# Deep Generative Models

Philip Schulz and Wilker Aziz

Generative Models

First Attempt: Log-linear Models

Second Attempt: Wake-Sleep

This is how we do: Variational Autoencoders

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# Recap: Generative Models

Joint distribution over observed data $x$ and latent variables $Z$.

$$p(x, z|\alpha) = \overbrace{p(x|z, \alpha)}^{\text{likelihood}} \underbrace{p(z|\alpha)}_{\text{prior}}$$

The likelihood and prior are often standard distributions (Gaussian, Bernoulli) with little dependence on side information.

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# Feature-rich Generative Models

Let us assume that $z$ has internal structure (features). How can we exploit that?

## First Idea

Make $p(x|z, \alpha)$ a log-linear model.

- Only discrete data
- Trainable with EM if we can efficiently enumerate $\mathcal{X}$ and $\mathcal{Z}$.

# Log-linear Model

Let us treat $z$ as observed.

$$p(x|z, \alpha = w) = \frac{\exp\left(w^\top f(x,z)\right)}{\sum_{x \in \mathcal{X}} \exp\left(w^\top f(x,z)\right)}$$

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## Weight Gradient

$$\frac{d}{dw} \log p(x|z,w) = f(x,z) - \mathbb{E}\left[f(X,z)|z,w\right]$$

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Updates need to be performed iteratively.

# Log-linear model with latent variables

Now let us treat $z$ as latent.

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## Model

$$p(x, z|w) = \underbrace{\frac{\exp\left(w^\top f(x, z)\right)}{\sum_{x \in \mathcal{X}} \exp\left(w^\top f(x, z)\right)}}_{p(x|z,w)} \times \underbrace{p(z)}_{arbitrary}$$

# Log-linear model with latent variables

## Posterior

$$p(z|x,w) = \frac{p(x,z|w)}{p(x|w)} = \frac{p(x,z|w)}{\sum_z p(x,z|w)} = \frac{\frac{\exp\left(w^\top f(x,z)\right)}{\sum_{x\in\mathcal{X}}\exp(w^\top f(x,z))} \times p(z)}{\sum_z \frac{\exp(w^\top f(x,z))}{\sum_{x\in\mathcal{X}}\exp(w^\top f(x,z))} \times p(z)}$$

# Log-linear model with latent variables

## Weight Gradient

$$\frac{d}{dw}\mathbb{E}_{p(z|x,w)}\left[\log p(x,z|w)\right] =$$
$$\frac{d}{dw}\sum_z p(z|x,w)\log p(x,z|w) =$$
$$\sum_z p(z|x,w)\frac{d}{dw}\log p(x,z|w)$$

# Log-linear model with latent variables

## Weight Gradient

$$\frac{d}{dw}\mathbb{E}_{p(z|x,w)}\left[\log p(x,z|w)\right] =$$

$$\frac{d}{dw}\sum_{z} p(z|x,w)\log p(x,z|w) =$$

$$\sum_{z} p(z|x,w)\ \underbrace{\frac{d}{dw}\log p(x,z|w)}_{\text{We've already solved this!}}$$

# Log-linear model with latent variables

## Weight Gradient

$$\frac{d}{dw}\mathbb{E}_{p(z|x,w)}\left[\log p(x,z|w)\right] =$$
$$\mathbb{E}_{p(z|x,w)}\left[f(x,Z)|x,w\right] - \mathbb{E}_{p(z|x,w)}\left[\mathbb{E}\left[(f(X,Z)|Z,w\right]\right]$$

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## Procedurally

$$\text{E\_count}(x,z) \text{ - } \text{E\_count}(x,z)\ \times\mathbb{E}\left[X|z,w\right]$$

# EM

E-step $p(z|x,w) = \frac{p(x,z|w)}{\sum_z p(x,z|w)}$ in $\mathcal{O}(|\mathcal{X}| \times |\mathcal{Z}|)$

M-step Iteratively optimise $w$ to match E_count$(x, z)$ with E_count$(x, z)$ $\times \mathbb{E}[X|z, w]$

## Restrictions

- Only log-linear models
- Scales badly

Generative Models

First Attempt: Log-linear Models

Second Attempt: Wake-Sleep

This is how we do: Variational Autoencoders

# Wake-sleep Algorithm

- Generalise latent variables to Neural Networks
- Train generative neural model
- Use variational inference! (kind of)

# Wake-sleep Architecture

2 Neural Networks:

- A generation network to model the data (the one we want to optimise) – parameters: $\theta$
- An inference (recognition) network (to model the latent variable) – parameters: $\lambda$
- Original setting: binary hidden units

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- A generation network to model the data (the one we want to optimise) – parameters: $\theta$
- An inference (recognition) network (to model the latent variable) – parameters: $\lambda$
- Original setting: binary hidden units
- Training is performed in a "hard EM" fashion

# Wake-sleep Training

## Wake Phase

- Use inference network to sample hidden unit setting $z$ from $q(z|x, \lambda)$
- Update generation parameters $\theta$ to maximize liklelihood of data given latent state $p(x|z, \theta)$

# Wake-sleep Training

## Wake Phase

- Use inference network to sample hidden unit setting $z$ from $q(z|x, \lambda)$
- Update generation parameters $\theta$ to maximize liklelihood of data given latent state $p(x|z, \theta)$

## Sleep Phase

- Produce dream sample $\tilde{x}$ from random hidden unit $z$
- Update inference parameters $\lambda$ to maximize probability of latent state $q(z|\tilde{x}, \lambda)$

# Wake Phase Objective

Assumes latent state $z$ to be fixed random draws from $q(z|x, \lambda)$.

$$\max_{\theta} \; \log p(x|z, \theta)$$

This is simply supervised learning with imputed latent data!

# Sleep Phase Objective

Assumes fake data $\tilde{x}$ and latent variables $z$ to be fixed random draw from $p(x, z|\theta)$.

$$\min_{\lambda} \; \mathbb{E}_{q(z|\tilde{x},\lambda)} [\log p(\tilde{x}, z|\theta)] + \mathbb{H}\left(q(z|\tilde{x}, \lambda)\right)$$

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## Advantages

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**Advantages**

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**Drawbacks**

- Inference and generative networks are trained on different objectives
- Inference weights $\lambda$ are updated on fake data $\tilde{x}$
- Generative weights are bad initially, giving wrong signal to the updates of $\lambda$

# Generative Model with NN Likelihood

## Goal

Define model $p(x, z|\theta) = p(x|z, \theta)p(z)$ where the likelihood $p(x|z, \theta)$ is given by a neural network. (We fix $p(z)$ for simplicity.)

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## Problem

$p(x) = \int p(x|z, \theta)p(z)\mathrm{dz}$ is hard to compute.

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## Problem

$p(x) = \int \underbrace{p(x|z, \theta)}_{\text{highly non-linear!}} p(z)\mathrm{d}z$ is hard to compute.

# Generative Model with NN Likelihood

## Solution: VI

$$\log p(x) \geq \overbrace{\mathbb{E}_{q(z|x,\lambda)}\left[\log p(x,z|\theta)\right] + \mathbb{H}\left(q(z|x,\lambda)\right)}^{\text{ELBO}}$$
$$= \mathbb{E}_{q(z|x,\lambda)}\left[\log p(x|z,\theta)\right] + \mathrm{KL}\left(p(z) \;||\; q(z|x,\lambda)\right)$$

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# Generative Model with NN Likelihood

## Solution: VI

$$
\begin{aligned}
\log p(x) &\geq \overbrace{\mathbb{E}_{q(z|x,\lambda)}\left[\log p(x, z|\theta)\right] + \mathbb{H}\left(q(z|x,\lambda)\right)}^{\text{ELBO}} \\
&= \underbrace{\mathbb{E}_{q(z|x,\lambda)}\left[\log p(x|z,\theta)\right]}_{\text{approximate by sampling}} + \underbrace{\mathrm{KL}\left(p(z) \mid\mid q(z|x,\lambda)\right)}_{\substack{\text{assume analytical} \\ \text{(true for exponential families)}}}
\end{aligned}
$$

# Generation Network Gradient

$$\frac{d}{d\theta}\mathbb{E}_{q(z|x,\lambda)}\left[\log p(x|z,\theta)\right]$$

$$= \mathbb{E}_{q(z|x,\lambda)}\left[\frac{d}{d\theta}\log p(x|z,\theta)\right]$$

$$\overset{\text{MC}}{\approx} \frac{1}{S}\sum_{i=1}^{S}\frac{d}{d\theta}\log p(x|z_i,\theta)$$

Note: $q(z|x,\lambda)$ does not depend on $\theta$.

# Inference Network Gradient

$$\frac{d}{d\lambda}\left[\mathbb{E}_{q(z|x,\lambda)}\left[\log p(x|z,\theta)\right] + \mathrm{KL}\left(p(z) \mid\mid q(z|x,\lambda)\right)\right]$$
$$= \frac{d}{d\lambda}\mathbb{E}_{q(z|x,\lambda)}\left[\log p(x|z,\theta)\right] + \underbrace{\frac{d}{d\lambda}\mathrm{KL}\left(p(z) \mid\mid q(z|x,\lambda)\right)}_{\text{analytical computation}}$$

The first term again requires approximation by sampling

# Inference Network Gradient

$$\frac{d}{d\lambda}\mathbb{E}_{q(z|x,\lambda)}\left[\log p(x|z,\theta)\right]$$
$$= \frac{d}{d\lambda}\int q(z|x,\lambda)\log p(x|z,\theta)\mathrm{dz}$$

## Problems for MC

- Sampling $z$ neglects $\frac{d}{d\lambda}q(z|x,\lambda)$
- Differentiating $q(z|x,\lambda)$ breaks the expectation

# Inference Network Gradient

$$= \frac{d}{d\lambda} \int q(z|x,\lambda) \log p(x|z,\theta) \mathrm{d}z$$

$$= \frac{d}{d\lambda} \int q(\epsilon) \log \left( p(x|\overbrace{h(\epsilon,\lambda)}^{=z},\theta) \times \underbrace{\left| \frac{d}{d\epsilon} h(\epsilon,\lambda) \right|}_{\text{constant if h linear}} \right) \mathrm{d}\epsilon$$

$$= \int q(\epsilon) \frac{d}{d\lambda} \log p(x|h(\epsilon,\lambda),\theta) \mathrm{d}\epsilon$$

$$= \mathbb{E}_{p(\epsilon)} \left[ \frac{d}{d\lambda} \log p(x|h(\epsilon,\lambda),\theta) \right] \overset{\text{MC}}{\approx} \frac{1}{S} \sum_{i=1}^{S} \frac{d}{d\lambda} \log p(x|h(\epsilon_i,\lambda),\theta)$$

# Gaussian Transformation

## Affine property

$$Ax + b \sim \mathcal{N}\left(\mu + b, A\Sigma A^{T}\right) \text{ for } x \sim \mathcal{N}\left(\mu, \Sigma\right)$$

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$$Ax + b \sim \mathcal{N}\left(b, AA^T\right) \text{ for } x \sim \mathcal{N}\left(0, \mathrm{I}\right)$$

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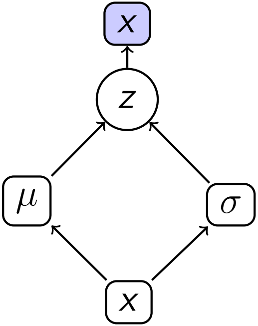
## Gaussian transformation

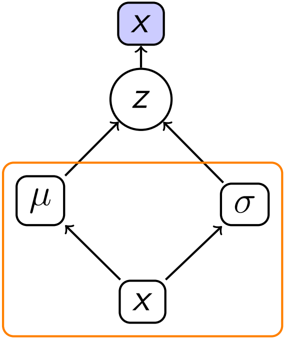
$$h(\epsilon, \lambda) = \mu(x, \lambda) + \mathrm{diag}(\sigma(x, \lambda)) \odot \epsilon \quad \epsilon \sim \mathcal{N}\left(0, \mathrm{I}\right)$$

# Computation Graph



# Computation Graph

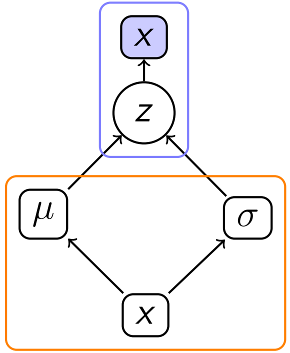
inference model



# Computation Graph

generation model

inference model



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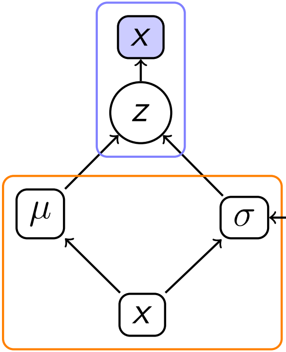


$\epsilon \sim \mathcal{N}(0, 1)$

# Example

- Data: binary mnist
- Likelihood: product of Bernoullis
  - Let $\phi = \sigma(\text{NN}(z))$
  - $\prod_{i=1}^{N} p(x_i|\phi) = \prod_{i=1}^{N} \phi^{x_i} \times (1-\phi)^{1-x_i}$
- Prior over $z$: $\mathcal{N}(0,1)$
- $q(z|x,\lambda) = \mathcal{N}\left(\mu(x,\lambda), \sigma(x,\lambda)^2\right)$
- $\mu(x,\lambda) = \text{NN}_{\mu}(x;\lambda)$
- $\sigma(x,\lambda) = \text{NN}_{\sigma}(x;\lambda)$

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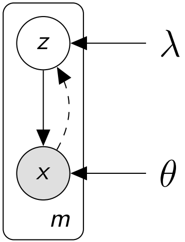
## Mean Field assumption

Variational approximation factorises over latent dimensions.

# Graphical Model




- approximate posterior
  $q(z|x, \lambda) = \mathcal{N}\left(\mu(x, \lambda), \sigma(x, \lambda)^2\right)$

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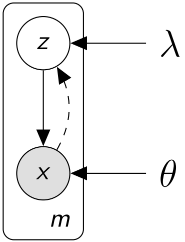


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    e.g. $\mu(x,\lambda) = W^{(u)}x + b^{(u)}$

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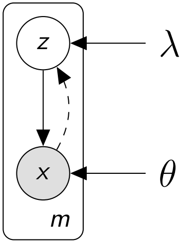


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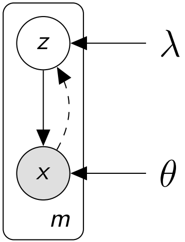
  - $\mu(x,\lambda) = \mathrm{NN}_\mu(x;\lambda)$
    e.g. $\mu(x,\lambda) = W^{(u)}x + b^{(u)}$
  - $\sigma(x,\lambda) = \exp(\mathrm{NN}_\sigma(x;\lambda))$
    e.g. $\sigma(x,\lambda) =$
    $\exp\left(\tanh(W^{(v)}x + b^{(v)})\right)$

# Graphical Model

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    e.g. $\sigma(x, \lambda) =$
    $\exp\left(\tanh(W^{(v)}x + b^{(v)})\right)$
  - $\lambda = \left(W^{(u)}, W^{(v)}, b^{(u)}, b^{(v)}\right)$

# Variational Autoencoder

**Advantages**

- Backprop training
- Easy to implement
- Posterior inference possible
- One objective for both NNs

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**Drawbacks**

- Discrete latent variables are difficult
- Optimisation may be difficult with several latent variables

# Summary

- When $|\mathcal{X}|$ and $|\mathcal{Z}|$ are not too large, we can do EM with features
- Otherwise use VI with simple approximation
- Wake-Sleep: train inference and generation networks with separate objectives
- VAE: train both networks with same objective
- Reparametrisation
  - Transform parameter-free variable $\epsilon$ into latent value $z$
  - Update parameters with stochastic gradient estimates

# Literature I

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