CS 132 – Spring 2020, Assignment 6

Answer 1.

If D is invertible, then $D \times D^{-1} = I$*, so, if we multiply each side by* D^{-1} *we get that*

$$\boxed{(B - C) \times D \times D^{-1} = 0 \times D^{-1} \equiv (B - C) = 0 \equiv B = C}$$

Answer 2.

Let $C = AB$*, then, if we know that both B and AB are invertible, then:*

$$C = AB \equiv C \times B^{-1} = A \times B \times B^{-1} \equiv C \times B^{-1} = A \equiv AB \times B^{-1} = A$$

$$(AB \times AB^{-1}) \times (B^{-1} \times B) = A \times (B \times AB^{-1}) \equiv \boxed{I = A \times (B \times AB^{-1})}$$

As such, if there is some x that multiplied by A gives the identity matrix then A is invertible. As we can see above that is the case.

Answer 3.

If P is invertible, then $P \times P^{-1} = I$*, so:*

$$A = P \times B \times P^{-1} \equiv P^{-1}A = (P \times P^{-1}) \times B \times P^{-1} \equiv P^{-1} \times A \times P = B \times (P^{-1} \times P)$$

$$\boxed{P^{-1} \times A \times P = B}$$

Answer 4.

$$\text{Let } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } A^{-1} = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \text{ so,}$$
$$A \times A^{-1} = I \equiv \begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now we need to get whatever x, y, z, w are, and since we know how to multiply matrices, we know that:

$$\begin{cases} ax + bz = 1 \\ cx + dz = 0 \end{cases} \text{ and } \begin{cases} ay + bw = 0 \\ cy + dw = 1 \end{cases}$$

If we solve the first equation we get that:

$$\begin{cases} ax + bz = 1 \\ z = -\frac{c}{d}x \end{cases} \equiv \begin{cases} ax - \frac{bc}{d}x = 1 \\ z = -\frac{c}{d}x \end{cases} \equiv \begin{cases} \frac{ad-bc}{d}x = 1 \\ z = -\frac{c}{d}x \end{cases}$$

As we can see, we are left with this equation $\frac{ad-bc}{d}x = 1$*, which is only impossible IF* $\frac{ad-bc}{d} = 0$*, which ONLY happens IF* $ad - bc = 0$*, so, the equation only has a solution when* $ad - bc \neq 0$

Answer 5.

Let $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ *and* $D = \begin{bmatrix} a & e \\ b & f \\ c & g \\ d & h \end{bmatrix}$ *applying basic rules of multiplication of matrices to* $A \times D = I$ *we get that:*

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} a & e \\ b & f \\ c & g \\ d & h \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

If we solve this to get a, b, c, ..., we get that:

$$\begin{cases} a + b + c = 1 \\ b + c + d = 0 \\ e + f + g = 0 \\ f + g + h = 1 \end{cases}$$

So if we attribute values that solve that, we get that:

$$\begin{cases} a = 1 \\ b = 0 \\ c = 0 \\ d = 0 \\ e = 0 \\ f = 0 \\ g = 0 \\ h = 1 \end{cases} \equiv \boxed{D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}}$$

It's impossible to find a matrix C that solves the problem, as it would need to be D^T*, since this is the only matrix that gives us at least the first column correctly, yet the math doesn't add up for the remaining ones. As if we solve* $D^T \times A =$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

Answer 6.

```
A = np.array([[-25, -9, -27], [546,  180, 537], [154, 50, 149]])
print("Let A = \n", A, "\n")
B = np.array([[0,0], [1,0],[0,1]])
print("Let B = \n", B, "\n")
C = np.array([[-25, -9, -27, 0, 0], [546,  180, 537, 1, 0], [154, 50, 149, 0, 1]])
print("Let C = [A B] = \n", C, "\n")

CEchelon = forwardElimination(C)
if (not inconsistentSystem(CEchelon)):
    CReducedEchelon = backsubstitution(CEchelon)

print("If we put C in reduced row echelon we get that C = \n", CReducedEchelon, "\n")
print("So the final two columns of the inverse of A = \n", CReducedEchelon[:,3:], "\n")
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```
Let A =
 [[-25  -9 -27]
 [546 180 537]
 [154  50 149]]

Let B =
 [[0 0]
 [1 0]
 [0 1]]

Let C = [A B] =
 [[-25  -9 -27   0   0]
 [546 180 537   1   0]
 [154  50 149   0   1]]

If we put C in reduced row echelon we get that C =
 [[  1.           0.           0.           1.5         -4.5       ]
 [  0.           1.           0.         -72.16666667 219.5       ]
 [ -0.          -0.           1.          22.66666667 -69.        ]]

So the final two columns of the inverse of A =
 [[  1.5         -4.5       ]
 [-72.16666667 219.5       ]
 [ 22.66666667 -69.        ]]
```

Answer 7.

By the IMT, if $Ax = v$ is consistent for every v, then A has at least one solution for each v, meaning that A is an invertible matrix.
This means that there is a C where $CA = I$, which can only happen when $Ax = 0$ ONLY has the trivial solution, and has this only has the trivial solution, all v's only have 1 solution.

Answer 8.

As per the IMT, if $Hx = c$ is inconsistent for some c, then $Hx = 0$ has more than the trivial solution.
This comes as when we simplify the equation, we will get a free variable, which then translates to further options when equaling $Hx = 0$.

Answer 9.

If L has the trivial solution to $Lx = 0$, then it must have n pivots, meaning its columns span R^n.

Answer 10.

If A^2's columns are linearly independent and A is a $n \times n$ matrix, then A^2 has n pivot columns, which means that it spans R^n.

Answer 11.

$$\textbf{\textit{A}} \ \boxed{\begin{bmatrix} .5 & .25 & .25 \\ .25 & .5 & .25 \\ .25 & .25 & .5 \end{bmatrix}}$$

B

We have the following starting matrix: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$*, so if we apply it to the stochastic matrix twice we get the following:*

$$\textit{First solution} = \begin{bmatrix} .5 \\ .25 \\ .25 \end{bmatrix}$$

$$\textit{Second solution} = \boxed{\begin{bmatrix} .375 \\ .3125 \\ .3125 \end{bmatrix}}$$

Answer 12.

$$\textbf{\textit{A}} \; \boxed{\begin{bmatrix} .6 & .4 & .4 \\ .3 & .3 & .5 \\ .1 & .3 & .1 \end{bmatrix}}$$

B

Our starting matrix is: $\begin{bmatrix} .5 \\ .5 \\ .0 \end{bmatrix}$ *so if we apply this to the stochastic matrix we get that:*

$$\boxed{x_1 = \begin{bmatrix} .5 \\ .3 \\ .2 \end{bmatrix}}$$

C

Our starting matrix is: $\begin{bmatrix} .0 \\ .4 \\ .6 \end{bmatrix}$ *so if we apply this to the stochastic matrix we get that:*

$$x_1 = \begin{bmatrix} .4 \\ .42 \\ .18 \end{bmatrix}$$

$$\boxed{x_2 = \begin{bmatrix} .48 \\ .336 \\ .184 \end{bmatrix}}$$

Answer 13.

For P to be a regular stochastic matrix, then P^2 must also be a regular stochastic matrix, so if we calculate P^2 we get that:

$$\boxed{\begin{bmatrix} 1 & .2 \\ 0 & .8 \end{bmatrix} \times \begin{bmatrix} 1 & .2 \\ 0 & .8 \end{bmatrix} = \begin{bmatrix} 1+0 & .2+.16 \\ 0+0 & 0+.64 \end{bmatrix} = \begin{bmatrix} 1 & .36 \\ 0 & .64 \end{bmatrix}}$$

As such, P is a regular stochastic matrix.

Answer 14.

After many trials, we would reach a steady state vector, which can be gathered from the following equations:
$Ax = x$*, with A being the stochastic matrix from the exercise. So if we start calculating:*
$Ax = x \equiv Ax - x = 0 \equiv (A - I)x = 0$ *So if we calculate* $(A - I)$*:*

$$\begin{bmatrix} .5 & .25 & .25 \\ .25 & .5 & .25 \\ .25 & .25 & .5 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -.5 & .25 & .25 \\ .25 & -.5 & .25 \\ .25 & .25 & -.5 \end{bmatrix}$$

If we call this matrix B, then we are left with calculating $Bx = 0$ *so if we calculate this we get:*

$$\begin{bmatrix} -.5 & .25 & .25 & 0 \\ .25 & -.5 & .25 & 0 \\ .25 & .25 & -.5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -.5 & -.5 & 0 \\ .25 & -.5 & .25 & 0 \\ .25 & .25 & -.5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -.5 & -.5 & 0 \\ 0 & -.375 & .375 & 0 \\ .25 & .25 & -.5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -.5 & -.5 & \\ 0 & -.375 & .375 & 0 \\ 0 & .375 & -.375 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -.5 & -.5 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So we get that : $\begin{cases} x_1 = .5x_2 + .5x_3 \\ x_2 = x_3 \\ x_3 - free \end{cases}$ *since we know this must be a probability vector that adds up to 1, then we can solve that:* $x_1 + x_2 + x_3 = 1 \equiv .5x_2 + .5x_3 + x_2 + x_3 = 1 \equiv .5x_3 + .5x_3 + x_3 + x_3 = 1 \equiv 3x_3 = 1 \equiv x_3 = \frac{1}{3}$

As such: $x_2 = \frac{1}{3}$ *and* $x_1 = .5 \times \frac{1}{3} + .5 \times \frac{1}{3} = \frac{1}{3}$

Answer: $\boxed{\begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}}$

Answer 15.

```
def getSteadyState(m):
    i = [[1,0,0],[0,1,0],[0,0,1]]
    m2 = [[0,0,0],[0,0,0],[0,0,0]]

    for j in range(len(i)):
        for k in range(len(i[0])):
            m2[j][k] = round(m[j][k]-i[j][k],2)

    return m2

M = [[.9,.01,.09],[.01,.9,.01],[.09,.09,.9]]
M2 = getSteadyState(M)

print("Our first matrix, A is:")
print(M)
print()

print("First, we need to get the steady state matrix, as the problem states it's on an USUAL day. \nSo we get that (A-I)x = 0, or
print(M2)
print()

M3 = [[-.1,.01,.09],[0,-.19,-.11],[0,0,0]]

print("Finally we solve Bx = 0:")
print(M3)
print("Or:")
print("x1 = 0.1x2 + 0.9x3 | x2 = (-.11/-.19)x3 | x3 - free")
print()

M4 = [.20,.29,.51]

print("If we calculate it so it creates a probability matrix we get:")
print(M4)
print()

M5 = [400,580,1020]

print("So now all we have to do is apply the starting matrix of cars to this one to get the average amount of cars per location i
print(M5)
print()

print("Finally we get our answer, which states that there will be on average 580 cars to be rented and ready to rent at the downt
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```
Our first matrix, A is:
[[0.9, 0.01, 0.09], [0.01, 0.9, 0.01], [0.09, 0.09, 0.9]]

First, we need to get the steady state matrix, as the problem states it's on an USUAL day.
So we get that (A-I)x = 0, or Bx = 0:
[[-0.1, 0.01, 0.09], [0.01, -0.1, 0.01], [0.09, 0.09, -0.1]]

Finally we solve Bx = 0:
[[-0.1, 0.01, 0.09], [0, -0.19, -0.11], [0, 0, 0]]
Or:
x1 = 0.1x2 + 0.9x3 | x2 = (-.11/-.19)x3 | x3 - free

If we calculate it so it creates a probability matrix we get:
[0.2, 0.29, 0.51]

So now all we have to do is apply the starting matrix of cars to this one to get the average
amount of cars per location in a typical day:
[400, 580, 1020]

Finally we get our answer, which states that there will be on average 580 cars to be rented
and ready to rent at the downtown location
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