E_{\mathrm{d}i}}{X'_{\mathrm{q}i}} \boldsymbol{V}_{\mathrm{q}i} + \left(\frac{1}{X'_{\mathrm{q}i}} - \frac{1}{X'_{\mathrm{d}i}}\right) \boldsymbol{V}_{\mathrm{d}i} \boldsymbol{V}_{\mathrm{q}i}, \\ Q_i & = \frac{E_{\mathrm{q}i}}{X'_{\mathrm{d}i}} \boldsymbol{V}_{\mathrm{q}i} + \frac{E_{\mathrm{d}i}}{X'_{\mathrm{q}i}} \boldsymbol{V}_{\mathrm{d}i} - \left(\frac{\boldsymbol{V}_{\mathrm{d}i}^2}{X'_{\mathrm{q}i}} + \frac{\boldsymbol{V}_{\mathrm{q}i}^2}{X'_{\mathrm{d}i}}\right) \end{cases}$$

Note that the state variables of the internal voltage have increased to two,

$E_{\mathrm{d}i}$ and $E_{\mathrm{q}i}$. This two-axis model matches the salient pole one-axis model asymptotically if the time constant $\tau_{\mathrm{q}i}$ is small enough. Specifically, assuming that $\tau_{\mathrm{q}i}$ is 0, the state variable $E_{\mathrm{d}i}$ satisfies the following relationship over time.

$$0 = -\frac{X_{\mathrm{q}i}}{X'_{\mathrm{q}i}} E_{\mathrm{d}i} + \left(\frac{X_{\mathrm{q}i}}{X'_{\mathrm{q}i}} - 1 \right) \boldsymbol{V}_{\mathrm{d}i}$$

Then, the active power and the active power are given by:

$$\begin{cases} P_i &= \frac{E_{\mathrm{q}i}}{X'_{\mathrm{d}i}} \boldsymbol{V}_{\mathrm{d}i} + \left(\frac{1}{X_{\mathrm{q}i}} - \frac{1}{X'_{\mathrm{d}i}} \right) \boldsymbol{V}_{\mathrm{d}i} \boldsymbol{V}_{\mathrm{q}i}, \\ Q_i &= \frac{E_{\mathrm{q}i}}{X'_{\mathrm{d}i}} \boldsymbol{V}_{\mathrm{q}i} - \left(\frac{\boldsymbol{V}_{\mathrm{d}i}^2}{X_{\mathrm{q}i}} + \frac{\boldsymbol{V}_{\mathrm{q}i}^2}{X'_{\mathrm{d}i}} \right) \end{cases}$$

Therefore, the dynamic characteristics of the generator model match those of the salient pole type uniaxial model. Furthermore, in the limit when the time constant $\tau_{\mathrm{d}i}$ is small enough:

$$\begin{cases} P_i &= \frac{V_{\mathrm{field}i}}{X_{\mathrm{d}i}} \boldsymbol{V}_{\mathrm{d}i} + \left(\frac{1}{X_{\mathrm{q}i}} - \frac{1}{X_{\mathrm{d}i}} \right) \boldsymbol{V}_{\mathrm{d}i} \boldsymbol{V}_{\mathrm{q}i}, \\ Q_i &= \frac{V_{\mathrm{field}i}}{X_{\mathrm{d}i}} \boldsymbol{V}_{\mathrm{q}i} - \left(\frac{\boldsymbol{V}_{\mathrm{d}i}^2}{X_{\mathrm{q}i}} + \frac{\boldsymbol{V}_{\mathrm{q}i}^2}{X_{\mathrm{d}i}} \right) \end{cases}$$

In particular, assuming the non-salient rotor type, that is, $X_{\mathrm{d}i}$ and $X_{\mathrm{q}i}$ are equal X_i, and the field input $V_{\mathrm{field}i}$ is constant and equal to $V_i^\star$, the above equations become:

$$P_i = \frac{V_i^\star |\boldsymbol{V}_i|}{X_i} \sin(\delta_i - \angle \boldsymbol{V}_i), \qquad Q_i = \frac{V_i^\star |\boldsymbol{V}_i|}{X_i} \cos(\delta_i - \angle \boldsymbol{V}_i) - \frac{|\boldsymbol{V}_i|^2}{X_i}$$

This matches the classic model. For the detailed derivation process, refer to [**?**, Section 5].

4 Mathematical model of load

4.1 Relational expression of current and voltage according to load characteristics

With regard to static load models, **constant impedance model**, **constant power model**, and **constant current model** or combinations of them are often used. These are all static models described by algebraic equations with respect to the current and voltage phasors. Let us assume that voltage phasor of a bus bar i connected with a load is $\boldsymbol{V}_i$ and the current phasor flowing from the load to the bus bar is $\boldsymbol{I}_i$ (14). Then, the constant impedance model is given by the following relationship:

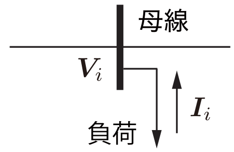


Fig. 14 Load connected to bus bar

$$\boldsymbol{I}_i = -\frac{\boldsymbol{V}_i}{z^\star_{\mathrm{load}i}} \tag{41a}$$

where, $z^\star_{\mathrm{load}i} \in \mathbb{C}$ is a constant expressing the impedance of the load.

The negative sign on the right side of Equation 41a indicates the load is grounded and the direction of the flow of thecurrent phasor $\boldsymbol{I}_i$ from the load to the bus bar i is defined as positive. Specifically:

$$\boldsymbol{I}_i = \frac{1}{z^\star_{\mathrm{load}i}}(0 - \boldsymbol{V}_i)$$

Incandescent lamps and electric heaters are common devices that can be expressed by the constant impedance model.

The constant current model is expressed by the following relationship, where $\boldsymbol{I}^\star_{\mathrm{load}i} \in \mathbb{C}$ is a constant current phasor.

$$\boldsymbol{I}_i = \boldsymbol{I}^\star_{\mathrm{load}i} e^{\boldsymbol{j}\angle \boldsymbol{V}_i} \tag{41b}$$

In other words, the following is true for the magnitude $|\boldsymbol{I}^\star_{\mathrm{load}i}|$ and phase $\angle \boldsymbol{I}^\star_{\mathrm{load}i}$:

$$|\boldsymbol{I}_i| = |\boldsymbol{I}^\star_{\mathrm{load}i}|, \qquad \angle \boldsymbol{I}_i = \angle \boldsymbol{I}^\star_{\mathrm{load}i} + \angle \boldsymbol{V}_i$$

The constant power model is given by the following relationship:

$$\boldsymbol{I}_i = \frac{P^\star_{\mathrm{load}i} - \boldsymbol{j}Q^\star_{\mathrm{load}i}}{\overline{\boldsymbol{V}}_i} \tag{41c}$$

where $P^\star_{\mathrm{load}i} \in \mathbb{R}$ and $Q^\star_{\mathrm{load}i} \in \mathbb{R}$ are constant active and reactive power supplied to the bus bar i. This is derived by considering the complex conjugate of:

$$P^\star_{\mathrm{load}i} + \boldsymbol{j}Q^\star_{\mathrm{load}i} = \boldsymbol{V}_i \overline{\boldsymbol{I}}_i$$

Power converters can be represented as constant-current models or constant power model, depending on their characteristics.

Depending on the purpose of the analysis, a dynamic load model might be used. Please see [**?**, Section 7.1.2] for further details.

4.2 Kron reduction of load bus bar

Even when there are generators and multiple types of loads connected in a power system, a mathematical model of the system can be obtained by defining the relationship between the current and voltage phasors of each device using Equations 21 and 41 and combining them with Equation 11. In this case, the constant impedance model in Equation 41a gives a linear relationship between the current and voltage phasors. On the other hand, the constant-current model in Equation 41b and the constant-power model in Equation 41c provide nonlinear relationships for current and voltage phasors, which generally makes mathematical analysis of the resulting power system model difficult. Let us illustrate this fact with the following example.

Example 1.4 Kron reduction of load bus bar

Let us assume that in the electrical power system of the Example 1.1, the generators are represented by devices 1 and 2, and the load is represented by device 3. The relationship between current and voltage phasors and the admittance matrix of the power grid is given by Equation 10. First, let us consider the case where the load is given by the constant impedance model. If $z^{\star}_{\mathrm{load3}}$ is the impedance of the load, the following Equation can be obtained:

$$\boldsymbol{I}_3 = -\frac{\boldsymbol{V}_3}{z^{\star}_{\mathrm{load3}}}$$

If this is substituted into Equation 10 to cancel $\boldsymbol{I}_3$:

$$\begin{bmatrix} \boldsymbol{I}_1 \\ \boldsymbol{I}_2 \\ 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{y}_{12} & -\boldsymbol{y}_{12} & 0 \\ -\boldsymbol{y}_{12} & \boldsymbol{y}_{12} + \boldsymbol{y}_{32} & -\boldsymbol{y}_{32} \\ 0 & -\boldsymbol{y}_{32} & \boldsymbol{y}_{32} + \frac{1}{z^{\star}_{\mathrm{load3}}} \end{bmatrix} \begin{bmatrix} \boldsymbol{V}_1 \\ \boldsymbol{V}_2 \\ \boldsymbol{V}_3 \end{bmatrix} \tag{42}$$

Please note that with the equation on the third row the voltage phasor $\boldsymbol{V}_3$ can be written in function of the voltage phasor $\boldsymbol{V}_2$ of the generator bus. Specifically:

$$\boldsymbol{V}_3 = \left(\boldsymbol{y}_{32} + \frac{1}{z^{\star}_{\mathrm{load3}}} \right)^{-1} \boldsymbol{y}_{32} \boldsymbol{V}_2$$

In addition, by replacing $\boldsymbol{V}_3$ using the above expression, the relationship between the current and voltage phasors of the generator bus bar group becomes:

$$\begin{bmatrix} \boldsymbol{I}_1 \\ \boldsymbol{I}_2 \end{bmatrix} = \boldsymbol{Y}_{\mathrm{Kron}} \begin{bmatrix} \boldsymbol{V}_1 \\ \boldsymbol{V}_2 \end{bmatrix}$$

The reduced admittance matrix of the load bus bar is obtained as:

$$\boldsymbol{Y}_{\mathrm{Kron}} := \begin{bmatrix} \boldsymbol{y}_{12} & -\boldsymbol{y}_{12} \\ -\boldsymbol{y}_{12} & \boldsymbol{y}_{12} + \boldsymbol{y}_{32} \end{bmatrix} - \begin{bmatrix} 0 \\ \boldsymbol{y}_{32} \end{bmatrix} \left(\boldsymbol{y}_{32} + \frac{1}{z^{\star}_{\mathrm{load3}}} \right)^{-1} \begin{bmatrix} 0 & \boldsymbol{y}_{32} \end{bmatrix}$$

Therefore, if the load is given as a constant impedance model, the load busbar and its variables can be eliminated by considering $\boldsymbol{Y}_{\mathrm{Kron}} \in \mathbb{C}^{2\times 2}$ as the admittance matrix of the new power grid, and the system can be equivalently transformed into a system of differential algebraic equations with only the generator connected to the busbar.

Next, let us consider a situation where the load is given as the constant current model. Let us assume that current phasor flows from the load to bus bar 3:

$$\boldsymbol{I}_3 = \boldsymbol{I}^\star_{\text{load3}} e^{\boldsymbol{j}\angle \boldsymbol{V}_3}$$

Then, following the same procedure as for the constant impedance model, and replacing $\boldsymbol{I}_3$ and $\boldsymbol{V}_3$ in Equation 10, the following is obtained:

$$\begin{bmatrix} \boldsymbol{I}_1 \\ \boldsymbol{I}_2 \end{bmatrix} = \boldsymbol{Y}'_{\text{Kron}} \begin{bmatrix} \boldsymbol{V}_1 \\ \boldsymbol{V}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -\boldsymbol{y}_{32} \end{bmatrix} \boldsymbol{y}_{32}^{-1} \boldsymbol{I}^\star_{\text{load3}} e^{\boldsymbol{j}\angle \boldsymbol{V}_3}$$

where:

$$\boldsymbol{Y}'_{\text{Kron}} := \begin{bmatrix} \boldsymbol{y}_{12} & -\boldsymbol{y}_{12} \\ -\boldsymbol{y}_{12} & \boldsymbol{y}_{12} + \boldsymbol{y}_{32} \end{bmatrix} - \begin{bmatrix} 0 \\ \boldsymbol{y}_{32} \end{bmatrix} \boldsymbol{y}_{32}^{-1} \begin{bmatrix} 0 & \boldsymbol{y}_{32} \end{bmatrix}$$

This means that, if the load is given as the constant current model, the relationship between the current and voltage phasors of the generators is affine. However, the phase $\angle \boldsymbol{V}_3$ of the voltage phasor of the load bus bar also has an impact on the current phasor of the generators. Generally, the phase $\angle \boldsymbol{V}_3$ of the voltage phasor is a nonlinear function of the voltage phasor $\boldsymbol{V}_3$.

Finally, let us consider a case wherein the load is given as the constant power model. When constant active power $P^\star_{\text{load3}}$ and reactive power $Q^\star_{\text{load3}}$ are supplied from the load to bus bar 3, the following relationship holds:

$$\boldsymbol{I}_3 = \frac{P^\star_{\text{load3}} - \boldsymbol{j} Q^\star_{\text{load3}}}{\overline{\boldsymbol{V}}_3}$$

Since this is a nonlinear relationship of $\boldsymbol{I}_3$ and $\boldsymbol{V}_3$, to cancel $\boldsymbol{V}_3$ from Equation 10, nonlinear calculation is necessary. Specifically, $\boldsymbol{V}_3$ is given as a solution for the quadratic equation related to complex variables:

$$\boldsymbol{y}_{32} \boldsymbol{V}_3 \overline{\boldsymbol{V}}_3 - \boldsymbol{y}_{32} \boldsymbol{V}_2 \overline{\boldsymbol{V}}_3 - P^\star_{\text{load3}} + \boldsymbol{j} Q^\star_{\text{load3}} = 0$$

Therefore, if the load of the constant power model is included in an electrical power system model, it is usually difficult to cancel the load bus bar through equivalent transformation.

As shown in Example 1.4, if the load is given as a static model expressed by Equation (41), the current phasor and voltage phasor of some or all load bus bars can be equivalently cancelled through algebraic calculation. This operation is called **Kron reduction of the load bus bar**. Specifically, when the load is given as the constant impedance model, Kron reduction of the load bus bar corresponds to reducing the dimension of the admittance matrix of the power grid by mathematically equivalent calculation.

If there is no device connected to the bus bar, such as a generator or load, we can assume a constant impedance model load with an infinite absolute value of $z^\star_{\text{load}i}$

connected to bus i in Equation 41a. This is equivalent to a virtual connection of a constant-current model load in which the current phasor between the load and the bus is always zero. Therefore, the above-described Kron reduction of the load bus bar allows for equivalent cancellation of bus bars without device.

4.3 Mathematical properties of the admittance matrix with reduced load bus bar

Let us analyze properties that hold for Kron reduction of the load bus bar when all loads are given by the constant impedance model. First, the procedure for Kron reduction in Example 1.4 is shown as a general form when the number of generator and load buses is arbitrary. The subscripts set of generator buses is $\mathcal{I}_{\mathrm{G}}$ and the subscripts set of load bus bars is $\mathcal{I}_{\mathrm{L}}$. Without loss of generality, the number of generator buses is assumed to be inferior to that of load buses. In other words:

$$\mathcal{I}_{\mathrm{G}} = \{1, \ldots, n\}, \qquad \mathcal{I}_{\mathrm{L}} = \{n+1, \ldots, n+m\}$$

where n and m are numbers of generator buses and load bus bars, respectively. By definition, $n+m$ is equal to the total number of bus bars N.

Let $\boldsymbol{I}_{\mathrm{G}} \in \mathbb{C}^n$ be the vector of current phasors of all generator buses, and $\boldsymbol{V}_{\mathrm{G}} \in \mathbb{C}^n$ be the vector of voltage phasors. Similarly, the vector of current phasors of all load buses is expressed as $\boldsymbol{I}_{\mathrm{L}} \in \mathbb{C}^m$, and the vector of voltage phasors is expressed as $\boldsymbol{V}_{\mathrm{L}} \in \mathbb{C}^m$. Then, the relationship between all current and voltage phasors with respect to the admittance matrix $\boldsymbol{Y} \in \mathbb{C}^{N\times N}$ of the power grid is:

$$\begin{bmatrix} \boldsymbol{I}_{\mathrm{G}} \\ \boldsymbol{I}_{\mathrm{L}} \end{bmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{Y}_{\mathrm{GG}} & \boldsymbol{Y}_{\mathrm{GL}} \\ \boldsymbol{Y}_{\mathrm{LG}} & \boldsymbol{Y}_{\mathrm{LL}} \end{bmatrix}}_{\boldsymbol{Y}} \begin{bmatrix} \boldsymbol{V}_{\mathrm{G}} \\ \boldsymbol{V}_{\mathrm{L}} \end{bmatrix} \tag{43}$$

Here, the relationship between the current phasor and the voltage phasor determined by the load in the constant impedance model is expressed using the admittance:

$$\boldsymbol{I}_{\mathrm{L}} = -\,\mathsf{diag}(\boldsymbol{y}^{\star}_{\mathrm{load}i})_{i\in\mathcal{I}_{\mathrm{L}}}\boldsymbol{V}_{\mathrm{L}}$$

Although $\boldsymbol{y}^{\star}_{\mathrm{load}i}$ is defined as a reciprocal of $\boldsymbol{z}^{\star}_{\mathrm{load}i}$ in Equation 41a and must have a non-zero value, since this is an expression of bus bars with no equipment connected, formally, $\boldsymbol{y}^{\star}_{\mathrm{load}i}$ is allowed to be 0. Substituting it in Equation 43 to cancel $\boldsymbol{V}_{\mathrm{L}}$, the followed is obtained.

$$\boldsymbol{I}_{\mathrm{G}} = \underbrace{\left\{\boldsymbol{Y}_{\mathrm{GG}} - \boldsymbol{Y}_{\mathrm{GL}}\left(\boldsymbol{Y}_{\mathrm{LL}} + \mathsf{diag}(\boldsymbol{y}^{\star}_{\mathrm{load}i})_{i\in\mathcal{I}_{\mathrm{L}}}\right)^{-1}\boldsymbol{Y}_{\mathrm{LG}}\right\}}_{\boldsymbol{Y}_{\mathrm{Kron}}}\boldsymbol{V}_{\mathrm{G}} \tag{44}$$

Note that in general, if the load consumes active and reactive power with respect to the admittance of the load, then the following holds as in transmission lines.

$$\mathrm{Re}[\boldsymbol{y}^{\star}_{\mathrm{load}i}] \geq 0, \qquad \mathrm{Im}[\boldsymbol{y}^{\star}_{\mathrm{load}i}] \leq 0, \qquad \forall i \in \mathcal{I}_{\mathrm{L}} \tag{45}$$

In addition, the conductance and susceptance matricces obtained by Kron reduction of the load bus, which are the real and imaginary parts of $\boldsymbol{Y}_{\mathrm{Kron}}$, are expressed as:

$$G_{\mathrm{Kron}} := \mathsf{Re}\big[\boldsymbol{Y}_{\mathrm{Kron}}\big], \qquad B_{\mathrm{Kron}} := \mathrm{i}\big[\boldsymbol{Y}_{\mathrm{Kron}}\big]$$

This introduces the following fact:

Theorem 1.2 (Properties of the Kron-reduced admittance matrix) *For the admittance matrix $\boldsymbol{Y} \in \mathbb{C}^{N\times N}$ of Equation 16, by dividing the block matrix of Equation 43, $\boldsymbol{Y}_{\mathrm{Kron}} \in \mathbb{C}^{n\times n}$ of Equation 44 can be obtained, corresponding to the admittance matrix of the system after the Kron reduction of the load bus bars. Additionally, the admittance $\boldsymbol{y}^{\star}_{\mathrm{load}i}$ of each load satisfies Equation 45. Then, the reduced conductance matrix G_{Kron} is positive semi-definite, and the reduced susceptance matrix B_{Kron} is symmetric. Furthermore, if the following is true for the generator buses:*

$$b_i = 0, \qquad \forall i \in \mathcal{I}_{\mathrm{G}} \tag{46a}$$

And the following is true for load bus bars:

$$\mathrm{Im}[\boldsymbol{y}^{\star}_{\mathrm{load}i}] + b_i \leq 0, \qquad \forall i \in \mathcal{I}_{\mathrm{L}} \tag{46b}$$

where b_i is a non-negative constant equivalent to the capacitance to ground of Equation 16, B_{Kron} is negative semidefinite.

Specifically, if the inequality of Equation 46b strictly holds for at least one load bus bar, then B_{Kron} is negative definite.

Proof Using Lemma 1.3 of the Mathematical Supplement at the end of the chapter, let us define:

$$\boldsymbol{Y}' := \begin{bmatrix} \boldsymbol{Y}_{\mathrm{GG}} & \boldsymbol{Y}_{\mathrm{GL}} \\ \boldsymbol{Y}_{\mathrm{LG}} & \boldsymbol{Y}_{\mathrm{LL}} + \mathsf{diag}(\boldsymbol{y}^{\star}_{\mathrm{load}i})_{i\in\mathcal{I}_{\mathrm{L}}} \end{bmatrix}$$

Since Equations **??** and **??** hold, the real part of $\boldsymbol{jY}'$ is symmetric, and the imaginary part is positive semidefinite. Therefore, Lemma 1.3 shows that the real part of $\boldsymbol{jY}_{\mathrm{Kron}}$ is symmetric and the imaginary part is semi-positive. This implies that the real part of $\boldsymbol{Y}_{\mathrm{Kron}}$, G_{Kron}, is positive semidefinite and the imaginary part, B_{Kron}, is symmetric.

Moreover, when Equation 46 holds, the imaginary part of $\boldsymbol{Y}'$ is negative semidefinite, which by the Lemma 1.3 implies that B_{Kron} is negative semidefinite. Similarly, if Equation 46b holds strictly for at least one load bus bar, then $\boldsymbol{Y}'$ and B_{Kron} are negative definite.

Theorem 1.2 shows that, if all loads are given by the constant impedance model, the conductance and susceptance matrix of related to the admittance matrix of Equation 16 remain positive semidefinite and symmetric, respectively, even if the load bus bars were Kron reduced. Therefore, it can be similarly applied to the electrical power system model, in which the generator buses and load bus bars in Section 3.3 are Kron reduced. Theorem 1.2 shows a case where all load bus bars are Kron reduced simultaneously. However, the same fact is shown when only some of load bus bars are Kron reduced.

As discussed above, while the definiteness of the admittance matrix is invariable to Kron reduction, the ”element signs” of the real and imaginary parts are not necessarily invariable. Let us confirm this with the next Example.

Example 1.5 Sign change of the admittance matrix by Kron reduction of bus bars

Regarding the admittance matrix $\boldsymbol{Y}_0$ when the ground capacitance in Equation 15 is negligible, the conductance matrix G_0 is positive semidefinite and the susceptance matrix B_0 is negative semidefinite. Then, the diagonal and non-diagonal elements of G_0 are non-negative and non-positive, respectively, and the diagonal and non-diagonal elements of B_0 are non-positive and non-negative, respectively. For example, For example, let bus 1 and bus 2 be the generator bus, bus 3 be the load bus. Then, the load bus admittance and the power grid admittance matrix are:

$$\boldsymbol{y}^{\star}_{\mathrm{load3}} = \alpha - \boldsymbol{j}\beta, \qquad \boldsymbol{Y} = \underbrace{\gamma \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{G_0} + \boldsymbol{j} \underbrace{\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}}_{B_0}$$

where α, β, and γ are non-negative constants. Please note that the signs of the real and imaginary parts of the load bus admittance of are the same as the signs of the diagonal element of the admittance matrix.

By Kron reducing the load bus bar, the reduced admittance matrix of Equation 44 becomes:

$$\boldsymbol{Y}_{\mathrm{Kron}} = \underbrace{\gamma \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{\alpha}{\alpha^2 + (2+\beta)^2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}_{G_{\mathrm{Kron}}}$$
$$+ \boldsymbol{j} \underbrace{\left(\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} + \frac{2+\beta}{\alpha^2 + (2+\beta)^2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right)}_{B_{\mathrm{Kron}}}$$

Here, for arbitrary non-negative constants, α, β, and γ, the diagonal element of G_{Kron} is non-negative and the diagonal and non-diagonal elements of B_{Kron} are negative and positive, respectively. However, the non-diagonal element of G_{Kron} is positive when:

$$\gamma < \frac{\alpha}{\alpha^2 + (2+\beta)^2}$$

Thus, when the laod bus bar is Kron reduced, the sign of the non-diagonal elements of the admittance matrix is not always invariable. The diagonal elements are invariable because of t he positive semidefinite nature of G_0 and the negative semidefinite nature of B_0.

A change of sign of an off-diagonal element can occur when the conductance and susceptance are both non-zero for the load or transmission line connected to the bus bar being Kron reduced. For example, in the above case, the admittance of the load is a pure imaginary number when $\alpha = 0$ and the admittance of the transmission line connected to bus bar 3; in other words, the elements of the third column and the third row of $\boldsymbol{Y}$, are all pure imaginary. In such a case, it can be shown that there is no sign change of the elements of the conductance and susceptance matrices.

The above-described Kron reduction has been known as a mathematical operation related to the derivation of an equivalent circuit [**?**]. Specifically, in the analysis of an electrical power system, it is applied to power flow calculations explained in Section **??**. Kron reduction has also been applied in graph theory, and has interesting mathematical properties [**?**].

Mathematical Supplement

Lemma 1.1 *Assume that the real matrix M is regular. Then, for real matrix N, the necessary and sufficient condition for $M + jN$ to be regular is that $M + NM - 1N$ is regular.*

Proof First, the fact that $M + \boldsymbol{j}N$ is regular is equivalent to the fact that there exist certain real square matrices P and Q such that $(M+\boldsymbol{j}N)(P+\boldsymbol{j}Q) = I$. If we rearrange these two equations for the real and imaginary parts, we get:

$$\underbrace{\begin{bmatrix} M & -N \\ N & M \end{bmatrix}}_{L} \begin{bmatrix} P & -Q \\ Q & P \end{bmatrix} = I \tag{47}$$

This means that the regularity of $M + \boldsymbol{j}N$ is equivalent to the regularity of L. Furthermore, using the properties of the determinant of block matrices:

$$\det L = \det M \det \left(M + NM^{-1}N \right)$$

From the assumption that $\det M \neq 0$, the regularity of L is equivalent to the regularity of $M + NM^{-1}N$. Then, the lemma is proven to be true. □

Lemma 1.2 *Let the real part of the complex matrix $\mathbf{Z}$ be positive definite. If the imaginary part of $\mathbf{Z}$ is symmetric, then the real part of $\mathbf{Z}^{-1}$ is positive definite and the imaginary part is symmetric. In particular, if the imaginary part of $\mathbf{Z}$ is positive semidefinite, then the imaginary part of $\mathbf{Z}^{-1}$ is negative semidefinite. Also,* *if the imaginary part of $\mathbf{Z}$ is positive definte, the imaginary part of $\mathbf{Z}^{-1}$ is negative definite.*

Proof Let $\mathbf{Z} = M + \boldsymbol{j}N$, where M is a real positive definite matrix and N is a symmetric matrix. Let P be the real part of $\mathbf{Z}^{-1}$ and Q be its imaginary part. From the Lemma 1.1, $M + NM^{-1}N$ is positive definite and therefore $\mathbf{Z}$ is regular. Therefore

$$(M + \boldsymbol{j}N)(P + \boldsymbol{j}Q) = I$$

The equations for the real and imaginary parts of L are equivalent to the Equation 47. This means that the diagonal and off-diagonal blocks of the inverse of L are P and Q. From the properties of the inverse of the block matrix

$$L^{-1} = \begin{bmatrix} (M + NM^{-1}N)^{-1} & M^{-1}N(M + NM^{-1}N)^{-1} \\ -(M + NM^{-1}N)^{-1}NM^{-1} & (M + NM^{-1}N)^{-1} \end{bmatrix}$$

Therefore,

$$P = (M + NM^{-1}N)^{-1}, \qquad Q = -M^{-1}N(M + NM^{-1}N)^{-1}$$

From the assumption that N is symmetric and M is positive definite, P is positive definite. Also, using the positive definite matrix $\sqrt{M^{-1}}$ such that $M^{-1} = \sqrt{M^{-1}}\sqrt{M^{-1}}$, Q is

$$Q = -\sqrt{M^{-1}}\underbrace{\sqrt{M^{-1}}N\sqrt{M^{-1}}\left(I + (\sqrt{M^{-1}}N\sqrt{M^{-1}})^2\right)^{-1}}_{X}\sqrt{M^{-1}}$$

where $\sqrt{M^{-1}}N\sqrt{M^{-1}}$ is symmetric, so it can be diagonalized by the orthogonal matrix V and

$$X = V\Lambda\left(I + \Lambda^2\right)^{-1}V^{\mathsf{T}}$$

However, Λ is a real diagonal matrix of eigenvalues of $\sqrt{M^{-1}}N\sqrt{M^{-1}}$. From this, since X is symmetric, Q is also symmetric. Furthermore, if N is semi-definite, then Λ is also semi-definite, and if N is positive, then Λ is also positive. Therefore, if N is semi-positive definite, then Q is semi-negative definite, and if N is positive definite, then Q is negative definite. From the above, the subject follows. □

Lemma 1.3 *Translated with DeepL Consider symmetric running matrices M and N. They are partitioned into block matrices*

$$\begin{bmatrix} \mathbf{Z}_{11} & \mathbf{Z}_{12} \\ \mathbf{Z}_{21} & \mathbf{Z}_{22} \end{bmatrix} := \underbrace{\begin{bmatrix} M_{11} & M_{12} \\ M_{12}^{\mathsf{T}} & M_{22} \end{bmatrix}}_{M} + \boldsymbol{j}\underbrace{\begin{bmatrix} N_{11} & N_{12} \\ N_{12}^{\mathsf{T}} & N_{22} \end{bmatrix}}_{N} \tag{48}$$

For $\mathbf{Z}_{11} - \mathbf{Z}_{12}\mathbf{Z}_{22}^{-1}\mathbf{Z}_{21}$ denote $\mathbf{Z}_{\mathrm{S}}$. Also assume that M is semi-definite and that M_{22} is positive definite. In this case, the real part of $\mathbf{Z}_{\mathrm{S}}$ is semipositive definite and the imaginary part is symmetric. In particular, if N is semipositive definite, then the *imaginary part of $\boldsymbol{Z}_\mathrm{S}$ is semipositive definite. Also, if N is positive definite, then the imaginary part of $\boldsymbol{Z}_\mathrm{S}$ is positive definite. Furthermore, if M is positive definite, then the real part of $\boldsymbol{Z}_\mathrm{S}$ is positive definite.*

Proof Translated with DeepL Denote by $\boldsymbol{Z}$ the matrix on the left-hand side of the Equation 48. First, if M is positive definite, then the real part of $\boldsymbol{Z}_\mathrm{S}$ is positive definite and the imaginary part is symmetric. Now, from the positive definiteness of M_{22}, $\boldsymbol{Z}_{22}$ is regular from the complement 1.1. Therefore, by the inverse property of the block matrix, $\boldsymbol{Z}_\mathrm{S}^{-1}$ coincides with the upper left block of $\boldsymbol{Z}^{-1}$. Here, from the complementation1.2, the real part of $\boldsymbol{Z}^{-1}$ is positive definite and the imaginary part is symmetric. This means that the real part of $\boldsymbol{Z}_\mathrm{S}^{-1}$ is positive definite and the imaginary part is symmetric. Therefore, applying the complementation 1.2 to $\boldsymbol{Z}_\mathrm{S}^{-1}$ shows that the real part of $\boldsymbol{Z}_\mathrm{S}$ is positive definite and the imaginary part is symmetric. Similarly, if N is semipositive definite, then from the complementation 1.2, the imaginary part of $\boldsymbol{Z}^{-1}$ is semidefinite, which means that the imaginary part of $\boldsymbol{Z}_\mathrm{S}^{-1}$ is semidefinite. Therefore, by applying the complementation 1.2 again to $\boldsymbol{Z}_\mathrm{S}^{-1}$, it can be shown that the imaginary part of $\boldsymbol{Z}_\mathrm{S}$ is semipositive definite.

Next, consider the case where M is semidefinite and N is symmetric. First, show that both the real and imaginary parts of $\boldsymbol{Z}_\mathrm{S}$ are symmetric. Now, since the real part M_{22} of $\boldsymbol{Z}_{22}$ is positive definite and the imaginary part N_{22} is symmetric, both real and imaginary parts of $\boldsymbol{Z}_{22}^{-1}$ are symmetric. Therefore,

$$\begin{aligned}\mathrm{Re}[\boldsymbol{Z}_\mathrm{S}] &= \mathrm{Re}[\boldsymbol{Z}_{11}] - \mathrm{Re}[\boldsymbol{Z}_{12}]\,\mathrm{Re}[\boldsymbol{Z}_{22}^{-1}]\,\mathrm{Re}[\boldsymbol{Z}_{21}] \\ &\quad + \mathrm{i}[\boldsymbol{Z}_{12}]\mathrm{i}[\boldsymbol{Z}_{22}^{-1}]\,\mathrm{Re}[\boldsymbol{Z}_{21}] + \mathrm{i}[\boldsymbol{Z}_{12}]\,\mathrm{Re}[\boldsymbol{Z}_{22}^{-1}]\mathrm{i}[\boldsymbol{Z}_{21}] \\ &\quad + \mathrm{Re}[\boldsymbol{Z}_{12}]\mathrm{i}[\boldsymbol{Z}_{22}^{-1}]\mathrm{i}[\boldsymbol{Z}_{21}] \\ &= M_{11} - M_{12}\,\mathrm{Re}[\boldsymbol{Z}_{22}^{-1}]M_{12}^\mathsf{T} + N_{12}\mathrm{i}[\boldsymbol{Z}_{22}^{-1}]M_{12}^\mathsf{T} + N_{12}\,\mathrm{Re}[\boldsymbol{Z}_{22}^{-1}]N_{12}^\mathsf{T} \\ &\quad + M_{12}\mathrm{i}[\boldsymbol{Z}_{22}^{-1}]N_{12}^\mathsf{T}\end{aligned}$$

From this we see that the real part of $\boldsymbol{Z}_\mathrm{S}$ is symmetric. Similarly,

$$\begin{aligned}\mathrm{i}[\boldsymbol{Z}_\mathrm{S}] &= N_{11} + N_{12}\mathrm{i}[\boldsymbol{Z}_{22}^{-1}]N_{12}^\mathsf{T} - N_{12}\,\mathrm{Re}[\boldsymbol{Z}_{22}^{-1}]M_{12}^\mathsf{T} \\ &\quad - M_{12}\mathrm{i}[\boldsymbol{Z}_{22}^{-1}]M_{12}^\mathsf{T} - M_{12}\,\mathrm{Re}[\boldsymbol{Z}_{22}^{-1}]N_{12}^\mathsf{T}\end{aligned}$$

From this we see that the imaginary part of $\boldsymbol{Z}_\mathrm{S}$ is also symmetric.

When M in the expression 48 is semi-definite, $\boldsymbol{Z}$ is not necessarily regular, since $\boldsymbol{Z}^{-1}$ is not necessarily regular. In contrast, in what follows, for any $\epsilon > 0$:

$$\boldsymbol{Z}^+ := \begin{bmatrix} \boldsymbol{Z}_{11} + \epsilon I & \boldsymbol{Z}_{12} \\ \boldsymbol{Z}_{21} & \boldsymbol{Z}_{22} + \epsilon I \end{bmatrix}$$

The real part of N, $M + \epsilon I$, is positive definite. Now, when N is semi-definite, for any $\epsilon > 0$ we have

$$\boldsymbol{Z}_\mathrm{S}^+ := \boldsymbol{Z}_{11} + \epsilon I - \boldsymbol{Z}_{12}(\boldsymbol{Z}_{22} + \epsilon I)^{-1}\boldsymbol{Z}_{21}$$

The real part of $(\boldsymbol{Z}_{22} + \epsilon I)^{-1}$ is positive definite, and the imaginary part is semipositive definite. Also, by expanding $(\boldsymbol{Z}_{22} + \epsilon I)^{-1}$ using the auxiliary theorem of the inverse matrix the following is obtained.

$$\boldsymbol{Z}_{\mathrm{S}}^{+} = \boldsymbol{Z}_{\mathrm{S}} + \underbrace{\epsilon \left\{ I + \boldsymbol{Z}_{12}\boldsymbol{Z}_{22}^{-1}(I + \epsilon \boldsymbol{Z}_{22}^{-1})^{-1}\boldsymbol{Z}_{22}^{-1}\boldsymbol{Z}_{21} \right\}}_{\Delta(\epsilon)}$$

As mentioned above, since the real and imaginary parts of $\boldsymbol{Z}_{\mathrm{S}}^{+}$ and $\boldsymbol{Z}_{\mathrm{S}}$ are symmetric, the real and imaginary parts of $\Delta(\epsilon)$ are also symmetric.

Next, we show that the imaginary part of $\boldsymbol{Z}_{\mathrm{S}}^{+}$ is semipositive definite if and only if $\boldsymbol{Z}_{\mathrm{S}}$ is semipositive definite. Note that the same argument can be used to show the semi- positive definiteness of the real part. For the following discussion

$$\mathcal{X} := \left\{ x : x^{\mathsf{T}} \mathrm{i}[\Delta(\epsilon)] x > 0, \quad \forall \epsilon > 0 \right\}$$

is defined as follows. From this definition of $\mathcal{X}$, for arbitrarily chosen $x \notin \mathcal{X}$, there exists some $\epsilon_0 > 0$ such that

$$x^{\mathsf{T}} \mathrm{i}[\Delta(\epsilon_0)] x \leq 0$$

Therefore, the imaginary part of $\boldsymbol{Z}_{\mathrm{S}}^{+}$ is semi-positive definite,

$$x^{\mathsf{T}} \mathrm{i}[\boldsymbol{Z}_{\mathrm{S}}] x + x^{\mathsf{T}} \mathrm{i}[\Delta(\epsilon_0)] x \geq 0$$

is derived. This leads to the following for all $x \notin \mathcal{X}$,

$$x^{\mathsf{T}} \mathrm{i}[\boldsymbol{Z}_{\mathrm{S}}] x \geq 0 \tag{49}$$

Furthermore, if the imaginary part of $\boldsymbol{Z}_{\mathrm{S}}^{+}$ is semi-positive definite, we show by contraposition that the expression 49 holds for all $x \in \mathcal{X}$. That is, for some $x \in \mathcal{X}$, we have

$$x^{\mathsf{T}} \mathrm{i}[\boldsymbol{Z}_{\mathrm{S}}] x < 0 \tag{50}$$

then the imaginary part of $\boldsymbol{Z}_{\mathrm{S}}^{+}$ is not semidefinite. Now, noting that the imaginary part of $\Delta(\epsilon)$ asymptotes to 0 in the limit of $\epsilon \to 0$, if for some $x \in \mathcal{X}$ the expression50

$$x^{\mathsf{T}} \mathrm{i}[\boldsymbol{Z}_{\mathrm{S}}] x + x^{\mathsf{T}} \mathrm{i}[\Delta(\epsilon_0)] x < 0$$

This means that the imaginary part of $\boldsymbol{Z}_{\mathrm{S}}^{+}$ is not semi- positive definite. The above facts show that the imaginary part of $\boldsymbol{Z}_{\mathrm{S}}$ is semi-positive definite.

Finally, we show that if N is positive definite, then the imaginary part of $\boldsymbol{Z}_{\mathrm{S}}$ is positive definite. For this purpose,

$$-\boldsymbol{j}\boldsymbol{Z} = N - \boldsymbol{j}M$$

is positive definite and the imaginary part is symmetric. Applying the results for the case where the real part is positive definite to this $-\boldsymbol{j}\boldsymbol{Z}_{\mathrm{S}}$, it can be shown that the real part of $-\boldsymbol{j}\boldsymbol{Z}_{\mathrm{S}}$ is positive definite. This implies that the imaginary part of $\boldsymbol{Z}_{\mathrm{S}}$ is positive definite. $\square$