Discrete Latent Variables: Variance Reduction

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VI Tutorial @ University of Alicante
https://vitutorial.github.io/tour/ua2020

Score Function Estimator

We are interested in computing

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$$= \underbrace{\mathbb{E}_{q(z|x,\lambda)}\left[\log p(x|z,\theta)\frac{\partial}{\partial\lambda}\log q(z|x,\lambda)\right]}_{\text{expected gradient :)}}$$

Score Function Estimator

We can now build an MC estimator

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&\overset{\text{MC}}{\approx} \frac{1}{S} \sum_{s=1}^{S} \log p(x|z^{(s)},\theta) \frac{\partial}{\partial \lambda} \log q(z^{(s)}|x,\lambda)
\end{aligned}
$$

where $z^{(s)} \sim q(z|x,\lambda)$

Score Function Estimator: Variance

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- the magnitude of $\log p(x|z,\theta)$ varies widely
- the model likelihood does not contribute to direction of gradient

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$$\approx E_{q(z|x,\lambda)}\left[\hat{r}(z)\frac{\partial}{\partial \lambda}\log q(z|x,\lambda)\right]$$

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- We can show that using these *baselines* do not bias the estimator.
- More generally, we can design more sophisticated *control variates* that further reduce the variance.

Control variates

Intuition

To estimate $\mathbb{E}[f(z)]$ via Monte Carlo we compute the empirical average of $\hat{f}(z)$ where $\hat{f}(z)$ is chosen so that $\mathbb{E}[\hat{f}(z)] = \mathbb{E}[f(z)]$ and $\mathrm{Var}(f) > \mathrm{Var}(\hat{f})$.

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- then for $\hat{f}(z) \triangleq f(z) - b(c(z) - \mathbb{E}[c(z)])$ it holds that $\mathbb{E}[\hat{f}(z)] = \mathbb{E}[f(z)]$
- and $\mathrm{Var}(\hat{f}) = \mathrm{Var}(f) - 2b\,\mathrm{Cov}(f, c) + b^2\,\mathrm{Var}(c)$

Choosing the control variate

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Of course, $\mathbb{E}[c(z)]$ must be known!

MC

We then use the estimate

$$\bar{f} \overset{\text{MC}}{\approx} \frac{1}{S}\left(\sum_{s=1}^{S} f(z^{(s)}) - bc(z^{(s)})\right) + b\bar{c}$$

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And recall that for us

$$f(z) = \log p(x|z,\theta)\frac{\partial}{\partial\lambda}\log q(z|x,\lambda)$$

and $z^{(s)} \sim q(z|x,\lambda)$

Expected score

The Expectation of the score function is 0.

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&= \int \frac{\partial}{\partial\lambda} q(z|x,\lambda)\mathrm{d}z \\
&= \frac{\partial}{\partial\lambda}\int q(z|x,\lambda)\mathrm{d}z \\
&= \frac{\partial}{\partial\lambda}1 = 0
\end{aligned}
$$

Baselines

With

$$f(z) = \log p(x|z,\theta)\frac{\partial}{\partial\lambda}\log q(z|x,\lambda)$$

and

$$c(z) = \frac{\partial}{\partial\lambda}\log q(z|x,\lambda)$$

we have

$$\hat{f}(z) =$$

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we have

$$\hat{f}(z) = (\log p(x|z,\theta) - b)\frac{\partial}{\partial\lambda}\log q(z|x,\lambda)$$

b is known as *baseline* in RL literature.

Examples of baselines

- Moving average of $\log p(x|z, \theta)$ based on previous batches

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- Moving average of $\log p(x|z, \theta)$ based on previous batches
- A trainable constant b
- A neural network prediction based on x e.g. $b(x; \omega)$
- The likelihood assessed at a deterministic point, e.g. $b(x) = \log p(x|z^\star, \theta)$ where $z^\star = \arg\max_z q(z|x, \lambda)$

Trainable baselines

Baselines are predicted by a regression model (e.g. a neural net).

The model is trained using an L_2-loss.

$$\min_{\omega} \left(b(x;\omega) - \log p(x|z,\theta)\right)^2$$

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- We can design control variates that reduce estimator variance, yet do not bias the estimator!

Literature I

John W. Paisley, David M. Blei, and Michael I. Jordan. Variational bayesian inference with stochastic search. In *ICML*, 2012. URL http://icml.cc/2012/papers/687.pdf.

Ronald J. Williams. Simple statistical gradient-following algorithms for connectionist reinforcement learning. *Machine Learning*, 8(3-4):229–256, 1992. URL https://doi.org/10.1007/BF00992696.