CS 132 – Spring 2020, Assignment 11

Answer A

$$x \times w = 18 + 2 - 15 = 5$$
$$x \times x = 36 + 4 + 9 = 49$$
$$\frac{x \times w}{x \times x} = \frac{5}{49}$$
$$\frac{5}{49} \times x = \begin{bmatrix} \frac{30}{49} \\ \frac{-10}{49} \\ \frac{15}{49} \end{bmatrix}$$

$$\boxed{\begin{bmatrix} \frac{30}{49} \\ \frac{-10}{49} \\ \frac{15}{49} \end{bmatrix}}$$

Answer B

$$\boxed{\|x\| = v \times v = \sqrt{6^2 + (-2)^2 + 3^2} = \sqrt{36 + 4 + 9} = \sqrt{49} = 7}$$

Answer C

Let our vector be $u = \begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix}$, so:

$$\|u\| = \sqrt{u \times u} = \sqrt{36 + 16 + 9} = \sqrt{61}$$

$$\frac{u}{\|u\|} = \frac{1}{\|u\|} \times u = \frac{1}{\sqrt{61}} \times u = \begin{bmatrix} \frac{-6}{\sqrt{61}} \\ \frac{4}{\sqrt{61}} \\ \frac{-3}{\sqrt{61}} \end{bmatrix}$$

$$\boxed{\begin{bmatrix} \frac{-6}{\sqrt{61}} \\ \frac{4}{\sqrt{61}} \\ \frac{-3}{\sqrt{61}} \end{bmatrix}}$$

Answer D

$$dist(u, z) = \|u - z\| = \sqrt{(u - z) \times (u - z)}$$

$$\sqrt{(u - z) \times (u - z)} = \sqrt{(0 + 4)^2 + (-5 + 1)^2 + (2 - 8)^2} = \sqrt{16 + 16 + 36} = \sqrt{68}$$

$$\boxed{\sqrt{68}}$$

Answer E

Two vectors are orthogonal if their product is zero, so, to check if these vectors are orthogonal we must verify if $u \times v = 0$:

$$u \times v = [12, 3, -5] \times \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix} = 24 - 9 - 15 = 0$$

$$\boxed{\text{Hence, since } u \times v = 0\text{, then these vectors are orthogonal.}}$$

Answer F

Two vectors are orthogonal if their product is zero, so, to check if these vectors are orthogonal we must verify if $y \times z = 0$:

$$y \times z = [-3, 7, 4, 0] \times \begin{bmatrix} 1 \\ -8 \\ 15 \\ -7 \end{bmatrix} = -3 - 56 + 60 + 0 = 1$$

$$\boxed{\text{Hence, since } y \times z \neq 0\text{, then these vectors are not orthogonal.}}$$

Answer G

a. $\boxed{\text{True}}$, as per theorem 1: $u \times v = v \times u$

c. $\boxed{\text{False}}$, as we saw earlier in the chapter: $\|cv\| = |c| \times \|v\|$

d. $\boxed{\text{True}}$, as per the Pythagorean Theorem

Answer H

For any u: $u \times u \geq 0$.

To prove this, we must consider the 3 different cases for the individual components of u: positive, negative and 0.

Since $u \times u = [u_1, u_2, ..., u_n] \times \begin{bmatrix} u_1 \\ u_2 \\ ... \\ u_n \end{bmatrix} = u_1^2 + u_2^2 + ... + u_n^2$, so, if an individual member of u, let it be u_x is positive, then u_x^2 is also positive. If u_x is negative, then u_x^2 will be positive, and finally, if u_x is zero, then u_x^2 will be 0.

$\boxed{\text{Hence, since no matter what } u_x \text{ is, } u_x^2 \text{ is always either positive or 0, then } u \times u \text{ MUST be equal to or greater than 0.}}$

$\boxed{u \times u \text{ will only be 0 if u is a 0 vector.}}$

Answer I

For any vector w in the span of u, v: $w = c_1 v + c_2 u$, so, if we calculate $y \times w$ we get that:

$$y \times w = y \times (c_1 v + c_2 u) = y \times (c_1 v) + y \times (c_2 u)$$

Since we know that for any v_x and u_x that are derived from v and u and multiplied by some scalar c then $y \times v_x = 0$ and $y \times u_x = 0$, then we have that:

$\boxed{y \times (c_1 v) + y \times (c_2 u) = 0 + 0 = 0, \text{ which means that y is orthogonal to any vector in the span of u and v.}}$

Answer J

For us to get x as a linear combination of u's, we must solve $[u_1 u_2 u_3] = x$:

$$\begin{bmatrix} 3 & 2 & 1 & 5 \\ -3 & 2 & 1 & -3 \\ 0 & -1 & 4 & 1 \end{bmatrix} \equiv \begin{bmatrix} 1 & 2/3 & 1/3 & 5/3 \\ 0 & 1 & 1/2 & 1/2 \\ 0 & 0 & 1 & 1/3 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 & 0 & 4/3 \\ 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & 1/3 \end{bmatrix}$$

$$\boxed{\text{x = 4/3 } \times u_1 + 1/3 \times u_2 + 1/3 \times u_3}$$

Answer K

Let $y = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $u = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ then: $\hat{y} = \frac{y \times u}{u \times u} \times u$

$y \times u = -1 - 3 = -4$ and $u \times u = 1 + 9 = 10$

$$\boxed{\text{So: } \hat{y} = \frac{-2}{5} \times u = \begin{bmatrix} 2/5 \\ -6/5 \end{bmatrix}}$$

Answer L

First we will calculate the orthogonal projection, let it be $\hat{y} = \frac{y \times u}{u \times u} \times u$:

$y \times u = 14 + 6 = 20$ and $u \times u = 49 + 1 = 50$ so: $\hat{y} = \frac{y \times u}{u \times u} \times u = \frac{2}{5} \times u = \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix}$

Since $\hat{y}$ is in the span of u, then we must calculate the remaining: $y - \hat{y} = \begin{bmatrix} -4/5 \\ 28/5 \end{bmatrix}$

$$\boxed{\text{y = } \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix} + \begin{bmatrix} -4/5 \\ 28/5 \end{bmatrix}}$$

Answer M

a. $\boxed{\text{True}}$, as, if the set contains the 0 vector it won't be linearly independent

b. $\boxed{\text{False}}$, this case describes an orthogonal set, not an orthonormal one. To be orthonormal the magnitudes of all vectors have to be 1.

c. $\boxed{\text{True}}$, as per theorem 7

d. $\boxed{\text{True}}$, as $\hat{y_2} = \frac{y \times uc_1}{uc_1 \times uc_1} \times uc_1 = \frac{c_1^2(y \times u)}{c_1^2(u \times u)} \times u = \frac{y \times u}{u \times u} \times u = \hat{y}$

e. $\boxed{\text{True}}$, as it will be a square matrix with linearly independent columns

Answer N

In order to prove that the inverse exists, we must see if the determinant is nonzero:

$det(UV) = det(U) \times det(V) = +/-1 \times +/-1 = +/-1$, so we can verify that it does have an inverse.

Now, in order to prove that $(UV)^T$ is the inverse of UV, we need to prove that $(UV) \times (UV)^T = (UV)^T \times (UV) = I$:

$$(UV) \times (UV)^T = UV \times V^T U^T = U \times I \times U^T = UU^T = I$$
$$(UV)^T \times (UV) = V^T U^T \times UV = V^T \times I \times V = V^T V = I$$

$\boxed{\text{So, since } (UV) \times (UV)^T = (UV)^T \times (UV) = I \text{, and UV is invertible, then UV is an orthogonal matrix.}}$

Answer O

In order to solve this, I multiplied each column by each other column and checked if all the results were 0.

In [6]:
```
import numpy as np
A = np.array([[-6, -3, 6,  1],
              [-1,  2, 1, -6],
              [ 3,  6, 3, -2],
              [ 6, -3, 6, -1],
              [ 2, -1, 2,  3],
              [-3,  6, 3,  2],
              [-2, -1, 2, -3],
              [ 1,  2, 1,  6]])

A1 = np.array([-6, -1, 3, 6, 2, -3, -2, 1])
A2 = np.array([-3, 2, 6, -3, -1, 6, -1, 2])
A3 = np.array([6, 1, 3, 6, 2, 3, 2, 1])
A4 = np.array([1, -6, -2, -1, 3, 2, -3, 6])

a12 = A1.dot(A2)
a13 = A1.dot(A3)
a14 = A1.dot(A4)

a21 = A2.dot(A1)
a23 = A2.dot(A3)
a24 = A2.dot(A4)

a31 = A3.dot(A1)
a32 = A3.dot(A2)
a34 = A3.dot(A4)

a41 = A4.dot(A1)
a42 = A4.dot(A2)
a43 = A4.dot(A3)

a1 = (a12 == 0 and a13 == 0 and a14 == 0)
a2 = (a21 == 0 and a23 == 0 and a24 == 0)
a3 = (a31 == 0 and a32 == 0 and a34 == 0)
a4 = (a41 == 0 and a42 == 0 and a43 == 0)
isOrtho = a1 and a2 and a3 and a4

print(isOrtho)
```

```
True
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