TUTORIAL HOUR - 19/10

1 Required

1.1 Material

- Slides by Professor Matthew Macaulay on choosing permutations that represent a finite group.
- A page about isomorphisms from *First-Semester Abstract Algebra: A Structural Approach* by Professor Jessica K. Sklar.

1.2 Exercises

1.2.1 Gallian

- Problems 50 - 56, Chapter 5, A First Course in Abstract Algebra
- Problems 34 - 40, Chapter 6, A First Course in Abstract Algebra

1.2.2 Programming

- Given a finite group G and a list of smaller groups $\{H_i\}$, write a program that checks if some H_i's can be identified as a subgroup of G.
  Identification here, is the same as checking if H_i is isomorphic to a subgroup if G.
- Investigate the group generated by $(1,12)(2,11)(3,10)(4,9)(5,8)(6,7)$ and $(2,12,7,4,11,6,10,8,9,5,3)$.
  - List out all its subgroups.
  - What is the maximal order an element can have?
  - Are there other ways of generating the same group?
- Search for a group G of order > 96, such that the set $H = \{ghg^{-1}h^{-1} | g, h \in G\}$ is not a group.

2 Additional

2.1 Two Towers

Consider the symmetric group S_{2^n} where n is a natural number.
We will generate a specific set of permutations using the following set of permutations:

$$Sy(2^n) := \left\{ \prod_{j=1}^{2^i} (j, j+2^i) \middle| i = 0, 1, \ldots, (n-1) \right\}$$

Observe:
$Sy(2^4) = \{(1,2),(1,3)(2,4),(1,5)(2,6)(3,7)(4,8),(1,9)(2,10)(3,11)(4,12)(5,13)(6,14)(7,15)(8,16)\}$.

Let $G(2^n)$ be the group generated by the permutations in $Sy(2^n)$.
How many elements does $G(2^n)$ contain?
What is the maximum order an element in $G(2^n)$?
Can you describe $G(2^2)$ using symmetries of an object?

2.2 The Smallest Member of the Happy Family

Consider the start string `ABCDEFGHIJK` and two functions `cycle` and `twist`.

`cycle` does the following:

$$\texttt{ABCDEFGHIJK} \rightarrow \texttt{KABCDEFGHIJ}$$

`twist` does:

$$\texttt{ABCDEFGHIJK} \rightarrow \texttt{ABHFJECKIDG}$$

How many different strings can this set-up generate?

Let X be the set of all possible finite length plans that contains these two functions in any order. Define an equivalence relation on X, two plans are equivalent if and only if they generate the same string from `ABCDEFGHIJK`.
Set $M_{11} := (X/\sim)$

How many subgroups does the group $(M_{11}, \star, [\phi])$ have?
What is the maximal order of an element in M_{11}?

Observe that the plan `twist, twist, twist, twist` is in the equivalence class $[\phi]$.
In fact ϕ is an element of minimal length[1] in the equivalence class.

Suppose that all representative elements of the equivalence classes are chosen to be members with the minimal length. What is the length of the largest representative element?

For instance the class [`twist`] has a its length of 1 which is bigger than the length of $[\phi]$.[2]

[1]Length of a plan is defined as the number of function calls it contains.
[2]We define the length of plan ϕ to be zero.