

TEST TWO ANSWERS

1 Mandatory Section

1.1 Question 1

Let $G = SL_2(\mathbb{Z})$, the group of invertible 2×2 matrices with values in \mathbb{Z} .
 Define a map $\phi : G \rightarrow G$ as $\phi(\sigma) = \tau\sigma\tau$, where τ is an element of order 2.
 Let $H := \{\sigma \in G \mid \phi(\sigma) = e\}$
 Is H a subgroup of G ?

Answer: **YES**

Closure:

If σ_1, σ_2 are in H , then $\tau\sigma_1\tau = \tau\sigma_2\tau = e$.

Observe, $(\tau\sigma_1\tau)(\tau\sigma_2\tau) = (\tau\sigma_1\sigma_2\tau) = e$ (as $\tau^2 = e$)

Thus $\sigma_1\sigma_2$ is in H .

Inverses:

If σ is in H , then $\tau\sigma\tau = e$.

Inverting both sides, $\tau\sigma^{-1}\tau = e$ ($\tau = \tau^{-1}$ as τ an element with an order of 2)

Thus σ^{-1} is in H . □

1.2 Question 2

Define a relation \sim on the group S_3 as:

$\tau \sim \sigma$ if and only if there exists a permutation $\mu \in S_3$ such that $\mu^{-1}\tau\mu = \sigma$.

If you think this is not an equivalence relation, type: **NA**

If you think it is, how many elements does (S_3/\sim) have?

Answer: **3**

Reflexive:

Picking μ to be the identity element, we have $\sigma \sim \sigma$.

Symmetric:

If $\tau \sim \sigma$ then $\mu^{-1}\tau\mu = \sigma$ for some element μ .

As a consequence, $\mu\sigma\mu^{-1} = \tau$. Thus $\sigma \sim \tau$.

Transitive:

If $\tau \sim \sigma$ then $\mu_1^{-1}\tau\mu_1 = \sigma$.

If $\sigma \sim \omega$ then $\mu_2^{-1}\sigma\mu_2 = \omega$.

Observe, $\mu_2^{-1}\mu_1^{-1}\tau\mu_1\mu_2 = (\mu_1\mu_2)^{-1}\tau(\mu_1\mu_2) = \omega$, thus $\tau \sim \omega$.

Equivalence Classes:

No.	class
1	$\{e\}$
2	$\{(1, 2), (1, 3), (2, 3)\}$
3	$\{(1, 2, 3), (1, 3, 2)\}$

1.3 Question 3

Which of these statements are true?

- A) The two elements $(1, 2)$, $(4, 3, 2, 1)$ generate the group S_4 .
- B) The two elements $(2, 5)$, $(2, 5, 4, 3, 1)$ generate the group S_5 .

Answer: Both statements are **TRUE**

Interestingly $(1, 2), (1, 2, \dots, n)$ generate S_n .

Lemma: The set of all (k, m) where $1 \leq k \neq m \leq n$ generate the group S_n

Let G be the group generated by $(1, 2)$ and $(1, 2, \dots, n)$.

Observe $(1, 2, \dots, n)^{k-1}(1, 2)(1, 2, \dots, n)^{n+1-k} = (k, k+1)$ for $2 \leq k \leq (n-1)$

Thus $(1, 2), (2, 3), \dots, (n-1, n) \in G$

Observe, $(1, 2)(2, 3)(1, 2) = (1, 3)$.

Working inductively, $(1, k)(k, k+1)(1, k) = (1, k+1)$.

Thus $(1, 2), (1, 3), \dots, (1, n) \in G$.

Finally, any (k, m) can be expressed as $(1, k)(1, m)(1, k)$. □

Note that $(4, 3, 2, 1)^{-1} = (1, 2, 3, 4)$. The above can be directly applied to (A).

(B)'s validity takes a little bit of additional work:

Let H be the group generated by $(2, 5)$ and $(2, 5, 4, 3, 1)$

Observe¹ $(2, 5, 4, 3, 1)(2, 5)(2, 5, 4, 3, 1)^{-1} = (1, 2) \in H$.

Further, $(1, 2)(2, 5, 4, 3, 1)^{-1}(1, 2) = (1, 2, 3, 4, 5) \in H$.

Using SageMath instead:

```
G = PermutationGroup([(1,2),(4,3,2,1)])
H = PermutationGroup([(2,5),(2,5,4,3,1)])
print(G.structure_description())
print(H.structure_description())
```

Out : 'S4'

Out : 'S5'

¹These evaluations were cross checked in SageMath which evaluates cycles from left to right.

2 Optional Section

2.1 Question 4

What is the largest order of an element in S_{20} ?

Answer: **420**

Theorem:

If a permutation σ can be decomposed into disjoint cycles σ_i for $1 \leq i \leq n$, then $order(\sigma) = lcm\{order(\sigma_1), \dots, order(\sigma_n)\}$

Thus the order question is equivalent to the following problem:

$$\text{Max: } \left\{ LCM\{\#(p_1), \dots, \#(p_i)\} \mid \text{all partitions } p_1, \dots, p_i \text{ of } \{1, \dots, 20\} \right\}$$

Note that $\#(p)$ is the cardinality of the partition p .

Exhaustively checking the elements, the maximal order in S_{20} is 420:

$(1, 2, 3, 4, 5, 6, 7)(8, 9, 10, 11, 12)(13, 14, 15, 16)(17, 18, 19)$ is one such element.

This can be solved using SageMath:

```
from sage.arith.functions import LCM_list
maxOrder = 0
for l in Partitions(20).list():
    temp = LCM_list(l)
    if maxOrder < temp:
        maxOrder = temp
print(maxOrder)
```

Out: '420'

Interestingly, the question of maximal order in S_n is answered by [Landau's function](#). Can you prove that this function is unbounded?

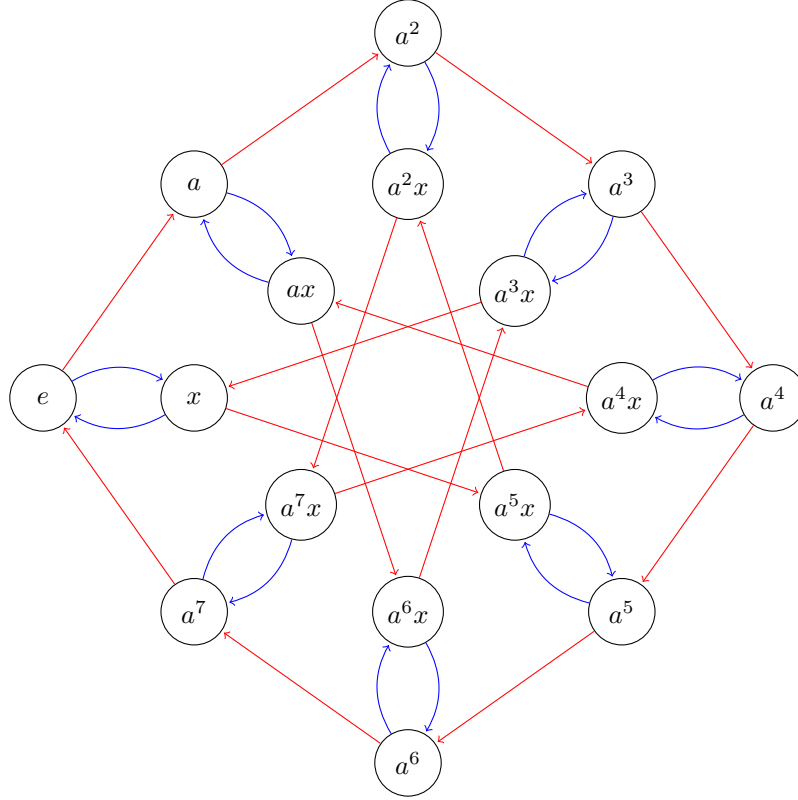
2.2 Question 5

Consider the following multiplication schema of a group G of order 16 containing elements:
 $\{e, a, a^2, a^3, a^4, a^5, a^6, a^7, x, ax, a^2x, a^3x, a^4x, a^5x, a^6x, a^7x\}$

The blue arrows denote right multiplication by x , that means $g \mapsto gx$

The red arrows denote right multiplication by a , that means $g \mapsto ga$

The nodes of the graph are labelled with the group elements that they represent.



How many cyclic subgroups does the group G have? (count the trivial one(s) too)

Answer: **8**

This question is equivalent to tracing specific closed paths in the graph.

The rule column depicts how the subgroup members are reached from the e node.

Different rules may generate the same subgroup, the listed rules are one among the many.

Cyclic subgroups:

No.	Rule	Cyclic subgroup
1	Do Nothing	$\{e\}$
2	R,R,R,R	$\{e, a^4\}$
3	B	$\{e, x\}$
4	R,R,R,R,B	$\{e, a^4x\}$
5	R,R	$\{e, a^2, a^4, a^6\}$
6	R,R,B	$\{e, a^2x, a^4, a^6x\}$
7	R	$\{e, a, a^2, a^3, a^4, a^5, a^6, a^7\}$
8	R,B	$\{e, ax, a^6, a^7x, a^4, a^5x, a^2, a^3x\}$

2.3 Question 6

Consider a puzzle that involves twelve colored beads and two operations that rearrange their positions. The puzzle is deemed solved if the colored beads have the following arrangement:



Figure 1: Start State

The below image shows how the two operations **invert** and **shuffle** change the arrangement of the colored beads from the start state.

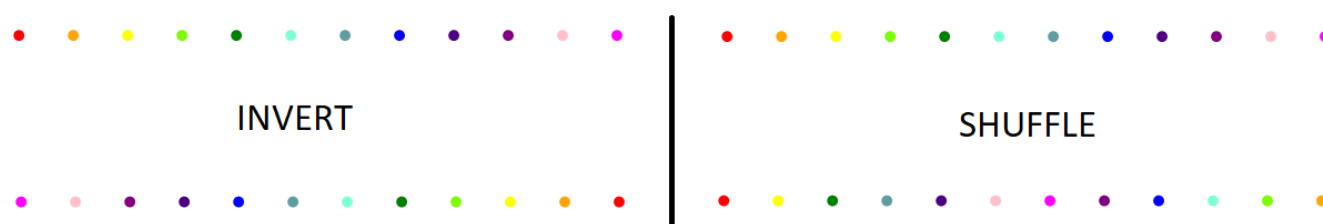


Figure 2: Operations

How many distinct arrangements can this puzzle reach from the start state by using a finite number of these two operations in any order?

Answer: **95040**

We can encode these two operations as permutations, similar to what SageMath did for the Rubik's Cube™.

Observe the **invert** operation can be encoded as $(1, 12)(2, 11)(3, 10)(4, 9)(5, 8)(6, 7)$.

The **shuffle** operation can be encoded as $(2, 12, 7, 4, 11, 6, 10, 8, 9, 5, 3)$.

The set of arrangements that the puzzle can achieve is precisely the group of permutations generated by these two permutations.

We can run a SageMath script to compute the cardinality:

```
G = PermutationGroup([[ (1,12), (2,11), (3,10), (4,9), (5,8), (6,7) ],
                      [ (2,12,7,4,11,6,10,8,9,5,3) ]])
print(G.order())
```

Out : '95040'

2.3.1 An Aside

This is a construction of one member of the 'Happy Family' of sporadic groups mentioned in 3Blue1Brown's [video](#) on group theory.