ANSWER KEY

Q1) No
This is **NOT** an equivalence relation. It violates transitive property.
Consider $f, g, h : [0,1] \to \mathbb{R}$ defined by $f(x) = 0$, $g(x) = x$, and $h(x) = 1$
Now $f \sim g$ as $(f-g)(0) = 0$ and $g \sim h$ as $(g-h)(1) = 0$
But $(f-h)(x) = -1$ which is never 0. Thus f is not related to h.

Q2) Well defined, Injective, Surjective
Well-defined:
Suppose we choose two different elements f and g from the same equivalence class.
Then $f \sim g$ then $f(0.5) = g(0.5)$. Thus under ϕ both $[f]$ and $[g]$ map to the same element in $\mathbb{R}$.

Injective:
Suppose we have two distinct equivalence classes $[f]$ and $[g]$, then $f(0.5) \neq g(0.5)$.
Under ϕ these classes map to different elements in R.

Surjective:
Given an element $r \in \mathbb{R}$ consider the class of the function $x - (0.5 - r)$.

Q3) Mobius Strip
Take a 15 cm $\times$ 1 cm strip of paper.
The equivalence relation $\sim$ glues the short edges of the paper together after twisting it once.

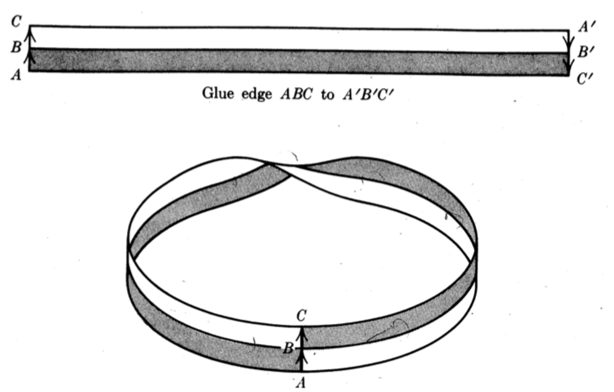

FIGURE 1.1. Constructing a Möbius strip.

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Q4) Closed, Identity, Inverses

Closed:

We note that $gcd(a,b)$ divides both a and b, thus $\dfrac{ab}{gcd(a,b)^2} \in \mathbb{N}$.

Identity:

Observe $n \circ 1 = 1 \circ n = \dfrac{n}{1} = n$

Inverses:

Let y be the inverse of x, we require $x \circ y = 1$

Now if $\dfrac{xy}{gcd(x,y)^2} = 1$ we obtain $gcd(x,y) = lcm(x,y)$

Thus the inverse of x is x itself.

NOT Associative:

Consider $2 \circ (4 \circ 8) = 2 \circ 2 = 1$

But $(2 \circ 4) \circ 8 = 2 \circ 8 = 4$

Q5) Closed

Closed:

The reflected position of the laser pointer will always beam on the circle.

NOT Associative:

Consider $((0,1) \star (-1,0)) \star (0,-1) = (0,-1) \star (0,-1) = (0,1)$

But $(0,1) \star ((-1,0) \star (0,-1)) = (0,1) \star (1,0) = (0,-1)$

NO IDENTITY:

Observe $(1,0) \star (0,1) = (-1,0)$

In fact, no point on the circle can be the identity element of this operation.

NO INVERSES:

A moot point.

Q6)

(a)

Let $(x_1, y_1) := (\frac{p_1}{q_1}, \frac{r_1}{s_1})$ and $(x_2, y_2) := (\frac{p_2}{q_2}, \frac{r_2}{s_2})$

Then under the binary operation we obtain:

$(x_1, y_1) \cdot (x_2, y_2) = \Big(\frac{p_1p_2s_1s_2 - r_1r_2q_1q_2}{q_1s_1q_2s_2}, \frac{p_1r_2q_2s_1 + p_2r_1q_1s_2}{q_1s_1q_2s_2}\Big)$

This is in $\mathbb{Q}^2$.

Now $\Big(\frac{p_1p_2s_1s_2 - r_1r_2q_1q_2}{q_1s_1q_2s_2}\Big)^2 + \Big(\frac{p_1r_2q_2s_1 + p_2r_1q_1s_2}{q_1s_1q_2s_2}\Big)^2 =$
$\frac{(p_1p_1s_1s_2)^2 + (r_1r_2q_1q_2)^2 + (p_1r_2q_2s_1)^2 + (p_2r_1q_1s_2)^2}{(q_1s_1q_2s_2)^2}$.

Observe that the numerator can be resolved into $(p_1^2s_1^2 + r_1^2q_1^2)(p_2^2s_2^2 + r_2^2q_2^2)$

Since (x_1, y_1) and (x_2, y_2) are on the circle, this expression evaluates to $(q_1q_2s_1s_2)^2$, thus the operation is indeed closed.

(b)

Consider $\Big((x_1, y_1) \cdot (x_2, y_2)\Big) \cdot (x_3, y_3) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1) \cdot (x_3, y_3)$

This turns out to be:

$\Big((x_1x_2x_3 - y_1y_2x_3 - x_1y_2y_3 - y_1x_2y_3), (x_1x_2y_3 - y_1y_2y_3 + x_1y_2y_3 + y_1x_2y_3)\Big)$

Consider $(x_1, y_1) \cdot \Big((x_2, y_2) \cdot (x_3, y_3)\Big) = (x_1, y_1) \cdot (x_2x_3 - y_2y_3, x_2y_3 + x_3y_2)$

This turns out to be:

$\Big((x_1x_2x_3 - y_1y_2x_3 - x_1y_2y_3 - y_1x_2y_3), (x_1x_2y_3 - y_1y_2y_3 + x_1y_2y_3 + y_1x_2y_3)\Big)$

Thus the operation is associative.

(c)

Let (x_1, y_1) be any element in $S^1(\mathbb{Q})$.

Observe $(x_1, y_1) \cdot (1, 0) = (x_1 - 0, y_1 + 0)$.

$(1, 0) \cdot (x_1, y_1) = (x_1 - 0, y_1 + 0)$.

Thus $(1, 0)$ is the identity element.

(d)

Let (x_1, y_1) be any element in $S^1(\mathbb{Q})$.

Observe $(x_1, y_1) \cdot (x_1, -y_1) = (x_1^2 + y_1^2, -x_1y_1 + x_1y_1) = (1, 0)$.

Observe $(x_1, -y_1) \cdot (x_1, y_1) = (x_1^2 + y_1^2, x_1y_1 - x_1y_1) = (1, 0)$.

Thus $(x_1, -y_1)$ is the inverse element.

This is in fact a group!

(e)

$$(\frac{3}{5}, \frac{4}{5}) \cdot (\frac{-12}{13}, \frac{5}{13}) = (\frac{-56}{65}, \frac{-33}{65})$$
$$(\frac{-56}{65}, \frac{-33}{65}) \cdot (\frac{8}{17}, \frac{-15}{17}) = (\frac{-943}{1105}, \frac{576}{1105})$$

Do $(3, 4, 5)$, $(-56, -33, 65)$ and $(-943, 576, 1105)$ seem familiar?

Can $(\frac{3}{5}, \frac{4}{5})^n := (\frac{3}{5}, \frac{4}{5}) \underbrace{\star \ldots \star}_{(n-1) times} (\frac{3}{5}, \frac{4}{5})$ generate these tuples infinitely?

Or does it loop back after a certain k operations?