Bri Miskovitz 2/24/16
Project 1: Tree Population Models Dynamical Systems

Model 1: $\frac{dT}{dt} = rT(1 - \frac{T}{K}) - H$

The T parameter represents the biomass of trees in a particular population, and it's a continuous parameter. The r parameter represents a rate of population growth with dimension $[trees]^{-1}$. Since the trees can only be removed by harvesting, the population cannot decay, therefore r must be positive. The K parameter represents the maximum carrying capacity of trees, and this parameter must also be positive. Model 1 and the logistic equation differ from each other by a parameter, H. It represents a harvesting rate with dimensions of $[\frac{trees}{time}]$. It's a continuous parameter because the population of the biomass of trees and time are both continuous. Continuous time and the continuous population also mean both T and t can take any real value, which can cause a problem that will later be mentioned..

This model is too simple to represent a tree population because it allows negative population because T can be any real value, and it allows for negative times for the same reason as T. The model doesn't help maintain the population. For example, if too many trees are being harvested and the population decreases too much, no part of the model will slow it down.

The model can be written in this dimensionless form:

$$\frac{dx}{d\tau} = x(1 - x) - h$$

$$\frac{dT}{dt} = rT(1 - \frac{T}{K}) - H$$

We can eliminate K from the logistic portion of the equation, and have a parameter involving H and K

$$T = Kx \qquad \Rightarrow x = \frac{T}{K};\ \frac{dx}{dT} = \frac{1}{K}$$

x is dimensionless, but the model still has dimensions of $time^{-1}$.

$$\frac{dx}{dt} = \frac{dx}{dT}\frac{dT}{dt}$$

Chain rule can be used to find dx/dt.

$$\frac{dx}{dt} = (\frac{1}{K})[rKx(1 - \frac{Kx}{K}) - H] \ = \ rx(1 - x) - \frac{H}{K}$$

$$\Gamma = r;\ \tau = \ \Gamma \dagger;$$

A time constant, Γ, can be represented as r because r has units of $time^{-1}$. Allow τ to be the new time variable and † have dimensions and units of time.

$$\frac{dx}{d\tau} = \frac{1}{r}\frac{dx}{d\tau} = \frac{r}{r}[x(1-x) - \frac{H}{rK}] = x(1-x) - \frac{H}{rK} = x(1-x) - h$$

$$\Rightarrow \; h = \frac{H}{rK}$$

The new dimensionless parameter is h, and now, the entire model is dimensionless.

To find this fixed points in terms of h, we set the system equal to zero.

x(1-x) - h = 0

$x^* = \frac{-1 \pm \sqrt{1-4h}}{-2h} \quad \Rightarrow \quad x^*_{1,2} = \frac{1}{2} \mp \frac{\sqrt{1-4h}}{2}$

There is an existence criteria for the fixed points: $h \leq \frac{1}{4}$

To determine the stability, we plug the fixed points into the derivative of the system.

f '(x) = 1 - 2x

$f'(x^*) = 1 - 1 \mp \sqrt{1-4h} = \mp\sqrt{1-4h}$

$h < 0$: $f'(x^*_1) < 0$ The first fixed point is stable when h is negative because the derivative is negative.

$f'(x^*_2) > 0$ The second fixed point is unstable when h is negative because the derivative is positive.

h = 0: $f'(x^*_1) = -1$ The first fixed point is again stable when h is zero because the derivative is negative.

$f'(x^*_2) = 1$ The second fixed point is again unstable when h is zero because the derivative is positive.

$0 < h < \frac{1}{4}$: $f'(x^*_1) < 0$ The first fixed point is again stable when h is greater than zero because the derivative is negative.

$f'(x^*_2) > 0$ The second fixed point is again unstable when h is zero because the derivative is negative.

$h = \frac{1}{4}$ $\quad f'(x^*_{1,2}) = 0$

The fixed points are semi-stable when $h = \frac{1}{4}$, so since the stability has changed, the critical value of h is one-fourth, $h_c = \frac{1}{4}$. To determine more information on the stability, one must use a different stability analysis approach, such as graphing. Shown in graph below, the fixed points at $h = \frac{1}{4}$ flow to the left.

There is a saddle bifurcation for $h_c = \frac{1}{4}$.

This model can be written into the saddle bifurcation normal form.

$f(x) = x - x^2 - h$

Form aimed for: $R - X^2$

$x - x^2 - 0.25 + 0.25$

Appropriate X can be found by completing the square.

-[(x - 0.5)(x - 0.5)] + 0.25

$0.25 - (x - 0.5)^2$

$f(x) = 0.25 - h - (x - 0.5)^2$

To achieve the normal form, new variables, R and X, can be defined and used.

R = 0.25 - h; $\quad X = (x - 0.5)^2$;

$f(x) = R - X^2$

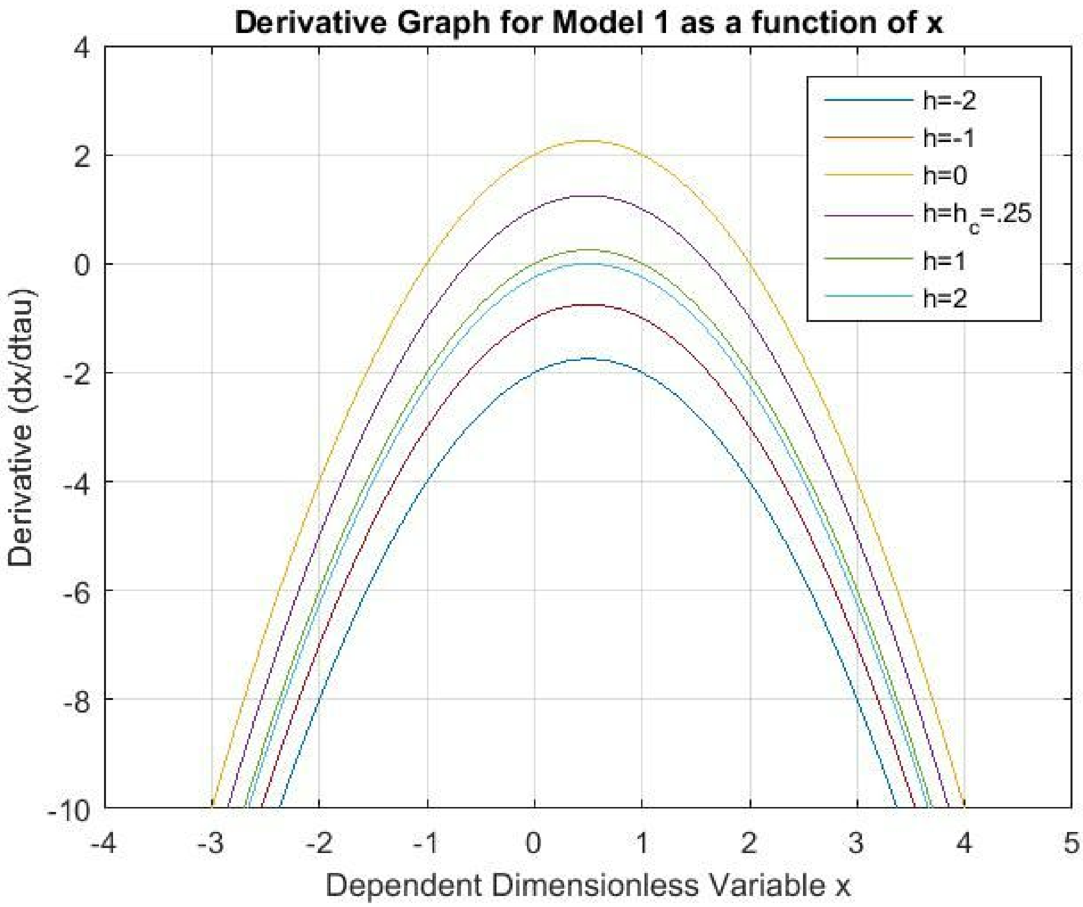
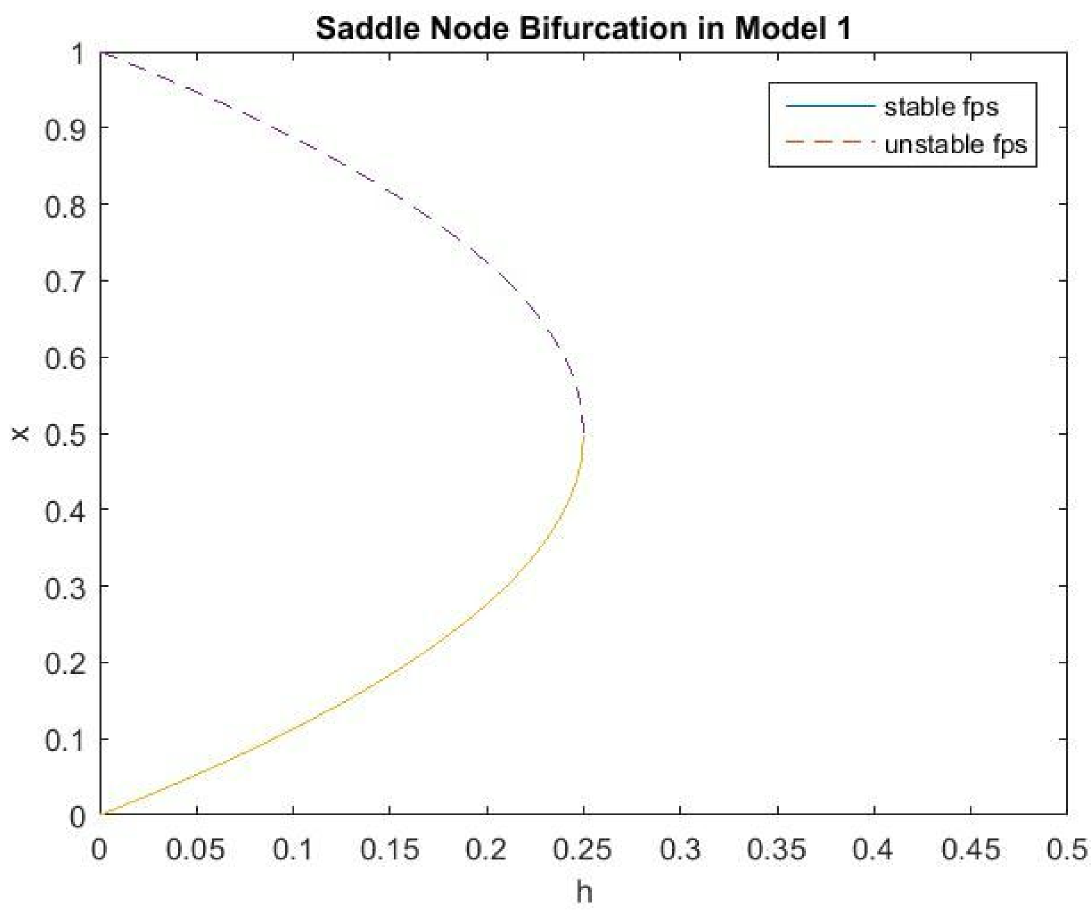


Figure 1. Derivative Graph for Model 1 in Dimensionless Form.

The graph proves that, in order for this model to have real equilibrium points, h must be at most $\frac{1}{4}$. When h is greater than $\frac{1}{4}$, the graph does not cross the x-axis. When $h = \frac{1}{4}$, the curve is tangent with the axis, so at that intersection, there is a semi-stable fixed point that flows toward the left. For h values less than $\frac{1}{4}$, the stability of fixed point one is consistent and the stability for fixed point two is consistent, and that agrees with the analytical work done earlier The fixed point that exists on the negative x-axis is unstable and the flow repels away from it. From the unstable fixed point, when x increased, the derivative also increases, and when x decreased, the derivative decreased. This isn't desired for a proper equilibrium of a population model, which is why this equilibrium point is unstable. The other fixed point exists on the positive x-axis and is stable. The flow attracts it. However, as x increased when the derivative starts as a negative value, the curve shows that the derivative will decrease. This is appropriate for a system striving for equilibrium, which is why this equilibrium point is stable.

Observing the fixed point on the left side of the graph, the flow repelling it shows that the population's rate will increase because the system aims for stable equilibrium points, and following the curve away from the unstable fixed point, the derivative increases. After the curve reaches its maximum, as x increases, the rate will decrease because the stable fixed points exist in direction of a decreasing derivative.

As h increases, the population growth rate of trees increases, resulting in more trees, and as h decreases to the critical h value, the population rate decreases, resulting fewer trees. This isn't biologically plausible because the model only allows tree removal from harvesting; we cannot have a negative population rate. When h is the critical value, the tree population rate is constant, so no more trees would grow. The population would just stay the same. This only seems plausible if the initial condition was zero because during the spring, the trees give off their offspring and more trees are created. Since outside factors are not included, this must occur if there are trees present.



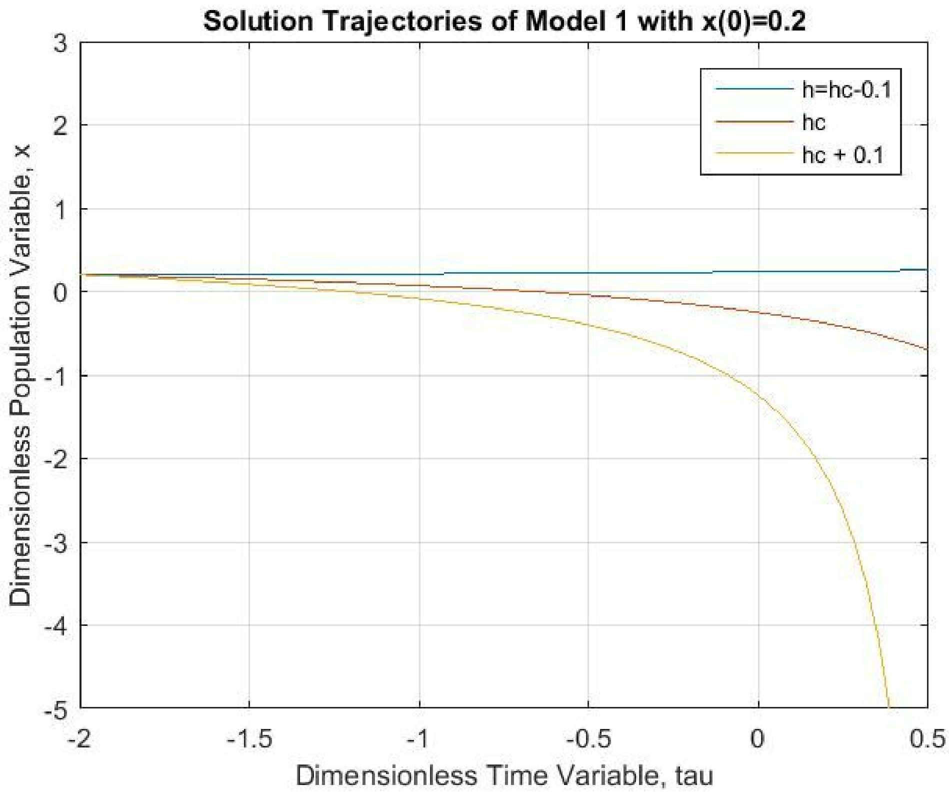


Figure 3. The Solution Trajectories when x(0) = 0.2

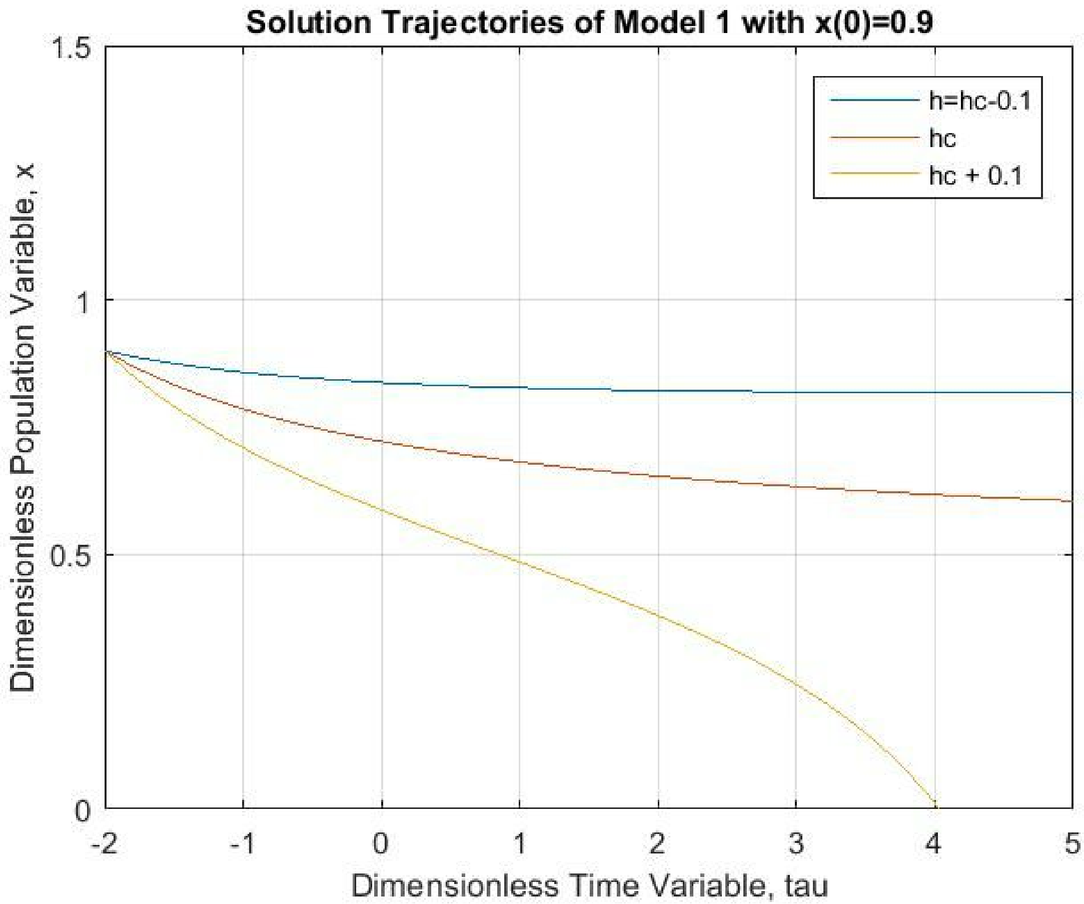


Figure 4. The Solution Trajectories when x(0) = 0.9

In Figure 3. and Figure 4, it is clear that this system depends its initial condition. Also, as tau increases, the population decreases into negative values. This data proves that this model is very flawed because a negative population cannot occur. Also, as analytically predicted, h_c remains relatively constant, and the h values less than the critical value also remain relatively constant. However, when h is greater than the critical value the population decreases fast. These results agree with analytical work as well.

Model 2: $\frac{dT}{dt} = rT(1-\frac{T}{K}) - HT$

The T, r, and K parameters have the same meaning, restrictions, and dimension as the first model. However, H now has dimensions equivalent to r, $time^{-1}$, and it is now a rate only affected by time. This model now does not allow negative populations because H is just a rate involving solely on time. The harvesting parameter is now H*T, and H will influence T. The H parameter can decrease the population or increase the population. Although this strategy is better suited than model one, it is still flawed. If the tree biomass population,T grows too large, then H will decrease, so the harvesting will slow down. Also, if the tree population becomes too small, the harvesting will speed up. These two situations cause a problem with the model. Although this model is flawed, it is still better fitted than model 1. As the harvesting dimensionless term decreases, in both models, the populations increase; however, this model increases much faster.

This model can be written in this dimensionless form:

$$\frac{dx}{d\tau} = x(1-x) - hx$$

T = Kx; dT = Kdx $x = \frac{T}{K}$ The x variable remains the same.

$$\frac{dx}{dt} = \frac{dT}{dt}\frac{dx}{dT} = \frac{1}{K}[rT(1-\frac{T}{K}) - HT] = rx - rx^2 - Hx$$

$\Gamma = r$; $\tau = \Gamma \dagger$;

A time constant, Γ, can be represented as r because r has units of $time^{-1}$. Allow τ to be the new time variable and $\dagger$ have dimensions and units of time.

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$$\frac{dx}{d\tau} = \frac{d\dagger}{d\tau}\frac{dx}{dt} = \frac{1}{r}[rx - rx^2 - Hx] = x(1-x) - \frac{H}{r}x$$

The term $\frac{H}{r}$ is dimensionless, and this will be the h term for this model. Now, the entire model is dimensionless.

$h = \frac{H}{r}$ $\Rightarrow$ $\frac{dx}{d\tau} = x(1-x) - hx$

The fixed points can be found by finding the zeroes of the system in terms of h.

$f(x) = x(1 - x - h) = 0 \quad \Rightarrow \quad x_1^* = 0; \quad x_2^* = 1 - h$

The second fixed point will depend on h, but there are no existence criteria.

Stability analysis can determine the stability of the fixed points.

$f'(x) = 1 - 2x - h$

$f'(x_1^*) = 1 - h$; $\quad f'(x_2^*) = h - 1$;

The fact that the fixed points have opposite stability is already obvious.

h < 1: $\quad f'(x_1^*) > 0$; $\quad f'(x_2^*) < 0$;

When h is less than 1, the first fixed point is unstable and the second fixed point in stable.

h = 1 $\quad f'(x_{1,2}^*) = 0$;

Both fixed points are semi-stable when h is one. According to Figure 5., the flow is to left.

h > 1: $\quad f'(x_1^*) < 0$; $\quad f'(x_2^*) > 0$;

When h is greater than 1, the first fixed point is stable, and the second fixed point is unstable.

Since the stability changes at h = 1, then the critical h value is 1. There is a transcritical bifurcation for the critical value of h.

$h_c = 1$

This model can be put into the transcritical normal formal, which proves that it's the correct classification.

$\frac{dx}{d\tau} = x - x^2 - hx \quad \Rightarrow \quad (1 - h)x - x^2$

The correct normal form is: $RX - X^2$. The variable x remains the same, but R is defined as a constant term.

X = x; $\quad$ R = 1 - h

$\frac{dx}{d\tau} = RX - X^2$

This therefore proves there is a transcritical bifurcation at the critical h value.

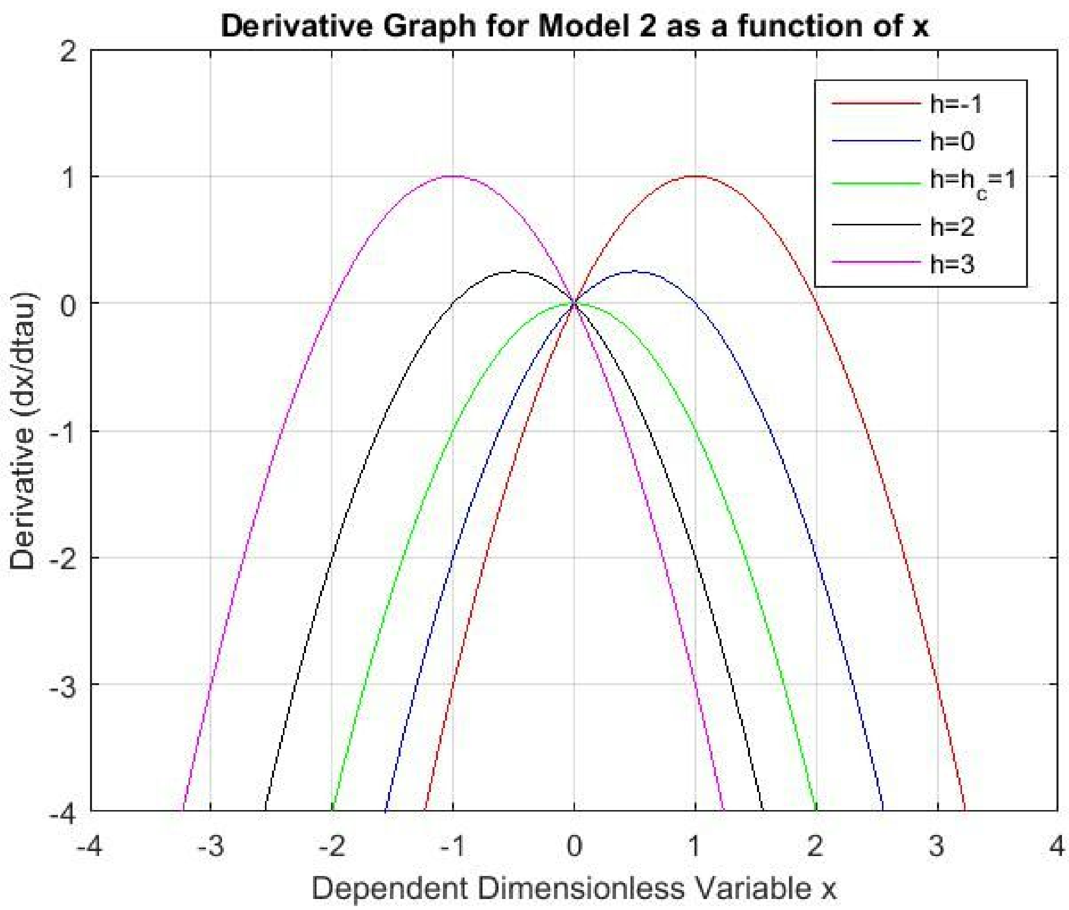

Figure 5. Derivative Graph for Model 2 in Dimensionless Form

Figure 2 shows that all curves have a fixed point at zero, but the second fixed point changes because it depends on h. When h is greater than 1, the first fixed point, $x^* = 0$ is stable because the system is attracted it and pulls x toward it. As x increases when the derivative starts as a negative value, the curve shows that the derivative will increase, and as x decreases when the derivative starts a negative value, the curve shows that the derivative will increase. The system decreased, so the rate increased to find equilibrium in the second scenario.

In the first scenario, the unstable fixed point will be crossed, and the pushed away from this point and toward the stable equilibrium point. From the unstable fixed point, when x increased, the derivative also increases, and when x decreased, the derivative decreased. This case isn't desired for a population model because if it was too large or small, it would only make matters worse, which is why it's considered the unstable equilibrium point. However, when h is less than 1, the curve crosses x = 0 and somewhere on the positive x-axis. The $x^* = 0$ fixed point will now be unstable, and the other fixed point will be stable. When the curve is followed and x increases through the first fixed point, the derivative increases. When x increases and the curve is followed through the second fixed point, the derivative decreases. When h is the critical value, there is one fixed point, and it is semistable. As x increases to $x^* = 0$, the derivative increases, but as x decreases toward it, the derivative also increases. Figure 2 agrees with analytical work.

When h decreases below the critical value of h, the unstable fixed point remains as $x^* = 0$, but the stable equilibrium point becomes a larger value of x. As this h parameter lowers, the size of the tree population needed to achieve equilibrium increases. The population will grow without bound. As h increases above the critical value of h, the unstable fixed points decrease, but the stable fixed point remains as zero, which means population needs to approach zero to achieve equilibrium. The population cannot decrease below the equilibrium point, $x^* = 0$, so the model cannot have a negative population. When h is the critical value, the rate is constant. If there was some arbitrary positive x value, and it decreased as it went along the curve, the derivative increases until it reaches the fixed point. However, if that arbitrary x is negative, the flow will repel x away from the fixed point, and as x decreases, the rate of population decreases. Therefore, as x decreases, depending on the arbitrary x value, the rate will be positive or negative. If the initial tree population is 0, the it will stay 0. If the initial condition is positive, it will remain positive, and if it's negative, the population will remain negative. However, we cannot have a negative initial condition, so this model for the critical value of h cannot have a negative population. A model that only allows positive, and empty, populations has been achieved, so it has been improved since the first model. This model does not affect how fast or slow the harvesting rate is, though.

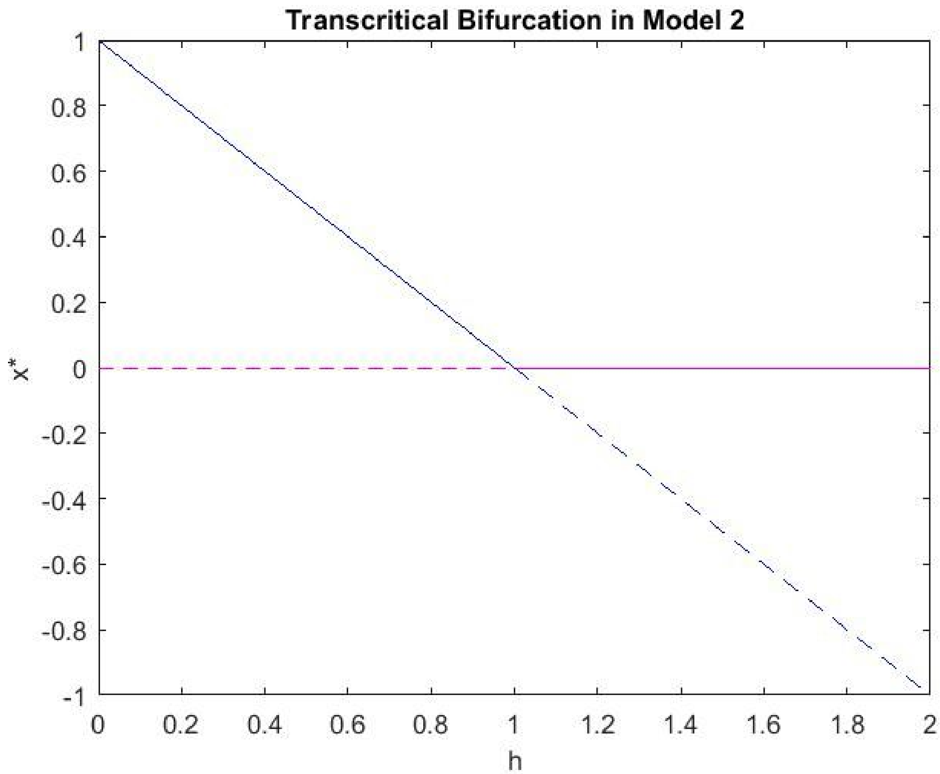

Figure 6. Transcritical Bifurcation Diagram in Model 2

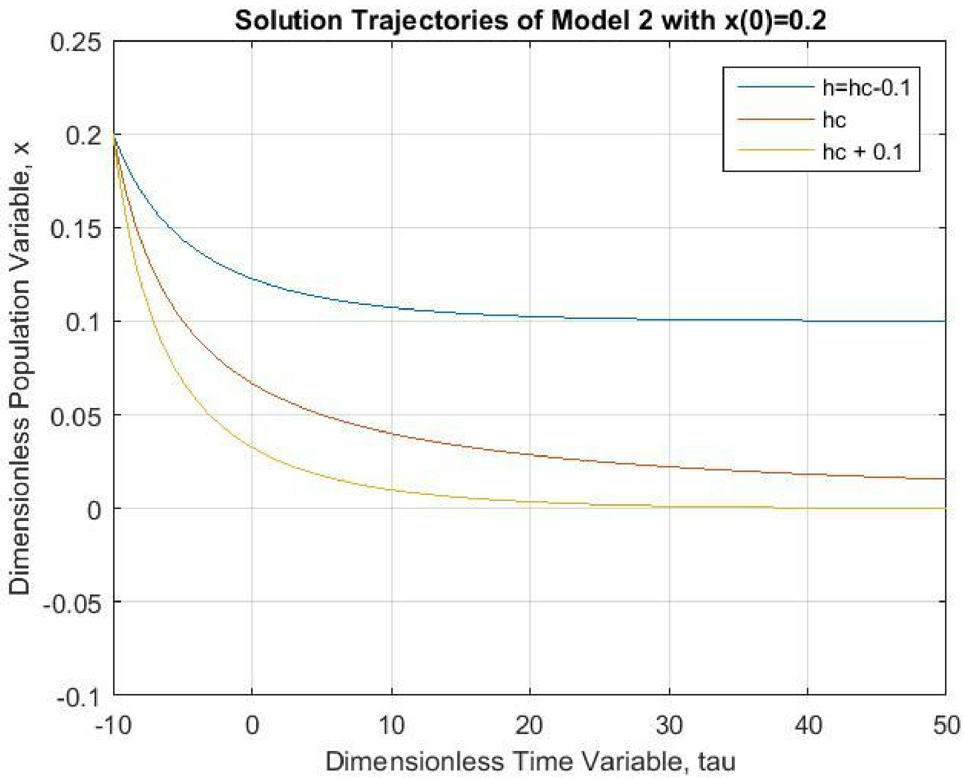

Figure 7. Solution Trajectories when x(0)=0.2 for model 2.

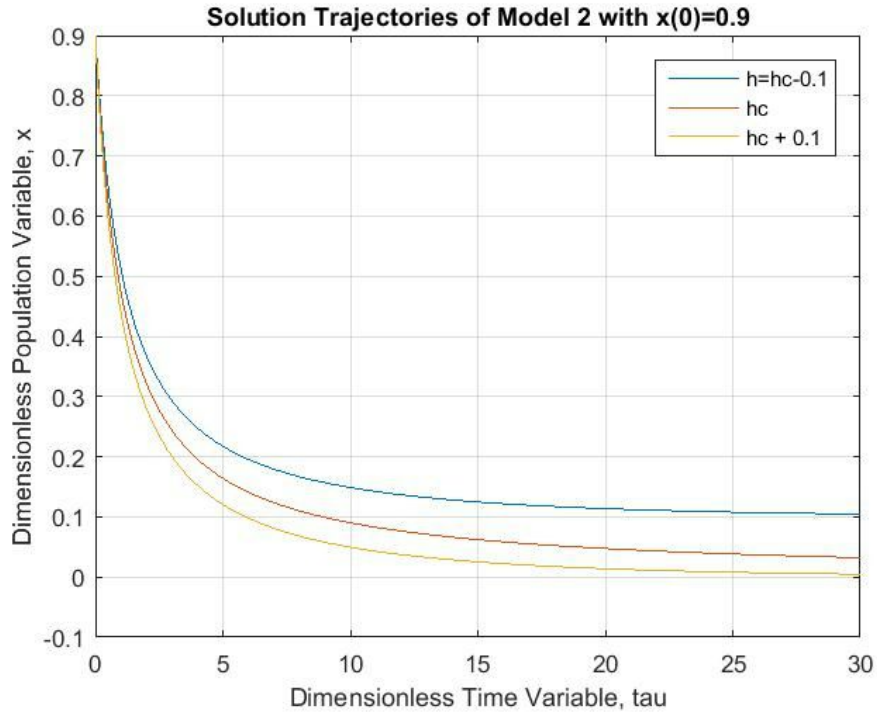

Figure 8. Solution Trajectories when x(0) = 0.9 for model 2.

In Figure 7. and Figure 8, the model no longer achieves a negative population, which was one of the significant differences from the first model. As h decreases, the rate of the decreasing population decreases. The graphs agree with the computations because as the dimensionless harvesting factor decreases, the greater the population's equilibrium will be. The initial condition solely affects the starting size of the population.

Model 3: $$\frac{dT}{dt} = rT(1 - \frac{T}{K}) - H\frac{T}{A + T}$$

The r, T, and K parameters all have the same dimensions, meaning, and restrictions. However, both T and A cannot be zero. The parameter, A, has dimensions of trees, while H has the same dimensions and meaning as the first model. A has the same restrictions as T. The best way to understand the importance of this new term, $H\frac{T}{T+A}$ and A is to observe the limits as T or A approach certain values. As T, the tree population, approaches zero, the term becomes $\frac{H}{A}$. The $\frac{1}{A}$ term causes the harvesting rate, H, to decrease. The tree population diminishes because too many trees were being harvested over a period of time, so this model will slow down the harvesting rate, and allow more trees to grow, which will allow the population to grow. As T approaches infinity, A becomes unnecessary and the term approaches H. If the tree population is growing fastly, the model should not lower the harvesting rate because the farmer(s) wishes to harvest as many trees as possible. As A approaches zero, the model becomes the same as from the first model, which allows for negative population and other problems, and as A approaches infinity, the model becomes the logistic equation, which doesn't include a harvesting factor. These limits portray A as a parameter that affects the harvesting rate, which solves some problems from the earlier models.

This model can be written in this dimensionless form:

$$\frac{dx}{d\tau} = x(1-x) - h\frac{x}{a+x}$$

$x = \frac{T}{K}$; K*dx = dT

The variable x remains the same.

$$\frac{dx}{dt} = \frac{dT}{dt}\frac{dx}{dT} \Rightarrow \frac{1}{K}[rKx(1-\frac{KX}{K}) - H\frac{Kx}{A+Kx} \Rightarrow rx(1-x) - \frac{Hx}{K(\frac{A}{K}+x)};$$

$\Gamma = r$; $\tau = \Gamma \dagger$;

A time constant, Γ, can be represented as r because r has units of $time^{-1}$. Allow τ to be the new time variable and $\dagger$ have dimensions and units of time.

$$\frac{dx}{d\tau} = \frac{1}{r}\frac{dx}{d\tau} = \frac{1}{r}[rx(1-x) - \frac{H}{K}\frac{x}{x+\frac{A}{K}}] = [x(1-x) - \frac{H}{rK}\frac{x}{x+\frac{A}{K}}]$$

The term $\frac{H}{rK}$ is dimensionless, so it represents h. The term $\frac{A}{K}$ is dimensionless, and is defined as a.Now, the entire model is dimensionless.

$$\frac{dx}{d\tau} = x(1-x) - h\frac{x}{x+a} \qquad h = \frac{H}{rK}; \qquad a = \frac{A}{K};$$

The fixed points are determined by finding the zeroes of the equation in terms of h and a.

$x[(1-x) - \frac{h}{x+a}] = 0 \Rightarrow x_1^* = 0$ The first fixed point is 0.

$(1 - x)(x + a) = h \quad \Rightarrow \quad x_{1,2}^* = \frac{(1-a)\pm\sqrt{(a+1)^2 - 4h}}{2}$

$h \leq \frac{(a+1)^2}{4}$

The second and third fixed point depend on h and a. There is an existence criteria for these fixed points.

When $h = \frac{(a+1)^2}{4}$,

There are 2 fixed points. There is the trivial point, and (1-a)/2

$h > \frac{(a+1)^2}{4}$: $\quad x_1^* = 0$

This condition violates the existence criteria for the second and third fixed point, but there is still a trivial fixed point.

$h < \frac{(a+1)^2}{4}$

There are 3 fixed points in this case, and the trivial point is stable.

When there is one fixed point, that trivial solution is semi-stable because it must be tangent to the x-axis.

$$f'(x) = 1 - 2x - \frac{h}{a+x} + \frac{hx}{(a+x)^2}$$

f '(0) = 1 - h/a -> h = a

When h = a, the trivial solution is semi-stable. When $h \neq a$, the trivial fixed point is stable when a is positive and unstable when a is negative. Since the stability depend on the sign of a, then h = a is a transcritical bifurcation.

h = $(a + 1)^2/4$

f '(0) = 1 - $\frac{(a+1)^2}{4}$ -> $1 = \frac{(a+1)^2}{4}$ -> a_c = 1

When a < 1, the trivial fixed point is unstable. When a = 1, it is semistable, and when a>1, it is stable. Since the stability changes as the value of a changes at the critical value of a, it is a saddle node bifurcation.

Conclusion:

The models definitely improved by adjusting the harvesting parameter in order to fix problems the previous one had. The first model depended too much on an initial condition, which isn't desired because the best scenario allows the smallest starting population and largest population over a period of time. Also, it allowed for negative populations, which isn't even possible. The second model was improved and only allowed positive populations. It didn't depend too much on it's initial condition seen from the solution trajectories, and that's another improvement. However, it doesn't affect the rate of the harvesting rate. In the third model, the new A parameter slows down the harvesting rate if needed, which will help the trees' chances of growing more and vast. In some cases in this model, there are 3 equilibrium points.

Although these models improved, they can still be improved. The real world cannot make certain parameters or factors such as pollution negligible to simplify the model. The environment affects trees in various and plentiful ways. These models are still too simple to describe a real tree population. However, the third model would be the best choice if necessary.

Collaborators:

Dr. Laura Munoz:	Described bifurcation conceptually in different perspectives
Dr. Elizabeth Cherry:	Explained the normal forms
Dr. Jacoby Jacob:	Clarified that discrete and continuous parameters can coexist in the same system.
MathWorks,:	Helped with troubleshooting
Ali Borden, Dominique Hall:	Explained new functions to help produce bifurcation diagrams
Desmos:	Showed graphs before plotting them numerically.
Wolfram Alpha:	Quickened crazy computational messes
Wikipedia:	Described less known bifurcation diagrams
Math Biology, J.D. Murray:	Showed what saddle node trajectories looked like, and explained phase portraits.