Discrete Variables in DGMs

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https://github.com/philschulz/VITutorial

What we know so far

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- Because the ELBO cannot be computed exactly, we need to sample latent values
- Main problem: the MC estimator is not differentiable
- Solution: reparametrisation gradient

Reparametrisation Gradient

Model Gradient

$$\frac{\partial}{\partial \theta}\mathbb{E}_{q(z|\lambda)}\left[\log p(x|z,\theta)\right] - \frac{\partial}{\partial \theta}\mathrm{KL}\left(q(z|\lambda) \mid\mid p(z|\theta)\right)$$

Reparametrisation Gradient

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Inference Network Gradient

$$\frac{\partial}{\partial \lambda} \mathbb{E}_{q(z|\lambda)} \left[\log p(x|z, \theta) \right] - \frac{\partial}{\partial \lambda} \mathrm{KL} \left(q(z|\lambda) \mid\mid p(z|\theta) \right)$$

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$$\mathbb{E}_{\phi(\epsilon)}\left[\frac{\partial}{\partial z}\log p(x|\overbrace{h^{-1}(\epsilon,\lambda)}^{z},\theta)\times\frac{\partial}{\partial\lambda}\overbrace{h^{-1}(\epsilon,\lambda)}^{z}\right]$$

Reparametrisation for Discrete Variables?

Revisiting the Inference Gradient

Control Variates and Baselines

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Revisiting the Inference Gradient

Control Variates and Baselines

Reparametrisation

In order to tranform variables, we need to compute the Jacobian (matrix of derivatives).

$$p(z) = \phi(h(z)) \left| \frac{d}{dz} h(z) \right|$$

The Jacobian is generally not available for discrete variables.

Cumulative Distribution Function

Insert picture here

Continuity

The outcome space of discrete variables is non-continuous. Thus, we cannot take derivatives with respect to real variables.

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$$
\begin{aligned}
&\frac{\partial}{\partial \lambda} \mathbb{E}_{q(z|\lambda)} [\log p(x|z, \theta)] = \\
&\frac{\partial}{\partial \lambda} \sum_z q(z|\lambda) \log p(x|z, \theta) = \\
&\quad \sum_z \frac{\partial}{\partial \lambda} q(z|\lambda) \log p(x|z, \theta)
\end{aligned}
$$

Back to Basic Calculus

$$\frac{d}{d\lambda} \log f(\lambda)$$

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$$\frac{d}{d\lambda} \log f(\lambda) = \frac{\frac{d}{d\lambda} f(\lambda)}{f(\lambda)}$$

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Consequence

$$\frac{d}{d\lambda} f(\lambda) = \frac{d}{d\lambda}\log f(\lambda) \times f(\lambda)$$

Score Function Estimator

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Apply this to the ELBO derivative.

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$$\sum_z q(z|\lambda) \frac{\partial}{\partial\lambda} \log q(z|\lambda) \times \log p(x|z,\theta) =$$

$$\mathbb{E}_{q(z|\lambda)} \left[\frac{\partial}{\partial\lambda} \log q(z|\lambda) \times \log p(x|z,\theta) \right]$$

Comparison Between Estimators

- Score function gradient

$$\mathbb{E}_{q(z|\lambda)}\left[\frac{\partial}{\partial\lambda}\log q(z|\lambda)\times\log p(x|z,\theta)\right]$$

- Reparametrisation gradient

$$\mathbb{E}_{\phi(\epsilon)}\left[\frac{\partial}{\partial\lambda}\log p(x|h^{-1}(\epsilon,\lambda),\theta)\right]$$

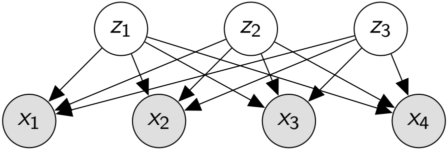
Example Model

Let us consider a latent factor model for topic modelling. Each document x consists of n i.i.d. categorical draws from that model. The categorical distribution in turn depends on the binary latent factors $z = (z_1, \ldots, z_k)$ which are also i.i.d.

$$\begin{aligned} z_j &\sim \text{Bernoulli}\,(\phi) && (1 \leq j \leq k) \\ x_i &\sim \text{Categorical}\,(g(z)) && (1 \leq i \leq n) \end{aligned}$$

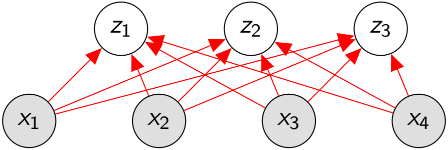
Here $g(\cdot)$ is a function computed by neural network with softmax output.

Example Model



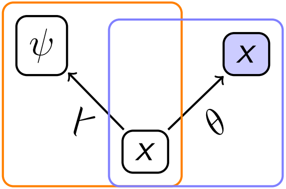
At inference time the latent variables are marginally dependent. For our variational distribution we are going to assume that they are not (recall: mean field assumption).

Inference Network



The inference network needs to predict k Bernoulli parameters ψ. Any neural network with sigmoid output will do that job.

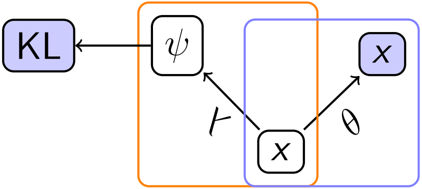
Computation Graph



inference model

generation model

Computation Graph



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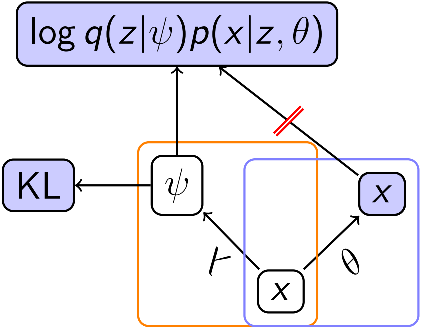

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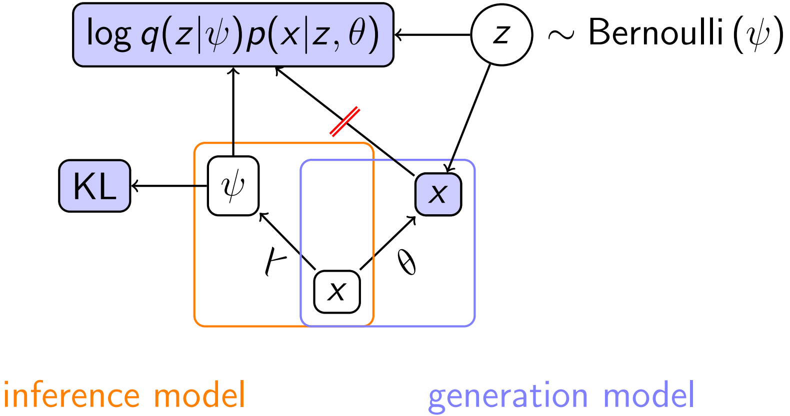
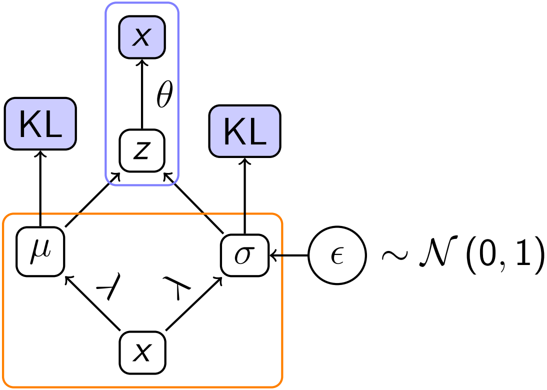


Reparametrisation Gradient

generation model

inference model



Pros and Cons

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  - Many libraries come with samplers for common distributions

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- Cons
  - High Variance!

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Control Variates and Baselines

Baselines

We attempt to centre the gradient estimate. To do this we learn a quantity C that we subtract from the reconstruction loss.

$$\log q(z|\lambda)\left(p(x|z,\theta) - C\right)$$

We call C a baseline. It does not change the expected gradient (Williams, 1992).

Baselines

We can make baselines input-depdendent to make them more flexible.

$$\log q(z|\lambda) \left(p(x|z, \theta) - C(x)\right)$$

However, baselines may not depend on the random value z! Quantities that may depend on the random value ($C(z)$) are called **control variates**. See Blei et al. (2012); Ranganath et al. (2014); Gregor et al. (2014).

Baselines

Baselines are predicted by a regression model (e.g. a neural net). The model is trained using an L_2-loss.

$$\min \left(C(x) - p(x|z,\theta) \right)^2$$

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- Reparametrisation not available for discrete variables.
- Use score function estimator.
- High variance.
- Always use baselines for variance reduction!

David M. Blei, Michael I. Jordan, and John W. Paisley. Variational bayesian inference with stochastic search. In *ICML*, 2012. URL http://icml.cc/2012/papers/687.pdf.

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