MOCK TEST ONE - 14/9

Notation:
$\mathbb{N}$ - natural numbers
$\mathbb{Z}$ - integers
$\mathbb{Q}$ - rational numbers
$\mathbb{R}$ - real numbers
$S^1(\mathbb{R}) := \big\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\big\}$
$S^1(\mathbb{Q}) := \big\{(x,y) \in \mathbb{Q}^2 \mid x^2 + y^2 = 1\big\}$

Q1.
Let $\mathbb{X}$ be the set of all functions $f : [0,1] \to \mathbb{R}$
Define a relation $\sim$ on $\mathbb{X}$ as $f \sim g$ if and only if $(f-g)(\alpha) = 0$ for some $\alpha \in [0,1]$
Note : $(f-g)(x) := f(x) - g(x)$
Is this an equivalence relation?

Q2.
Let $\mathbb{Y}$ be the set of all continuous functions $f : [0,1] \to \mathbb{R}$
Define a relation $\sim$ on the set $\mathbb{Y}$ as $f \sim g$ if and only if $f(0.5) = g(0.5)$
Consider the map $\phi : (\mathbb{Y}/\sim) \to \mathbb{R}$ that sends $[f]$ to $f(0.5)$.
Is ϕ well defined? If so, is it injective? Is it surjective?

Q3.
Let $R = [0,15] \times [0,1]$, let $\sim$ be an equivalence relation on R

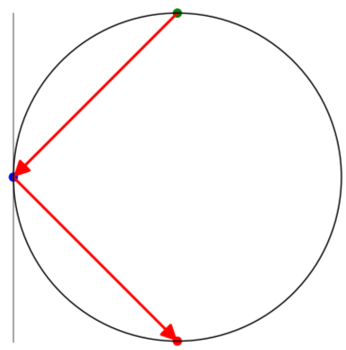
$$(x_1, y_1) \sim (x_2, y_2) \Leftrightarrow \begin{cases} x_1 = x_2 & y_1 = y_2 \\ x_1 = 0; x_2 = 15 & y_1 + y_2 = 1 \\ x_1 = 15; x_2 = 0 & y_1 + y_2 = 1 \end{cases}$$

It is clear R is a rectangle, what does $(R/\sim)$ look like?

Q4.
Define the binary operation $\circ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ as $a \circ b = \dfrac{ab}{gcd(a,b)^2}$
What group properties does the tuple $(\mathbb{N}, \circ, 1)$ satisfy?
You don't have to consider commutativity.

Q5.

Define the binary operation $\star : S^1(\mathbb{R}) \times S^1(\mathbb{R}) \to S^1(\mathbb{R})$ as
$(x_1, y_1) \star (x_2, y_2) = (x_3, y_3)$
A person stands at the location (x_1, y_1) and shines a laser pointer at (x_2, y_2) where the tangent acts as a mirror and reflects the laser to the point (x_3, y_3)



Green Point $\star$ Blue Point = Red Point

What group properties does the tuple $\left(S^1, \star, (1,0)\right)$ satisy?
You don't have to consider commutativity.

Q6.

Define the binary operation $\cdot : S^1(\mathbb{Q}) \times S^1(\mathbb{Q}) \to S^1(\mathbb{Q})$ as
$(x_1, y_1) \cdot (x_2, y_2) = \Big((x_1x_2 - y_1y_2), (x_1y_2 + x_2y_1)\Big)$

(a) Verify the closure this operation.
(b) Verify the associative property of this operation.
(c) Is there an element (x, y) that is an identity element of this operation.
(d) If so, do all elements have inverses?
(e) Compute $\Big((\frac{3}{5}, \frac{4}{5}) \cdot (\frac{-12}{13}, \frac{5}{13})\Big) \cdot (\frac{8}{17}, \frac{-15}{17})$