UNIVERSITÀ DEGLI STUDI DI MILANO
FACOLTÀ DI SCIENZE E TECNOLOGIE

Intrinsic dimension estimation for locally undersampled data

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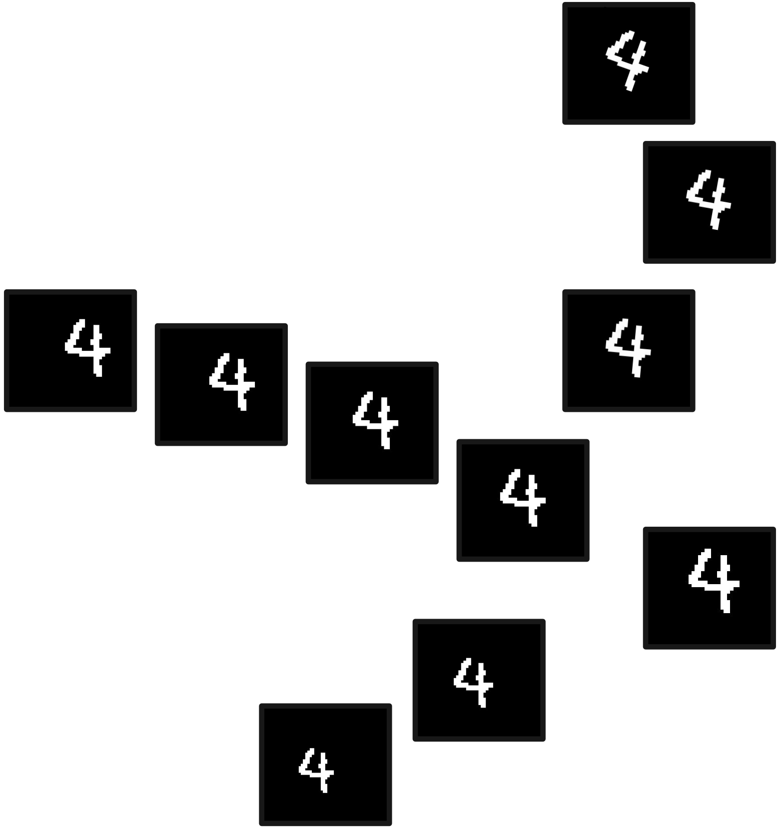
What is the structure underlying complex data?



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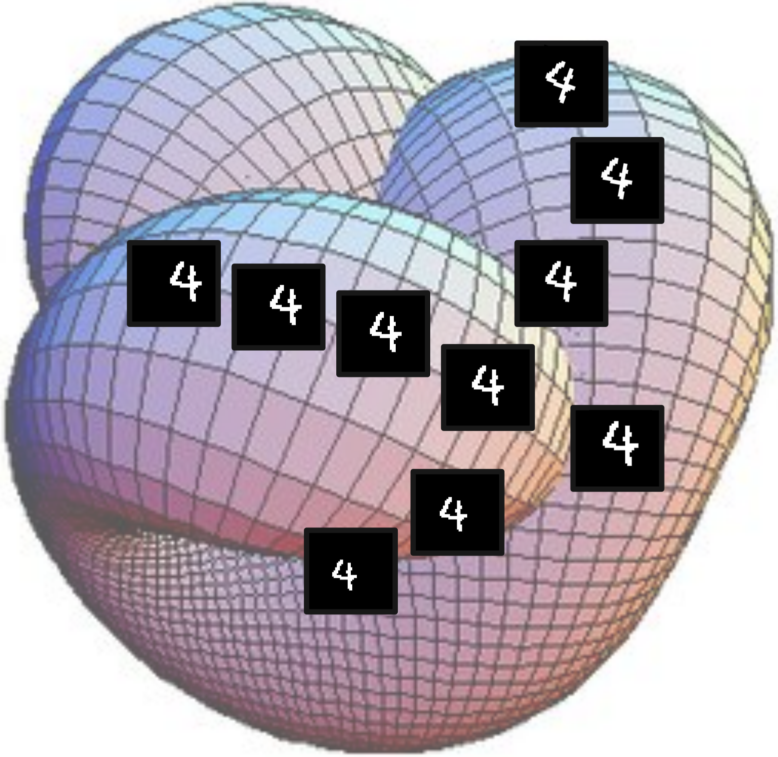

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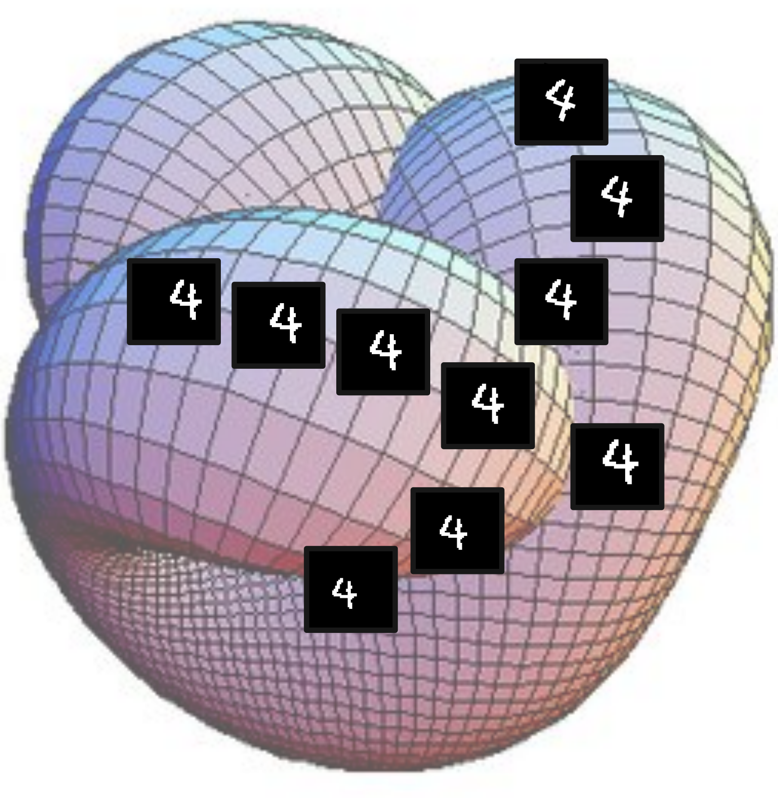
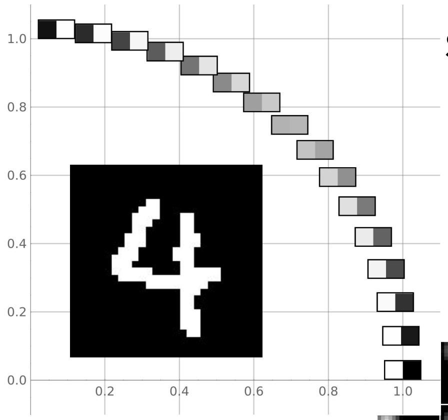
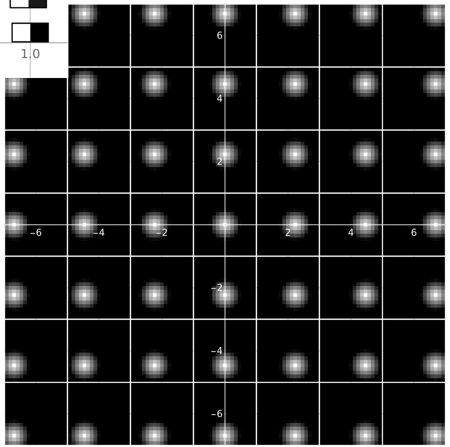




Simple embeddings for humans are complex for maths

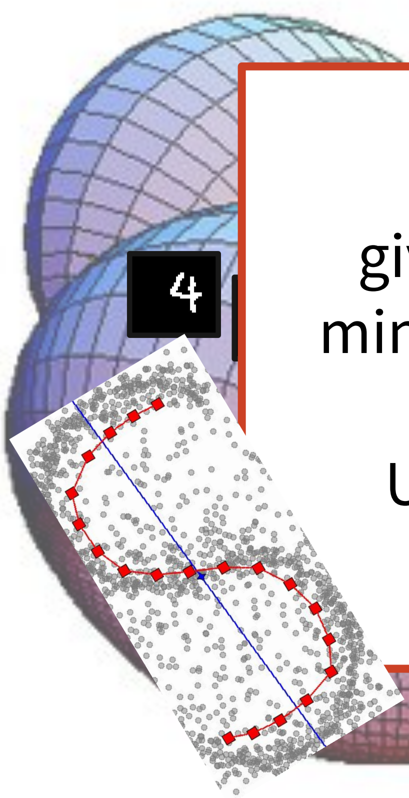


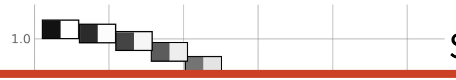
pixel space is huge!

$$\mathbb{R}^{28\times 28}$$

High dimensional embeddings are redundant: **Intrinsic Dimensionality**

What is the structure underlying complex data?



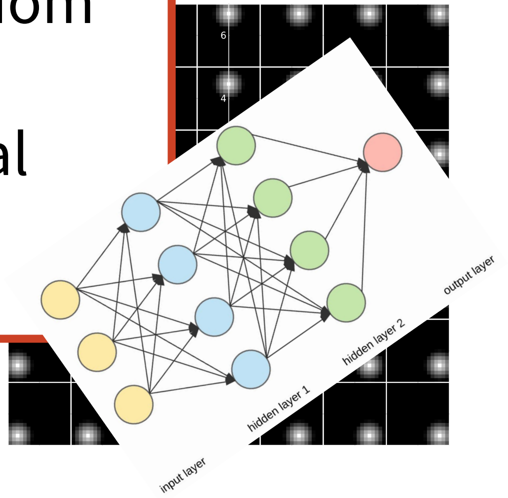


Simple embeddings for
humans are
plex for maths

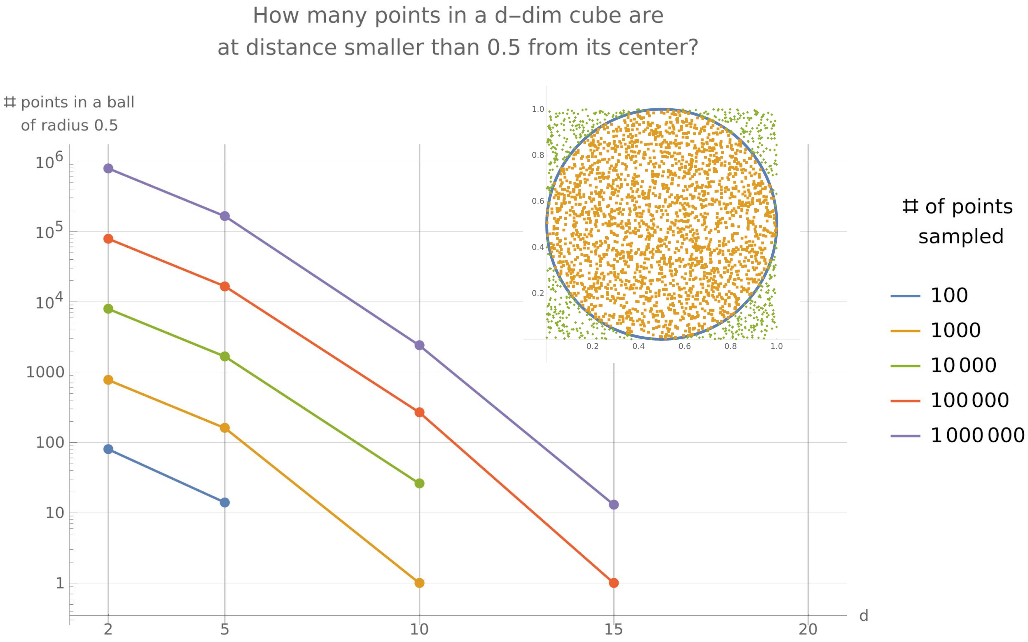
Intrinsic dimension estimation:
given the embedded data, retrieve the minimum number of degrees of freedom

Unsupervised learning, dimensional reduction, feature extraction

embeddings are redundant:
Intrinsic Dimensionality



Curse of dimensionality: exponential undersampling in high dimension

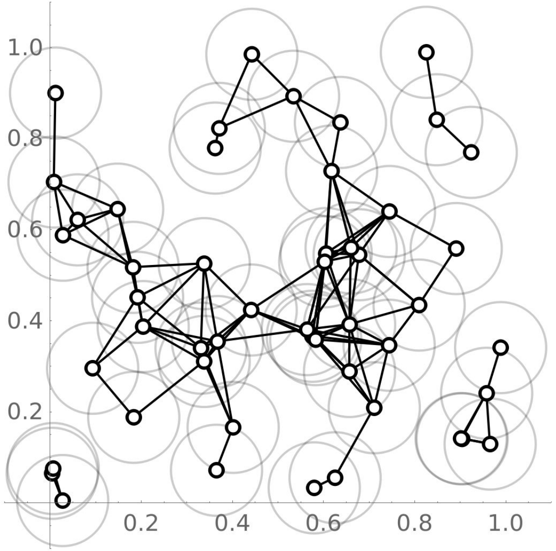


High dimension
d > 6

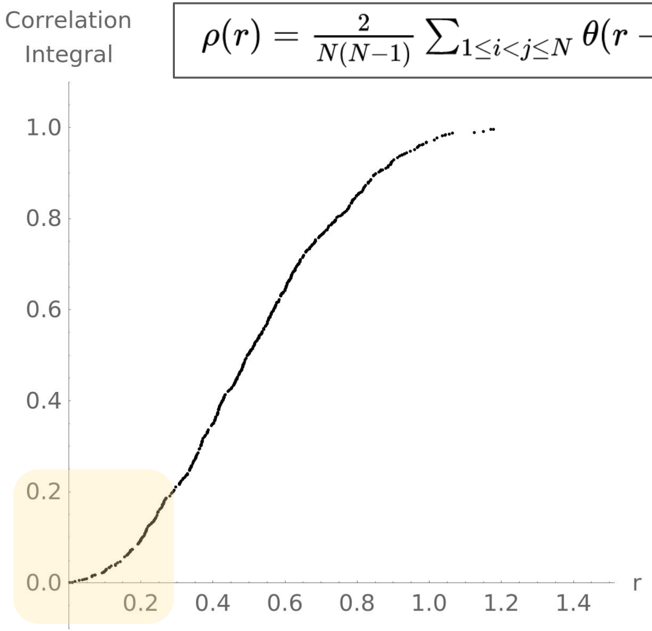
Eckmann, J-P., and David Ruelle. "Fundamental limitations for estimating dimensions and Lyapunov exponents in dynamical systems." *Physica D: Nonlinear Phenomena* 56.2-3 (1992): 185-187.

Geometric estimators: look at the local structure of data

Locally, datasets are linear ⇒ Local number of neighbours scales as r^d

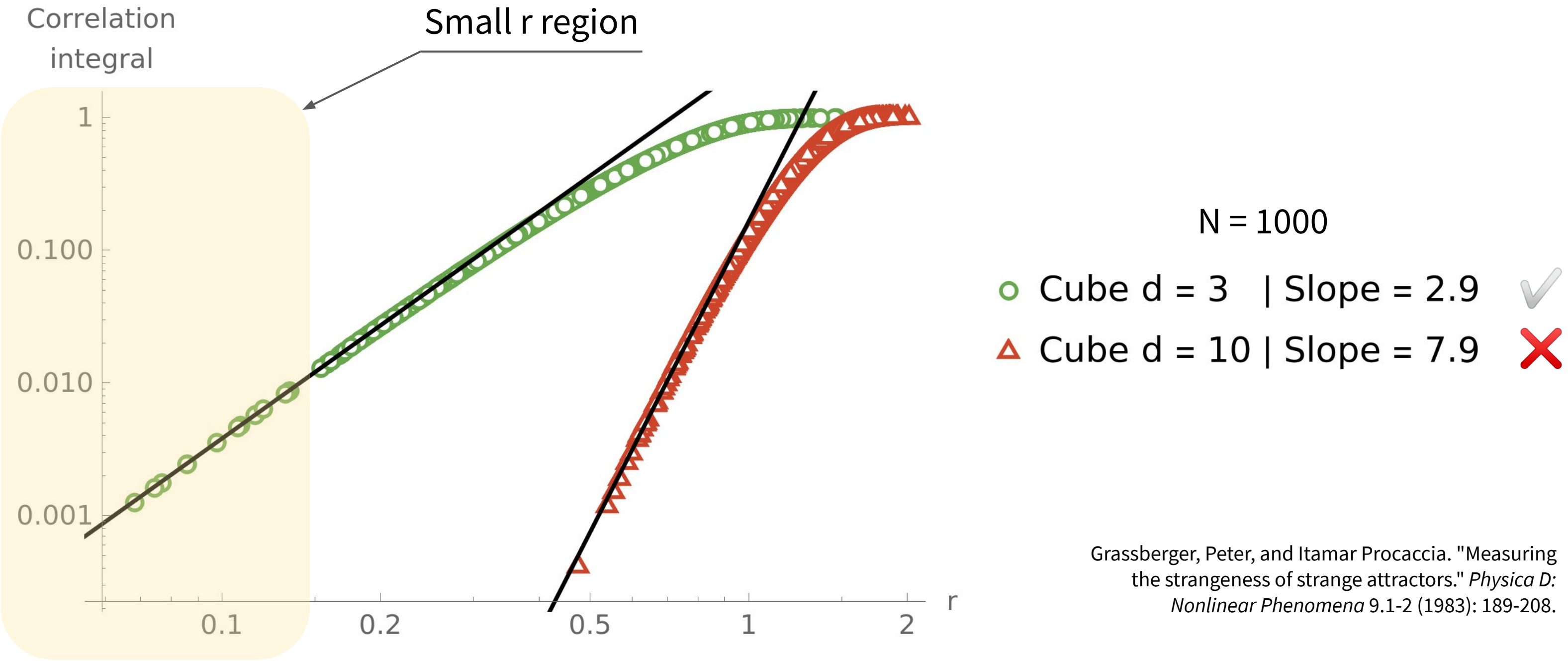


$$\rho(r) = \frac{2}{N(N-1)} \sum_{1 \le i < j \le N} \theta(r - ||\vec{x}_i - \vec{x}_j||)$$



Data undersampling ⇒ ID underestimation

The intrinsic dimension can be extracted by a linear fit on the log-log plot in the region of small r



Grassberger, Peter, and Itamar Procaccia. "Measuring the strangeness of strange attractors." *Physica D: Nonlinear Phenomena* 9.1-2 (1983): 189-208.

A tradeoff between non-linearity and undersampling

	Geometric local estimators	Projective global estimators	???
Non linear	✓	✗	✓
High dimension	✗ (exp d)	✓ (d log d)	✓

Grassberger, Peter, and Itamar Procaccia. "Measuring the strangeness of strange attractors." *Physica D: Nonlinear Phenomena* 9.1-2 (1983): 189-208.

Ceruti, Claudio, et al. "Danco: An intrinsic dimensionality estimator exploiting angle and norm concentration." *Pattern recognition* 47.8 (2014): 2569-2581.

Hein, Matthias, and Jean-Yves Audibert. "Intrinsic dimensionality estimation of submanifolds in R d." *Proceedings of the 22nd international conference on Machine learning*. ACM, 2005.

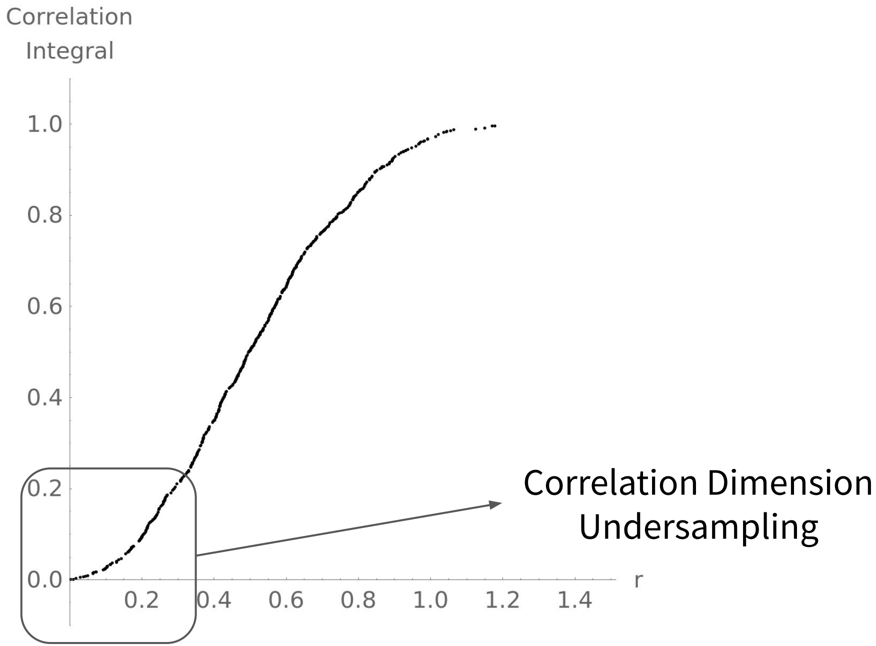
Correlation dimension
DANCO
Hein
...

Principal Component Analysis

Pearson, Karl. "LIII. On lines and planes of closest fit to systems of points in space." *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science* 2.11 (1901): 559-572.

Anna V. Little, Jason Lee, Yoon-Mo Jung, and Mauro Maggioni. Estimation of intrinsic dimensionality of samples from noisy low-dimensional manifolds in high dimensions with multiscale SVD. In 2009 IEEE/SP 15th Workshop on Statistical Signal Processing. IEEE, 2009.

The Full Correlation Integral Estimator (FCI)



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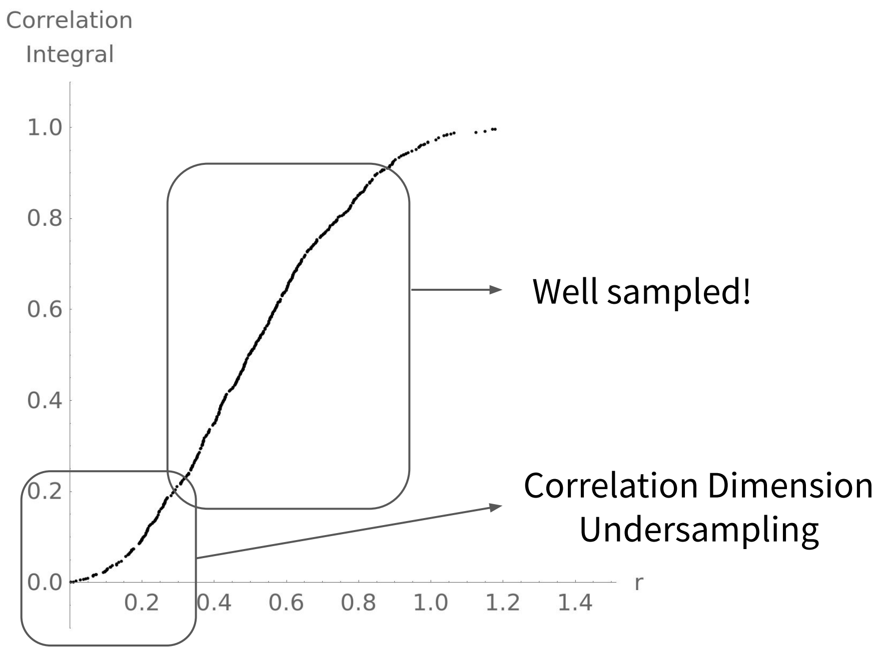

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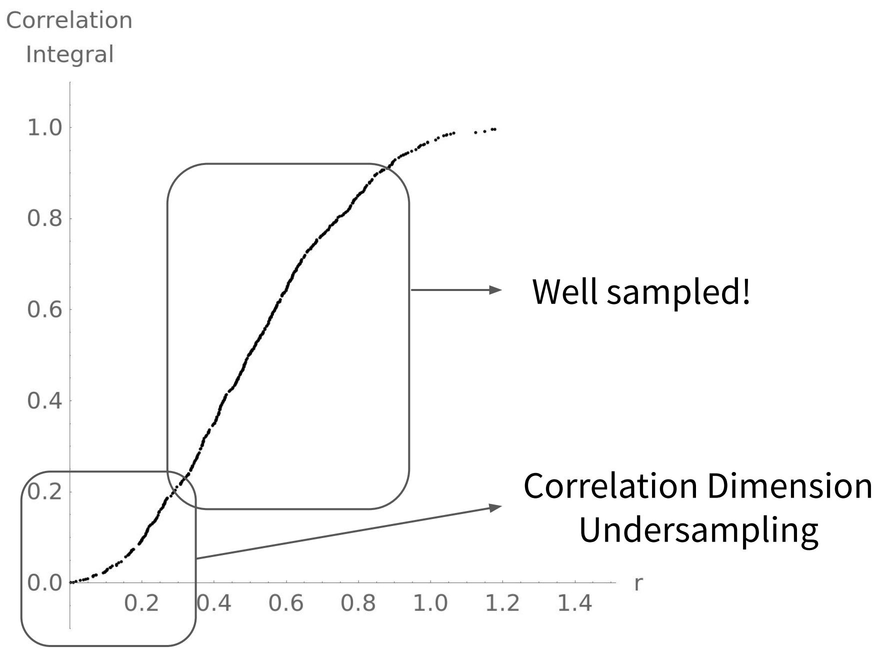
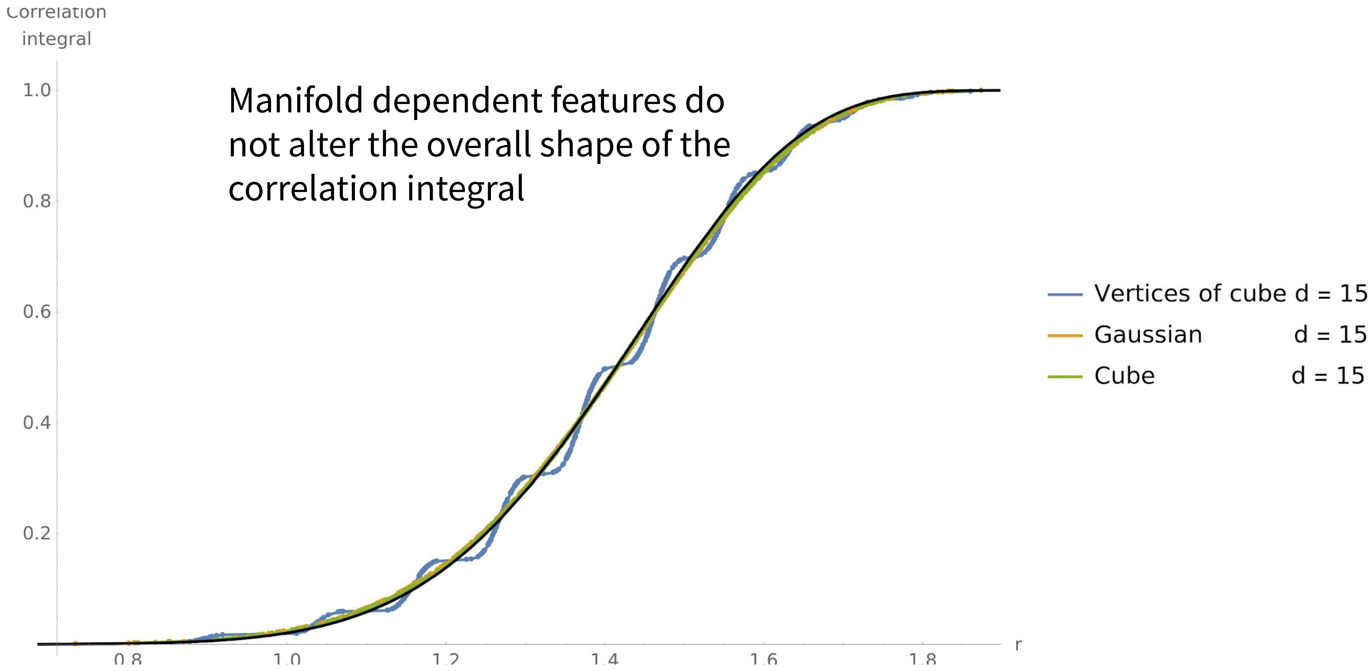


Linear manifolds
Isotropic sampling measure
Linear embeddings
$\Rightarrow$

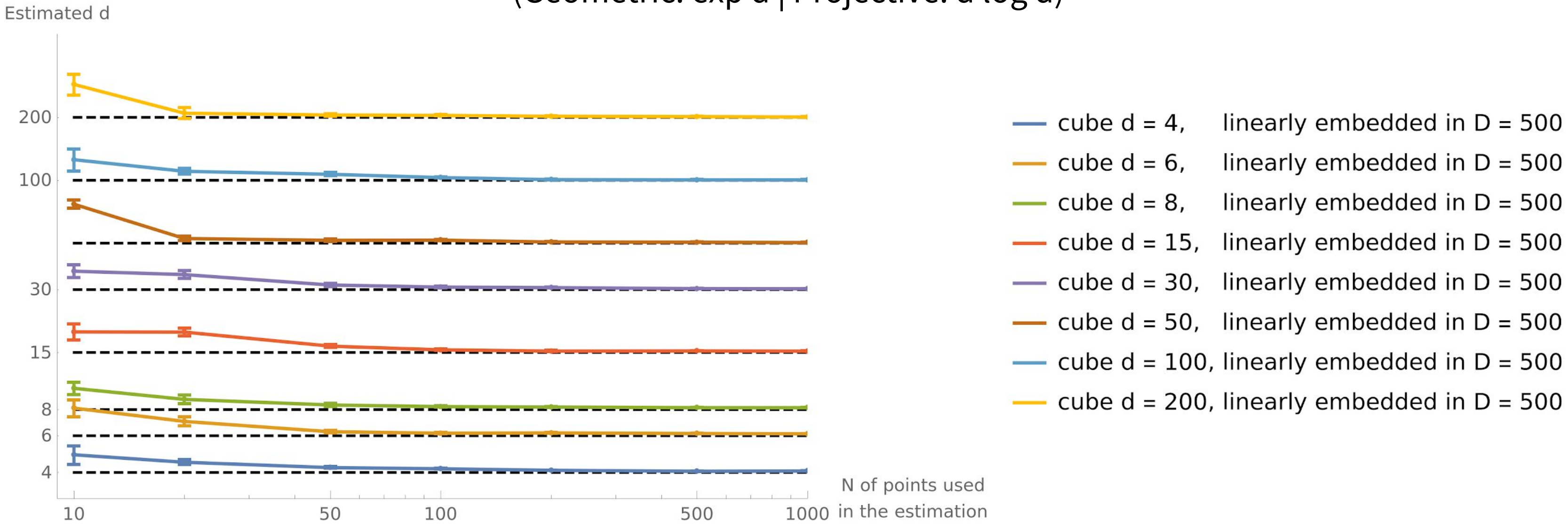
$$\rho(r;d) = \frac{1}{2} + \frac{\Omega_{d-1}}{\Omega_d}(r^2-2)\,{}_2F_1\left(\begin{matrix}\frac{1}{2}, 1-\frac{d}{2} \\ \frac{3}{2}\end{matrix}\middle|(r^2-2)^2\right)$$

The FCI estimator is robust to non idealities



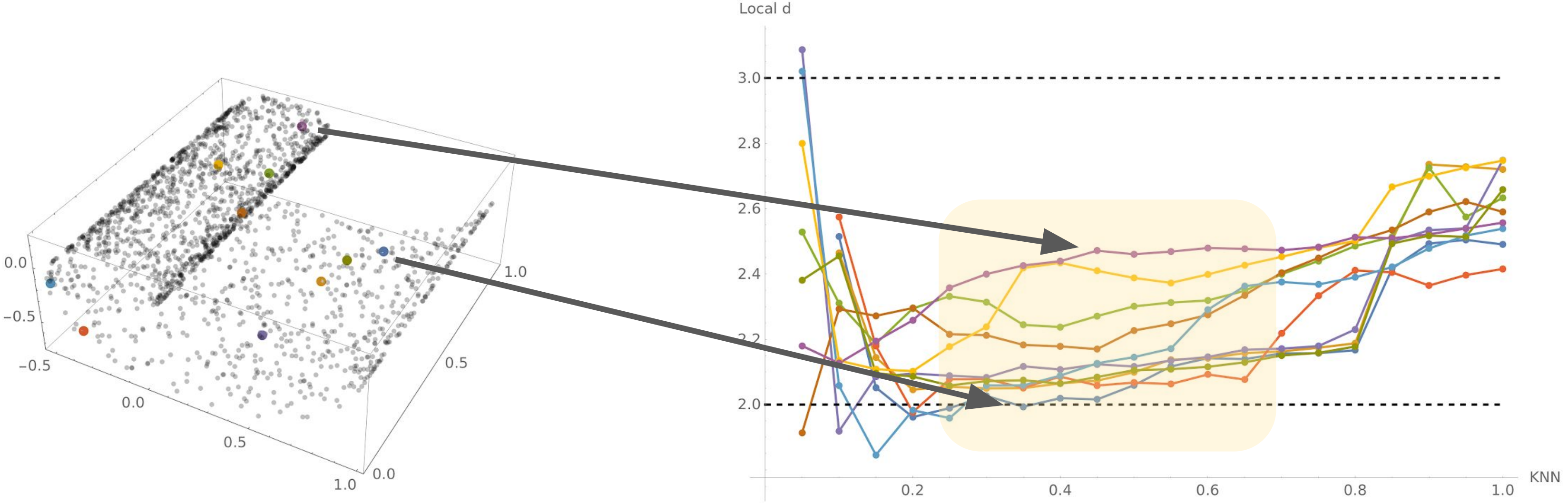
The FCI estimator is robust to undersampling

Able to estimate in the extreme
undersampled regime N < d
(Geometric: exp d | Projective: d log d)



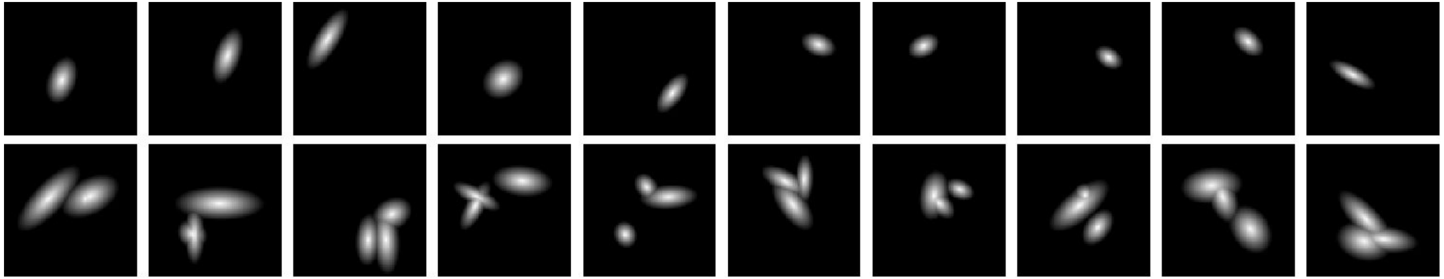
A multiscale generalization of the FCI

Thanks to robustness + extreme undersampling



Use the “most persistent” minimum as the estimator of the Intrinsic Dimension

The multiscale FCI estimator can deal with complex bitmap images

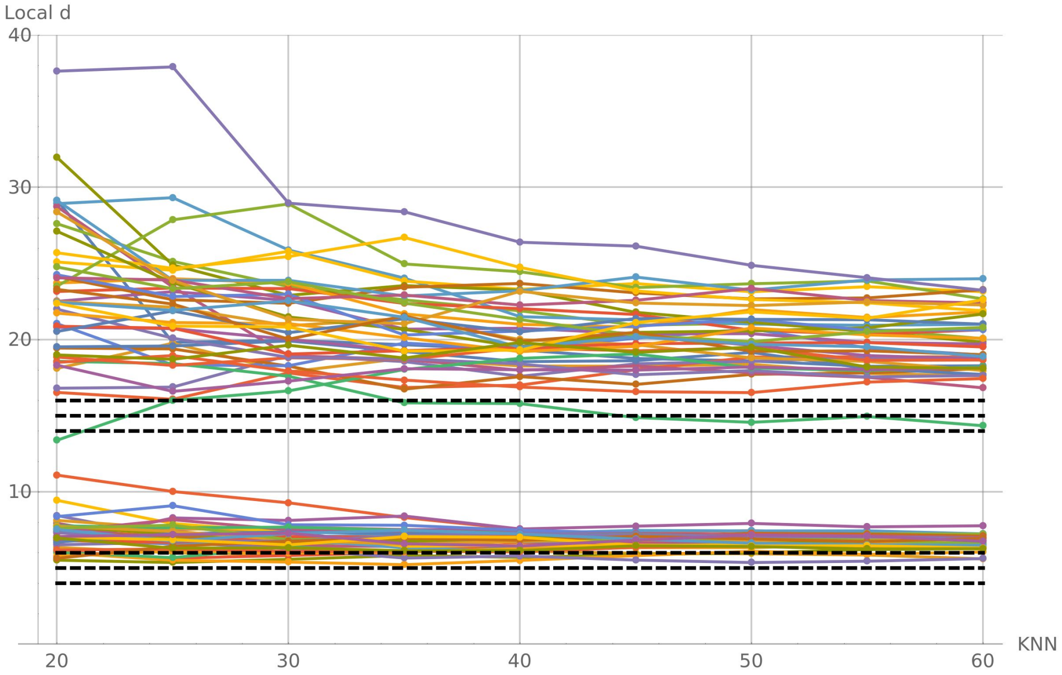


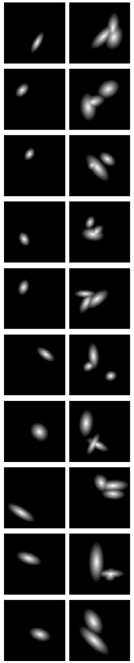
d=5
$D=81^2$

d=15
$D=81^2$

5 degrees of freedom per blob:
translation x, translation y, eccentricity,
scale, tilt

The multiscale FCI estimator can deal with complex bitmap images





The multiscale FCI is more versatile than other estimators

Estimator	$\mathcal{SR}_{2,3}$	$\mathcal{H}_{20,50} \cup \mathcal{H}_{30,50}$	$\mathcal{C}_{6,12}$	$\mathcal{B}_{5,81^2}$	$\mathcal{B}_{15,81^2}$
CorrDim [8]	1.98	12.53	5.93	5	13.5
Takens [10]	1.97	12.01	5.77	N.A.	N.A.
Hein et al. [13]	2	13	6	N.A.	N.A.
PCA	3	20 & 30	12	40	40
mPCA [24]	3	20 & 30	[9,12]	[2,10]	[6,31]
Multiscale FCI	2	20 & 30	6	5	15
Non linear	✓	✗	✓	✓	✓
High dimension	✗	✓	✗	✗	✓
Multidimensional	✗	✓	✗	✗	✗

Hybrid between local geometric methods and projective global methods

Easy multiscale generalization

Performant in a wide variety of situations

Thank you for your attention!

Learn more:

V.E., Marco Gherardi, Pietro Rotondo, "Intrinsic dimension estimation for locally undersampled data"
https://arxiv.org/abs/1906.07670 (accepted in Scientific Reports)