Szeminárium - 2022. 04. 21.

1.) hogyan szerkesztünk modellezésre alkalmas bázisfüggvényeket

interp. feltételek
$$f_{n,i}^{(j)}(\alpha)=0,\ j=0,1,\dots i-1,\quad f_{n,i}^{(i)}(\alpha)>0$$
$$f_{n,n-i}^{(j)}(\beta)=0,\ j=0,1,\dots i-1,\quad (-1)^i f_{n,n-i}^{(i)}(\beta)>0$$

LinearCombination3 :: UpdateDataForInterpolation

$$\mathcal{F}_n=\{f_{n,i}(u): u\in[\alpha,\beta]\}_{i=0}^{n}\subset C^n([\alpha,\beta],[0,1])$$

j - deriválási rend

$$f_{n,i}(u)=f_{n,n-i}(\alpha+\beta-u),\ \forall u\in[\alpha,\beta],\ \forall i=\overline{0,n}$$ ← intervallum tükrözése

↳ szimm. tul., nemnegatív, egységfelbontást alkotó

$$\sum_{i=0}^{n}$$

- nem érdekel az algebrai alak

☆ igazoljuk, hogy lineárisan független!

$$F=\{F_i: [a,b]\to\mathbb{R}\}_{i=0}^{n}$$

$$\sum_{i=0}^{n}\lambda_i F_i(u)=0\quad \forall u\in[a,b]$$

~~- redu~~

- reductio ad absurdum:

1) a.) tegyük fel, hogy az adott függv. rendszer $\{f_{n,i}(u) : u \in [\alpha, \beta]\}_{i=0}^{n}$ lin. függő, vagyis léteznek nem azonosan nulla $\{\lambda_i\}_{i=0}^{n} \in \mathbb{R}$ valós skalárok, amelyre fennáll

$$\lambda(u) := \sum_{i=0}^{n} \lambda_i f_{n,i}(u) = 0 \quad \forall u \in [\alpha, \beta]$$

ez az azonosság

• vegyük észre

$$0 = \lambda^{(0)}(\alpha) = \sum_{i=0}^{n} \lambda_i f_{n,i}(\alpha) \overset{(1)}{=}$$

$$\overset{(1)}{=} \lambda_0 f_{n,0}(\alpha) + \lambda_1 \cdot 0 + \lambda_2 \cdot 0 + \ldots + \lambda_n \cdot 0 =$$

$$= \lambda_0 \cdot \underbrace{f_{n,0}^{(0)}(\alpha)}_{>0} \Rightarrow \lambda_0 = 0 \Rightarrow$$

$$\overset{(1)}{\Rightarrow} \lambda(u) = \sum_{i=1}^{n} \lambda_i f_{n,i}(u) \Rightarrow$$

$$\Rightarrow \lambda^{(1)}(u) = \sum_{i=1}^{n} \lambda_i f^{(1)}_{n,i}(u)$$

• vegyük észre, hogy

$$\lambda^{(1)}(\alpha) = \lambda_1 \cdot f^{(1)}_{n,1}(\alpha) + \lambda_2 \cdot 0 + \ldots + \lambda_n \cdot 0 =$$

$$= \lambda_1 \cdot \underbrace{f^{(1)}_{n,1}(\alpha)}_{>0} \overset{(1)}{\Rightarrow}$$

$$\Rightarrow \lambda_1 = 0$$

$\Rightarrow$ iteratívan folytatva az eljárást, az utolsó előtti ($(n-1).$) lépésben a $\lambda(u) = \sum_{i=n}^{n} \lambda_i f_{n,i}(u)$ alakot kapjuk, amelyet n-edrendben deriválva,

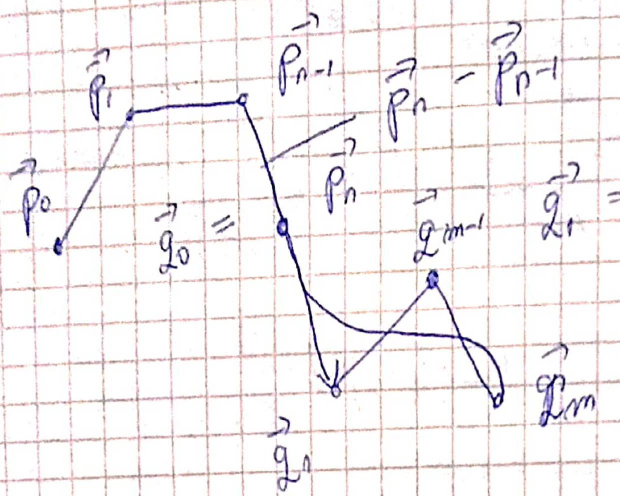
$$0 = \lambda^{(n)}(\alpha) = \lambda_n \cdot \underbrace{f^{(n)}_{n,n}(\alpha)}_{>0} \Rightarrow \lambda_n = 0$$

• azaz ellentmondáshoz jutottunk, ⇒ rossz a feltétel ⇒
⇒ lin függetlenek ⇒ $\{f_{n,i}(u) : u \in [\alpha, \beta]\}_{i=0}^{n}$

más $A(u) = \sum_{i=0}^{n} a_{n-i} f_{n,n-i}(u)$

$$A^{(0)}(\beta) = \sum_{i=0}^{n} a_{n-i} \cdot f_{n,n-i}^{(0)}(\beta) = a_n \cdot f_{n,n}^{(0)}(\beta) + 0 + \ldots + 0$$

$\Rightarrow a_n = 0$



$\vec{q}_1 = \vec{q}_0 + (\vec{p}_n - \vec{p}_{n-1}) = 2\vec{p}_n - \vec{p}_{n-1}$

• $g_m = \{g_{m,i}(u) : u \in [\gamma, \delta]\}_{i=0}^{m}$

$$l_n(u) = \sum_{i=0}^{n} p_i f_{n,i}(u),\ u \in [\alpha, \beta]$$

$$r_m(u) = \sum_{i=0}^{m} q_i g_m(u),\ u \in [\gamma, \delta]$$

$$\begin{cases} C^0: \vec{l}_n(\beta) = r_m(\gamma) \\ C^1: \vec{l}_n^{(1)}(\beta) = \vec{r}_m(\gamma) \end{cases}$$

$$\vec{l}_n(\beta) = \sum_{i=0}^{n} \vec{p}_i f_{n,i}(\beta) = \sum_{i=0}^{n} \vec{p}_{n-i} \cdot f_{n,n-i}(\beta) =$$

$$\overset{(2)}{=} \vec{p}_n \cdot f_{n,n}(\beta) + \vec{p}_{n-1} \underbrace{f_{n,n-1}(\beta)}_{0} + \ldots + \vec{p}_0 \underbrace{f_{n,0}(\beta)}_{0} =$$

$$= \vec{p}_n \cdot f_{n,n}(\beta) \overset{\text{egys. f. tul.}}{=} 1$$

$$\vec{r}_m(\mu) = \sum_{i=0}^{m} \vec{q}_i \cdot g_{m,i}(\mu) = \vec{p}_0$$

$$= \vec{q}_0 \cdot \underbrace{g_{m,0}(\mu)}_{1} + \vec{0} + \dots + \vec{0} =$$

$$= \vec{q}_0$$

$$\vec{l}_n(\beta) \overset{\vec{p}_n}{=} \vec{r}_m(\mu) = \vec{q}_0$$

$$\vec{l}_n^{(1)}(\beta) = \sum_{i=0}^{n} \vec{p}_{n-i} \cdot f_{n,n-i}^{(1)}(\beta) =$$

$$= \vec{p}_n \cdot f_{n,n}^{(1)}(\beta) + \vec{p}_{n-1} \cdot \underbrace{f_{n,n-1}^{(1)}(\beta)}_{\neq 0} + \vec{p}_{n-2} \cdot \underbrace{f_{n,n-2}^{(1)}(\beta)}_{0} + \dots$$

$$+ \vec{p}_0 \cdot f_{n,0}^{(1)}(\beta) = (*)$$

$$= \sum_{i=0}^{n} f_{n,n-i}(u) = 1 \quad \forall u \in [\alpha, \beta]$$

$$\sum_{i=0}^{n} f_{n,n-i}^{(1)}(u) = 0$$

$$u = \beta: \quad f_{n,n}^{(1)}(\beta) + f_{n,n-1}^{(1)}(\beta) + 0 + \dots + 0 = 0 \Rightarrow$$

$$\Rightarrow f_{n,n}^{(1)}(\beta) = - \underbrace{f_{n,n-1}^{(1)}(\beta)}_{>0}$$

$$(*) = \vec{p}_n \left(- f_{n,n-1}^{(1)}(\beta)\right) + \vec{p}_{n-1} \cdot f_{n,n-1}^{(1)}(\beta) =$$

$$= \underbrace{- f_{n,n-1}^{(1)}(\beta)}_{>0} \cdot (\vec{p}_n - \vec{p}_{n-1})$$

...

$$g_{m,0}^{(1)}(\mu) = - g_{m,1}^{(1)}(\mu) \Rightarrow$$

$$\Rightarrow \vec{r}_m^{(1)}(\mu) = g_{m,1}^{(1)}(\mu) \cdot (\vec{q}_1 - \vec{q}_0)$$

Scanned with CamScanner

$$-f^{(1)}_{n,n-1}(\beta)\,(\vec{p}_n-\vec{p}_{n-1}) = g^{(1)}_{m,1}\,(\vec{q}_1-\underbrace{\vec{q}_0}_{\vec{p}_n})$$

① → osztás után

$$\boxed{\vec{p}_n-\frac{f^{(1)}_{n,n-1}(\beta)}{g^{(1)}_{m,1}(\gamma)}(\vec{p}_n-\vec{p}_{m-1})=\vec{q}_1}$$

helyzetvektor a $\vec{q}_1$-re nézve

ugyanolyan függvénynél:

$n=m$

$[\alpha,\beta]=[\gamma,\delta]$

$f_{n,i}(u)=g_{m,i}(u)\quad \forall u\in[\alpha,\beta]=[\gamma,\delta]$

(szimm) $g^{(1)}_{m,1}(\alpha)=-g^{(1)}_{m,m-1}(\underbrace{\alpha+\beta-\alpha}_{\beta})=-f^{(1)}_{n,n-1}(\beta)$

$\Rightarrow \vec{q}_1=\vec{q}_0\,(\vec{p}_n-\vec{p}_{n-1})=2\vec{p}_n-\vec{p}_{n-1}\Rightarrow$

$\Rightarrow \vec{p}_m-\frac{f^{(1)}_{n,n-1}(\beta)}{g^{(1)}_{m,1}(\gamma)}(\vec{p}_m-\vec{p}_{n-1})=$

$=\vec{p}_m+\vec{p}_m-\vec{p}_{m-1}=\boxed{2\vec{p}_m-\vec{p}_{n-1}=\vec{q}_1}$

- felhasználási lehetőség:

$$\begin{bmatrix}1\\u\\u^2\\u^3\end{bmatrix}=\begin{bmatrix}\quad & \quad & \quad & \quad\end{bmatrix}\begin{bmatrix}(1-u)^3\\3u(1-u)^2\\3u^2(1-u)\\u^3\end{bmatrix}\quad \forall u\in[0,1]$$

1.) Hat meg a kontrollpontokat

$\vec{p}_0(x_0, y_0, z_0)$, $\vec{p}_1(x_1, y_1, z_1)$, $\vec{p}_2(x_2, y_2, z_2)$, $\vec{p}_3(x_3, y_3, z_3)$

$\{[u+(1-u)]^3 = 1^3 = 1 \quad \forall u \in [0,1]\}$

és az $F_0(u) = (1-u)^3$

$F_1(u) = 3u(1-u)^2$

$F_2(u) = 3u^2(1-u)$

$F_3(u) = u^3 \qquad u \in [0,1]$

bázis függvények lineáris kombinációjával leírt görbének az $u = \frac{1}{2}$ paraméteréhez tartozó

a) görbepontokat

b) érintővektort és az érintő egyenes egyenletét

(1) Megoldás

a.) görbepont

$$\vec{c}(u) = \left(\vec{p_0}\, F_0(u) + \vec{p_1}\, F_1(u) + \vec{p_2}\, F_2(u) + \vec{p_3}\, F_3(u)\right)\Big|_{u=\frac{1}{2}}$$

$$= \vec{p_0}\,\frac{1}{8} + \vec{p_1}\,\frac{3}{8} + \vec{p_2}\,\frac{3}{8} + \vec{p_3}\,\frac{1}{8}$$

$$= \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}\frac{1}{8} + \dots = \begin{bmatrix} c_x \\ c_y \\ c_z \end{bmatrix} \qquad F_0 + F_1 + F_2 + F_3 = 1$$

b.)

$$\frac{d}{du}F_0(u)\Big|_{u=\frac{1}{2}} = \frac{d}{du}(1-u)^3 = \left|-3(1-u)^2\right|_{u=\frac{1}{2}} = -\frac{3}{4}$$

$$\frac{d}{du}F_1(u) = \frac{d}{du}\left[3u(1-u)^2\right] = 3\left[(1-u)^2 - 2u(1-u)\right]\Big|_{u=\frac{1}{2}} = -\frac{3}{4}$$

$$\frac{d}{du}F_2(u) = \frac{d}{du}\left[3u^2(1-u)\right] = 3\left(2u(1-u) - u^2\right)\Big|_{u=\frac{1}{2}} = \frac{3}{4}$$

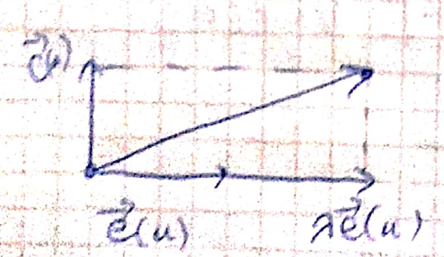
$$\frac{d}{du}F_3(u) = \left|3u^2\right|_{u=\frac{1}{2}} = \frac{3}{4}$$

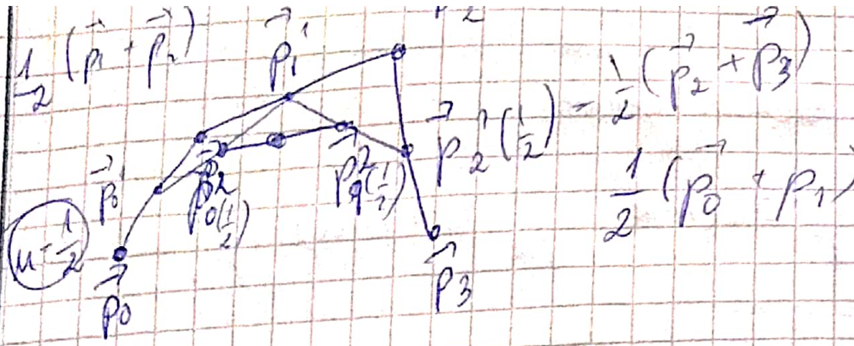
$$F_0 + F_1 + F_2 + F_3 = 0$$

$$\vec{e}(u) = \vec{p_0}\frac{d}{du}F_0(u) + \vec{p_1}\frac{d}{du}F_1(u) + \vec{p_2}\frac{d}{du}F_2(u) + \vec{p_3}\frac{d}{du}F_3(u) =$$

$$= -\frac{3}{4}\vec{p_0} - \frac{3}{4}\vec{p_1} + \frac{3}{4}\vec{p_2} + \frac{3}{4}\vec{p_3} = \dots = \begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix} = \vec{c}(u) + \lambda\vec{e}(u)$$

$\lambda \in \mathbb{R}$





pl.

$$(1-u)\vec{p} + u\vec{q}\,\Big|_{u=\frac{1}{3}} = \qquad u\in[0,1]$$

$$\frac{1}{2}\left(\vec{p}_0^{\,1}\left(\frac{1}{2}\right) + \vec{p}_1^{\,1}\left(\frac{1}{2}\right)\right) = \frac{1}{4}\vec{p}_0$$

$$\hookrightarrow \frac{1}{2}\left(\frac{1}{2}(\vec{p}_0+\vec{p}_1) + \frac{1}{2}(\vec{p}_1+\vec{p}_2)\right) =$$

$$= \frac{1}{4}\vec{p}_0 + \frac{2}{4}\vec{p}_1 + \frac{1}{4}\vec{p}_2$$

$$\vec{p}_0^{\,3}\left(\frac{1}{2}\right) = \frac{1}{8}\vec{p}_0 + \frac{3}{8}\vec{p}_1 + \frac{3}{8}\vec{p}_2 + \frac{1}{8}\vec{p}_3$$

2.) Tekintsük az $u_0=0<u_1=\frac{1}{3}<u_2=\frac{2}{3}<u_3=1$ csomóértékekhez tartozó interpolálandó $\vec{d}_0(x_0,y_0,z_0)$, $\vec{d}_1(x_1,y_1,z_1)$, $\vec{d}_2(x_2,y_2,z_2)$ és $d_3(x_3,y_0,z_{03})$ adatpontokat.

Hat. meg annak a $[0,1]$-es harmadfokú Bézier görbének a kontrollpontjait, amely teljesíti a megadott interpolációs feltételeket.

Megoldás

harmadfokú Bézier görbe

$$\vec{c}(u) = \vec{p}_0(1-u)^3 + \vec{p}_1 3u(1-u)^2 + \vec{p}_2 3u^2(1-u) + \vec{p}_3 u^3$$

$$\vec{c}(0) = \vec{d}_0 = \vec{p}_0$$

$$\vec{c}\left(\frac{1}{3}\right) = \vec{d}_1 = \frac{8}{27}\vec{p}_0 + \frac{12}{27}\vec{p}_1 + \frac{6}{27}\vec{p}_2 + \frac{1}{27}\vec{p}_3$$

$$\vec{c}\left(\frac{2}{3}\right) = \vec{d}_2 = \frac{1}{27}\vec{p}_0 + \frac{6}{27}\vec{p}_1 + \frac{12}{27}\vec{p}_2 + \frac{8}{27}\vec{p}_3$$

$$\vec{c}(1) = \vec{d}_3 = \vec{p}_3$$

$$\begin{cases} \vec{d}_0 = \vec{p}_0 \\ \vec{d}_3 = \vec{p}_3 \end{cases} \quad HF$$

LinearCombination3::UpdateDataForInterpolation

3)

$$\vec{s}(u,v) = \left[F_0(u)\ F_1(u)\ F_2(u)\ F_3(u)\right] \begin{bmatrix} P_{00} & P_{01} & P_{02} & P_{03} \\ P_{10} & P_{11} & P_{12} & P_{13} \\ P_{20} & P_{21} & P_{22} & P_{23} \\ P_{30} & P_{31} & P_{32} & P_{33} \end{bmatrix} \begin{bmatrix} F_0(v) \\ F_1(v) \\ F_2(v) \\ F_3(v) \end{bmatrix}$$

harmadfokú Bézier folt

TensorProductSurface3::CalculatePartialDerivatives

::GenerateImage

a.) felületi pont

b.) u és v irányú parciális deriváltak

c.) normálvektor és érintősík

a.) $\vec{s}\left(\frac{1}{2},\frac{1}{2}\right) = \begin{bmatrix} \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{bmatrix} \cdot P \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{3}{8} \\ \frac{3}{8} \\ \frac{1}{8} \end{bmatrix} =$

$= \frac{1}{64} \begin{bmatrix} 1 & 3 & 3 & 1 \end{bmatrix} P \cdot \begin{bmatrix} 1 \\ 3 \\ 3 \\ 1 \end{bmatrix} = \dots = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$

b.)

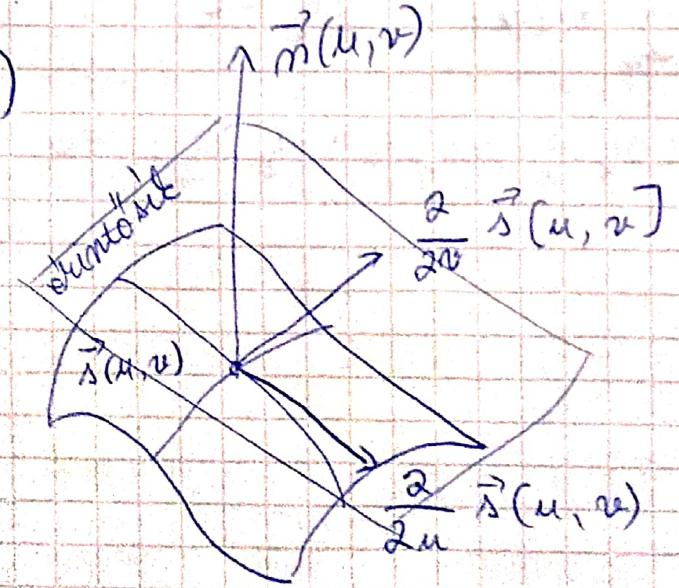
$\frac{\partial}{\partial u} \vec{s}\left(\frac{1}{2},\frac{1}{2}\right) = \begin{bmatrix} -\frac{3}{4} & -\frac{3}{4} & \frac{3}{4} & \frac{3}{4} \end{bmatrix} \cdot P \begin{bmatrix} \frac{1}{8} \\ \frac{3}{8} \\ \frac{3}{8} \\ \frac{1}{8} \end{bmatrix} =$

$= \frac{3}{32} \begin{bmatrix} -1 & -1 & 1 & 1 \end{bmatrix} P \begin{bmatrix} 1 \\ 3 \\ 3 \\ 1 \end{bmatrix} = \dots = \begin{bmatrix} s^u_x \\ s^u_y \\ s^u_z \end{bmatrix}$

hasonlóan / tükrözve: felcserélt

$\frac{\partial}{\partial v} \vec{s}\left(\frac{1}{2},\frac{1}{2}\right) = \frac{3}{32} \begin{bmatrix} 1 & 3 & 3 & 1 \end{bmatrix} P \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \dots = \begin{bmatrix} s^v_x \\ s^v_y \\ s^v_z \end{bmatrix}$

c.)



$$\vec{n}(u,v) = \left| \frac{\partial}{\partial u} \vec{s}(u,v) \times \frac{\partial}{\partial u} \vec{s}(u,v) \right|_{\left(\frac{1}{2},\frac{1}{2}\right)} =$$

$$= \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ s^u_x & s^u_y & s^u_z \\ s^v_x & s^v_y & s^u_z \end{pmatrix} = \vec{i}\,\underbrace{(s^u_y \cdot s^u_z - s^u_z \cdot s^v_y)}_{n_x} + \vec{j}\, n_y + \vec{k} \cdot n_z = \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}$$

érintő sík:

$$\left\langle \begin{bmatrix} x - s_x \\ y - s_y \\ z - s_z \end{bmatrix}, \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} \right\rangle = 0$$, mert ~~a~~ merőlegesek egymásra

$$\alpha x + \beta y + \mu z + \delta = 0$$

$$(x - sx)nx + (y - sy)\, n_y + (z - s_z)\, n_z = 0$$

$$\underset{\alpha}{x n_x} + \underset{\beta}{y n_y} + \underset{\mu}{z n_z} \underset{\delta}{- (n_x s_x + n_y s_y + n_z \cdot s_z)} = 0$$

Scanned with CamScanner