Quantum Entanglement and Quantum Computing ISA 2 Assignment Nov 2020

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I. Problem Statement

Let σ_x, σ_y and σ_z be the Pauli spin matrices.

$$\sigma_x := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \sigma_y := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \sigma_z := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Find
1. $R_{1x}(\alpha) := e^{-i\alpha(\sigma x \otimes I2)}$ and $R_{1y}(\alpha) := e^{-i\alpha(\sigma y \otimes I2)}$ where $\alpha \in \mathbb{R}$ and I_2 denotes 2x2 unit matrix.
2. Consider special case $R_{1x}(\alpha = \pi/2)$ and $R_{1y}(\alpha = \pi/4)$. Calculate $R_{1x}(\alpha = \pi/2)R_{1y}(\alpha = \pi/4)$. Discuss.
3. Confirm the phase implications of the two functions by displaying in the Bloch sphere.

II. Introduction

In mathematical physics and mathematics, the Pauli matrices are a set of three 2 × 2 complex matrices which are Hermitian and unitary. Usually indicated by the Greek letter sigma (σ), they are occasionally denoted by tau (τ) when used in connection with isospin symmetries. They are:

$$\sigma_1 = \sigma_x := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\sigma_2 = \sigma_y := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$\sigma_3 = \sigma_z := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

where $\imath$ is the usual complex number imaginary unit. Let I be a 2 × 2 matrix. We see that the Pauli matrices obey the following properties:

$$\sigma_1^2 = I; \sigma_2^2 = I; \sigma_3^2 = I$$

The trace of a matrix is the sum of the diagonal terms. For all three Pauli spin matrices the trace is zero.
The Pauli matrices when exponentiated, give rise to the rotation operators, which rotate the Bloch vector ρ , about the $\widehat{x}$, $\widehat{y}$ and $\widehat{z}$ axes, by a given angle θ:

$$R_x(\theta) = e^{-\imath \frac{\theta}{2} X}; R_y(\theta) = e^{-\imath \frac{\theta}{2} Y}; R_z(\theta) = e^{-\imath \frac{\theta}{2} Z}$$

Now, if operator A satisfies $A^2 = I$, it can be seen that,

$$e^{i\theta A} = cos(\theta)I + isin(\theta)A$$

As the Pauli matrices satisfy $X^2 = Y^2 = Z^2 = I$, the rotation matrices are expanded as:

$$R_x(\theta) = e^{-i\frac{\theta}{2}X} = cos\tfrac{\theta}{2}I - isin\tfrac{\theta}{2}X$$
$$= \begin{bmatrix} cos(\frac{\theta}{2}) & -isin(\frac{\theta}{2}) \\ -isin(\frac{\theta}{2}) & cos(\frac{\theta}{2}) \end{bmatrix}$$

$$R_y(\theta) = e^{-i\frac{\theta}{2}Y} = cos\tfrac{\theta}{2}I - isin\tfrac{\theta}{2}Y$$
$$= \begin{bmatrix} cos(\frac{\theta}{2}) & -sin(\frac{\theta}{2}) \\ sin(\frac{\theta}{2}) & cos(\frac{\theta}{2}) \end{bmatrix}$$

$$R_z(\theta) = e^{-i\frac{\theta}{2}Z} = cos\tfrac{\theta}{2}I - isin\tfrac{\theta}{2}Z$$
$$= \begin{bmatrix} e^(i\frac{\theta}{2}) & 0 \\ 0 & e^(i\frac{\theta}{2}) \end{bmatrix}$$

The representation of the states can be done using a Bloch Sphere. In quantum mechanics and computing, the Bloch sphere is a geometrical representation of the pure state space of a two-level quantum mechanical system (qubit). It is a unit 2-sphere, with antipodal points corresponding to a pair of mutually orthogonal state vectors. The north and south poles of the Bloch sphere are typically chosen to correspond to the standard basis vectors $\mid 0 >$ and $\mid 1 >$ respectively.

III. Solution

A. Part I

As mentioned in section II, we can express the rotation matrices in exponential form as $R_x(\theta) = e^{-\imath\frac{\theta}{2}X}$. But in the problem statement, the rotational matrix is defined as $R_{1x}(\alpha) = e^{-i\alpha(\sigma x \otimes I2)}$.
Thus, comparing the two equations, we can conclude that α and θ are notations for rotation angle and are related as

$$\alpha = \frac{\theta}{2}$$

The required Kronecker products as follows:

$$\sigma x \otimes I_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\sigma y \otimes I_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix}$$

$$\sigma z \otimes I_2 = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & -1 & 0\\0 & 0 & 0 & -1\end{bmatrix}$$

Using these matrices, we can find the vaues of $R_{1x}(\alpha)$ and $R_{1y}(\alpha)$, substituting the matrices, we get:

$$R_{1x}(\alpha) = \begin{bmatrix}0 & 0 & e^{-i\alpha} & 0\\0 & 0 & 0 & e^{-i\alpha}\\e^{-i\alpha} & 0 & 0 & 0\\0 & e^{-i\alpha} & 0 & 0\end{bmatrix}$$

$$R_{1y}(\alpha) = \begin{bmatrix}0 & 0 & -ie^{-i\alpha} & 0\\0 & 0 & 0 & -ie^{-i\alpha}\\ie^{-i\alpha} & 0 & 0 & 0\\0 & ie^{-i\alpha} & 0 & 0\end{bmatrix}$$

B. Part II

1) Method I: It is given that $R_{1x}(\alpha) := e^{-i\alpha(\sigma x\otimes I_2)}$ and $R_{1y}(\alpha) := e^{-i\alpha(\sigma y\otimes I_2)}$ where $\alpha \in \mathbb{R}$ and I_2 denotes 2x2 unit matrix. Substituting we get:

$$R_{1x}(\alpha = \pi/2) = \begin{bmatrix}0 & 0 & e^{-i\pi/2} & 0\\0 & 0 & 0 & e^{-i\pi/2}\\e^{-i\pi/2} & 0 & 0 & 0\\0 & e^{-i\pi/2} & 0 & 0\end{bmatrix}$$

$$R_{1y}(\alpha = \pi/4) = \begin{bmatrix}0 & 0 & -ie^{-i\pi/4} & 0\\0 & 0 & 0 & -ie^{-i\pi/4}\\ie^{-i\pi/4} & 0 & 0 & 0\\0 & ie^{-i\pi/4} & 0 & 0\end{bmatrix}$$

It is known that $e^{-i\phi} = \cos\phi - i\sin\phi$. Therefore:

$$R_{1x}(\pi/2) = \begin{bmatrix}0 & 0 & -i & 0\\0 & 0 & 0 & -i\\-i & 0 & 0 & 0\\0 & -i & 0 & 0\end{bmatrix}$$

$$R_{1y}(\pi/4) = \begin{bmatrix}0 & 0 & \frac{-1-i}{\sqrt{2}} & 0\\0 & 0 & 0 & \frac{-1-i}{\sqrt{2}}\\\frac{i+1}{\sqrt{2}} & 0 & 0 & 0\\0 & \frac{i+1}{\sqrt{2}} & 0 & 0\end{bmatrix}$$

Multiplying we get:

$$R_{1x}(\pi/2)R_{1y}(\pi/4) = -i(\tfrac{i+1}{\sqrt{2}})\begin{bmatrix}0 & 0 & -i & 0\\0 & 0 & 0 & -i\\-i & 0 & 0 & 0\\0 & -i & 0 & 0\end{bmatrix} *$$

$$\begin{bmatrix}0 & 0 & \frac{-1-i}{\sqrt{2}} & 0\\0 & 0 & 0 & \frac{-1-i}{\sqrt{2}}\\\frac{i+1}{\sqrt{2}} & 0 & 0 & 0\\0 & \frac{i+1}{\sqrt{2}} & 0 & 0\end{bmatrix} = (\tfrac{i-1}{\sqrt{2}})\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & -1 & 0\\0 & 0 & 0 & -1\end{bmatrix}$$

$$R_{1x}(\pi/2)R_{1y}(\pi/4) = e^{-i\frac{\pi}{4}(\sigma z\otimes I_2)}$$

$$R_{1x}(\pi/2)R_{1y}(\pi/4) = R_{1z}(\pi/4)$$

We can see that on multipying $R_{ix}(\pi/2)$ with $R_{iy}(\pi/4)$, the resulting matrix can be represented as $R_{1z}(\pi/4)$. From this we can conclude that the phase of the resulting matrix is in the z direction with a rotation of $\theta = 2\alpha = \pi/2$ for the same reasons mentioned above.

2) Method 2 : As discussed earlier the three Pauli matrices are usually given as 2x2 matrices σ_x, σ_y and σ_z where 'i' is the usual complex number imaginary unit. From complex number theory it is well known that arbitrary complex number $a + ib$ where $a, b \in \mathbb{R}$ can be given as:

$$\begin{bmatrix}a & -b\\b & a\end{bmatrix} \text{ and } i = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$$

We can expand the Pauli matrices to a set of 4x4 equivalent matrices containing only real entries from the set $\{-1, 0, 1\}$.

$$0 = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}; 1 = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}; -1 = \begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}; i = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$$

By substituting each element of the 2x2 Pauli matrices, we get the 4x4 representations as:

$$\sigma_x = \begin{bmatrix}0 & 0 & 1 & 0\\0 & 0 & 0 & 1\\1 & 0 & 0 & 0\\0 & 1 & 0 & 0\end{bmatrix} = a;\ \sigma_y = \begin{bmatrix}0 & 0 & 0 & 1\\0 & 0 & -1 & 0\\0 & -1 & 0 & 0\\1 & 0 & 0 & 0\end{bmatrix} = b$$

$$\sigma_z = \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & -1 & 0\\0 & 0 & 0 & -1\end{bmatrix} = c$$

$$\text{Consider, } \sigma_y \otimes I_2 =, \begin{bmatrix}0 & -i\\i & 0\end{bmatrix} \otimes \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}0 & -i\\i & 0\end{bmatrix} \otimes I = \begin{bmatrix}0I & -iI\\iI & 0I\end{bmatrix}$$

$$\text{Thus, } \sigma_y \otimes I_2 = \begin{bmatrix}0 & 0 & 0 & 1\\0 & 0 & -1 & 0\\0 & -1 & 0 & 0\\1 & 0 & 0 & 0\end{bmatrix} = b$$

Similarly, we get $\sigma_x \otimes I_2$ = a and $\sigma_z \otimes I_2$ = c.
Therefore,

$$R_x(\pi/2)R_y(\pi/4) = \begin{bmatrix}0 & 0 & -i & 0\\0 & 0 & 0 & -i\\-i & 0 & 0 & 0\\0 & -i & 0 & 0\end{bmatrix}\begin{bmatrix}0 & 0 & 0 & \frac{1-i}{\sqrt{2}}\\0 & 0 & \frac{i-1}{\sqrt{2}} & 0\\0 & \frac{i-1}{\sqrt{2}} & 0 & 0\\\frac{1-i}{\sqrt{2}} & 0 & 0 & 0\end{bmatrix}$$

$$= -i(\tfrac{1-i}{\sqrt{2}})\begin{bmatrix}0 & 0 & 1 & 0\\0 & 0 & 0 & 1\\1 & 0 & 0 & 0\\0 & 1 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & 0 & 0 & 1\\0 & 0 & -1 & 0\\0 & -1 & 0 & 0\\1 & 0 & 0 & 0\end{bmatrix}$$

$$= (\tfrac{-1-i}{\sqrt{2}})\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & -1 & 0\\0 & 0 & 0 & -1\end{bmatrix} = (\tfrac{-1-i}{\sqrt{2}})\begin{bmatrix}i & 0\\0 & -i\end{bmatrix}$$

$$= i(\tfrac{-1-i}{\sqrt{2}})\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} = e^{-i\frac{\pi}{4}\sigma_z}$$

It is seen that we can bring down the 4x4 representation we got back to a 2x2 equivalent representation. This will be useful to plot the Bloch sphere representation as seen in further sections.

C. Part III

We can represent the rotational matrices on the IMB Quantum Lab using Python Language. We use the rotational gates R_x and R_y to plot the rotational gates and see the outputs as mentioned in Part II.
For the Rotation Matrix $R_x(\pi)$ we take $R_x(\pi)$ as we need to plot values for $\theta = 2\alpha = \pi$

```
State Vectors:
[0.000000e+00+0.j 6.123234e-17-1.j]
```

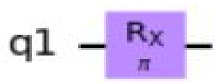

Fig 1. State Vectors and Circuit Simulation for $R_x(\pi)$

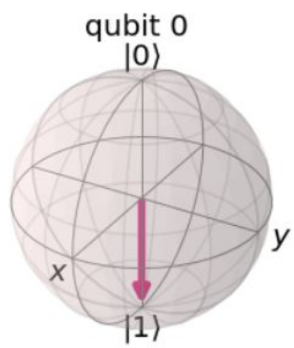

Fig 2. Bloch Sphere for $R_x(\pi)$

For the Rotation Matrix $R_y(\frac{\pi}{2})$ we take $R_y(\frac{\pi}{2})$ as we need to plot values for $\theta = 2\alpha = \frac{\pi}{2}$

```
State Vectors:
[0.70710678+0.j 0.70710678+0.j]
```

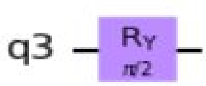

Fig 3. State Vectors and Circuit Simulation for $R_y(\frac{\pi}{2})$

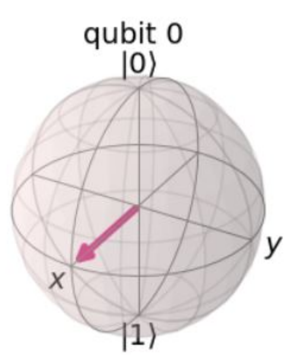

Fig 4. Bloch Sphere for $R_y(\frac{\pi}{2})$

For the Rotation Matrix $R_x(\pi)R_y(\frac{\pi}{2})$

```
State Vectors:
[ 0.70710678+0.00000000e+00j -0.70710678-8.65956056e-17j]
```

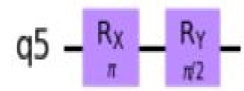

Fig 5. State Vectors and Circuit Simulation for $R_x(\pi)R_y(\frac{\pi}{2})$

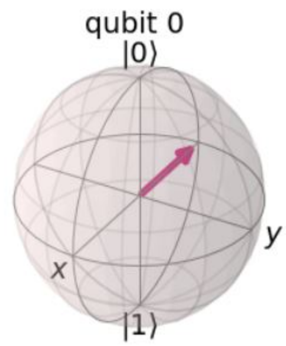

Fig 6. Bloch Sphere for $R_x(\pi)R_y(\frac{\pi}{2})$

IV. Results

Thus, we have delved into the special case of $R_x(\alpha = \pi/2)$ and $R_y(\alpha = \pi/4)$ and the phase implications on multiplying them. We have also visualised the state vectors and rotation on the Bloch sphere using IBM Quantum Lab.

V. Verification and Errors

We can see the results of the theoretical problem are corroborated with the Bloch sphere visualizations. Angles must be taken in radians (not degrees) to obtain correct output.

References

[1] Steeb, Willi-Hans. Hilbert Spaces, Wavelets, Generalised Functions and Modern Quantum Mechanics. Dordrecht: Kluwer, 1998. Print.

[2] HARRIS, J. R. "Forming Groups with 4 × 4 Matrices." The Mathematical Gazette, vol. 94, no. 531, 2010, pp. 426–429. JSTOR, www.jstor.org/stable/25759726. Accessed 3 Dec. 2020.

[3] 'Qiskit: An Open-source Framework for Quantum Computing', 2019, 10.5281/zenodo.2562110,'https://qiskit.org/textbook/ch-states/single-qubit-gates.html'