CS 132 – Spring 2020, Assignment 12

Answer A

Firstly, we need to check if u_1 and u_2 are orthogonal, to do this we need multiply them and get:

$$\boxed{\mathrm{u}_1 \times u_2 = [3,4,0] \times \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix} = -12 + 12 + 0 = 0}$$

Thus, we have that u_1, u_2 is an orthogonal set.

With that, to find the projection of y in this set, we have that:

$$\hat{y} = \frac{y \times u_1}{u_1 \times u_1} \times u_1 + \frac{y \times u_2}{u_2 \times u_2} \times u_2$$

$$y \times u_1 = [6,3,-2] \times \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} = 18 + 12 + 0 = 30$$

$$y \times u_2 = [6,3,-2] \times \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix} = -24 + 9 + 0 = -15$$

$$u_1 \times u_1 = [3,4,0] \times \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} = 9 + 16 + 0 = 25$$

$$u_2 \times u_2 = [-4,3,0] \times \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix} = 16 + 9 + 0 = 25$$

$$\hat{y} = \frac{30}{25} \times u_1 + \frac{-15}{25} \times u_2 = \begin{bmatrix} 18/5 \\ 24/5 \\ 0 \end{bmatrix} + \begin{bmatrix} 12/5 \\ -9/50 \end{bmatrix} = \boxed{\begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}}$$

Answer B

Since we know that W is an orthogonal set by $\{u_1, u_2\}$:

$$\hat{y} = \frac{y \times u_1}{u_1 \times u_1} \times u_1 + \frac{y \times u_2}{u_2 \times u_2} \times u_2$$

$$y \times u_1 = [-1, 4, 3] \times \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = -1 + 4 + 3 = 6$$

$$y \times u_2 = [-1, 4, 3] \times \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix} = 1 + 12 + -6 = 7$$

$$u_1 \times u_1 = [1, 1, 1] \times \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 3$$

$$u_2 \times u_2 = [-1, 3, -2] \times \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix} = 1 + 9 + 4 = 14$$

$$\hat{y} = 2 \times u_1 + \frac{1}{2} \times u_2 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} + \begin{bmatrix} -1/2 \\ 3/2 - 1 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 7/2 \\ 1 \end{bmatrix}$$

Now, we know that $y = \hat{y} + y_2$, so $y_2 = y - \hat{y} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} - \begin{bmatrix} 3/2 \\ 7/2 \\ 1 \end{bmatrix} = \begin{bmatrix} -5/2 \\ 1/2 \\ 2 \end{bmatrix}$

$$\boxed{\text{y} = \begin{bmatrix} 3/2 \\ 7/2 \\ 1 \end{bmatrix} + \begin{bmatrix} -5/2 \\ 1/2 \\ 2 \end{bmatrix}}$$

Answer C

The closest point to y in the subspace spanned by $\{v_1, v_2\}$ will be the orthogonal projection of y in that subspace, so let this be $\hat{y}$ then:

$$\hat{y} = \frac{y \times v_1}{v_1 \times v_1} \times v_1 + \frac{y \times v_2}{v_2 \times v_2} \times v_2$$

$$y \times v_1 = [3, -1, 1, 13] \times \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix} = 3 + 2 + -1 + 26 = 30$$

$$y \times v_2 = [3, -1, 1, 13] \times \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix} = -12 + -1 + 0 + 39 = 26$$

$$v_1 \times v_1 = [1, -2, -1, 2] \times \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix} = 1 + 4 + 1 + 4 = 10$$

$$v_2 \times v_2 = [-4, 1, 0, 3] \times \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix} = 16 + 1 + 0 + 9 = 26$$

$$\hat{y} = 3 \times v_1 + v_2 = \begin{bmatrix} 3 \\ -6 \\ -3 \\ 6 \end{bmatrix} + \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix} = \boxed{\begin{bmatrix} -1 \\ -5 \\ -3 \\ 9 \end{bmatrix}}$$

Answer D

The closest approximation to z by the vectors that form $c_1 v_1 + c_2 v_2$ is the projection of z in the Subspace spanned by v_1, v_2, let this be $\hat{z}$, then:

$$\hat{z} = \frac{z \times v_1}{v_1 \times v_1} \times v_1 + \frac{z \times v_2}{v_2 \times v_2} \times v_2$$

$$z \times v_1 = [2, 4, 0, -1] \times \begin{bmatrix} 2 \\ 0 \\ -1 \\ -3 \end{bmatrix} = 4 + 0 + 0 + 3 = 7$$

$$z \times v_2 = [2, 4, 0, -1] \times \begin{bmatrix} 5 \\ -2 \\ 4 \\ 2 \end{bmatrix} = 10 - 8 + 0 - 2 = 0$$

$$v_1 \times v_1 = [2, 0, -1, -3] \times \begin{bmatrix} 2 \\ 0 \\ -1 \\ -3 \end{bmatrix} = 4 + 0 + 1 + 9 = 14$$

$$v_2 \times v_2 = [5, -2, 4, 2] \times \begin{bmatrix} 5 \\ -2 \\ 4 \\ 2 \end{bmatrix} = 25 + 4 + 16 + 4 = 49$$

$$\hat{z} = \frac{1}{2} \times v_1 = \boxed{\begin{bmatrix} 2 \\ 0 \\ -1/2 \\ -3/2 \end{bmatrix}}$$

Answer E

Since we know that the least-squares solution of $Ax = b$ satisfies $A^T Ax = A^T b$ then we need to start by calculating $A^T A$ and $A^T b$:

$$A^T \times A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix}$$

$$A^T \times b = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix} \times \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$$

Now we've gotten the equation: $\boxed{\begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix} \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}}$

If we solve this we get:

$$\begin{bmatrix} 3 & 3 & 6 \\ 3 & 11 & 14 \end{bmatrix} \equiv \begin{bmatrix} 1 & 1 & 2 \\ 3 & 11 & 14 \end{bmatrix} \equiv \begin{bmatrix} 1 & 1 & 20 & 1 & 1 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \equiv \boxed{\hat{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

Answer F

Since we know that any least-squares solution of $Ax = b$ satisfies $A^T Ax = A^T b$ then we need to start by calculating $A^T A$ and $A^T b$:

$$A^T \times A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 3 & 3 \\ 3 & 3 & 0 \\ 3 & 0 & 3 \end{bmatrix}$$

$$A^T \times b = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 7 \\ 2 \\ 3 \\ 6 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 27 \\ 12 \\ 15 \end{bmatrix}$$

If we now solve the equation we get:

$$\begin{bmatrix} 6 & 3 & 3 & 27 \\ 3 & 3 & 0 & 12 \\ 3 & 0 & 3 & 15 \end{bmatrix} \equiv \begin{bmatrix} 3 & 3 & 0 & 12 \\ 3 & 3 & 0 & 12 \\ 3 & 0 & 3 & 15 \end{bmatrix} \equiv \begin{bmatrix} 0 & 0 & 0 & 0 \\ 3 & 3 & 0 & 12 \\ 3 & 0 & 3 & 15 \end{bmatrix} \equiv \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 4 \\ 1 & 0 & 1 & 5 \end{bmatrix} \equiv \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 4 \\ 0 & 1 & -1 & -1 \end{bmatrix} \equiv$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 5 \\ 0 & 1 & -1 & -1 \end{bmatrix} \equiv \begin{cases} x_1 = 5 - 1x_3 \\ x_2 = -1 + x_3 \\ x_3 - free \end{cases} \equiv \boxed{\begin{bmatrix} 5 - x_3 \\ x_3 - 1 \\ x_3 \end{bmatrix}}$$

Answer G

To start, the projection of b in ColA would be given by: $\hat{b} = \frac{b \times a_1}{a_1 \times a_1} \times a_1 + \frac{b \times a_2}{a_2 \times a_2} \times a_2$

So, we must first calculate $b \times a_1$, $b \times a_2$, $a_1 \times a_1$ and $a_2 \times a_2$:

$$b \times a_1 = [3, -1, 5] \times \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 3 + 1 + 5 = 9$$

$$b \times a_2 = [3, -1, 5] \times \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} = 6 - 4 + 10 = 12$$

$$a_1 \times a_1 = [1, -1, 1] \times \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 1 + 1 + 1 = 3$$

$$a_2 \times a_2 = [2, 4, 2] \times \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} = 4 + 16 + 4 = 24$$

So now if we solve the equation we get:

$$\hat{b} = 3 \times a_1 + \tfrac{1}{2} \times a_2 \equiv \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 4 \\ -1 \\ 4 \end{bmatrix}}$$

From here, we could easily solve $A\hat{x} = \hat{b}$, however, due to the calculations we've made previously, we already know what weights are assign to each of the columns of A in order to produce $\hat{b}$, so, it's clear that the answer is:

$$\hat{x} = \boxed{\begin{bmatrix} 3 \\ 1/2 \end{bmatrix}}$$

Answer H

Calculating Au we get that: $Au = \begin{bmatrix} 3 \\ 8 \\ 2 \end{bmatrix}$, and if we calculate Av we get that: $Av = \begin{bmatrix} 7 \\ 2 \\ 8 \end{bmatrix}$

If we calculate $b - Au$ and $b - Av$, we get that: $b - Au = \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix}$, and that $b - Av = \begin{bmatrix} -2 \\ 2 \\ -4 \end{bmatrix}$

From here we can calculate $\|b - Au\|$ and $\|b - Av\|$ and get that: $\|b - Au\|\sqrt{24}$ and $\|b - Av\| = \sqrt{24}$.
Since the least-square solution for a linearly independent system such as A is the unique point that is closest to b in Col A.

$\boxed{\text{Then neither of these points can be the least square solution.}}$

Answer I

The columns of A are linearly independent if and only if $Ax = 0$ has only the trivial solution (x = 0).
Since we know that $A^T A$ is invertible, then if we suppose that $Ax = 0$ for some x, we get that: $A^T Ax = A^T 0 = 0$, so x will be in $Nul(A^T A)$, but since $A^T A$ is invertible, the only vector in it's null-space is 0, meaning that $x = 0$.

$\boxed{\text{As such, if } A^T A \text{ is invertible, then if } Ax = 0\text{, for some x, then } x = 0}$

Answer J

In [12]:
```
import numpy as np

A = np.array([[0, .7, 1],
              [-.7, 0, .7],
              [-1, -.7, 0],
              [-.7, -1, -.7],
              [0, -.7, -1],
              [.7, 0, -.7],
              [1, -.7, 0],
              [.7, 1, .7],
              [0, -.7, 1],
              [.7, 0, -.7],
              [-1, .7, 0],
              [.7, -1, .7],
              [0, .7, -1],
              [-.7, 0, .7],
              [1, -.7, 0],
              [-.7, 1, -.7]])

b = [.7, 0, -.7, -1, -.7, 0, .7, 1, 0, 0, 0, 0, 0, 0, 0, 0]

AT = np.transpose(A)

ALPHA = AT.dot(A)

BETA = AT.dot(b)


x = np.linalg.solve(ALPHA,BETA)

print("x = ", np.round(x,4))
```

```
x =  [0.4336 0.4529 0.3535]
```

Answer K

The linear model that leads to a least-squares fit of the equation above is $y = X\beta + \varepsilon$ or:

$$\boxed{\begin{bmatrix} x_1 & x_1^2 & x_1^3 \\ \vdots & \vdots & \vdots \\ x_n & x_n^2 & x_n^3 \end{bmatrix} \times \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_n \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}}$$

For the given data we have:

In [29]:
```
import numpy as np

X = np.array([[4, 4**2, 4**3],
              [6, 6**2, 6**3],
              [8, 8**2, 8**3],
              [10, 10**2, 10**3],
              [12, 12**2, 12**3],
              [14, 14**2, 14**3],
              [16, 16**2, 16**3],
              [18, 18**2, 18**3]])

y = [1.58, 2.08, 2.5, 2.8, 3.1, 3.4, 3.8, 4.32]

x = np.linalg.lstsq(X,y,rcond=None)
beta = np.round(x[0],4)

print("The least-square curve formed by this data is: ", beta[0], "x +", beta[1], "x^2 +", beta[2], "x^3")
```

```
The least-square curve formed by this data is:  0.5132 x + -0.0335 x^2 + 0.001 x^3
```

Answer L

The linear model that leads to a least-squares fit of the equation above is $y = X\beta + \varepsilon$ or:

$$\boxed{\begin{bmatrix} e^{-.02t_1} & e^{-.07t_1} & \\ \vdots & \vdots & \vdots \\ e^{-.02t_n} & e^{-.07t_n} & \end{bmatrix} \times \begin{bmatrix} M_A \\ M_B \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_n \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}}$$

For the given data we have:

In [37]:
```
import numpy as np
import math

e = math.e

X = np.array([[e**(-.02*10), e**(-.07*10)],
              [e**(-.02*11), e**(-.07*11)],
              [e**(-.02*12), e**(-.07*12)],
              [e**(-.02*14), e**(-.07*14)],
              [e**(-.02*15), e**(-.07*15)]])

y = [21.34, 20.68, 20.05, 18.87, 18.30]

x = np.linalg.lstsq(X,y,rcond=None)
beta = np.round(x[0],4)

print("The least-square curve formed by this data is: e^(-0.02*", beta[0], ") + e^(-.07*", beta[1], ")")
```

```
The least-square curve formed by this data is: e^(-0.02* 19.9411 ) + e^(-.07* 10.1015 )
```

Answer M

To start off, we need to calculate the eigenvalues of A, which we get by solving the characteristic equation of A, which would be:

$$\begin{bmatrix} 6 & -2 \\ -2 & 9 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 6-\lambda & -2 \\ -2 & 9-\lambda \end{bmatrix}$$

So we get that: $(6-\lambda) \times (9-\lambda) - 4 = \lambda^2 - 15\lambda + 50 = 0$
If we solve this we get that: $\lambda = 5 | \lambda = 10$.

From here we can calculate each of the eigenvectors v_1 and v_2:

$$\begin{bmatrix} 6 & -2 \\ -2 & 9 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \equiv \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \equiv \begin{cases} x_1 = 2x_2 \\ x_2 - free \end{cases} \equiv \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 6 & -2 \\ -2 & 9 \end{bmatrix} - \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix} \equiv \begin{bmatrix} 0 & 0 \\ 1 & 1/2 \end{bmatrix} \equiv \begin{cases} x_1 = -1/2x_2 \\ x_2 - free \end{cases} \equiv \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$$

If we now normalize both v_1 and v_2 into u_1 and u_2 we get that:

$$u_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \times \frac{1}{\|v_1\|} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \times \frac{1}{\sqrt{5}}$$

$$u_2 = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} \times \frac{1}{\|v_2\|} = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} \times \frac{1}{\sqrt{5}}$$

As such, we get that:

$$\boxed{P = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1/2 \\ 1 & 1 \end{bmatrix}} \boxed{D = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}}$$

Answer N

Firstly, we need to establish the matrix of the quadratic form, which in this case would be $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$. With the matrix defined, we now need to find it's eigenvalues, which would be:

$$\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix}$$

So we get that: $(2-\lambda) \times (-6-\lambda) - 9 = \lambda^2 + 4\lambda - 21 = 0$
If we solve this we get that: $\lambda = -7 | \lambda = 3$.

From here we can calculate each of the eigenvectors v_1 and v_2:

$$\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} - \begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix} = \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} \equiv \begin{bmatrix} 1 & 1/3 \\ 0 & 0 \end{bmatrix} \equiv \begin{cases} x_1 = -1/3x_2 \\ x_2 - free \end{cases} \equiv \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix} \equiv \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} \equiv \begin{cases} x_1 = 3x_2 \\ x_2 - free \end{cases} \equiv \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

If we take a careful look at these two vectors, we'll notice that they have the same norm, hence, after normalizing both v_1 and v_2, we get that $\boxed{P = \frac{1}{\sqrt{10}} \begin{bmatrix} -1/3 & 3 \\ 1 & 1 \end{bmatrix}}$ and $D = \begin{bmatrix} -7 & 0 \\ 0 & 3 \end{bmatrix}$.

$\boxed{\text{With these, we have the variable change } x = Py.}$

$\boxed{\text{As for the new quadratic form, we have that: } x^T A x = (Py)^T A (Py) = y^T P^T A P y = y^t D y = 3y_2^2 + 7y_1^2}$

Answer O

Firstly, we need to establish the matrix of the quadratic form, which in this case would be $A = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$. With the matrix defined, we now need to find it's eigenvalues, which would be:

$$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -1-\lambda & -1 \\ -1 & -1-\lambda \end{bmatrix}$$

So we get that: $(-1-\lambda) \times (-1-\lambda) - 1 = \lambda^2 + 2\lambda = 0$
If we solve this we get that: $\lambda = -2 | \lambda = 0$.

From here we can calculate each of the eigenvectors v_1 and v_2:

$$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \equiv \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \equiv \begin{cases} x_1 = x_2 \\ x_2 - free \end{cases} \equiv \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \equiv \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \equiv \begin{cases} x_1 = -x_2 \\ x_2 - free \end{cases} \equiv \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

If we take a careful look at these two vectors, we'll notice that they have the same norm, hence, after normalizing both v_1 and v_2, we get that $\boxed{P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}$ and $D = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}$.

$\boxed{\text{With these, we have the variable change } x = Py.}$

$\boxed{\text{As for the new quadratic form, we have that: } x^T Ax = (Py)^T A(Py) = y^T P^T APy = y^t Dy = -2y_1^2}$

Answer P

Firstly, we need to establish the matrix of the quadratic form, which in this case would be $A = \begin{bmatrix} 3 & 4 \\ 4 & 9 \end{bmatrix}$. With the matrix defined, we now need to find it's eigenvalues, which would be:

$$\begin{bmatrix} 3 & 4 \\ 4 & 9 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 3-\lambda & 4 \\ 4 & 9-\lambda \end{bmatrix}$$

So we get that: $(3-\lambda) \times (9-\lambda) - 16 = \lambda^2 - 12\lambda + 11 = 0$
If we solve this we get that: $\lambda = 1 | \lambda = 11$.

From here we can calculate the eigenvector for $\lambda = 11$, let this be v_1:

$$\begin{bmatrix} 3 & 4 \\ 4 & 9 \end{bmatrix} - \begin{bmatrix} 11 & 0 \\ 0 & 11 \end{bmatrix} = \begin{bmatrix} -8 & 4 \\ 4 & -2 \end{bmatrix} \equiv \begin{bmatrix} 2 & -1/2 \\ 0 & 0 \end{bmatrix} \equiv \begin{cases} x_1 = 1/2x_2 \\ x_2 - free \end{cases} \equiv \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$$

If we normalize this vector we get: $u_1 = \frac{1}{\sqrt{1.25}} \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$

a. The maximum value of the quadratic form subject to the constraint $x^T x = 1$ is 11.

b. This value is attained at $u_1 = \frac{1}{\sqrt{1.25}} \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$.

c. The maximum value of the quadratic form subject to the constraint $x^T x = 1$ and $x^T u_1 = 0$ is 1.