TEST TWO ANSWERS

1 Mandatory Section

1.1 Question 1

Let $G = SL_2(Z)$, the group of invertible 2×2 matrices with values in $\mathbb{Z}$.
Define a map $\phi : G \to G$ as $\phi(\sigma) = \tau\sigma\tau$, where τ is an element of order 2.
Let $H := \{\sigma \in G | \phi(\sigma) = e\}$
Is H a subgroup of G?

Answer: **YES**
Closure:
If σ_1, σ_2 are in H, then $\tau\sigma_1\tau = \tau\sigma_2\tau = e$.
Observe, $(\tau\sigma_1\tau)(\tau\sigma_2\tau) = (\tau\sigma_1\sigma_2\tau) = e$ (as $\tau^2 = e$)
Thus $\sigma_1\sigma_2$ is in H.

Inverses:
If σ is in H, then $\tau\sigma\tau = e$.
Inverting both sides, $\tau\sigma^{-1}\tau = e$ ($\tau = \tau^{-1}$ as τ an element with an order of 2)
Thus σ^{-1} is in H. □

1.2 Question 2

Define a relation $\sim$ on the group S_3 as:
$\tau \sim \sigma$ if and only if there exists a permutation $\mu \in S_3$ such that $\mu^{-1}\tau\mu = \sigma$.
If you think this is not an equivalence relation, type: `NA`
If you think it is, how many elements does $(S_3/\sim)$ have?

Answer: **3**
Reflexive:
Picking μ to be the identity element, we have $\sigma \sim \sigma$.

Symmetric:
If $\tau \sim \sigma$ then $\mu^{-1}\tau\mu = \sigma$ for some element μ.
As a consequence, $\mu\sigma\mu^{-1} = \tau$. Thus $\sigma \sim \tau$.

Transitive:
If $\tau \sim \sigma$ then $\mu_1^{-1}\tau\mu_1 = \sigma$.
If $\sigma \sim \omega$ then $\mu_2^{-1}\sigma\mu_2 = \omega$.
Observe, $\mu_2^{-1}\mu_1^{-1}\tau\mu_1\mu_2 = (\mu_1\mu_2)^{-1}\tau(\mu_1\mu_2) = \omega$, thus $\tau \sim \omega$.

Equivalence Classes:

No.	class
1	$\{e\}$
2	$\Big\{(1,2),(1,3),(2,3)\Big\}$
3	$\Big\{(1,2,3),(1,3,2)\Big\}$

1.3 Question 3

Which of these statements are true?
A) The two elements $(1,2)$, $(4,3,2,1)$ generate the group S_4.
B) The two elements $(2,5)$, $(2,5,4,3,1)$ generate the group S_5.

Answer: Both statements are **TRUE**
Interestingly $(1,2),(1,2,\dots,n)$ generate S_n.
Lemma: The set of all (k,m) where $1 \le k \ne m \le n$ generate the group S_n
Let G be the group generated by $(1,2)$ and $(1,2,\dots,n)$.
Observe $(1,2,\dots,8)^{k-1}(1,2)(1,2,\dots,8)^{n+1-k} = (k,k+1)$ for $2 \le k \le (n-1)$
Thus $(1,2),(2,3),\dots,(n-1,n) \in G$

Observe, $(1,2)(2,3)(1,2) = (1,3)$.
Working inductively, $(1,k)(k,k+1)(1,k) = (1,k+1)$.
Thus $(1,2),(1,3),\dots,(1,n) \in G$.

Finally, any (k,m) can be expressed as $(1,k)(1,m)(1,k)$. $\square$

Note that $(4,3,2,1)^{-1} = (1,2,3,4)$.The above can be directly applied to (A).

(B)'s validity takes a little bit of additional work:
Let H be the group generated by $(2,5)$ and $(2,5,4,3,1)$
Observe[1] $(2,5,4,3,1)(2,5)(2,5,4,3,1)^{-1} = (1,2) \in H$.
Further, $(1,2)(2,5,4,3,1)^{-1}(1,2) = (1,2,3,4,5) \in H$.

Using SageMath instead:

```
G = PermutationGroup([(1,2),(4,3,2,1)])
H = PermutationGroup([(2,5),(2,5,4,3,1)])
print(G.structure_description())
print(H.structure_description())
```

```
Out :  'S4'
Out :  'S5'
```

[1] These evaluations were cross checked in SageMath which evaluates cycles from left to right.

2 Optional Section

2.1 Question 4

What is the largest order of an element in S_{20}?

Answer: **420**
Theorem:
If a permutation σ can be decomposed into disjoint cycles σ_i for $1 \leq i \leq n$, then
$order(\sigma) = lcm\{order(\sigma_1), \ldots, order(\sigma_n)\}$

Thus the order question is equivalent to the following problem:

$$\text{Max: } \left\{ LCM\left\{\#(p_1), \ldots, \#(p_i)\right\} \middle| \text{ all partitions } p_1, \ldots, p_i \text{ of } \{1, \ldots, 20\} \right\}$$

Note that $\#(p)$ is the cardinality of the partition p.
Exhaustively checking the elements, the maximal order in S_{20} is 420:
$(1, 2, 3, 4, 5, 6, 7)(8, 9, 10, 11, 12)(13, 14, 15, 16)(17, 18, 19)$ is one such element.

This can be solved using SageMath:

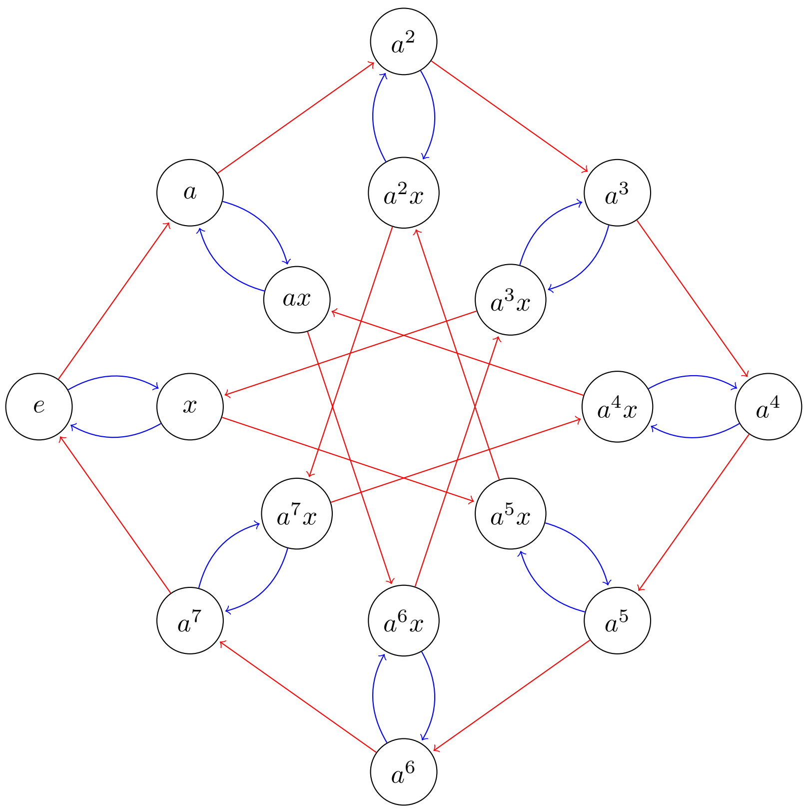
```
from sage.arith.functions import LCM_list
maxOrder = 0
for l in Partitions(20).list():
    temp = LCM_list(l)
    if maxOrder < temp:
        maxOrder = temp
print(maxOrder)
```

```
Out: '420'
```

Interestingly, the question of maximal order in S_n is answered by Landau's function.
Can you prove that this function is unbounded?

2.2 Question 5

Consider the following multiplication schema of a group G of order 16 containing elements:
$\{e, a, a^2, a^3, a^4, a^5, a^6, a^7, x, ax, a^2x, a^3x, a^4x, a^5x, a^6x, a^7x\}$
The blue arrows denote right multiplication by x, that means g$\mapsto$gx
The red arrows denote right multiplication by a, that means g$\mapsto$ga
The nodes of the graph are labelled with the group elements that they represent.



How many cyclic subgroups does the group G have? (count the trivial one(s) too)

Answer: **8**
This question is equivalent to tracing specific closed paths in the graph.
The rule column depicts how the subgroup members are reached from the e node.
Different rules may generate the same subgroup, the listed rules are one among the many.

Cyclic subgroups:

No.	Rule	Cyclic subgroup
1	Do Nothing	$\{e\}$
2	R,R,R,R	$\{e, a^4\}$
3	B	$\{e, x\}$
4	R,R,R,R,B	$\{e, a^4x\}$
5	R,R	$\{e, a^2, a^4, a^6\}$
6	R,R,B	$\{e, a^2x, a^4, a^6x\}$
7	R	$\{e, a, a^2, a^3, a^4, a^5, a^6, a^7\}$
8	R,B	$\{e, ax, a^6, a^7x, a^4, a^5x, a^2, a^3x\}$

2.3 Question 6

Consider a puzzle that involves twelve colored beads and two operations that rearrange their positions. The puzzle is deemed solved if the colored beads have the following arrangement:

Figure 1: Start State

The below image shows how the two operations `invert` and `shuffle` change the arrangement of the colored beads from the start state.

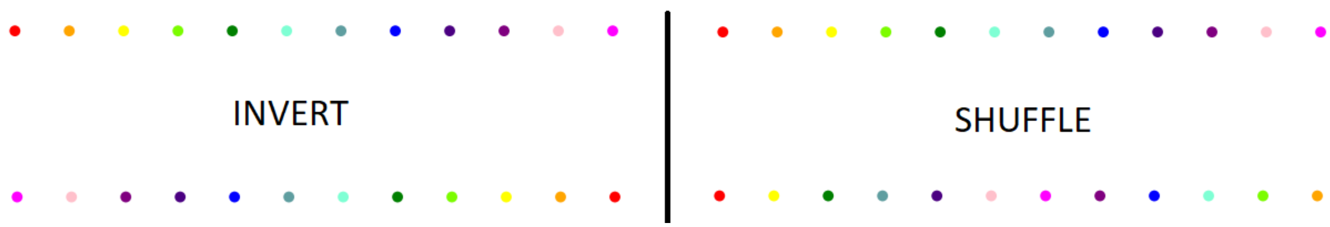


Figure 2: Operations

How many distinct arrangements can this puzzle reach from the start state by using a finite number of these two operations in any order?

Answer: **95040**
We can encode these two operations as permutations, similar to what SageMath did for the Rubik's Cube™.
Observe the `invert` operation can be encoded as $(1,12)(2,11)(3,10)(4,9)(5,8)(6,7)$.
The `shuffle` operation can be encoded as $(2,12,7,4,11,6,10,8,9,5,3)$.

The set of arrangements that the puzzle can achieve is precisely the group of permutations generated by these two permutations.

We can run a SageMath script to compute the cardinality:

```
G = PermutationGroup([[(1,12),(2,11),(3,10),(4,9),(5,8),(6,7)],
                      [(2,12,7,4,11,6,10,8,9,5,3)]])
print(G.order)
```

```
Out :  '95040'
```

2.3.1 An Aside

This is a construction of one member of the 'Happy Family' of sporadic groups mentioned in 3Blue1Brown's video on group theory.