TEST THREE ANSWERS

1 Questions

1.1 Question One

Let G be the group generated by distinct symbols a, b and their symbolic inverses a^{-1}, b^{-1}.
Let H be the commutator subgroup of G. Which diagram is generated by aH and bH?

We define a diagram of a group G with generators $\{g_i\}$ as a picture that consists of points representing distinct elements of G and lines connecting points if the two underlying elements x, y satisfy $xg_i = y$ for some generator g_i.
Line lengths can be re-scaled to make a prettier picture (if deemed necessary).

Answer: **A 2D integer lattice**
We note $aba^{-1}b^{-1} \in H$, thus $abH = baH$.
Every element in G/H has the form $a^n b^m H$ for $n, m \in \mathbb{Z}$.
We can define a group homomorphism $\phi : G/H \to \mathbb{R}^2$ as $\phi(aH) = (1, 0)$ and $\phi(bH) = (0, 1)$.
The image of G/H in $\mathbb{R}^2$ under ϕ forms an integer lattice.

1.2 Question Two

Let $\mathbb{X} :=$ All subgroups of $\mathbb{Z}$.
Define a map $\alpha : \mathbb{Z} \times \mathbb{X} \to \mathbb{X}$ as $(g, H) \mapsto$ the smallest subgroup containing g and H.
Is this a group action?

Answer: **NO**
Group actions must be compatible with the group operation.
Consider $\alpha(3, \alpha(5, 8\mathbb{Z})) = \alpha(3, \mathbb{Z}) = \mathbb{Z}$.
However, $\alpha(3 + 5, 8\mathbb{Z}) = 8\mathbb{Z}$.

Thus α is not a group action.

1.3 Question Three

Let $\mathbb{X} :=$
$\{< (1, 2, 3, 4, 5) >, < (1, 2, 3, 5, 4) >, < (1, 2, 4, 3, 5) >, < (1, 2, 4, 5, 3) >, < (1, 2, 5, 3, 4) >, < (1, 2, 5, 4, 3) >\}$
be a set of cyclic groups H_i.

Define an action $\beta : S_5 \times \mathbb{X} \to \mathbb{X}$ as $(\sigma, H_i) \mapsto \sigma^{-1} H_i \sigma$.
Recall $\sigma^{-1} H \sigma := \{\sigma^{-1} h \sigma | h \in H\}$.

What is the order of $\bigcap_{i=1}^{6} \mathrm{Stab}(H_i)$?

Answer: `1`

We claim $\bigcap_{i=1}^{6} H_i$ is a normal subgroup.

Let h be an element in $\bigcap_{i=1}^{6} H_i$, then $h^{-1}H_ih = H_i$.

Let g be any element in S_5, it suffices to show ghg^{-1} stabilizes all H_i.

Let $g^{-1}H_ig = H_j$.

Observe, $gh^{-1}g^{-1}H_ighg^{-1} = gh^{-1}H_jhg^{-1} = gH_jg^{-1} = H_i$.

Observe that the order of $\bigcap_{i=1}^{6} H_i$ is atmost 20.

This can be computed with SageMath:

```
G = CyclicPermutationGroup(5)
g = G.gens()[0]
S = SymmetricGroup(5)

stab = []
for h in S:
   if PermutationGroup([h^-1*g*h]) == G:
      stab.append(h)
print(len(stab))
```

`Out : '20'`

Lemma: The only two normal subgroups of S_5 are A_5 and the trivial group.

With the computed bound, we have $\bigcap_{i=1}^{6} H_i = \{e\}$.

1.4 Question Four

Let $\mathbb{X} := \Big(\underbrace{S_5 \times S_5 \times \cdots \times S_5}_{5 \text{ times}}\Big)$.

Define $\mathbb{X}_\tau := \{(\sigma_1, \sigma_2, \ldots, \sigma_5) \in \mathbb{X} | \sigma_1\sigma_2 \ldots \sigma_5 = \tau\}$ for each $\tau \in S_5$.

Define an action $\delta : (\mathbb{Z}/5\mathbb{Z}) \times \mathbb{X} \to \mathbb{X}$ as $\Big(1, (\sigma_1, \sigma_2, \ldots, \sigma_5)\Big) \mapsto (\sigma_2, \sigma_3, \ldots, \sigma_1)$.

For how many values of τ is $\mathbb{X}_\tau$ invariant under the above action?

Answer: **1**
Let τ be any non-identity element of S_5.
There are non-trivial elements $\sigma_1, \sigma_2 \in S_5$ such that $\sigma_1\sigma_2 = \tau$. Observe $(\sigma_1, e, e, e, \sigma_2) \in \mathbb{X}_\tau$.
We note that $\Big(1, (\sigma_1, e, e, e, \sigma_2)\Big) = (e, e, e, \sigma_2, \sigma_1)$ is not in $\mathbb{X}_\tau$ as $Z(S_5) = \{e\}$.

$\mathbb{X}_e$ is invariant under this action.
If $(\sigma_1, \sigma_2, \ldots, \sigma_5) \in \mathbb{X}_e$, then $\sigma_1\sigma_2 \ldots \sigma_5 = e$.
Observe $\sigma_2\sigma_3 \ldots \sigma_1 = \sigma_1^{-1}(\sigma_1\sigma_2 \ldots \sigma_5)\sigma_1 = \sigma_1^{-1}e\sigma_1 = e$. Thus $(\sigma_2, \sigma_3, \ldots, \sigma_1) \in \mathbb{X}_e$.

1.5 Question Five

Let $\mathbb{X}$ be the set of all subgroups of S_6.
Define a group action $\psi : Aut(S_6) \times \mathbb{X} \to \mathbb{X}$ as $(\phi, H) \mapsto \phi(H)$.
What is the order of $\mathrm{Stab}(A_6)$?

Answer: **1440**
Lemma: A_6 is the only subgroup of order 360 in S_5.
Suppose there was another subgroup H of the same order.
H has an index of 2, thus H is a normal subgroup.
Since the only normal subgroups of S_6 are A_6 and $\{e\}$, H must be A_6.

Every automorphism ϕ is a bijection, thus $\phi(H)$ has the same order as H.
Applying the above lemma, $\mathrm{Stab}(A_6) = Aut(S_6)$.
We can compute the order of $Aut(S_6)$ by interfacing with GAP:

```
print(gap(``Size(AutomorphismGroup(SymmetricGroup(6)))''))
```

```
Out : `1440'
```

Note that there are 720 inner automorphisms but 1440 automorphisms in total.
Can you construct an automorphism that is not an inner automorphism?[1]

Interestingly, S_6 and S_2 are the only symmetric groups with automorphisms that are not inner.

[1] It is related to the action defined in question three.