On open and closed convex codes

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Abstract

Combinatorial codes, i.e. subsets of the power set, naturally arise as the language of neural networks in the brain and silicon. A number of combinatorial codes in the brain arise as a pattern of intersections of convex sets in a Euclidean space. Such codes are called convex. Not every code is convex. What makes a code convex and what determines its embedding dimension is still poorly understood. Here we establish that the codes that arise from open convex sets and the codes that arise from closed convex sets are distinct classes of codes. We prove that open convexity is inherited from a sub-code with the same simplicial complex. Finally we prove that codes that contain all intersections of its maximal codewords are both open and closed convex, and provide an upper bound for their embedding dimension.

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1 Introduction

The brain represents information via patterns of neural activity often referred to as neural codes. Neural codes can be approximated by combinatorial patterns, where each 'codeword' σ is a subset

$$\sigma \subseteq [n] \stackrel{\text{def}}{=} \{1, \ldots, n\},$$

of n neurons that are active at a given point of time[1]. Stimuli are encoded by a population of neurons via a *combinatorial code* $\mathcal{C}$, a collection of codewords

$$\mathcal{C} \subset 2^{[n]} \stackrel{\text{def}}{=} \{\text{all subsets of } [n]\}.$$

Neurons in the brain represent stimuli that can be modeled as a stimulus space $X \subset \mathbb{R}^d$. The *receptive field* of a neuron is the subset U of the stimulus space X in which the neuron is active. A population of neurons exhibits *convex receptive fields* if each neuron's receptive field is convex. A number of areas in mammalian brains possess neurons with convex receptive fields. A paradigmatic example is hippocampal *place cells* [12], a class of cells in the mammalian hippocampus which act as position sensors where the relevant stimulus space $X \subset \mathbb{R}^2$ is the animal's two-dimensional environment. Receptive fields can be easily computed when both the neuronal activity data and the relevant stimulus space are available. Once receptive fields of a neuronal population are known, the stimuli can be easily 'decoded' from neural activity [5].

Inferring any information about the stimulus space X from the neural activity alone, i.e. without any a priori knowledge about the space of stimuli, is a much harder problem. The convexity of receptive fields has been exploited to infer topological features of the space of stimuli [4, 7, 13]. These results inevitably rely on the nerve lemma [1], which guarantees that for any finite cover $\mathcal{U} = \{U_i\}_{i\in[n]}$ by convex sets $U_i \subset \mathbb{R}^d$ that are either all open or all closed[2], its *nerve*[3], i.e. the abstract simplicial complex

$$\text{nerve}(\mathcal{U}) \stackrel{\text{def}}{=} \{\sigma \subseteq [n] \text{ such that } \bigcap_{i\in\sigma} U_i \neq \varnothing\} \subset 2^{[n]},$$

is homotopy equivalent to the underlying space $X = \cup_{i\in[n]} U_i$. While this allows for inference of the homotopy type of the underlying space, it is often desirable to infer less coarse information about the stimulus space.

More detailed information about the underlying space X is contained in the *code* of a cover $\mathcal{U}$ [3, 6]. If $\sigma \subseteq [n]$, then the *atom* of $\mathcal{U}$ corresponding to σ is the set $A^{\mathcal{U}}_\sigma$ defined as

$$A^{\mathcal{U}}_\sigma \stackrel{\text{def}}{=} \left(\bigcap_{i\in\sigma} U_i\right) \setminus \bigcup_{j\notin\sigma} U_j. \tag{1}$$

The *combinatorial code* of $\mathcal{U}$ is

$$\text{code}(\mathcal{U}, X) \stackrel{\text{def}}{=} \{\sigma \subseteq [n] \text{ such that } A^{\mathcal{U}}_\sigma \neq \varnothing\}, \tag{2}$$

where $A^{\mathcal{U}}_\varnothing \stackrel{\text{def}}{=} X \setminus (\cup_{i=1}^n U_i)$ and $\varnothing \notin \text{code}(\mathcal{U}, X)$ if and only if $X = \cup_{i=1}^n U_i$. Note that unlike the nerve, the code of a cover may fail to be an abstract simplicial complex (see e.g. Figure 1.1). Any combinatorial code $\mathcal{C} \subset 2^{[n]}$ can be *completed* to an abstract simplicial complex $\Delta(\mathcal{C})$, the minimal simplicial complex containing $\mathcal{C}$, that we call *the simplicial complex of the code* $\mathcal{C}$. A combinatorial code $\mathcal{C}$ can thus be thought of as its simplicial complex $\Delta(\mathcal{C})$ that is 'missing' some of its non-maximal faces. Moreover, $\Delta(\text{code}(\mathcal{U}, X)) = \text{nerve}(\mathcal{U})$.

A code of a convex cover of $X \subset \mathbb{R}^d$ contains information beyond the homotopy type of the stimulus space X. For example, as first observed in [3], the combinatorial code may impose constraints on the embedding dimension d, even if the nerve does not. Thus, to infer such properties of the stimulus space X from neural activity one needs to understand the embedding properties of combinatorial

[1]Note that this notion of neural code describes only the set of possible response patterns of a network, but does not include the "dictionary" of relationships between response patterns and network inputs.

[2]The nerve lemma remains valid for a finite collection of closed sets [1].

[3]The term "nerve" predated any neuroscience application of this object.

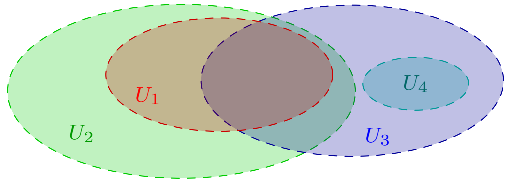


Figure 1.1: An example of a cover and its code. This cover has code $\mathcal{C} = \{\varnothing, 2, 3, 12, 23, 34, 123\}$. Since $13 \notin \mathcal{C}$ but $13 \subset 123$, $\mathcal{C}$ is not a simplicial comlex. Note that we often denote the codeword $\{i_1, i_2, \dots, i_k\} \in \mathcal{C}$ by the string $i_1 i_2 \dots i_k$; for example, the codeword $\{1, 2, 3\}$ is abbreviated to 123.

codes. There is currently little understanding of the embedding dimension properties beyond those stemming from the nerve of the cover [3, 9, 10, 14]. Moreover, the problem of intrinsically describing what makes a combinatorial code convex is currently poorly understood [2].

In this work, we investigate the questions of convexity and embedding dimension of combinatorial codes. In Section 2.3 we first establish that open convex codes (i.e. codes realizable by open convex covers), and closed convex codes (i.e. codes realizable by closed convex covers) are distinct overlapping classes and then find a non-degeneracy condition on the cover that guarantees that the appropriate code is both open convex and closed convex. Non-degenerate covers are thus *natural* in the neuroscience context, where receptive fields (i.e. the sets U_i) are intrinsically noisy. We propose that convex codes that arise from non-degenerate covers should serve as the standard definition for the convex codes in neuroscience-related context.

In Section 3 we prove the *monotonicity* of open convex codes, where open convexity is preserved for every supra-code of an open convex code within the same simplicial convex and the embedding dimension of such supra-code can increase only by one. In Section 4 we prove that *maximal intersection complete* codes are both open and closed convex[4] and also derive an upper bound for embedding dimension of these codes.

2 Convex codes

We first observe that without sufficiently strong assumptions about the cover $\mathcal{U} = \{U_i\}$, any code can be realized as $\text{code}(\mathcal{U}, X)$.

Lemma 2.1. Every code $\mathcal{C} \subset 2^{[n]}$ can be obtained as $\mathcal{C} = \text{code}(\mathcal{U}, X)$ for a collection of (not necessarily convex) $U_i \subset \mathbb{R}^1$.

Proof. It suffices to consider the case where each $i \in [n]$ appears in some codeword $\sigma \in \mathcal{C}$. For each $\sigma \in \mathcal{C}$, choose points $x_\sigma \in \mathbb{R}^1$ such that $x_\sigma \neq x_\tau$ if $\sigma \neq \tau$. Define $U_i = \{x_\sigma \mid i \in \sigma\}$ and $\mathcal{U} = \{U_i\}_{i\in[n]}$. If $\varnothing \in \mathcal{C}$, then $\mathcal{C} = \text{code}(\mathcal{U}, \mathbb{R}^1)$. Otherwise, $\mathcal{C} = \text{code}(\mathcal{U}, X)$, where $X = \cup_{\sigma\in\mathcal{C}}\{x_\sigma\}$. $\square$

The sets U_i in the above proof are finite subsets of $\mathbb{R}^1$. However, even if one requires that the sets U_i be open and connected, almost all codes still can arise as the code of such cover.

Lemma 2.2. Any code $\mathcal{C} \subset 2^{[n]}$ that contains all singleton codewords, i.e. $\forall i \in [n], \{i\} \in \mathcal{C}$, can be obtained as $\mathcal{C} = \text{code}(\mathcal{U}, X)$ for a collection of open connected subsets $U_i \subset \mathbb{R}^3$.

Proof. Similar to the proof of Lemma 2.1, one can place disjoint open balls $B_\sigma \subset \mathbb{R}^3$ for each $\sigma \in \mathcal{C}$ and define $U_i = (\cup_{i\in\sigma} B_\sigma) \cup T_i$, where each $T_i \subset \mathbb{R}^3$ is a collection of open "narrow tubes" that connect

[4] Open convexity of maximal intersection complete codes was first hypothesized in [2].

all the balls B_σ with $\sigma \ni i$. Because these sets are embedded in $\mathbb{R}^3$, the "tubes" T_i can always be arranged so that for each $i \neq j$ the intersections $T_i \cap T_j$ are contained in the union of balls B_σ. By construction, these U_i are connected and open and $\mathcal{C} = \text{code}\,(\mathcal{U}, (\cup_{i=1}^{n} U_i) \cup B_\varnothing)$. □

The condition of having all singleton words can *not* be relaxed without any further assumptions. For example, it can be easily shown that the code $\mathcal{C} = \{\varnothing, 1, 2, 13, 23\}$, previously described in [3,6] can not be realized as a code of a cover by open connected sets[5].

We consider two kinds of *convex codes*: (i) open and (ii) closed.

Definition 2.3. We say that a code $\mathcal{C} \subseteq 2^{[n]}$ is open (closed) convex if there is a collection $\mathcal{U}$ of open (closed) convex subsets of $\mathbb{R}^d$ for some $d \geq 1$ and an open (closed) subset $X \subseteq \mathbb{R}^d$, such that $\text{code}(\mathcal{U}, X) = \mathcal{C}$.

2.1 Local obstructions to convexity

Not every code arises from a closed convex or open convex cover. For example, the code $\mathcal{C} = \{\varnothing, 1, 2, 13, 23\}$ above can not be an open (or closed) convex code. The failure of this code to be convex is "local" in that it is missing the codeword 3, and adding new codewords which do not include $i = 3$ would not make this code convex.

Definition 2.4. For any $\sigma \subset [n]$ the *link* of $\mathcal{C}$ at σ is the code $\text{link}_\sigma\,\mathcal{C} \subseteq 2^{[n]\setminus\sigma} \subset 2^{[n]}$ on the same set of neurons, defined as

$$\text{link}_\sigma\,\mathcal{C} \stackrel{\text{def}}{=} \{\tau \,|\, \tau \cup \sigma \in \mathcal{C} \text{ and } \tau \cap \sigma = \varnothing\}\,.$$

Note that the link of a code is typically *not* a simplicial complex, but the simplicial complex of a link is the usual $\text{link}_\sigma\,\Delta = \{\nu \in \Delta \,|\, \nu \cup \sigma \in \Delta, \text{ and } \nu \cap \sigma = \varnothing\}$ of the appropriate simplicial complex:[6]

$$\Delta\,(\text{link}_\sigma\,\mathcal{C}) = \text{link}_\sigma\,\Delta\,(\mathcal{C})\,.$$

Moreover, it is easy to see that if $\mathcal{C} = \text{code}(\mathcal{U}, X)$, then for every non-empty $\sigma \in \Delta(\mathcal{C})$

$$\text{link}_\sigma\,\mathcal{C} = \text{code}\left(\{U_j \cap U_\sigma\}_{j\in[n]\setminus\sigma}, U_\sigma\right), \qquad \text{where } U_\sigma = \bigcap_{i\in\sigma} U_i.$$

Since any intersection of convex sets is convex, we thus observe

Lemma 2.5. If $\mathcal{C}$ is an open (or closed) convex code, then for any $\sigma \in \Delta\,(\mathcal{C})$, $\text{link}_\sigma\,\mathcal{C}$ is also an open (or closed) convex code.

Note that

$$\sigma \in \Delta(\mathcal{C}) \setminus \mathcal{C} \iff \varnothing \notin \text{link}_\sigma\,\mathcal{C}. \tag{3}$$

We call the faces of $\Delta(\mathcal{C})$, that are "missing" from the code, *simplicial violators* of $\mathcal{C}$. If a code $\mathcal{C}$ is convex, and σ is a simplicial violator, then the convex code $\text{link}_\sigma\,\mathcal{C} = \text{code}(\{V_i\}, U_\sigma)$ is special in that the convex sets $V_j = U_j \cap U_\sigma$ cover another convex set U_σ. A simple corollary of Lemma 2.5 is the following observation (which first appeared in [6]) that provides a class of 'local' obstructions to being an open (or closed) convex code.

Proposition 2.6. Let $\sigma \neq \varnothing$ be a simplicial violator of a code $\mathcal{C}$. If $\text{link}_\sigma\,\Delta(\mathcal{C})$ is not a contractible simplicial complex, then $\mathcal{C}$ is not an open (or closed) convex code.

[5] Indeed, assuming converse, it follows that $U_3 = (U_1 \cap U_3) \cup (U_2 \cap U_3)$ and, since $U_1 \cap U_2 = \varnothing$, U_3 is a union of two disjoint open sets, which yields a contradiction.

[6] This is because both $\text{link}_\sigma\mathcal{C}$ and $\text{link}_\sigma\Delta$ have the same set of maximal elements.

Proof. Assume converse, i.e. $\mathcal{C}$ is open (or closed) convex and σ satisfies (3). Then the sets $U_j \cap U_\sigma$ cover a convex and open (or closed) set U_σ, and thus by the nerve lemma the simplicial complex

$$\text{nerve}\left(\{U_j \cap U_\sigma\}_{j\in[n]\setminus\sigma}\right) = \Delta(\text{link}_\sigma \mathcal{C}) = \text{link}_\sigma \Delta(\mathcal{C})$$

is contractible. □

Example 2.7. Let $\mathcal{C} = \{\varnothing, 1, 2, 3, 4, 123, 124\}$. Then $\sigma = 12$ is a simplicial violator of $\mathcal{C}$ and $\text{link}_\sigma \mathcal{C} = \{3, 4\}$. Since $\Delta(\text{link}_\sigma \mathcal{C})$ is not contractible, the code $\mathcal{C}$ is not the code of an open (or closed) convex cover. This is perhaps the minimal example of a non-convex code that can be still realized by an open connected cover[7].

Note that if the condition that all sets are open, or alternatively all sets are closed, is dropped, then (at the time of this writing) there are no known obstructions for a code to arise as a code of a convex cover. For instance, if one set is allowed to be of the "wrong kind", the code $\mathcal{C} = \{\varnothing, 1, 2, 13, 23\}$ above can be realized by intervals on a line, such as $U_1 = (0, 2)$, $U_2 = [2, 4]$, $U_3 = [1, 3]$. For this reason, we only consider either open or closed convex codes.

2.2 Do truly "non-local" obstructions via nerve lemma exist?

The "local" obstructions to convexity in Proposition 2.6 equally apply to any open (or closed) good cover, i.e. a cover where each non-empty intersection $U_\sigma = \cap_{i\in\sigma} U_i$ is contractible. Since this property stems from applying the nerve lemma to the cover of U_σ by the other contractible sets, it is natural to define a more general "non-local" obstruction to convexity that also stems from the nerve lemma.

Definition 2.8. We say that a non-empty subset $\sigma \subseteq [n]$ *covers* a code $\mathcal{C} \subseteq 2^{[n]}$ if for every $\tau \in \mathcal{C}$, $\tau \cap \sigma \neq \varnothing$.

Note that any code covered by at least one non-empty set σ does not have the empty set. Moreover, σ covers $\mathcal{C} = \text{code}\left(\{U_i\}_{i\in[n]}, \bigcup_{i\in[n]} U_i\right)$ if and only if $\bigcup_{i\in[n]} U_i = \bigcup_{j\in\sigma} U_j$.

Lemma 2.9. If there exist two non-empty subsets $\sigma_1, \sigma_2 \subseteq [n]$, that both cover the code $\mathcal{C} \subseteq 2^{[n]}$, but the codes $\mathcal{C} \cap \sigma_a \stackrel{\text{def}}{=} \{\tau \cap \sigma_a | \tau \in \mathcal{C}\} \subseteq 2^{\sigma_a}$ have simplicial complexes $\Delta(\mathcal{C} \cap \sigma_a)$ that are *not* homotopy equivalent, then $\mathcal{C}$ is not a code of a good cover by open (or closed) sets.

Proof. If such good cover existed, then the condition that each of the non-empty subsets σ_a covers the code $\mathcal{C}$ implies that that $\cup_{i\in[n]} U_i = \cup_{j\in\sigma_a} U_j$ for each $a \in \{1, 2\}$. Thus, by the nerve lemma, $\Delta(\mathcal{C})$ has the same homotopy type as each of the complexes $\Delta(\mathcal{C} \cap \sigma_a)$. This yields a contradiction. □

The above obstruction to convexity can be thought as "non-local" because it is conditioned on the homotopy type of a subset that covers the entire code. While there is a plethora of combinatorial codes with these "non-local" obstructions, we found that every such code that we've considered[8] inevitably possesses a local obstruction for convexity. Perhaps the smallest such example is the code $\mathcal{C} = \{23, 14, 123\}$ that meets the conditions of Lemma 2.9 with $\sigma_1 = \{12\}$, and $\sigma_2 = \{34\}$, but also has a local obstruction for the simplicial violator $\sigma = \{1\}$. The exact reason for the significant difficulty of finding a "truly non-local" obstruction to having a good open (or closed) cover is still unclear. It is likely that any code $\mathcal{C} \subset 2^{[n]}$ that has a "non-local" obstruction (i.e. the conditions of Lemma 2.9 are met) must also have a "local" obstruction, i.e. a simplicial violator $\sigma \in \Delta(\mathcal{C}) \setminus \mathcal{C}$ such that $\Delta(\text{link}_\sigma \mathcal{C})$ is not a contractible simplicial complex.

[7]In fact, all the non-convex codes on three neurons (these were classified in [3]) can not be realized by open (or closed) connected sets. This is because the only obstruction to convexity is the "disconnection" of one set, similar to the case of the code $\mathcal{C} = \{\varnothing, 1, 2, 13, 23\}$.

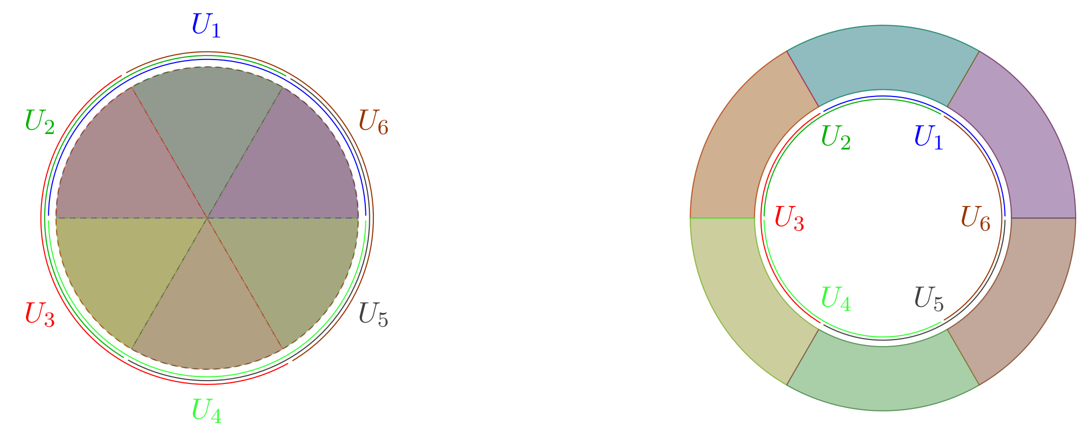
[8]This included computer-assisted search among random codes.

2.3 The difference between open and closed convex codes

The homotopy-type obstructions via the nerve lemma are obstructions to being a code of a good cover (as opposed to convex sets) and equally apply to both open and closed versions of the Definition 2.3. However, it turns out that the open and the closed convex codes are in fact distinct classes of codes. Perhaps a minimal example of an open convex code that is not closed convex is the code

$$\mathcal{C} = \{123, 126, 156, 456, 345, 234, 12, 16, 56, 45, 34, 23, \emptyset\} \subseteq 2^{[6]}. \quad (4)$$

This code is realizable by open convex cover (Figure 2.1a) and also by an open or closed good cover (Figure 2.1b).



(a) An open convex realization of $\mathcal{C}$

(b) A closed good cover realization of $\mathcal{C}$

Figure 2.1: Two different realizations of the code $\mathcal{C}$ in (4). In both realizations, each set U_i is covered by the others, and is indicated by the colored arcs external to the sets; the colors of regions are combinations of the colors of the constituent sets. For example, in (a), U_1 is the open upper half-disk, while in (b) U_1 is the top right closed annular section.

Lemma 2.10. The code (4) is not closed convex.

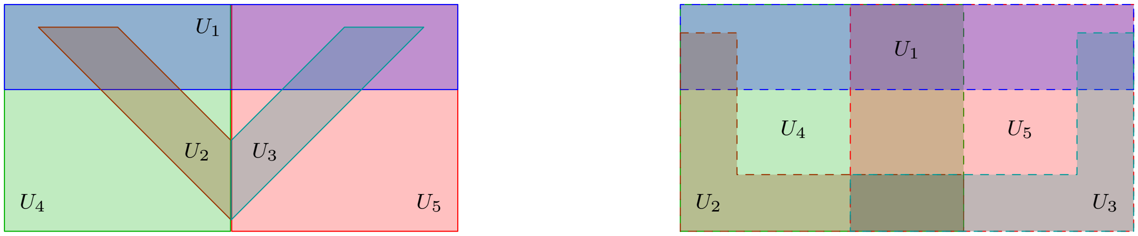
The proof is given in the Appendix, section 5.1. The following example was originally considered in [11], where it was proved that it is not open convex, but possesses a realization by a good open cover, thus does not have any "local obstructions" to convexity stemming from Proposition 2.6. Consider a code

$$\mathcal{C} = \{2345, 124, 135, 145, 14, 15, 24, 35, 45, 4, 5\} \subseteq 2^{[5]}. \quad (5)$$

It was shown in [11], that this code can not be realized by open convex cover. However, it can be obtained as a code of a closed convex cover (Figure 2.2a) and also an open good cover (Figure 2.2b).

The examples in (4) and (5) show that open convex and closed convex are distinct classes of codes. Moreover, they illustrate that one cannot generally "convert" an open convex code into a closed convex code or vice versa by simply taking closures of sets of an open cover or interiors of sets in a closed cover. Nevertheless, it is rather intuitive that a "sufficiently non-degenerate" cover should have the same code regardless of taking interiors or closures.

Definition 2.11. We say a finite cover $\mathcal{U}$ is non-degenerate if the following two conditions hold:



(a) A closed convex realization of $\mathcal{C}$

(b) An open good cover realization of $\mathcal{C}$

Figure 2.2: Two different realizations of the code $\mathcal{C} = \{2345, 124, 135, 145, 14, 15, 24, 35, 45, 4, 5\}$.

(i) For all $\sigma \in \text{code}(\mathcal{U}, \mathbb{R}^d)$, the atoms $A^{\mathcal{U}}_\sigma$ are *top-dimensional*, i.e. any non-empty intersection with an open set $B \subseteq \mathbb{R}^d$ has non-empty interior:

$$B \text{ is open and } A^{\mathcal{U}}_\sigma \cap B \neq \varnothing \implies \text{int}(A^{\mathcal{U}}_\sigma \cap B) \neq \varnothing.$$

(ii) For all $\sigma \subseteq [n]$, $\bigcap_{i\in\sigma} \partial U_i \subseteq \partial\left(\bigcap_{i\in\sigma} U_i\right)$.

Note that if the sets U_i are closed and convex, then condition (i) implies condition (ii) (see Lemma 5.2 in section 5.2). Whereas if the sets U_i are *open* and convex, then condition (i) does not imply condition (ii). For example, the atoms of the cover by the open convex sets $U_1 = \{(x, y) \in \mathbb{R}^2 \,|\, y > x^2\}$ and $U_2 = \{(x, y) \in \mathbb{R}^2 \,|\, y < -x^2\}$ do not satisfy condition (ii).

For an open cover $\mathcal{U} = \{U_i\}$, we denote by $\text{cl}(\mathcal{U})$ the cover by the closures $V_i = \text{cl}(U_i)$. Similarly, for a closed cover $\mathcal{U} = \{U_i\}$ we denote by $\text{int}(\mathcal{U})$ the cover by the interiors $V_i = \text{int}(U_i)$. Recall that if a set is convex, then both its closure and its interior are convex.

Theorem 2.12. Assume that $\mathcal{U} = \{U_i\}$ is a convex and non-degenerate cover, then

$$\begin{aligned} U_i \text{ are open} \quad &\implies \quad \text{code}(\mathcal{U}, \mathbb{R}^d) = \text{code}\left(\text{cl}\left(\mathcal{U}\right), \mathbb{R}^d\right); \\ U_i \text{ are closed} \quad &\implies \quad \text{code}(\mathcal{U}, \mathbb{R}^d) = \text{code}\left(\text{int}\left(\mathcal{U}\right), \mathbb{R}^d\right). \end{aligned}$$

The proof is given in section 5.2. This theorem guarantees that if an open convex code is realizable by a non-degenerate cover, then it is also closed convex; similarly if a closed convex code is realizable by a non-degenerate cover, then it is also open convex.

Non-degenerate covers are thus *natural* in the neuroscience context, where receptive fields (i.e. the sets U_i) are intrinsically noisy and should not change their code after taking closure or interior. This suggests that convex codes that arise from non-degenerate covers should serve as the "standard definition" for the "convex codes" in neuroscience-related contexts. Note that the existence of a non-degenerate convex cover realization is *extrinsic* in that it is not defined in terms of the combinatorics of the code alone. A combinatorial description of such codes is unknown at the time of this writing.

3 Monotonicity of open convex codes

The set of all codes $\mathcal{C} \subseteq 2^{[n]}$ with a prescribed simplicial complex $K = \Delta(C)$ forms a poset. It is easy to see that if $\mathcal{C}$ is a convex code than its sub-code can be non-convex. For example any non-convex code is a sub-code of of its simplicial complex, and every simplicial complex is both an open and closed convex code (this follows from Theorem 4.4 in Section 4). It turns out that open convexity is a monotone increasing property.

Theorem 3.1. Assume that a code $\mathcal{C} \subset 2^{[n]}$ is open convex with embedding dimension $d = \operatorname{embdim} \mathcal{C}$. Then every code $\mathcal{D}$ that satisfies $\mathcal{C} \subsetneq \mathcal{D} \subseteq \Delta(\mathcal{C})$ is also open convex with embedding dimension $\operatorname{embdim} \mathcal{D} \leq \operatorname{embdim} \mathcal{C} + 1$.

Note that the above bound on the embedding dimension is sharp. For example, the open convex code $\mathcal{C} = \{123, 12, 1\}$ has embedding dimension $\operatorname{embdim} \mathcal{C} = 1$, but its simplicial complex $\mathcal{D} = \Delta(\mathcal{C})$ has embedding dimension $\operatorname{embdim} \mathcal{D} = 2$. To prove this theorem we shall use the following lemma. Let $M(\mathcal{C})$ denote the facets of the simplicial complex $\Delta(\mathcal{C})$.

Lemma 3.2. Let $\mathcal{U} = \{U_i\}$ be an open convex cover in $\mathbb{R}^d$, $d \geq 2$, with $\operatorname{code}(\mathcal{U}, X) = \mathcal{C}$. Assume that there exists an open Euclidean ball $B \subset \mathbb{R}^d$ such that $\operatorname{code}(\{B \cap U_i\}, B \cap X) = \mathcal{C}$, and for every maximal set $\alpha \in M(\mathcal{C})$, its atom has non-empty intersection with the $(d-1)$-sphere: $\partial B \cap A_\alpha^{\mathcal{U}} \neq \varnothing$. Then for every $\mathcal{D}$ such that $\mathcal{C} \subsetneq \mathcal{D} \subseteq \Delta(\mathcal{D})$, there exists an open convex cover $\mathcal{V} = \{V_i\}$ with $V_i \subseteq U_i$, such that $\mathcal{D} = \operatorname{code}(\mathcal{V}, B \cap X)$. Moreover, if the cover $\mathcal{U}$ is non-degenerate, then the cover $\mathcal{V}$ can also be chosen to be non-degenerate.

The proof of this lemma is given in Section 5.3. Intuitively, the reason why this lemma holds is that one can "chip away" small chords from the ball B inside the atoms $A_\alpha^{\mathcal{U}}$ to uncover only the atoms corresponding to the codewords in $\mathcal{D} \setminus \mathcal{C}$.

Proof of Theorem 3.1. Assume that $\mathcal{U}$ is an open convex cover in $\mathbb{R}^d$ with $\operatorname{code}(\mathcal{U}, X) = \mathcal{C}$. Since there are only finitely many codewords, there exists a radius $r > 0$ and an open Euclidean ball $B_r^d \subset \mathbb{R}^d$, of radius r, centered at the origin, that satisfies $\operatorname{code}(\{B_r^d \cap U_i\}, B_r^d \cap X) = \mathcal{C}$. Let $\pi : \mathbb{R}^{d+1} \to \mathbb{R}^d$ be the standard projection. Let $B \overset{\text{def}}{=} B_r^{d+1}$ denote the open ball in $\mathbb{R}^{d+1}$, centered at the origin and of the same radius r. Define $\tilde{U}_i = \pi^{-1}(U_i)$. By construction, $\tilde{\mathcal{U}} = \{\tilde{U}_i\}$ is an open, convex cover, such that each of its atoms has non-empty intersection with the sphere ∂B. Moreover, $\operatorname{code}(\{B \cap \tilde{U}_i\}, B \cap \pi^{-1}(X)) = \mathcal{C}$. Thus the conditions of Lemma 3.2 are satisfied for the cover $\tilde{\mathcal{U}}$, and $\mathcal{D}$ is an open convex code with $\operatorname{embdim} \mathcal{D} \leq \operatorname{embdim} \mathcal{C} + 1$. ☐

Note that the proof of Lemma 3.2 (see Appendix 2 in Section 5.3) breaks down if one assumes that the convex sets U_i are closed. Moreover, it is currently not known if the monotonicity property holds in the setting of the closed convex codes. The differences between the open convex and the closed convex codes (described in the previous section) leave enough room for either possibility.

4 Maximal intersection complete codes are open and closed convex

Here we prove that maximal intersection complete codes are both open convex and closed convex. The open convexity of maximal intersection complete codes was first hypothesized in [2].

Definition 4.1. The *intersection completion of a code* $\mathcal{C}$ is the code that consists of all non-empty intersections of codewords in $\mathcal{C}$:

$$\widehat{\mathcal{C}} = \{\sigma \mid \sigma = \bigcap_{\nu \in \alpha} \nu \text{ for some nonempty } \alpha \in 2^{\mathcal{C}}\}. \tag{6}$$

Note that the intersection completion satisfies $\mathcal{C} \subseteq \widehat{\mathcal{C}} \subseteq \Delta(\mathcal{C})$.

Definition 4.2. Let $\mathcal{C} \subset 2^{[n]}$ be a code, and denote by $M(C) = \{\sigma_1, \sigma_2, \ldots \sigma_k\} \subset \mathcal{C}$ the subcode consisting of all maximal codewords[9] of $\mathcal{C}$. A code $\mathcal{C}$ is said to be

- *intersection complete* if $\widehat{\mathcal{C}} = \mathcal{C}$;

[9] I.e. the facets of $\Delta(\mathcal{C})$.

- *maximal intersection complete* if $\widehat{M(\mathcal{C})} \subseteq \mathcal{C}$.

Note that any simplicial complex (i.e. $\mathcal{C} = \Delta(\mathcal{C})$) is intersection complete and any intersection complete code is maximal intersection complete. Intersection complete codes allow a simple construction of a closed convex realization that we describe in Section 5.4 (see Lemma 5.7). However, in order to prove that maximal intersection complete codes are both open and closed convex, we need the following

Proposition 4.3. Let $\mathcal{M}(\mathcal{C}) = \widehat{M(C)}$ be the intersection completion of $M(C)$. Then there exists an open convex and non-degenerate cover $\mathcal{U}$ in $d = (k-1)$-dimensional space with $\mathcal{M}(\mathcal{C}) = \mathrm{code}(\mathcal{U}, \mathbb{R}^d)$.

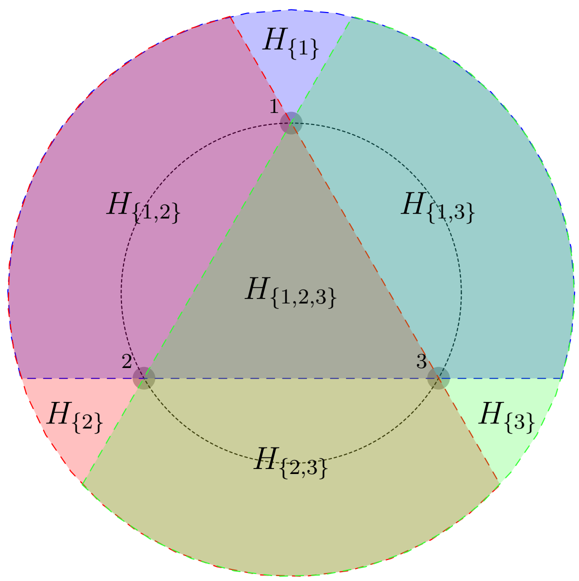


Figure 4.1: The oriented hyperplane arrangement $\{P_a\}$ and its chambers H_ρ.

Proof. If $k = 1$ this statement is trivially true. Assume $k \geq 2$ and consider a regular geometric $(k-1)$-simplex Δ^{k-1} in $\mathbb{R}^{k-1}$ constructed by evenly spacing vertices $[k]$ on the unit sphere $S^{k-2} \subseteq \mathbb{R}^{k-1}$. Construct a collection of hyperplanes $\{P_a\}_{a=1}^k$ in $\mathbb{R}^{k-1}$ by taking P_a to be the plane through the facet of $\partial\Delta^{k-1}$ which does not contain vertex a. Denote by H_a^+ the *closed* half-space containing the vertex a bounded by P_a and by H_a^- the complementary *open* half-space. Observe that this arrangement splits $\mathbb{R}^{k-1}$ into $2^k - 1$ disjoint, non-empty, convex chambers

$$H_\rho = \bigcap_{a \in \rho} H_a^+ \cap \bigcap_{b \notin \rho} H_b^-,$$

indexed by all non-empty[10] subsets $\rho \subseteq [k]$.

For every $i \in [n]$ consider $\rho(i) \stackrel{\text{def}}{=} \{a \in [k] \,|\, \sigma_a \ni i\} \subset [k]$, i.e. the collection of indices of maximal faces σ_a that contain i, and construct a collection of convex open sets $\mathcal{U} = \{U_i\}$

$$U_i \stackrel{\text{def}}{=} \coprod_{\rho \subseteq \rho(i)} H_\rho. \tag{7}$$

[10]The empty set is not included because under this definition, $H_\varnothing = \varnothing$.

To show that the sets U_i are convex and open, observe that the above construction implies that we have the disjoint unions

$$\mathbb{R}^{k-1} = \coprod_{\rho \neq \varnothing} H_\rho \quad \text{and} \quad H_b^+ = \coprod_{\rho \ni b} H_\rho,$$

thus

$$\mathbb{R}^{k-1} \setminus U_i = \left(\coprod_{\rho \neq \varnothing} H_\rho \right) \setminus \left(\coprod_{\rho \subseteq \rho(i)} H_\rho \right) = \coprod_{\rho \not\subseteq \rho(i)} H_\rho = \bigcup_{b \notin \rho(i)} \left(\coprod_{\rho \ni b} H_\rho \right) = \bigcup_{b \notin \rho(i)} H_b^+.$$

Therefore, by de Morgan's Law,

$$U_i = \mathbb{R}^{k-1} \setminus \left(\bigcup_{b \notin \rho(i)} H_b^+ \right) = \bigcap_{b \notin \rho(i)} H_b^-. \tag{8}$$

This is an intersection of open convex sets, and therefore open and convex. Note that if $\rho(i) = [k]$, this is an intersection over an empty index, and we interpret this set as all of $\mathbb{R}^{k-1}$.

To show that $\text{code}(\mathcal{U}, \mathbb{R}^{k-1}) = \mathcal{M}(\mathcal{C})$, observe that because the chambers of the hyperplane arrangement satisfy $H_\rho \cap H_\nu \neq \varnothing$ iff $\rho = \nu$, the atoms of the cover $\{U_i\}$ take the form

$$A_\sigma^{\mathcal{U}} = \bigcap_{i \in \sigma} U_i \setminus \left(\bigcup_{j \notin \sigma} U_j \right) = \left(\bigcup_{\rho \in \cap_{i \in \sigma} R_i} H_\rho \right) \setminus \left(\bigcup_{\nu \in \cup_{j \notin \sigma} R_j} H_\nu \right),$$

where each $R_i \stackrel{\text{def}}{=} \{\rho \subseteq \rho(i)\} \subseteq 2^{[k]} \setminus \varnothing$ is the collection of the non-empty subsets of $\rho(i)$, and therefore

$$\text{code}(\{U_i\}, \mathbb{R}^{k-1}) = \text{code}(\{R_i\}, 2^{[k]} \setminus \varnothing). \tag{9}$$

Now, observe that

$$\rho \in \bigcap_{i \in \sigma} R_i \iff \forall i \in \sigma,\, \rho \subseteq \rho(i) \iff \forall i \in \sigma,\, \forall a \in \rho,\, i \in \sigma_a \iff \sigma \subseteq \bigcap_{a \in \rho} \sigma_a,$$

and also that,

$$\rho \notin \bigcup_{j \notin \sigma} R_j \iff \forall j \notin \sigma, \rho \not\subseteq \rho(i) \iff \forall j \notin \sigma, \exists\, a \in \rho \text{ such that } j \notin \sigma_a \iff \sigma \supseteq \bigcap_{a \in \rho} \sigma_a$$

Therefore, $\rho \in \bigcap_{i \in \sigma} R_i \setminus \left(\bigcup_{j \notin \sigma} R_j \right)$ if and only if $\sigma = \bigcap_{a \in \rho} \sigma_a$ and thus

$$\mathcal{M}(\mathcal{C}) = \text{code}(\{R_i\}, 2^{[k]} \setminus \varnothing) = \text{code}(\{U_i\}, \mathbb{R}^{k-1}).$$

Lastly, we show the cover $\mathcal{U}$ is non-degenerate. Each atom of $\mathcal{U}$ is the union of H_ρ, thus they are all top-dimensional. We must also show that for any $\sigma \subset [n]$, $\bigcap_{i \in \sigma} \partial U_i \subseteq \partial \left(\bigcap_{i \in \sigma} U_i \right)$. First, we show that for any non-empty $\tau \subseteq [k]$,

$$\bigcap_{a \in \tau} \text{cl}(H_a^-) = \text{cl} \left(\bigcap_{a \in \tau} H_a^- \right). \tag{10}$$

Indeed, the inclusion $\bigcap_{a \in \tau} \text{cl}(V_a) \supseteq \text{cl}(\bigcap_{a \in \tau} V_a)$ holds for any collection of sets V_a (see Lemma 5.1 in the appendix). To prove the opposite direction, let's embed $\mathbb{R}^{k-1}$ as the subspace $\mathbb{R}^{k-1} \cong \{x = (x_1, \ldots, x_k) \in \mathbb{R}^k \mid \sum_{a=1}^k x_a = 1\}$ and parameterize $H_a^- = \{x \in \mathbb{R}^k \mid \sum_{b=1}^k x_b = 1,\, x_a < 0\}$. Let

$y \in \bigcap_{a\in\tau} \mathrm{cl}(H_a^-)$, then for each $a \in \tau$, $y_a \leq 0$. If $\tau = [k]$, both sides of (10) are empty, thus we can assume $\tau \neq [k]$, choose some $b \notin \tau$ and define an element $y^\varepsilon \in \mathbb{R}^{k-1} \subset \mathbb{R}^k$ as

$$y_a^\varepsilon = \begin{cases} y_a - \varepsilon & \text{if } a \in \tau, \\ y_a + \varepsilon|\tau| & \text{if } a = b, \\ y_a & \text{otherwise,} \end{cases}$$

where $|\tau| > 0$ is the number of elements in τ. By construction $y^\varepsilon \in \bigcap_{a\in\tau} H_a^-$, and $\lim_{\varepsilon\to 0} y^\varepsilon = y$. Thus $y \in \mathrm{cl}(\bigcap_{a\in\tau} H_a^-) \supseteq \bigcap_{a\in\tau} \mathrm{cl}(H_a^-)$ and the equality (10) holds.

We can now combine the equality (8) with (10) to obtain

$$\mathrm{cl}(\bigcap_{i\in\sigma} U_i) = \mathrm{cl}(\bigcap_{i\in\sigma} \bigcap_{a\notin\rho(i)} H_a^-) = \bigcap_{i\in\sigma} \bigcap_{a\notin\rho(i)} \mathrm{cl}(H_a^-) = \bigcap_{i\in\sigma} \mathrm{cl}(\bigcap_{a\notin\rho(i)} H_a^-) = \bigcap_{i\in\sigma} \mathrm{cl}(U_i). \tag{11}$$

Therefore, since U_i are open we obtain

$$\begin{aligned} \bigcap_{i\in\sigma} \partial U_i &= \bigcap_{i\in\sigma} (\mathrm{cl}(U_i) \setminus U_i) \subseteq \bigcap_{i\in\sigma} \left(\mathrm{cl}(U_i) \setminus \bigcap_{i\in\sigma} U_i \right) = \left(\bigcap_{i\in\sigma} \mathrm{cl}(U_i) \right) \setminus \bigcap_{i\in\sigma} U_i \\ &= \mathrm{cl}(\bigcap_{i\in\sigma} U_i) \setminus \bigcap_{i\in\sigma} U_i = \partial(\bigcap_{i\in\sigma} U_i). \end{aligned}$$

Therefore the cover $\mathcal{U}$ is non-degenerate. □

As a corollary we obtain the following result:

Theorem 4.4. Suppose $\mathcal{C} \subset 2^{[n]}$ is a maximum intersection complete code. Then $\mathcal{C}$ is both open convex and closed convex with the embedding dimension $d \leq \max\{2, (k-1)\}$, where k is the number of facets of the complex $\Delta(\mathcal{C})$.

Proof. Note that the case of $k = 1$, i.e. $M(\mathcal{C}) = \{[n]\}$, was proved in [2]. We first consider the case when the number of maximal codewords is $k \geq 3$ and begin by constructing convex regions $\{H_\rho\}_{\rho\in 2^{[k]}\setminus\varnothing}$ and the open convex cover $\{U_i\}_{i=1}^n$ as in the proof of Proposition 4.3 (see Figure 4.1). In this cover, every atom that corresponds to a maximal codeword is unbounded, therefore we can apply Lemma 3.2 using the open ball of radius 2 centered at the origin. This yields an open convex and non-degenerate cover, thus by Theorem 2.12 the code $\mathcal{C}$ is both open convex and closed convex.

If $1 \leq k < 3$, we formally append $3 - k$ empty maximal codewords $\{\gamma_j\}_{j=1}^{3-k}$ to $M(\mathcal{C})$ and apply the same construction. Because the γ_i are empty, they serve only to "lift" the construction to $\mathbb{R}^2$. The sets U_i are contained entirely in $\bigcap_{j=1}^{3-k} H_{\gamma_j}^-$, but the γ_i have no other effect on their composition. This allows to carry out the rest of the above proof in the same way as in the case of $k \geq 3$. □

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5 Appendix: supporting proofs.

5.1 Proof of Lemma 2.10

Proof. Consider the code $\mathcal{C}$ in eq. (4) and assume that there exists a closed convex cover $\mathcal{U} = \{U_i\}$ in $\mathbb{R}^d$, with $\mathrm{code}(\mathcal{U}, \mathbb{R}^d) = \mathcal{C}$. Without loss of generality, we can assume that the U_i are compact[11]. Let A_σ denote the atom associated to a σ. Because U_i are compact and convex one can pick points x_{123}, x_{345}, and x_{156} in the closed sets A_{123}, A_{345} and A_{156} respectively so that for every $a \in A_{123}$, its distance to the closed line segment $M = \overline{x_{345}x_{156}}$ satisfies[12] $\mathrm{dist}(a, M) \geq \mathrm{dist}(x_{123}, M) \neq 0$, i.e. x_{123} minimizes the distance to the line segment M. Moreover, the points $x_{123}, x_{156}, x_{345}$ can not be collinear. For the rest of this proof we will consider only the convex hull of these three points (Figure 5.1).

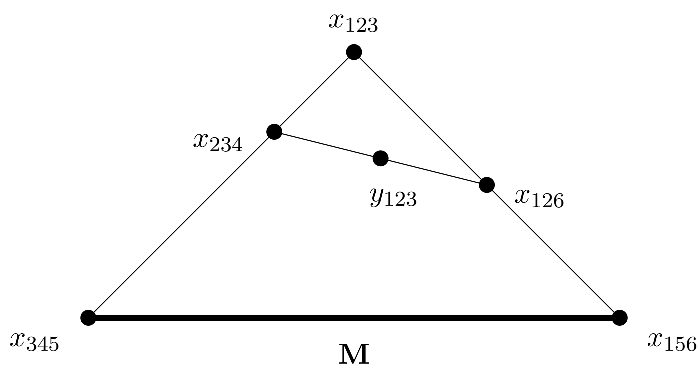

Figure 5.1

Consider the closed line segment $L = \overline{x_{123}x_{156}}$. Because U_1 is convex, $L \subset U_1$, therefore the code (4) of the cover imposes that

$$L \subset A_{123} \sqcup A_{12} \sqcup A_{126} \sqcup A_{16} \sqcup A_{156}.$$

Because each of the atoms above is contained in either U_2 or U_6, $L \subset U_2 \cup U_6$. Since L is connected and the sets $U_2 \cap L$ and $U_6 \cap L$ are closed and non-empty, we conclude that $U_2 \cap U_6 \cap L \subset A_{126}$ must be nonempty, thus there exists a point $x_{126} \in A_{126} \cap L$ that lies in the interior of L. By the same argument, there also exist points

$$x_{234} \in A_{234} \text{ in the interior of } \overline{x_{123}x_{345}} \subset U_3, \text{ covered by } U_2 \text{ and } U_4.$$
$$y_{123} \in A_{123} \text{ in the interior of } \overline{x_{234}x_{126}} \subset U_2, \text{ covered by } U_1 \text{ and } U_3.$$

and these points must lie on the interiors of their respective line segments (Figure 5.1).

Finally we observe that because the point $y_{123} \in A_{123}$ lies in the interior of a line segment $\overline{x_{234}x_{126}}$, it also lies in the interior of the closed triangle $\triangle(x_{123}, x_{156}, x_{345})$, and thus $d(y_{123}, M) < d(x_{123}, M)$. This yields a contradiction, since we chose $x_{123} \in A_{123}$ to have the minimal distance to the line segment M. $\square$

5.2 Proof of Theorem 2.12

We shall need the following several lemmas. The following lemma is well-known (see e.g. [8], exercises in Chapter 1), nevertheless we give its proof for the sake of completeness.

[11] If U_i are not compact, then one can intersect them with a ball of large enough radius to obtain the same code.

[12] Because, U_5 is convex and contains the endpoints of M, $x_{123} \notin M$. Moreover, since both M and A_{123} are compact, the function $f(a) = \mathrm{dist}(a, M)$ achieves its minimum on A_{123}.

Lemma 5.1. For any finite cover $\mathcal{U} = \{U_i\}_{i=1}^n$ and a subset $\sigma \subseteq [n]$, the following hold:

$$\mathrm{cl}(\bigcup_{i\in\sigma} U_i) = \bigcup_{i\in\sigma} \mathrm{cl}(U_i), \tag{12}$$

$$\mathrm{cl}(\bigcap_{i\in\sigma} U_i) \subseteq \bigcap_{i\in\sigma} \mathrm{cl}(U_i), \tag{13}$$

$$\mathrm{int}(\bigcap_{i\in\sigma} U_i) = \bigcap_{i\in\sigma} \mathrm{int}(U_i), \tag{14}$$

$$\mathrm{int}(\bigcup_{i\in\sigma} U_i) \supseteq \bigcup_{i\in\sigma} \mathrm{int}(U_i). \tag{15}$$

Proof. Observe that since $U_i \subseteq \mathrm{cl}(U_i)$, we have $\bigcup_{i\in\sigma} U_i \subseteq \bigcup_{i\in\sigma} \mathrm{cl}(U_i)$ and thus

$$\mathrm{cl}(\bigcup_{i\in\sigma} U_i) \subseteq \mathrm{cl}\left(\bigcup_{i\in\sigma} \mathrm{cl}(U_i)\right) = \bigcup_{i\in\sigma} \mathrm{cl}(U_i). \tag{16}$$

Similarly, we find the inclusion (13). Using $U_i \supseteq \mathrm{int}(U_i)$, one also obtains the inclusion (15) and the inclusion

$$\mathrm{int}(\bigcap_{i\in\sigma} U_i) \supseteq \bigcap_{i\in\sigma} \mathrm{int}(U_i). \tag{17}$$

Observe that for any $j \in \sigma$, $\mathrm{cl}(U_j) \subseteq \mathrm{cl}(\bigcup_{i\in\sigma} U_i)$ and $\mathrm{int}(U_j) \supseteq \mathrm{int}(\bigcap_{i\in\sigma} U_i)$, thus we obtain $\bigcup_{i\in\sigma} \mathrm{cl}(U_i) \subseteq \mathrm{cl}(\bigcup_{i\in\sigma} U_i)$ and $\bigcap_{i\in\sigma} \mathrm{int}(U_i) \supseteq \mathrm{int}(\bigcap_{i\in\sigma} U_i)$. These combined with (16) and (17) yields (12) and (14) respectively. $\square$

Lemma 5.2. Assume that every atom of the cover $\mathcal{U} = \{U_i\}$ is full-dimensional, i.e. any non-empty intersection with an open set $B \subseteq \mathbb{R}^d$ has non-empty interior, and the subsets U_i are closed and convex, then for any non-empty $\tau \subseteq [n]$,

$$\bigcap_{i\in\tau} \partial U_i \subseteq \partial\left(\bigcup_{i\in\tau} U_i\right), \tag{18}$$

$$\bigcap_{i\in\tau} \partial U_i \subseteq \partial\left(\bigcap_{i\in\tau} U_i\right). \tag{19}$$

Proof. To show (18) assume the converse, then there exist $x \in \left(\bigcap_{i\in\tau} \partial U_i\right) \bigcap \mathrm{int}\left(\bigcup_{i\in\tau} U_i\right)$. Thus there exists an open ball $B \ni x$, such that $B \subseteq \bigcup_{i\in\tau} U_i$. First, let us show that these assumptions imply that

$$B \cap \bigcap_{i\in\tau} \mathrm{int}(U_i) = \varnothing. \tag{20}$$

Indeed, if there existed a point $y \in B \cap \bigcap_{i\in\tau} \mathrm{int}(U_i)$, then for every $\varepsilon > 0$ such that $z = x + \varepsilon(x-y) \in B$ and every $i \in \tau$, $z \notin U_i$ by convexity of U_i[13]. This implies $B \not\subseteq \bigcup_{i\in\tau} U_i$, a contradiction, thus (20) holds.

Denote by $\rho \supseteq \tau$ the element of code $\left(\{U_i\}, \mathbb{R}^d\right)$ such that $x \in A^{\mathcal{U}}_\rho = \bigcap_{i\in\rho} U_i \setminus \bigcup_{j\notin\rho} U_j$. Because the sets U_j are closed, we can choose the open ball $B \ni x$, that satisfies (20) so that it is disjoint from $\bigcup_{j\notin\rho} U_j$. Therefore, using (14), we obtain

$$\mathrm{int}(B \cap A^{\mathcal{U}}_\rho) = \mathrm{int}(B \cap \bigcap_{i\in\rho} U_i) \subseteq \mathrm{int}(B \cap \bigcap_{i\in\tau} U_i) = B \cap \bigcap_{i\in\tau} \mathrm{int}(U_i) = \varnothing.$$

[13]This is because $y \in \mathrm{int}(U_i)$, $x \in \partial U_i$ and U_i is convex, thus for every $\varepsilon > 0$, $z = x + \varepsilon(x-y) \notin U_i$.

Since $x \in B \cap A_\rho^{\mathcal{U}}$, this contradicts the non-degeneracy of $\mathcal{U}$, and thus finishes the proof of (18).

To prove (19), consider $x \in \bigcap_{i\in\tau} \partial U_i \subseteq \bigcap_{i\in\tau} U_i$. Because of (18), any open neighborhood $O \ni x$ satisfies $O \not\subseteq \bigcup_{i\in\tau} U_i$ and thus $O \not\subseteq \bigcap_{i\in\tau} U_i$. Therefore $x \in \partial\left(\bigcap_{i\in\tau} U_i\right)$. $\square$

Note that if the condition that the sets U_i are convex is violated, then the conclusions of the above lemma need not hold. For example, the sets $U_1 = \{(x,y) \in \mathbb{R}^2 \,|\, y \leq x^2\}$ and $U_2 = \{(x,y) \in \mathbb{R}^2 \,|\, y \geq -x^2\}$ do not satisfy the inclusion (18).

Lemma 5.3. If the cover $\mathcal{U} = \{U_i\}_{i\in[n]}$ is non-degenerate, then for every non-empty subset $\sigma \subseteq [n]$

$$U_i \text{ are closed and convex } \implies \operatorname{int}(\bigcup_{i\in\sigma} U_i) = \bigcup_{i\in\sigma} \operatorname{int}(U_i), \tag{21}$$

$$U_i \text{ are open and convex } \implies \operatorname{cl}(\bigcap_{i\in\sigma} U_i) = \bigcap_{i\in\sigma} \operatorname{cl}(U_i). \tag{22}$$

Proof. First, we show that if the cover $\mathcal{U}$ is non-degenerate and closed convex, then

$$\operatorname{int}(\bigcup_{i\in\sigma} U_i) \subseteq \bigcup_{i\in\sigma} \operatorname{int}(U_i). \tag{23}$$

It suffices to show that if $x \notin \bigcup_{i\in\sigma} \operatorname{int}(U_i)$, then $x \in \partial(\bigcup_{i\in\sigma} U_i) \bigcup \left(\mathbb{R}^d \setminus \bigcup_{i\in\sigma} U_i\right)$. If $x \notin \bigcup_{i\in\sigma} U_i$, then this is true, thus we can assume that the set $\tau \overset{\text{def}}{=} \{i \in \sigma \,|\, x \in U_i\}$ is non-empty, and since $x \notin \bigcup_{i\in\sigma} \operatorname{int}(U_i)$, we conclude that $x \in \bigcap_{i\in\tau} \partial U_i$. Thus by Lemma 5.2 (eq. (18)), $x \in \partial(\bigcup_{i\in\tau} U_i)$. Now observe that $\bigcup_{i\in\sigma} U_i = A \cup B$ with $A \overset{\text{def}}{=} \bigcup_{i\in\tau} U_i$ and $B \overset{\text{def}}{=} \bigcup_{j\in\sigma\setminus\tau} U_j$. Since $x \notin B$, and B is closed, there exists an open neighborhood $O \ni x$ with $O \cap B = \varnothing$. Therefore, using (14) we obtain that

$$O \cap \operatorname{int}(A) = \operatorname{int}(O \cap A) = \operatorname{int}(O \cap (A \cup B)) = O \cap \operatorname{int}\,(A \cup B)\,,$$

and thus we conclude

$$x \in \partial A \cap O = (A \setminus \operatorname{int} A) \cap O = ((A \cup B) \setminus (\operatorname{int}\,(A \cup B) \cap O)) \cap O = \partial\,(A \cup B) \cap O.$$

Thus, $x \in \partial\,(A \cup B) = \partial\left(\bigcup_{i\in\sigma} U_i\right)$, which proves (23). Combined with (15) in Lemma 5.1, this finishes the proof of (21).

To prove (22), taking into account (13), we need to show that $\operatorname{cl}(\bigcap_{i\in\sigma} U_i) \supseteq \bigcap_{i\in\sigma} \operatorname{cl}(U_i)$. Assume converse, then there exists $x \in \bigcap_{i\in\sigma} \operatorname{cl}(U_i)$ and $r > 0$ such that

$$\forall \varepsilon \in (0,r) \text{ the open } \varepsilon\text{-ball } B_\varepsilon(x) \text{ satisfies } B_\varepsilon(x) \cap \bigcap_{i\in\sigma} U_i = \varnothing. \tag{24}$$

Denote $\tau \overset{\text{def}}{=} \{i \in \sigma \,|\, x \in \partial U_i\}$; we can assume that τ is non-empty (otherwise, $x \in \operatorname{cl}\left(\bigcap_{i\in\sigma} U_i\right)$). Using the condition (ii) of Definition 2.11 we conclude $x \in \bigcap_{i\in\tau} \partial U_i \subseteq \partial\left(\bigcap_{i\in\tau} U_i\right)$, thus for every open ε-ball $B_\varepsilon(x)$ centered at x, $B_\varepsilon(x) \cap \bigcap_{i\in\tau} U_i \neq \varnothing$. Because $x \in \bigcap_{j\in\sigma\setminus\tau} U_j$, and U_j are open, for a sufficiently small ε, $B_\varepsilon(x) \subset \bigcap_{j\in\sigma\setminus\tau} U_j$. Thus $B_\varepsilon(x) \cap \bigcap_{i\in\sigma} U_i \neq \varnothing$, which contradicts (24). This finishes the proof of (22). $\square$

Finally, we give the proof of Theorem 2.12

Proof. We need to show that if $\mathcal{U}$ is convex and non-degenerate, then the cover of closures $\operatorname{cl}(\mathcal{U}) \overset{\text{def}}{=} \{\operatorname{cl}(U_i)\}$ and the cover of interiors $\operatorname{int}(\mathcal{U}) \overset{\text{def}}{=} \{\operatorname{int}(U_i)\}$ have the same code as $\mathcal{U}$. First, we show that

$\mathrm{code}(\mathcal{U}) = \mathrm{code}(\mathrm{cl}(\mathcal{U}))$. Let $A_\sigma^{\mathcal{U}}$ denote an atom of $\mathcal{U}$ and $A_\sigma^{\mathrm{cl}(\mathcal{U})}$ denote the corresponding atom of $\mathrm{cl}(\mathcal{U})$. If $A_\sigma^{\mathcal{U}} = \emptyset$, then using (22) and (12) we conclude that

$$\bigcap_{i\in\sigma} U_i \subseteq \bigcup_{j\notin\sigma} U_j \implies \mathrm{cl}(\bigcap_{i\in\sigma} U_i) \subseteq \mathrm{cl}(\bigcup_{j\notin\sigma} U_j) \implies \bigcap_{i\in\sigma} \mathrm{cl}(U_i) \subseteq \bigcup_{j\notin\sigma} \mathrm{cl}(U_j),$$

and thus $A_\sigma^{\mathrm{cl}(\mathcal{U})} = \emptyset$. Therefore, $\mathrm{code}(\mathrm{cl}(\mathcal{U})) \subseteq \mathrm{code}(\mathcal{U})$. On the other hand, using (12) we obtain

$$\begin{aligned} A_\sigma^{\mathrm{cl}(\mathcal{U})} &= \bigcap_{i\in\sigma} \mathrm{cl}(U_i) \setminus \bigcup_{j\notin\sigma} \mathrm{cl}(U_j) = \bigcap_{i\in\sigma} \mathrm{cl}(U_i) \setminus \mathrm{cl}(\bigcup_{j\notin\sigma} U_j) \\ &= \left(\bigcap_{i\in\sigma} \mathrm{cl}(U_i) \setminus \bigcup_{j\notin\sigma} U_j\right) \setminus \left(\mathrm{cl}(\bigcup_{j\notin\sigma} U_j) \setminus \bigcup_{j\notin\sigma} U_j\right) \supseteq A_\sigma^{\mathcal{U}} \setminus \partial\left(\bigcup_{j\notin\sigma} U_j\right). \end{aligned}$$

Thus, if $A_\sigma^{\mathcal{U}}$ is nonempty, since it is top dimensional while $\partial\left(\bigcup_{i\notin\sigma} U_i\right)$ is of non-zero codimension, $A_\sigma^{\mathcal{U}} \not\subseteq \partial\left(\bigcup_{j\notin\sigma} U_j\right)$, implying $A_\sigma^{\mathrm{cl}(\mathcal{U})} \neq \emptyset$, and thus, $\mathrm{code}(\mathcal{U}) = \mathrm{code}(\mathrm{cl}(\mathcal{U}))$.

Next, we show that $\mathrm{code}(\mathrm{int}(\mathcal{U})) = \mathrm{code}(\mathcal{U})$. Let $A_\sigma^{\mathcal{U}}$ be an atom of $\mathcal{U}$ and $A_\sigma^{\mathrm{int}(\mathcal{U})}$ be the corresponding atom of $\mathrm{int}(\mathcal{U})$. If $A_\sigma^{\mathcal{U}} = \emptyset$, then using (14) and (21) we conclude that

$$\bigcap_{i\in\sigma} U_i \subseteq \bigcup_{j\notin\sigma} U_j \implies \mathrm{int}(\bigcap_{i\in\sigma} U_i) \subseteq \mathrm{int}(\bigcup_{j\notin\sigma} U_j) \implies \bigcap_{i\in\sigma} \mathrm{int}(U_i) \subseteq \bigcup_{j\notin\sigma} \mathrm{int}(U_j),$$

which implies $A_\sigma^{\mathrm{int}(\mathcal{U})} = \varnothing$. Therefore, $\mathrm{code}(\mathrm{int}(\mathcal{U})) \subseteq \mathrm{code}(\mathcal{U})$. On the other hand, using (14) we obtain

$$\begin{aligned} A_\sigma^{\mathrm{int}(\mathcal{U})} &= \bigcap_{i\in\sigma} \mathrm{int}(U_i) \setminus \bigcup_{j\notin\sigma} \mathrm{int}(U_j) \supset \mathrm{int}(\bigcap_{i\in\sigma} U_i) \setminus \bigcup_{j\notin\sigma} U_j = \\ &= \left(\bigcap_{i\in\sigma} U_i \setminus \bigcup_{j\notin\sigma} U_j\right) \setminus \partial\left(\bigcap_{i\in\sigma} U_i\right) = A_\sigma^{\mathcal{U}} \setminus \partial\left(\bigcap_{i\in\sigma} U_i\right). \end{aligned}$$

Thus, if $A_\sigma^{\mathcal{U}}$ is nonempty, since it is top dimensional while $\partial\left(\bigcap_{i\in\sigma} A_i\right)$ is of non-zero codimension, $A_\sigma^{\mathrm{int}(\mathcal{U})} \neq \emptyset$. Therefore, $\mathrm{code}(\mathcal{U}) = \mathrm{code}(\mathrm{int}(\mathcal{U}))$. $\square$

5.3 Proof of Lemma 3.2

In order to prove Lemma 3.2 we will need the following two lemmas.

Lemma 5.4. Let $\mathcal{W} = \{W_i\}$ be a collection of sets, $W_i \subseteq X$, and $\mathcal{C} = \mathrm{code}(\mathcal{W}, X)$. Assume that Q is a proper subset of some atom of $\mathcal{W}$, i.e. $\varnothing \neq Q \subsetneq A_\alpha^{\mathcal{W}}$, for a non-empty $\alpha \in \mathcal{C}$. Then for any $\sigma_0 \subsetneq \alpha$, the cover $\mathcal{V} = \{V_i\}$ by the sets

$$V_i = \begin{cases} W_i, & \text{if } i \in \sigma_0, \\ W_i \setminus Q, & \text{if } i \notin \sigma_0 \end{cases} \tag{25}$$

adds the codeword σ_0 to the original code, i.e. $\mathrm{code}(\mathcal{V}, X) = \mathrm{code}(\mathcal{W}, X) \cup \sigma_0$.

Proof. Since $Q \subsetneq A_\alpha^{\mathcal{W}}$, $\mathrm{code}(\{V_i \cap (X \setminus Q)\}, X \setminus Q) = \mathrm{code}(\mathcal{W}, X)$. Moreover, because $\sigma_0 \subset \alpha$, $\mathrm{code}(\{V_i \cap Q\}, Q) = \sigma_0$ by construction. Finally, observe that if $X = Y \sqcup Z$, then $\mathrm{code}(\mathcal{V}, X) = \mathrm{code}(\{V_i \cap Y\}, Y) \cup \mathrm{code}(\{V_i \cap Z\}, Z)$, therefore we obtain

$$\mathrm{code}(\mathcal{V}, X) = \mathrm{code}(\{V_i \cap (X \setminus Q)\}, X \setminus Q) \cup \mathrm{code}(\{V_i \cap Q\}, Q) = \mathrm{code}(\mathcal{W}, X) \cup \sigma_0.$$

$\square$

Recall that $M(\mathcal{C}) \subset \mathcal{C}$ denotes the set of maximal codewords of $\mathcal{C}$. A subset $A \cap B$ of a topological space is called *relatively open in* B if it is an open set in the induced topology of the subset B.

Lemma 5.5. Let $\mathcal{U} = \{U_i\}$ be an open convex cover in $\mathbb{R}^d$, $d \geq 2$, with $\operatorname{code}(\mathcal{U}, X) = \mathcal{C}$. Assume that there exists an open Euclidean ball $B \subset \mathbb{R}^d$ such that $\operatorname{code}(\{B \cap U_i\}, B \cap X) = \mathcal{C}$, and for every maximal set $\alpha \in M(\mathcal{C})$, the set $\partial B \cap \operatorname{cl}\left(\bigcap_{i\in\alpha} U_i\right)$ is non-empty and is relatively open in ∂B. Then for any simplicial violator $\sigma_0 \in \Delta(\mathcal{C}) \setminus \mathcal{C}$, there exists an open convex cover $\mathcal{V} = \{V_i\}$ with $V_i \subseteq U_i$, so that $\operatorname{code}(\mathcal{V}, B \cap X) = \mathcal{C} \cup \sigma_0$, and the cover $\mathcal{V}$ satisfies the same condition above with the same open ball B. Moreover, if the cover $\mathcal{U} = \{U_i\}$ is non-degenerate, then the cover $\mathcal{V}$ can also be chosen to be non-degenerate.

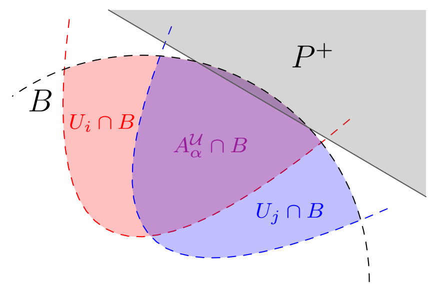

Figure 5.2: The oriented half space P^+ is chosen to intersect the ball B inside $A_\alpha^{\mathcal{U}}$.

Proof. Choose a facet $\alpha \in M(\mathcal{C})$ such that $\alpha \supsetneq \sigma_0$. Because α is a facet of $\Delta(\mathcal{C})$, the atom $A_\alpha^{\mathcal{U}} = \cap_{i\in\alpha(\sigma)} U_i$ is convex open and (by assumption) has a non-empty relatively open intersection with the Euclidean sphere ∂B. This implies that we can always select an oriented and *closed* half-space $P^+ \subset \mathbb{R}^d$ such that $P^+ \cap B \subset A_\alpha^{\mathcal{U}}$, and $\left(A_\alpha^{\mathcal{U}} \cap B\right) \setminus P^+ \neq \varnothing$ has relatively open intersection with the sphere ∂B (see Figure 5.2).

We define two open covers, $\mathcal{W} = \{W_i\}$, with $W_i \stackrel{\text{def}}{=} U_i \cap B$ and $\mathcal{V} = \{V_i\}$ via the equation (25), with $Q = B \cap P^+$. We thus can use Lemma 5.4, and conclude that $\operatorname{code}(\mathcal{V}, X \cap B) = \operatorname{code}(\{B \cap U_i\}, B \cap X) \cup \sigma_0 = \mathcal{C} \cup \sigma_0$. Note that by construction the sets V_i are open and convex, moreover, the cover $\mathcal{V}$ automatically satisfies the same condition on the atoms of facets of $\Delta(\mathcal{C})$.

Finally, If $\mathcal{U}$ is non-degenerate, then is $\mathcal{V}$ also non-degenerate. Indeed, because $A_\alpha^{\mathcal{U}}$ is open, the only two atoms that were changed, $A_\alpha^{\mathcal{V}} = \left(A_\alpha^{\mathcal{U}} \cap B\right) \setminus P^+$ and $A_{\sigma_0}^{\mathcal{V}} = P^+ \cap B$ are also top-dimensional. Moreover, since the only new pieces of boundaries of $V_i \subseteq U_i$ are introduced on the chord $\partial P^+ \cap B$ and on the sphere ∂B, if the condition that for all $\sigma \subseteq [n]$, $\bigcap_{i\in\sigma} \partial U_i \subseteq \partial\left(\bigcap_{i\in\sigma} U_i\right)$ holds then the same condition should hold for the sets V_i. $\square$

A consecutive application of the above lemma to all the codewords in $\mathcal{D} \setminus \mathcal{C}$ for any supra-code $\mathcal{D}$ with the same simplicial complex yields Lemma 3.2.

Proof of Lemma 3.2. Let $\mathcal{U} = \{U_i\}$ be an open convex cover in $\mathbb{R}^d$, $d \geq 2$, with $\operatorname{code}(\mathcal{U}, X) = \mathcal{C}$. Assume that there exists an open Euclidean ball $B \subset \mathbb{R}^d$ such that $\operatorname{code}(\{B \cap U_i\}, B \cap X) = \mathcal{C}$, and for every maximal set $\alpha \in M(\mathcal{C})$, its atom has non-empty intersection with the $(d-1)$-sphere ∂B. Let $\mathcal{C} \subsetneq \mathcal{D} \subseteq \Delta(\mathcal{D})$ and denote $\mathcal{D} \setminus \mathcal{C} = \{\sigma_1, \sigma_2, \ldots, \sigma_l\}$. Let $\sigma_1 \subsetneq \alpha \in M(\mathcal{C})$. Since $A_\alpha^{\mathcal{U}}$ is open, $\partial B \cap \operatorname{cl}\left(A_\alpha^{\mathcal{U}}\right)$ is relatively open in ∂B. We can now apply Lemma 5.5 to the "missing" codeword σ_1, and obtain a new cover $\mathcal{V}^{(1)}$ that again satisfies the condition of Lemma 5.5. Consecutively applying Lemma 5.5 with $\sigma_0 = \sigma_j$, $j = 2, 3, \ldots l$, we obtain covers $\mathcal{V}^{(j)}$, so that the last cover, $\mathcal{V} \stackrel{\text{def}}{=} \mathcal{V}^{(l)}$ is the desired cover of Lemma 3.2. $\square$

5.4 A closed convex realization for an intersection complete code

Here we provide an explicit construction of a closed convex cover of an intersection complete code. Intersection complete codes are maximal intersection complete, and thus Theorem 4.4 ensures that intersection complete codes are both open convex and closed convex. Nevertheless, a different construction below is may be useful for certain applications due to its simplicity.

Definition 5.6. The *potential cover* of the code $\mathcal{C}$, is a collection $\mathcal{V} = \{V_i\}_{i\in[n]}$ of closed convex sets $V_i \subset \mathbb{R}^{|\mathcal{C}|}$, defined as follows. For each non-empty codeword $\sigma \in \mathcal{C}$ let e_σ be a unit vector in $\mathbb{R}^{|\mathcal{C}|}$ so that $\{e_\sigma\}$ is a basis for $\mathbb{R}^{|\mathcal{C}|}$. For each $i \in [n]$, we define V_i as the convex hull

$$V_i \stackrel{\text{def}}{=} \operatorname{conv}\{e_\sigma \mid \sigma \in \mathcal{C},\ \sigma \ni i\}.$$

Since this is a cover by convex closed sets, the code of the potential cover is closed convex. Note however, that this cover is *not* non-degenerate (Definition 2.11), and can not be easily extended to an open convex cover.

Lemma 5.7. Let $\mathcal{V} = \{V_i\}$ denote the potential cover of $\mathcal{C}$, and $X \stackrel{\text{def}}{=} \operatorname{conv}\{e_\sigma \mid \sigma \in \mathcal{C},\ \sigma \neq \varnothing\}$. Then the code of the potential cover of $\mathcal{C}$ is the intersection completion of that code: $\operatorname{code}(\mathcal{V}, X) = \widehat{\mathcal{C}}$.

Proof. Note that because the vectors e_σ are linearly independent,

$$\varnothing \notin \widehat{C} \iff \exists i \in [n], V_i = X \iff X = \bigcup_{i\in[n]} V_i \iff \varnothing \notin \operatorname{code}(\mathcal{V}, X).$$

Moreover,

$$\bigcap_{i\in\sigma} V_i = \operatorname{conv}\{e_\tau | \, \tau \in \mathcal{C},\ \tau \supseteq \sigma\}, \tag{26}$$

in particular, $\operatorname{code}(\mathcal{V}, X) \subseteq \Delta(\mathcal{C})$. To show that $\operatorname{code}(\mathcal{V}, X) \subseteq \widehat{\mathcal{C}}$, assume that a non-empty $\sigma \in \operatorname{code}(\mathcal{V}, X)$, i.e. $A_\sigma^{\mathcal{V}} = \left(\bigcap_{i\in\sigma} V_i\right) \setminus \bigcup_{j\notin\sigma} V_j$ is nonempty. If there exists a vertex $j \in \left(\bigcap_{\sigma\subseteq\tau\in\mathcal{C}} \tau\right) \setminus \sigma$, then by eq. (26), $\bigcap_{i\in\sigma} V_i \subset V_j$, which contradicts $\sigma \in \operatorname{code}(\mathcal{V}, X)$. Hence $\sigma = \bigcap_{\mathcal{C}\ni\tau\supseteq\sigma} \tau \in \widehat{\mathcal{C}}$. Conversely, assume that a nonempty $\sigma \in \widehat{\mathcal{C}}$ and let $\sigma_1, \ldots, \sigma_k \in \mathcal{C}$ be code elements such that $\sigma = \bigcap_{\ell=1}^k \sigma_\ell$. Then the point $\frac{1}{k}\sum_{\ell=1}^k e_{\sigma_\ell} \in \left(\bigcap_{i\in\sigma} V_i\right) \setminus \bigcup_{j\notin\sigma} V_j$. Hence $\sigma \in \operatorname{code}(\mathcal{V}, X)$. ☐

An immediate corollary is that any intersection complete code is a closed convex code.

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