TUTORIAL HOUR - 9/11

1 Required

1.1 Material

- A MAA article about visualizing the first group isomorphism theorem.
- Visually constructing homomorphisms.
- Lecture slides by Professor Matthew Macauley on group homomorphisms.

1.2 Exercises

1.2.1 Gallian

- Chapter 6, 40 - 50, A First Course in Abstract Algebra

1.2.2 Herstein

- Section 2.7, 1 - 3, 15 - 16, Topics in Algebra
- Section 2.8, 1 - 4, 10 - 11, 16, Topics in Algebra

1.2.3 Programming

- Given a natural number n less than 200, output the number of group structures on a set of n elements.
- Define $gHg^{-1} := \{ghg^{-1} | h \in H\}$. Given a group G, how many subgroups H of G satisfy the property $gHg^{-1} = H$ for all $g \in G$?
- Read more about the `PermutationGroupHomomorphism_from_gap()` function and construct a homomorphism from $Z/2Z$ to the Klein Four Group.

2 Additional

2.1 Maximal slackers

We call a subgroup H of a group G maximal if there exists no subgroup H' such that $H \subsetneq H'$. Define $J(G) := \bigcap_{H \text{ is maximal}} H$.

Compute the following and compare $J(G)$ with $NotGen(G)$[1]:

- $J(S_3)$
- $J(\mathbb{Z})$
- $J(\mathbb{Z}/4\mathbb{Z})$

[1] Refer to the material provided on 25/10

2.2 A generalized 15 puzzle

Consider a rectangular $m \times n$ tiled grid with an unoccupied bottom-right space.

1	2	$\dots$	m
$m+1$	$m+2$	$\dots$	$2m$
$\vdots$	$\vdots$	$\ddots$	$\vdots$
$m(n-1)+1$	$m(n-1)+2$	$\dots$	B

The puzzle behaves like the 15 puzzle - tiles adjacent to the blank space can be pushed into it. Let $G_{m\times n}$ be the set of reachable states where the blank tile remains in the bottom-right corner. $G_{m\times n}$ forms a group under composition.

For $2 \leq m, n \leq 5$ find the minimal set of moves that generate $G_{m\times n}$ and express them in terms of the basic `pushLeft, pushRight, pushDown, pushUp` moves.
Observe that the move `M` defined as `pushLeft pushDown pushRight pushUp` generates $G_{2\times 2}$

What group is $G_{m\times n}$ isomorphic to?

2.3 Commutator Chain

Given a group G define the commutator subgroup $Comm(G)$ as the subgroup generated by all elements $\{ghg^{-1}h^{-1} | g, h \in G\}$.
Given a finite group G, we define a commutator chain $\{G, Comm(G), Comm(Comm(G)), \dots\}$.

We terminate the chain once a stationary point is reached. A stationary point is defined as $Comm^{(n)}(G)$ such that $Comm^{(n)}(G) \simeq Comm^{(n+k)}(G)$ for all $k \in \mathbb{N}$.

What is the commutator chain of a finite abelian group?
What is the commutator chain of:

- S_4
- S_5
- Mathieu Group M_{12}