# CS 480 – Fall 2021, PSet 3

**Answer 1**

(a) Firstly, let's extend what we are given:

$$f(x,y,z) = (x-c_x)^2 + (y-c_y)^2 + z^2 + r^2 =$$
$$= x^2 + c_x^2 - 2\times x\times c_x + y^2 + c_y^2 - 2\times y\times c_y + z^2 + r^2 =$$
$$= x^2 - 2\times x\times c_x + y^2 - 2\times y\times c_y + z^2 + r^2 + c_x^2 + c_y^2 =$$
$$= x^2(1-\tfrac{2\times c_x}{x}) + y^2(1-\tfrac{2\times c_y}{y}) + z^2 + r^2 + c_x^2 + c_y^2$$

This would give us $Q$ in $p^TQp$ to be:

$$\begin{bmatrix} (1-\frac{2\times c_x}{x}) & 0 & 0 & 0 \\ 0 & (1-\frac{2\times c_y}{y}) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & r^2+c_x^2+c_y^2 \end{bmatrix}$$

(b) We know that the Unit Normal Vector $n(x,y,z) = \bigtriangledown f(x,y,z)$, using the final form of the equation calculated above we get that:

$$n(x,y,z) = \bigtriangledown f(x,y,z)$$
$$\bigtriangledown f(x,y,z) =$$
$$\begin{bmatrix} \frac{d}{dx}f(x,y,z) \\ \frac{d}{dy}f(x,y,z) \\ \frac{d}{dz}f(x,y,z) \end{bmatrix}$$
$$\begin{bmatrix} \frac{d}{dx}x^2(1-\frac{2\times c_x}{x}) + y^2(1-\frac{2\times c_y}{y}) + z^2 + r^2 + c_x^2 + c_y^2 \\ \frac{d}{dy}x^2(1-\frac{2\times c_x}{x}) + y^2(1-\frac{2\times c_y}{y}) + z^2 + r^2 + c_x^2 + c_y^2 \\ \frac{d}{dz}x^2(1-\frac{2\times c_x}{x}) + y^2(1-\frac{2\times c_y}{y}) + z^2 + r^2 + c_x^2 + c_y^2 \end{bmatrix}$$
$$\begin{bmatrix} 2\times(x-c_x) \\ 2\times(y-c_y) \\ 2\times z \end{bmatrix}$$

(c) The general format for the equation of a hyperboloid is:

$f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = r$

Given this equation, the usual parametric equations are:

$$f(u, v) = \begin{bmatrix} a \times \sqrt{r + u^2} \times cos(v) \\ b \times \sqrt{r + u^2} \times sin(v) \\ c \times u \end{bmatrix}$$

If we adapt this to our equation we have:

$f(x, y, z) = (x - c_x)^2 + (y - c_y)^2 + z^2 + r^2 = 0 \equiv$
$\equiv \frac{(x-c_x)^2}{1^2} + \frac{(y-c_y)^2}{1^2} - \frac{(z)^2}{-(1^2)} = -r$

So we get:

$$f(u, v) = \begin{bmatrix} \sqrt{-r + u^2} \times cos(v) \\ \sqrt{-r + u^2} \times sin(v) \\ -u \end{bmatrix}$$

We know that $z = -u$. Furthermore, we know by the definition of the formula that $x - c_x = \sqrt{-r + u^2} \times cos(v)$ and $y - c_y = \sqrt{-r + u^2} \times sin(v)$ so, solving all of this we get:

$x - c_x = \sqrt{-r + u^2} \times cos(v) \equiv x = \sqrt{-r + u^2} \times cos(v) + c_x$
$y - c_y = \sqrt{-r + u^2} \times sin(v) \equiv y = \sqrt{-r + u^2} \times sin(v) + c_y$
$z = -u$

Putting this in terms of $z$ and $\theta$ we get:

$$f(\theta, z) = \begin{bmatrix} \sqrt{-r + z^2} \times cos(\theta) + c_x \\ \sqrt{-r + z^2} \times sin(\theta) + c_y \\ z \end{bmatrix}$$

Limiting our parameters to $-\pi \leq \theta \leq \pi$ and $z_{min} \leq z \leq z_{max}$. We know that the $z_{min}$ is either going to be 0, or whatever the coordinate is for the point where $x = y = 0$, if we calculate this we get:

$$z_{min} = max(0, zeroPoint)$$

$$\begin{aligned}
&zeroPoint : f(0,0,z) = 0 \equiv (x - c_x)^2 + (y - c_y)^2 + z^2 + r^2 = 0 \equiv \\
&\equiv (0 - c_x)^2 + (0 - c_y)^2 + z^2 + r^2 = 0 \equiv \\
&\equiv -c_x^2 - c_y^2 + z^2 + r^2 = 0 \equiv \\
&\equiv z^2 = c_x^2 + c_y^2 - r^2 \equiv \\
&\equiv z = \sqrt{c_x^2 + c_y^2 - r^2} \\
&z_{min} = max(0, \sqrt{c_x^2 + c_y^2 - r^2})
\end{aligned}$$

As for $z_{max}$, there is no actual cap, as the Hyperboloid could go on forever in the positive direction, so $z_{max} = \infty$. This means that, in all:

$$f(\theta, z) = \begin{bmatrix} \sqrt{z^2 - r} \times cos(\theta) + c_x \\ \sqrt{z^2 - r} \times sin(\theta) + c_y \\ z \end{bmatrix}$$

$$\wedge -\pi \leq \theta \leq \pi$$

$$\wedge\, max(0, \sqrt{c_x^2 + c_y^2 - r^2}) \leq z \leq \infty$$

(d) We know from lecture that the normal of a surface can be obtained by deriving from the parametric equations. When given two variables, we know that the normal will be the multiplication of the derivation of each of the variables, so in this case:

$$n(\theta, z) = \frac{df}{d\theta} \times \frac{df}{dz} = \begin{bmatrix} -\sqrt{z^2 - r} \times sin(\theta) \\ \sqrt{z^2 - r} \times cos(\theta) \\ 0 \end{bmatrix} \times \begin{bmatrix} \frac{cos(\theta) \times z}{\sqrt{z^2 - r}} \\ \frac{sin(\theta) \times z}{\sqrt{z^2 - r}} \\ 1 \end{bmatrix} = \begin{bmatrix} -z \times sin(\theta) \times cos(\theta) \\ z \times sin(\theta) \times cos(\theta) \\ 0 \end{bmatrix}$$

**Answer 2**

(a) We know that the parametric line equation is given by $p(u) = p_0 + u(p_1 - p_0)$ so we know that for this line segment, the parametric equation in $u$ is:

$$p(u) = \begin{bmatrix} -2 \times u + 2 \\ 0 \\ 6 \times u \end{bmatrix}$$

(b) Given the equation above, and the equations for the ellipse at $z = 0$, we can assume that the equation of the point $P(u, v)$ is:

$$P(u, v) = \begin{bmatrix} (2 + r_x \times cos(2\pi v)) - 2 \times u \\ r_y \times sin(2\pi v) \\ 6u \end{bmatrix}$$

(c) We know that, to get the Normal, we must use the gradient of the function, since we already have $P(u, v)$, we can simply take the gradient of that, for $u$ and for $v$ and then multiply it (as seen in lecture). So we have:

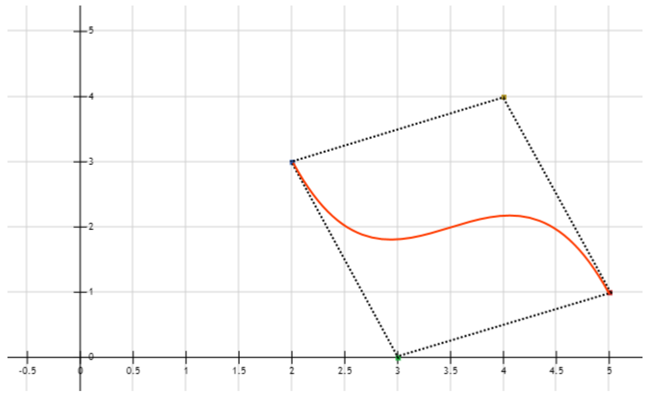
$$N(u, v) = \frac{dP}{du} \times \frac{dP}{dv} = \begin{bmatrix} -2 \\ 0 \\ 6 \end{bmatrix} \times \begin{bmatrix} -2 \times \pi \times r_x \times sin(2 \times \pi \times v) \\ 2 \times \pi \times r_y \times cos(2 \times \pi \times v) \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \times \pi \times r_x \times sin(2 \times \pi \times v) \\ 0 \\ 0 \end{bmatrix}$$

Finally, we need to make a unit vector, to do this we take the vector and divide it by its length $n = \frac{n}{\|n\|}$:

$$N(u, v) = \begin{bmatrix} 4 \times \pi \times r_x \times sin(2 \times \pi \times v) \\ 0 \\ 0 \end{bmatrix} /(4 \times \pi \times r_x \times sin(2 \times \pi \times v)) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

**Answer 3**

(a)




(b) We know from class that a Cubic Bezier Curve is defined by:

$$P(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} M_{BEZ} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

As such, $P'(u)$ would be:

$$P'(u) = \begin{bmatrix} 3u^2 & 2u & 1 & 0 \end{bmatrix} M_{BEZ} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

With that said, we can now solve for $P'(0)$:

$$P'(0) = \begin{bmatrix} 3u^2 & 2u & 1 & 0 \end{bmatrix} M_{BEZ} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} =$$

$$= \begin{bmatrix} 3u^2 & 2u & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} =$$

$$= \begin{bmatrix} -3u^2 + 6u - 3 & 9u^2 - 12u + 3 & -9u^2 + 6u & 3u^2 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} =$$

$$= \begin{bmatrix} -3 & 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} =$$

$$-3 \times p_0 + 3 \times p_1 = 3(p_1 - p_0) =$$
$$= (3, -9)$$

(c) As seen in lecture, $c_1$ continuity is given when $curve'_1(u = 1) = curve'_2(u = 0)$, since both of these curves are Cubic Bezier Curves, and as seen above and in lecture, they both follow the format below:

$$curve(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} M_{BEZ} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

$$curve'(u) = \begin{bmatrix} 3u^2 & 2u & 1 & 0 \end{bmatrix} M_{BEZ} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} -3u^2 + 6u - 3 & 9u^2 - 12u + 3 & -9u^2 + 6u & 3u^2 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

If we first calculate $curve'_1(u=1)$ we get:

$$curve'_1(1) = \begin{bmatrix}-3u^2+6u-3 & 9u^2-12u+3 & -9u^2+6u & 3u^2\end{bmatrix} \begin{bmatrix}p_0\\p_1\\p_2\\p_3\end{bmatrix} =$$

$$= \begin{bmatrix}0 & 0 & -3 & 3\end{bmatrix} \begin{bmatrix}p_0\\p_1\\p_2\\p_3\end{bmatrix} =$$

$$= -3 \times p_2 + 3 \times p_3 = 3(p_3 - p_2) =$$
$$= (3, -9)$$

With that said, we can calculate $curve'_2(u=0)$ we get:

$$curve'_2(0) = \begin{bmatrix}-3u^2+6u-3 & 9u^2-12u+3 & -9u^2+6u & 3u^2\end{bmatrix} \begin{bmatrix}p_0\\p_1\\p_2\\p_3\end{bmatrix} =$$

$$= \begin{bmatrix}-3 & 3 & 0 & 0\end{bmatrix} \begin{bmatrix}p_0\\p_1\\p_2\\p_3\end{bmatrix} =$$

$$= -3 \times p_2 + 3 \times p_3 = 3(p_3 - p_2) =$$
$$= (-6, 15)$$

As such, we can clearly say that these two segments do not have C1 continuity.

(d) To obtain C1 Continuity, we could simply change $p_1$ in the second curve to be $(1, -4)$.

**Answer 4**

(a) We know that the curve for the cubic spline section above is determined by some equation $P(u) = a \times u^3 + b \times u^2 + c \times u + d$. Furthermore, we are given the boundary conditions for this equation $P(0) = p_k$ and $P(1) = p_{k+1}$, if we apply the equation, we get:

$$P(0) = a \times 0 + b \times 0 + c \times 0 + d = d = p_k$$
$$P(1) = a \times 1 + b \times 1 + c \times 1 + d = a + b + c + d = p_{k+1}$$
$$P(0)' = 3 \times a \times 0 + 2 \times b \times 0 + c = c = \tfrac{1}{2}[(1+b)(p_k - p_{k-1}) + (1-b)(p_{k+1} - p_k)]$$
$$P(1)' = 3 \times a \times 1 + 2 \times b \times 1 + c = 3 \times a + 2 \times b + c = \tfrac{1}{2}[(1+b)(p_{k+1} - p_k) + (1-b)(p_{k+2} - p_{k+1})]$$

With we now turn these into matrices we get:

$$\begin{bmatrix} p_k \\ p_{k+1} \\ \frac{1}{2}[(1+b)(p_k - p_{k-1}) + (1-b)(p_{k+1} - p_k)] \\ \frac{1}{2}[(1+b)(p_{k+1} - p_k) + (1-b)(p_{k+2} - p_{k+1})] \end{bmatrix} = \begin{bmatrix} d \\ a+b+c+d \\ c \\ 3 \times a + 2 \times b + c \end{bmatrix} \equiv$$

$$\equiv \begin{bmatrix} p_k \\ p_{k+1} \\ \frac{1}{2}[(1+b)(p_k - p_{k-1}) + (1-b)(p_{k+1} - p_k)] \\ \frac{1}{2}[(1+b)(p_{k+1} - p_k) + (1-b)(p_{k+2} - p_{k+1})] \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \equiv$$

$$\equiv \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{2}(1+b) & b & \frac{1}{2}(1-b) & 0 \\ 0 & -\frac{1}{2}(1+b) & b & \frac{1}{2}(1-b) \end{bmatrix} \times \begin{bmatrix} p_{k-1} \\ p_k \\ p_{k+1} \\ p_{k+2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \equiv$$

$$\equiv \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{2}(1+b) & b & \frac{1}{2}(1-b) & 0 \\ 0 & -\frac{1}{2}(1+b) & b & \frac{1}{2}(1-b) \end{bmatrix} \times \begin{bmatrix} p_{k-1} \\ p_k \\ p_{k+1} \\ p_{k+2} \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \equiv$$

$$\equiv \begin{bmatrix} \frac{1}{2}(-b-1) & \frac{1}{2}(-b-1) + b + 2 & \frac{1-b}{2} + b - 2 & \frac{1-b}{2} \\ b+1 & -2b + \frac{b+1}{2} - 3 & 2 & \frac{b-1}{2} \\ \frac{1}{2}(-b-1) & b & \frac{1-b}{2} & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} p_{k-1} \\ p_k \\ p_{k+1} \\ p_{k+2} \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

From here, let us call the first matrix in the multiplication $A$, then we have that

$$P(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \times \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \times A^{-1} \times \begin{bmatrix} p_{k-1} \\ p_k \\ p_{k+1} \\ p_{K+2} \end{bmatrix}$$

(b) If we take $A^{-1}$ from the prior exercise, and calculate it, we get:

$$M_c = A^{-1} = \begin{bmatrix} \frac{1-b}{b+1} & \frac{1-b}{b+1} & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ \frac{2(b-3)}{b-1} & \frac{2(b-2)}{b-1} & 2 & 1 \end{bmatrix}$$

We also know that $P(u) = B_0(u) \times p_{k-1} + B_1(u) \times p_k + B_2(u) \times p_{k+1} + B_3(u) \times p_{k+2}$. As we saw in lecture, there is a very easy way to determine these, by looking at each of the columns of $M_c$, in this case we get:

$$B_0(u) = \frac{1-b}{b+1} \times u^3 + u + \frac{2(b-3)}{b-1}$$
$$B_1(u) = \frac{1-b}{b+1} \times u^3 + u + \frac{2(b-2)}{b-1}$$
$$B_2(u) = -(u^3) + u + 2$$
$$B_3(u) = u^3 + u^2 + u + 1$$

(c) Let us take the following two adjacent segments: $p_k, p_{k+1}$ and $p_{k+1}, p_{k+2}$, we can use these points to check if the Derivative of $P_1'(1)$ and $P_2'(0)$ have the same direction:

$$P_1'(1) = \frac{1}{2}[(1+b)(p_{k+1} - p_k) + (1-b)(p_{k+2} - p_{k+1})]$$
$$P_2'(0) = \frac{1}{2}[(1+b)(p_{k+1} - p_k) + (1-b)(p_{k+2} - p_{k+1})]$$

As we can see, the derivatives of the curve at these two points is exactly the same, and, as such, these two points have $C_1$ continuity.

(d) To check if the changes to $b$ affect the tangent direction, and not just the magnitude, of the curve, we must look at the derivatives of the Blending Functions:

$$B_0'(u) = \frac{3-3b}{b+1} \times u^2 + 1$$
$$B_1'(u) = \frac{3-3b}{b+1} \times u^2 + 1$$
$$B_2'(u) = -3 \times u^2 + 1$$
$$B_3'(u) = 3 \times u^2 + 2 \times u + 1$$

As you can see, both $B_0$ and $B_1$ Blending Functions have $b$ as a factor of $u$. This means that, a change in the sign of $b$ would, in fact, change the tangent direction, and not just the magnitude.