

## 1 Scissors Congruence

MIT Media Labs has an interesting web [applet](#) that shows how two polygons are scissors congruent<sup>1</sup>.

## 2 An equivalence relation on the real numbers

Define the equivalence relation  $\sim$  on the set  $\mathbb{R}$  as  $a \sim b \Leftrightarrow (a - b) \in \mathbb{Z}$ .

Let  $\phi : \mathbb{R} \rightarrow (\mathbb{R}/\sim)$ ,  $\phi(r) = [r]$  be the associated canonical map<sup>2</sup>.

You can run `equivalence.txt` on [SAGE](#)'s online platform to visualize the action of  $\sim$  on  $\mathbb{R}$ .

The *Coil* mode converts the real line into a helix.

The *Classes* mode shows how elements are sent to their equivalence classes under  $\phi$ .

## 3 Assignment

**This is not evaluative.**

Let  $\mathbb{X}$  be the set of all finite words generated by the letters  $\{x, y\}$  and their symbolic inverses  $\{x^{-1}, y^{-1}\}$ .

Elements of  $\mathbb{X}$  have the form  $x^{n_1}y^{n_2}x^{n_3}\dots y^{n_k}$  where  $n_1, n_2, \dots, n_k$  are integers.<sup>3</sup> When all exponents are zero, you get the empty word  $\varepsilon$ .

Define the binary operation  $\cdot : \mathbb{X} \times \mathbb{X} \rightarrow \mathbb{X}$  and let  $w_1 \cdot w_2$  be the word  $w$  obtained after  $w_2$  is concatenated to  $w_1$ . Some operations are listed below:

- $x \cdot xy = x^2y$
- $xyx \cdot \varepsilon = xyx$
- $x \cdot x^{-1} = \varepsilon$

Verify:  $(\mathbb{X}, \cdot, \varepsilon)$  is a group.

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<sup>1</sup>A [paper](#) published by the applet's developers.

<sup>2</sup>A canonical map is a surjective function that sends an element to its equivalence class.

<sup>3</sup>set  $x^n := \underbrace{x \cdot x \cdot \dots \cdot x}_{n \text{ times}}$  and  $x^{-n} := \underbrace{x^{-1} \cdot x^{-1} \cdot \dots \cdot x^{-1}}_{n \text{ times}}$

Construct a function  $f : \mathbb{X} \rightarrow \mathbb{Z}$  inductively:

*Base Case:* Set it's initial values.

- $f(\varepsilon) = 0$
- $f(x) = 2$  and  $f(x^{-1}) = -2$
- $f(y) = -3$  and  $f(y^{-1}) = 3$

*Inductive Hypothesis:*  $f$  has been defined for all words of length  $< n$ .

*Inductive Step:* Consider a word  $w$  of length  $n$ .

Rewrite  $w$  as it's first letter  $l$  and a word  $v$  of length  $n - 1$ . Thus  $w = l \cdot v$ .

Define  $f(w) = f(l) + f(v)$

Verify:  $f$  is a surjective function.

What is  $f(xyxyx)$  ?

Define a relation  $\sim$  on the set  $\mathbb{X}$  as  $w \sim v \Leftrightarrow f(w) = f(v)$ .

Verify:  $\sim$  is an equivalence relation.

Using the sort of procedure that was done to  $\star$  to generate a group<sup>4</sup>, can the set  $(\mathbb{X}/\sim)$  be made into a group?

If so, can you find a group that behaves exactly like it?

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<sup>4</sup>Look at the material provided on 3/9 for more details.