Home Exercises 1

1/

$$A = \begin{bmatrix} 2 & \boxed{5} & -1 & 3 & 6 \\ 1 & 0 & \boxed{0} & -2 & 0 \\ 4 & 1 & -2 & \boxed{0} & 7 \\ 0 & 3 & 5 & 1 & \boxed{-1} \end{bmatrix}$$

A is a 4×5 matrix.

$A(2,3) = \underline{\underline{0}}$

$A(3,4) = a_{34} = \underline{\underline{0}}$

$A(1,2) = \underline{\underline{5}}$

$A(4,5) = a_{45} = \underline{\underline{-1}}$

2) Let A be a 4×3 matrix defined

$$A(i,j) = \begin{cases} i - j & , \quad i < j \\ 0 & , \quad i = j \\ i + 2j & , \quad i > j \end{cases}$$

Write A in matrix form.

$$A = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \quad j \\ \begin{array}{c} i \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1-2 & 1-3 \\ 2+2\cdot 1 & 0 & 2-3 \\ 3+2\cdot 1 & 3+2\cdot 2 & 0 \\ 4+2\cdot 1 & 4+2\cdot 2 & 4+2\cdot 3 \end{bmatrix} \end{array} = \begin{bmatrix} 0 & -1 & -2 \\ 4 & 0 & -1 \\ 5 & 7 & 0 \\ 6 & 8 & 10 \end{bmatrix}$$

3/ a/ $1 + 2 + 3 + \cdots + 100 = \sum_{k=1}^{100} k = \left[\sum_{k=0}^{99} (k+1) \right]$

$\left[= 50 \cdot 101 = 5050 \right]$

b/ $\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots + \frac{1}{10}$

$= \sum_{k=2}^{10} \frac{(-1)^k}{k}$

$\boxed{\begin{array}{l} a_{33} = A(3,3) \\ a_{11,11} = A(11,11) \end{array}}$

c/ $a_{11} + a_{22} + \cdots + a_{nn} = \sum_{i=1}^{n} a_{ii}$

d) $a_{k1} x_1 + a_{k2} x_2 + a_{k3} x_3 + \cdots + a_{kn} x_n$

$$= \sum_{j=1}^{n} a_{kj} x_j$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{k1} & a_{k2} & \cdots & a_{kn} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \\ x_n \end{bmatrix} = \begin{bmatrix} \vdots \\ \sum_{j=1}^{n} a_{kj} x_j \\ \vdots \end{bmatrix}$$

e) $a_{k1} b_{1p} + a_{k2} b_{2p} + a_{k3} b_{3p} + \dots + a_{kn} b_{np}$

$$= \sum_{j=1}^{n} a_{kj} \, b_{jp}$$

$$\begin{array}{c} \\ k \end{array}
\begin{bmatrix} & \vdots & & \\ a_{k1} & a_{k2} & \dots & a_{kn} \\ & \vdots & & \end{bmatrix}
\begin{array}{c} p \\ \begin{bmatrix} & b_{1p} & \\ & b_{2p} & \\ \dots & & \dots \\ & \vdots & \\ & b_{np} & \end{bmatrix} \end{array}
= \begin{bmatrix} \boxed{\ } \end{bmatrix}$$

3)

$$A = \begin{bmatrix} 2 & 5 & -1 & 3 & 6 \\ 1 & 0 & 0 & -2 & 0 \\ 4 & 1 & -2 & 0 & 7 \\ 0 & 3 & 5 & 1 & -1 \end{bmatrix}$$

a) $\sum_{j=1}^{5} A(3,j) = 4 + 1 - 2 + 0 + 7 = \underline{\underline{10}}$

b) $\sum_{k=1}^{4} A(k,k) = 2 + 0 - 2 + 1 = \underline{\underline{1}}$

c) $\sum_{i=1}^{4} A(i,1)\,A(i,3) = 2\cdot(-1) + 1\cdot 0 + 4\cdot(-2) + 0\cdot 5$

$= -2 + 0 - 8 + 0 = \underline{\underline{-10}}$

5)

$$A = \begin{bmatrix} 1 & -3 \\ 2 & 4 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} a & 3 \\ 1 & -2 \\ x & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & -5 & x \end{bmatrix}$$

a) $$A + B = \begin{bmatrix} 1+a & -3+3 \\ 2+1 & 4-2 \\ 0+x & -1+2 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 1+a & 0 \\ 3 & 2 \\ x & 1 \end{bmatrix}}}$$

b) $$5A = \begin{bmatrix} 5\cdot 1 & 5\cdot(-3) \\ 5\cdot 2 & 5\cdot 4 \\ 5\cdot 0 & 5\cdot(-1) \end{bmatrix} = \underline{\underline{\begin{bmatrix} 5 & -15 \\ 10 & 20 \\ 0 & -5 \end{bmatrix}}}$$

c, $\underset{3\times 1}{C} + \underset{1\times 3}{D}$ not defined

d, $\underset{3\times 2}{A} + \underset{3\times 1}{C}$ not defined

e, $$-B = \begin{bmatrix} -a & -3 \\ -1 & -(-2) \\ -x & -2 \end{bmatrix} = \begin{bmatrix} -a & -3 \\ -1 & 2 \\ -x & -2 \end{bmatrix}$$

f, $$B - A = \begin{bmatrix} a-1 & 3-(-3) \\ 1-2 & -2-4 \\ x-0 & 2-(-1) \end{bmatrix} = \begin{bmatrix} a-1 & 6 \\ -1 & -6 \\ x & 3 \end{bmatrix}$$

6)

$$A = \underbrace{\begin{bmatrix} 1 & 3 & 5 & 1 \end{bmatrix}}_{1 \times 4}, \quad B = \underbrace{\begin{bmatrix} -1 \\ 3 \\ 2 \\ 4 \end{bmatrix}}_{4 \times 1}$$

$$AB = \underbrace{\begin{bmatrix} 1 & 3 & 5 & 1 \end{bmatrix}}_{1 \times 4} \underbrace{\begin{bmatrix} -1 \\ 3 \\ 2 \\ 4 \end{bmatrix}}_{4 \times 1}$$

$$= \begin{bmatrix} 1 \cdot (-1) + 3 \cdot 3 + 5 \cdot 2 + 1 \cdot 4 \end{bmatrix} = \begin{bmatrix} -1 + 9 + 10 + 4 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 22 \end{bmatrix}}}$$

$$BA = \begin{bmatrix} -1 \\ 3 \\ 2 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1\cdot 1 & -1\cdot 3 & -1\cdot 5 & -1\cdot 1 \\ 3\cdot 1 & 3\cdot 3 & 3\cdot 5 & 3\cdot 1 \\ 2\cdot 1 & 2\cdot 3 & 2\cdot 5 & 2\cdot 1 \\ 4\cdot 1 & 4\cdot 3 & 4\cdot 5 & 4\cdot 1 \end{bmatrix} = \underline{\begin{bmatrix} -1 & -3 & -5 & -1 \\ 3 & 9 & 15 & 3 \\ 2 & 6 & 10 & 2 \\ 4 & 12 & 20 & 4 \end{bmatrix}}$$

7)

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -3 & 2 & -1 \\ -2 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1\cdot 1 - 1\cdot 2 + 1\cdot 1 & 1\cdot 2 - 1\cdot 4 + 1\cdot 2 & 1\cdot 3 - 1\cdot 6 + 1\cdot 3 \\ -3\cdot 1 + 2\cdot 2 - 1\cdot 1 & -3\cdot 2 + 2\cdot 4 - 1\cdot 2 & -3\cdot 3 + 2\cdot 6 - 1\cdot 3 \\ -2\cdot 1 + 1\cdot 2 + 0\cdot 1 & -2\cdot 2 + 1\cdot 4 + 0\cdot 2 & -2\cdot 3 + 1\cdot 6 + 0\cdot 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

$$BA = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ -3 & 2 & -1 \\ -2 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1\cdot 1 + 2\cdot(-3) + 3\cdot(-2) & 1\cdot(-1) + 2\cdot 2 + 3\cdot 1 & 1\cdot 1 + 2\cdot(-1) + 3\cdot 0 \\ 2\cdot 1 + 4\cdot(-3) + 6\cdot(-2) & 2\cdot(-1) + 4\cdot 2 + 6\cdot 1 & 2\cdot 1 + 4\cdot(-1) + 6\cdot 0 \\ 1\cdot 1 + 2\cdot(-3) + 3\cdot(-2) & 1\cdot(-1) + 2\cdot 2 + 3\cdot 1 & 1\cdot 1 + 2\cdot(-1) + 3\cdot 0 \end{bmatrix}$$

$$= \begin{bmatrix} -11 & 6 & -1 \\ -22 & 12 & -2 \\ -11 & 6 & -1 \end{bmatrix} \neq 0$$

Transpose

Ex

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

$$\boxed{A^T(i, j) = A(j, i)}$$

$$\boxed{\begin{aligned} &\text{(i)} \quad (A+B)^T = A^T + B^T \\ &\text{(ii)} \quad (A^T)^T = A \\ &\text{(iii)} \quad (AB)^T = B^T A^T \end{aligned}}$$

Ex.

$$\underbrace{\big(\underbrace{\underset{3\times 2}{A} \quad \underset{2\times 5}{B}}_{3\times 5}\big)^T}_{5\times 3} = \underbrace{\underset{5\times 2}{B^T} \quad \underset{2\times 3}{A^T}}_{5\times 3}$$

Inverse Matrix

Identity matrix I

$$\boxed{AI = IA = A \quad \text{for any matrix } A}$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\underline{I}(i,j) = \begin{cases} 1 & , \text{ if } i = j \\ 0 & , \text{ if } i \neq j \end{cases}$$

<u>Ex</u> a/ $\frac{5}{3} = 5 \cdot \frac{1}{3} = 5 \cdot 3^{-1}$

b/ $3 \cdot 3^{-1} = 3 \cdot \frac{1}{3} = \frac{3}{3} = 1$

the inverse number of 3

Let A be a square matrix. If there exists a matrix B such that

$$\boxed{AB = BA = I}$$

then B is the inverse of A and we denote $B = A^{-1}$.

If a square matrix A has an inverse matrix, then the inverse is unique.

If $AB = BA = I$ and $AC = CA = I$, then

$$B = BI = B(AC) = (BA)C = IC = C \quad \blacksquare$$

Example

$$\begin{cases} 3x + 2y + z = 1 \\ -x + 5y + 2z = 9 \\ 2x + y - z = 5 \end{cases}$$

$$A = \begin{bmatrix} 3 & 2 & 1 \\ -1 & 5 & 2 \\ 2 & 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 9 \\ 5 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$AX = \begin{bmatrix} 3 & 2 & 1 \\ -1 & 5 & 2 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 3x + 2y + z \\ -x + 5y + 2z \\ 2x + y - z \end{bmatrix}}_{3 \times 1} \overset{\downarrow}{=} \underbrace{\begin{bmatrix} 1 \\ 9 \\ 5 \end{bmatrix}}_{3 \times 1} = B$$

$$\begin{cases} 3x + 2y + z = 1 \\ -x + 5y + 2z = 9 \\ 2x + y - z = 5 \end{cases}$$

$$\Leftrightarrow \boxed{AX = B} \quad \Big| A^{-1} \cdot$$

$$\Leftrightarrow A^{-1}AX = A^{-1}B$$

$$\Leftrightarrow IX = A^{-1}B$$

$$\Leftrightarrow \boxed{X = A^{-1}B}$$

If A has an inverse A^{-1}

$$\boxed{3x = 5} \quad \Big| \frac{1}{3} \cdot$$

$$\Leftrightarrow \frac{1}{3} \cdot 3x = \frac{1}{3} \cdot 5$$

$$\Leftrightarrow 1 \cdot x = \frac{5}{3}$$

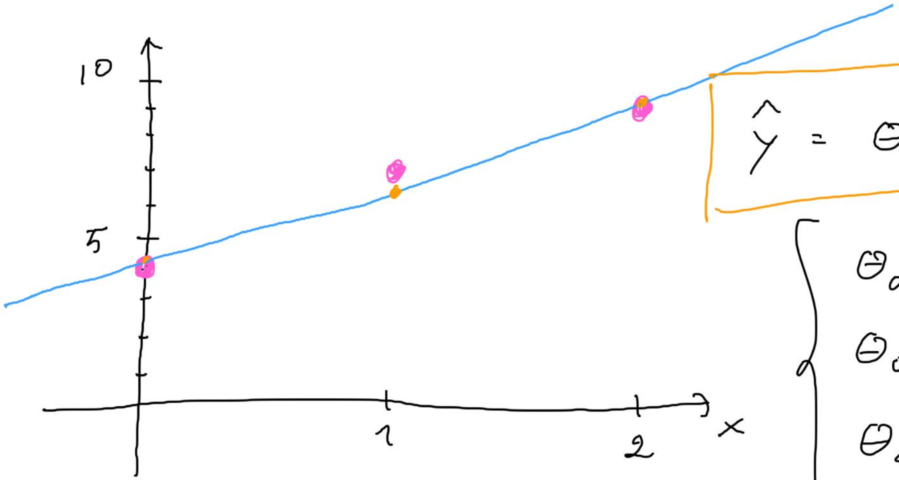
$$\Leftrightarrow \boxed{x = \frac{5}{3}}$$

Function	Python
$\sin(x)$	np.sin(x)
$\cos(x)$	np.sin(x)
x^3	x**3
e^x	np.exp(x)
$\ln(x)$	np.log(x)
$\sqrt{x}$	np.sqrt(x)
$\lvert x \rvert$	np.abs(x)

Linear Regression

Example

i	1	2	3
x_i	0	1	2
y_i	4	7	9



$$\boxed{\hat{y} = \theta_0 + \theta_1 x}$$

$$\begin{cases} \theta_0 + \theta_1 x_1 = y_1 \\ \theta_0 + \theta_1 x_2 = y_2 \\ \theta_0 + \theta_1 x_3 = y_3 \end{cases}$$

$$\begin{cases} \theta_0 \cdot 1 + \theta_1 x_1 = y_1 \\ \theta_0 \cdot 1 + \theta_1 x_2 = y_2 \\ \theta_0 \cdot 1 + \theta_1 x_3 = y_3 \end{cases}$$

$$\Leftrightarrow \underbrace{\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \end{bmatrix}}_{X} \underbrace{\begin{bmatrix} \theta_0 \\ \theta_1 \end{bmatrix}}_{M} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}}_{y}$$

$$\Leftrightarrow XM = y$$

X is not a square matrix, and therefore it does not have an inverse.

$$\underset{3\times 2}{X}\,\underset{2\times 1}{M} = \underset{3\times 1}{y} \qquad \Big|\, \underset{2\times 3}{X^T}\cdot$$

$$\Rightarrow \underbrace{\underset{2\times 3}{X^T}\,\underset{3\times 2}{X}}_{2\times 2}\,\underset{2\times 1}{M} = \underset{2\times 3}{X^T}\,\underset{3\times 1}{y} \qquad \Big|\, (X^T X)^{-1}\cdot$$

$$\Leftrightarrow (X^T X)^{-1}(X^T X)M = (X^T X)^{-1}X^T y$$

$$\Leftrightarrow I M = (X^T X)^{-1}X^T y$$

$$\Leftrightarrow \boxed{M = (X^T X)^{-1}X^T y}$$

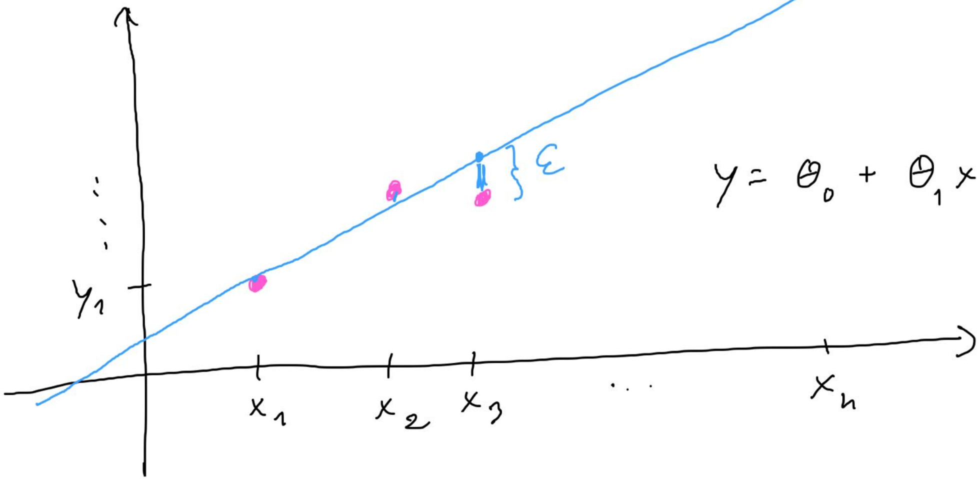
i	x_i	y_i
1	0	4
2	1	7
3	2	9

$$X = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad y = \begin{bmatrix} 4 \\ 7 \\ 9 \end{bmatrix}$$

Design matrix

$$\boxed{M = (X^T X)^{-1} X^T y}$$

Linear Regression



Random error

$$y = \theta_0 + \theta_1 x + \underbrace{\varepsilon}_{\text{Random error}}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

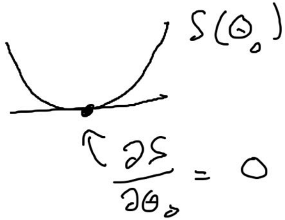
We try to find a functions

$$h(x) = \theta_0 + \theta_1 x$$

which minimizes the square error

$$\boxed{S = \sum_{i=1}^{n} \left(h(x_i) - y_i \right)^2 = \sum_{i=1}^{n} \left(\theta_0 + \theta_1 x_i - y_i \right)^2}$$

$$\begin{cases} \dfrac{\partial}{\partial \theta_0} \sum_{i=1}^{n} (\theta_0 + \theta_1 x_i - y_i)^2 = 0 \\ \dfrac{\partial}{\partial \theta_1} \sum_{i=1}^{n} (\theta_0 + \theta_1 x_i - y_i)^2 = 0 \end{cases}$$



$\vdots$

$$\begin{cases} \theta_0 = ? \\ \theta_1 = ? \end{cases}$$

$$
\begin{cases}
\theta_0 + \theta_1 x_1 = y_1 \\
\theta_0 + \theta_1 x_2 = y_2 \\
\theta_0 + \theta_1 x_3 = y_3 \\
\vdots \\
\theta_0 + \theta_1 x_n = y_n
\end{cases}
$$

$$
\boxed{X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}} \quad M = \begin{bmatrix} \theta_0 \\ \theta_1 \end{bmatrix} \quad \boxed{Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix}}
$$

$$XM = \begin{bmatrix} 1 \cdot \theta_0 + x_1 \theta_1 \\ 1 \cdot \theta_0 + x_2 \theta_1 \\ \vdots \\ 1 \cdot \theta_0 + x_n \theta_1 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = Y$$

You can't solve this exactly because X is not a square matrix and therefore does not have an inverse.

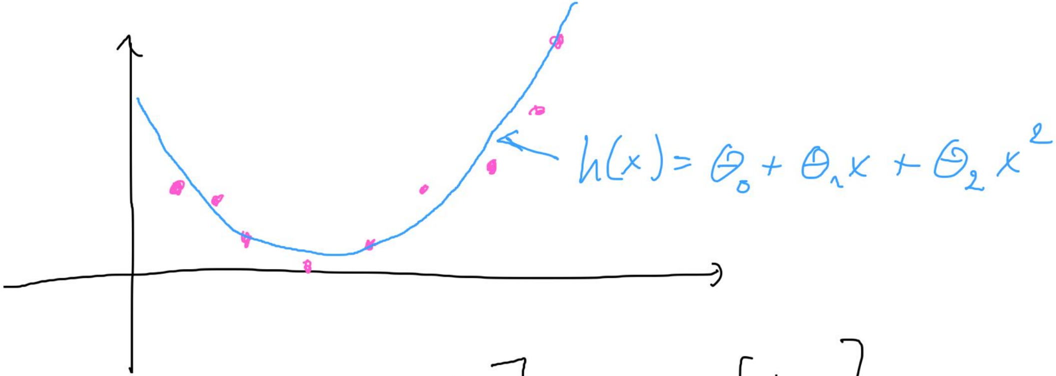
$$XM = y \qquad \mid X^T \cdot$$

$$\Rightarrow \quad X^T X M = X^T y \qquad \mid (X^T X)^{-1} \cdot$$

$$\Leftrightarrow \quad \underbrace{(X^T X)^{-1} (X^T X)}_{I} M = (X^T X)^{-1} X^T y$$

$$\Leftrightarrow \quad \boxed{M = (X^T X^{-1}) X^T y}$$

Regression



$$M = \begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \end{bmatrix} \quad X = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \\ \vdots & \vdots & \vdots \\ 1 & x_n & x_n \end{bmatrix} \quad y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix}$$

$$XM = y$$

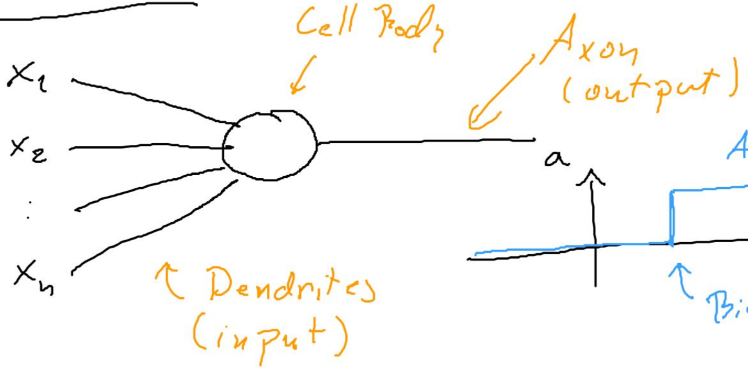
$$\Rightarrow \quad X^T X M = X^T y$$

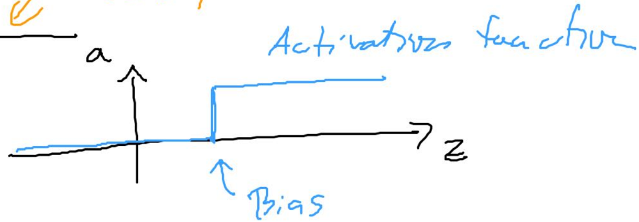
$$\Leftrightarrow \quad (X^T X)^{-1}(X^T X) M = (X^T X)^{-1} X^T y$$

$$\Leftrightarrow \quad \boxed{M = (X^T X)^{-1} X^T y}$$

Neural Networks

Simplified Neuron

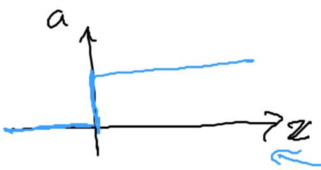


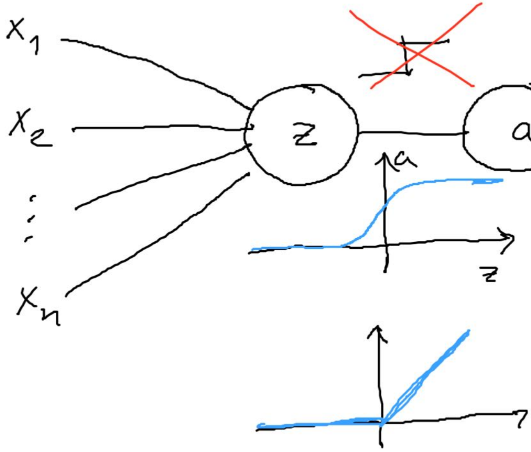


$$z = w_1 x_1 + w_2 x_2 + \dots + w_n x_n$$

$$z = w_0 + w_1 x_1 + w_2 x_2 + \dots + w_n x$$

Bias

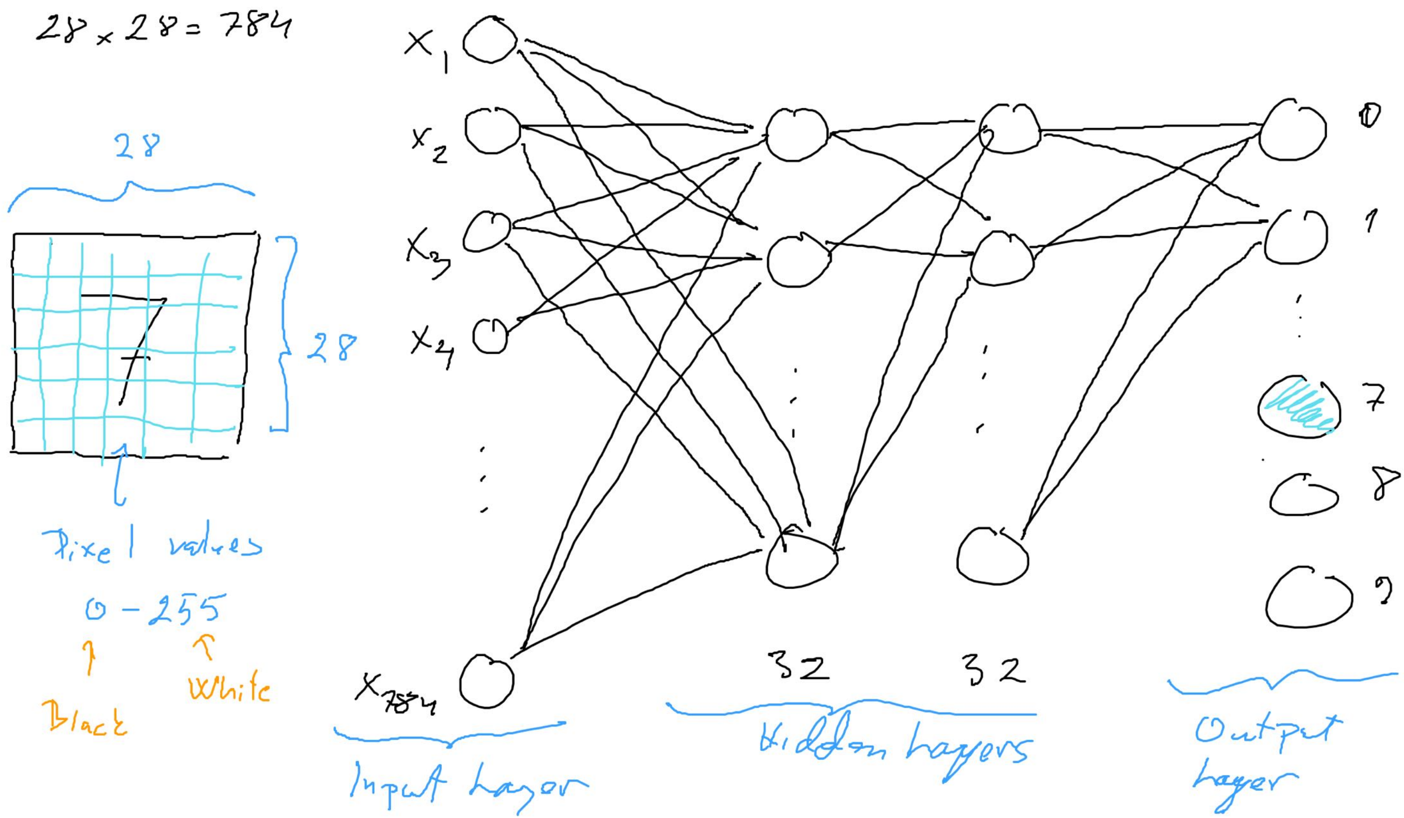




Sigmoid $g(x) = \frac{1}{1 - e^{-z}}$

ReLU $g(x) = \begin{cases} 0 & , x < 0 \\ x & , x \geq 0 \end{cases}$

$$\boxed{\begin{aligned} z &= \Theta_0 + \Theta_1 x_1 + \Theta_2 x_2 + \cdots + \Theta_n x_n \\ a &= g(z) = \frac{1}{1 - e^{-x}} \end{aligned}}$$



28 x 28 = 784
28
28
Pixel values
0 - 255
Black
White
X1
X2
X3
X4
X784
32
32
0
1
7
8
9
Input Layer
Hidden Layers
Output Layer