DSP - Laboratory 3

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Abstract

Third laboratory experience [1] for the course Digital Signal Processing (DSP) a.y. 21/22. Topic: z-transform.

Before starting with the exercises make sure you have installed **Signal Processing Toolbox** (https://www.mathworks.com/products/signal.html.).

1 Definition of the z-transform

A very important category of LTI systems is described by difference equations of the following type

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

From which, through z-transform we obtain

$$H(z) = \frac{\sum\limits_{k=0}^{M} b_k z^{-k}}{\sum\limits_{k=0}^{N} a_k z^{-k}}$$

where $H(z)$ is the transfer function of the system. In Matlab notation, as indexes must start from 1, if we consider the vectors a and b of the coefficients of the polynomials at numerator and denominator, after posing $n_b = length(b)$, $n_a = length(a)$, we will have a representation of $H(z)$ according to

$$Y(z) = \frac{b_0 + b_1 z^{-1} + ... + b_M z^{-M}}{a_0 + a_1 z^{-1} + ... + a_N z^{-N}} X(z)$$

The transfer function $H(z)$ is represented by means of the vectors a and b in several Matlab functions, as described in the following.

2 Matlab functions

2.1 zplane

The function zplane creates a plot of the positions of zeros and poles in the plane of the complex variable z, with the unit circle for reference, starting from the coefficients a and b. Each zero is represented with a 'o' and each pole with a 'x' on the plot. Multiple zeros and poles are indicated by the multiplicity number shown to the upper right of the zero or pole. The function is called as: zplane(b,a) where b and a are row vectors. It uses the function roots to calculate the roots of numerator and denominator of the transfer function.

Example 1: plot the positions of zeros and poles of the transfer function

$$H(z) = \frac{2 + 2z^{-1} + z^{-2}}{1 - 0.8z^{-1}}$$

[1] All information and material presented here is intended to be used for educational or informational purposes only.

2.2 impz

The function impz computes the impulse response of a system starting from the coefficients b and a. [h,t]=impz(b,a) produces the impulse response in vector h and the time axis in vector t. If the output arguments h and t are omitted, a plot of the impulse response is directly displayed. If the impulse response is of infinite length, only its initial part is computed.

Example 1: compute the impulse response of the system

$$H(z) = \frac{1}{1 - 0.9z^{-1}}$$

2.3 freqz

The function freqz is used to compute the frequency response of systems expressed by difference equations or rational transfer functions.

[H,w]=freqz(b,a,N); where N is a positive integer, returns the frequency response H and the vector w with the N angular frequencies at which H has been calculated (i.e. N equispaced points on the unit circle, between 0 and π). If N is omitted, a default value of 512 is assumed. If no output argument is specified, the amplitude plot and the phase plot of the frequency response are directly displayed.

[H,w]=freqz(b,a,w); where w is a vector of frequencies (in radians, e.g. w=-π:1/100:π;) computes the frequency response at the frequencies specified by w. This function can be used to evaluate the DTFT of a sequence x on any desired set of frequencies w, e.g. with the command [X,w]=freqz(x,1,w); See help freqz for a complete reference.

Example 1: compute the frequency response of

$$H(z) = \frac{1}{1 - 0.9z^{-1}}$$

that is a system with exponentially decaying impulse response $h[n] = (0.9)^n u[n]$.

Example 2: compute the frequency response of

$$h[n] = u[n] - u[n - L].$$

Note that if L is odd, $h[n + (L - 1)/2]$ is a real and symmetric sequence.

Exercise 3: compute the frequency response of

$$H(z) = \frac{1 - 0.9^8 z^{-8}}{1 - 0.9z^{-1}}$$

that is a system with truncated exponentially decaying impulse $h[n] = 0.9^n(u[n] - u[n - 8])$.

2.4 filter

The function filter implements the filtering of an input sequence x, starting from a transfer function H(z) expressed as ratio between polynomials in z^{-1} with coefficients given by vectors b (numerator) and a (denominator). The filter is applied to an input sequence x with the Matlab command y=filter(b,a,x);

Example 1: filter a delta pulse of length 100 with the filter

$$H(z) = \frac{1}{1 - 0.9z^{-1}}$$

that is a system with exponentially decaying impulse response $h[n] = (0.9)^n u[n]$.

2.5 tf2zp

The command [z,p,k]=tf2zp(b,a) finds zeros, poles and gain of the transfer function associated to coefficients b and a.

2.6 zp2tf

The command [b,a]=zp2tf(z,p,k) finds the coefficients b and a of the associated transfer function, given a set of zero locations in vector z, a set of pole locations in vector p, and a gain in scalar k.

2.7 Exercises

- Consider a low pass filter using pole-zero placement for which the poles are $p = [0.5; 0.45 + 0.5j; 0.45 - 0.5j]$ and the zeros are $z = [-1; i; -j]$:
  - Convert the pole-zero representation to a rational transfer function representation
  - Make a plot of the desired magnitude and phase response
- Design of a simple all-pass filter with complex conjugate poles $z_0 = 0.9e^{(j*0.1*\pi)}; z_1 = z_0'$.
- If we have two polynomials $X_1(z) = 2+3z^{-1}+4z^{-2}$ and $X_2(z) = 3+4z^{-1}+5z^{-2}+6z^{-3}$, express their product $X_3(z) = X_1(z)X_2(z)$ by means of the convolution of the sequences corresponding to the inverse Z-transforms, i.e. $x_1 = 2, 3, 4$ and $x_2 = 3, 4, 5, 6$

3 Partial fraction expansion

Given a rational transfer function

$$X(z) = \frac{b_0 + b_1 z^{-1} + ... + b_M z^{-M}}{a_0 + a_1 z^{-1} + ... + a_M z^{-M}} = \frac{B(z)}{A(z)}$$

$X(z)$ can be expressed, by means of partial fraction expansion, as

$$X(z) = \sum_{k=0}^{M-N} C_k z^{-k} + \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}}$$

We can derive such a partial fraction expansion by means of the Matlab function residuez.

[A,p,C]=residuez(b,a) computes the constants on the numerator (A_k, known as also residues), poles (p_k), and direct terms (C_k) of X(z). The returned column vector A contains the residues, column vector p contains the pole locations, and row vector C contains the polynomials terms when $\geq N$.

Example 1: Derive analytically and plot the impulse response of the system with transfer function

$$H(z) = \frac{z^{-1}}{3 - 4z^{-1} + z^{-2}} = \frac{0 + z^{-1}}{3 - 4z^{-1} + z^{-2}} |z| > 1$$

4 Exercises

4.1 Exercise 1

Given a causal system specified by the difference equation

$$y[n] = x[n-1] - 1.2x[n-2] + x[n-3] + 1.3y[n-1] - 1.04y[n-2] + 0.222y[n-3]$$

- plot the frequency response (amplitude and phase) using freqz;
- check zero and pole positions using zplane;
- determine zeros and poles using roots, or using [z,p,k]=tf2zp(b,a);
- plot the impulse response using impz.

4.2 Exercise 2

Given the difference equation

$$y[n] = -0.4y[n-1] + 0.12y[n-2] + x[n] + 2x[n-1]$$

- compute:
  - the corresponding transfer function $H(z)$
  - the partial-fraction expansion of H(z), using [r,p,k]=residuez(b,a)
  - the consequent analytic expression of the impulse response h[n]
- compare the plot of the computed h[n] with the one produced by means of impz;
- compare also with the sequence y produced by x=[1 zeros(1,N)]; y=filter(b,a,x).

4.3 Exercise 3

Compute and plot the frequency response of

$$H(z) = \frac{0.15(1 - z^{-2})}{0.7 - 0.7z^{-1} + 0.7z^{-2}}$$

for $0 \geq \omega \leq \pi$. What type of filter does it represent? Check what changes if instead we use

$$H_!(z) = \frac{0.15(1 - z^{-2})}{0.7 - 0.5z^{-1} + z^{-2}}$$

Make use of freqz, zplane, impz.