TUTORIAL HOUR - 7/9

1 Scissors Congruence

MIT Media Labs has an interesting web applet that shows how two polygons are scissors congruent[1].

2 An equivalence relation on the real numbers

Define the equivalence relation $\sim$ on the set $\mathbb{R}$ as $a \sim b \Leftrightarrow (a-b) \in \mathbb{Z}$.
Let $\phi : \mathbb{R} \to (\mathbb{R}/\sim)$, $\phi(r) = [r]$ be the associated canonical map[2].

You can run `equivalence.txt` on SAGE's online platform to visualize the action of $\sim$ on $\mathbb{R}$.
The *Coil* mode converts the real line into a helix.
The *Classes* mode shows how elements are sent to their equivalence classes under ϕ.

3 Assignment

This is not evaluative.

Let $\mathbb{X}$ be the set of all finite words generated by the letters $\{x, y\}$ and their symbolic inverses $\{x^{-1}, y^{-1}\}$.
Elements of $\mathbb{X}$ have the form $x^{n_1}y^{n_2}x^{n_3}\dots y^{n_k}$ where $n_1, n_2, \dots, n_k$ are integers.[3] When all exponents are zero, you get the empty word ε.

Define the binary operation $\cdot : \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ and let $w_1 \cdot w_2$ be the word w obtained after w_2 is concatenated to w_1. Some operations are listed below:

- $x \cdot xy = x^2y$
- $xyx \cdot \varepsilon = xyx$
- $x \cdot x^{-1} = \varepsilon$

Verify: $(\mathbb{X}, \cdot, \varepsilon)$ is a group.

[1] A paper published by the applet's developers.
[2] A canonical map is a surjective function that sends an element to its equivalence class.
[3] set $x^n := \underbrace{x \cdot x \cdot \ldots \cdot x}_{n\ times}$ and $x^{-n} := \underbrace{x^{-1} \cdot x^{-1} \cdot \ldots \cdot x^{-1}}_{n\ times}$

Construct a function $f : \mathbb{X} \to \mathbb{Z}$ inductively:

Base Case: Set its initial values.

- $f(\varepsilon) = 0$
- $f(x) = 2$ and $f(x^{-1}) = -2$
- $f(y) = -3$ and $f(y^{-1}) = 3$

Inductive Hypothesis: f has been defined for all words of length$< n$.

Inductive Step: Consider a word w of length n.
Rewrite w as it's first letter l and a word v of length $n - 1$. Thus $w = l \cdot v$.
Define $f(w) = f(l) + f(v)$

Verify: f is a surjective function.
What is $f(xyxyx)$?

Define a relation $\sim$ on the set $\mathbb{X}$ as $w \sim v \Leftrightarrow f(w) = f(v)$.
Verify: $\sim$ is an equivalence relation.

Using the sort of procedure that was done to $\star$ to generate a group[4], can the set $(\mathbb{X}/\sim)$ be made into a group?
If so, can you find a group that behaves exactly like it?

[4] Look at the material provided on 3/9 for more details.