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Properties of moments of a family of GARCH processes

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Abstract

This paper considers the moments of a family of first-order GARCH processes. First, a general condition for the existence of any integer moment of the absolute values of the observations is given. Second, a general expression for this moment as a function of lower-order moments is derived. Third, the kurtosis and the autocorrelation function of the squared and absolute-valued observations are derived. The results apply to a number of different GARCH parameterizations. Finally, the existence, or lack thereof, of the theoretical counterpart to the so-called Taylor effect in some members of this GARCH family is discussed. Possibilities of extending the results to higher-order GARCH processes are indicated and potential applications of the statistical theory proposed. © 1999 Elsevier Science S.A. All rights reserved.

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1. Introduction

Modelling time series with leptokurtic observations and volatility clusters with GARCH models has become a growth industry, and this has led to

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a number of modifications and extensions of the original model of Bollerslev (1986) and Taylor (1986, pp. 78–79). For a recent overview of GARCH models see, for example, Bollerslev et al. (1994). Several papers such as Bollerslev (1986, 1988), Fornari and Mele (1997), Granger and Ding (1996), He and Teräsvirta (1997, 1999), Nelson (1990, 1991) and Sentana (1995) have paid attention to statistical properties of some of those GARCH models. However, a unified framework for considering statistical properties of various GARCH processes has been lacking in the literature. Hentschel (1995) provided one that appears suitable for model selection purposes. On the other hand, obtaining existence conditions for moments and analytic expressions for existing moments of various GARCH processes as special cases of a single unifying model would also be useful. This would make it possible to investigate what at least seemingly different models actually contribute in practice, i.e., how well they are able to satisfy stylized facts in the observed series. These observational facts include leptokurtosis and nonlinear dependence, of which the latter shows as positive autocorrelations of squared and absolute-valued observations.

This paper provides a unifying framework for considering statistical properties of many, if not all, GARCH models. The discussion is restricted to the GARCH(1, 1) model mainly because this is the most commonly applied parameterization in practice. Possible extensions are mentioned in the last section. The structure of the present paper is as follows. The general family of GARCH(1, 1) models is defined in Section 2. The existence of moments of the sequence of random variables obeying a GARCH(1, 1) model that belongs to this family and the expressions for the existing moments are discussed in Section 3. The fourth moment and the kurtosis receive special attention. Sections 4–6 deal with the autocorrelation function of the squared and absolute-valued sequences of observations from a GARCH(1, 1) process. These results make it possible to discuss the relationship between members of our GARCH family and the so-called Taylor effect (Granger and Ding, 1995a, b), which is done in Section 7. Section 8 concludes.

2. The general GARCH(1, 1) model

Consider the following general class of GARCH(1, 1) models:

$$\varepsilon_t = z_t h_t, \tag{1}$$

where $\{z_t\}$ is a sequence of independent identically distributed random variables with zero mean. For the existence of the nth moment of ε_t it is necessary to assume that $\mathsf{E}|z_t|^n < \infty$. Furthermore,

$$h_t^k = g(z_{t-1}) + c(z_{t-1})h_{t-1}^k, \quad k = 1 \text{ or } 2, \tag{2}$$

where $g_t = g(z_t)$ and $c_t = c(z_t)$ are well-defined functions of z_t. Finally,

$$\Pr\{h_t^k > 0\} = 1 \tag{3}$$

for all t. As for $\{g_t\}$ and $\{c_t\}$, these functions are either constants or sequences of independent identically distributed random variables such that they are stochastically independent of h_t^k. Both g_t and c_t contain parameters that affect the moment structure of ε_t. Constraints on the parameters in g_t and c_t are necessary to guarantee that condition (3) holds. Note that conditions for $\{g_t\}$ can be relaxed in such a way that $\{g_t\}$ is merely a sequence of uncorrelated identically distributed random variables. This case will be discussed later.

Appropriate choices of k, g_t and c_t yield, among others, the following GARCH(1,1) models:

- The GARCH(1,1) (Bollerslev, 1986; Taylor, 1986, pp. 78–79) model for $k = 2$ and $g_{t-1} \equiv \alpha_0$, $c_{t-1} = \beta + \alpha_1 z_{t-1}^2$.
- The absolute value GARCH (AVGARCH(1,1)) model (Taylor, 1986, pp. 78–79; Schwert, 1989) for $k = 1$ and $g_{t-1} \equiv \alpha_0$, $c_{t-1} = \beta + \alpha_1 |z_{t-1}|$.
- The GJR-GARCH(1,1) (Glosten et al. (1993)) model for $k = 2$ and $g_{t-1} \equiv \alpha_0$, $c_{t-1} = \beta + (\alpha_1 + \omega I(z_{t-1})) z_{t-1}^2$ where $I(z_{t-1}) = 1$ if $z_{t-1} < 0$, and $I(z_{t-1}) = 0$ otherwise.
- The nonlinear GARCH(1,1) (NLGARCH(1,1,k); Engle, 1990) model. If $k = 2$ then $g_{t-1} \equiv \alpha_0$, $c_{t-1} = \beta + \alpha_1(1 - 2\eta \operatorname{sgn}(z_{t-1}) + \eta^2) z_{t-1}^2$. If $k = 1$ then $g_{t-1} \equiv \alpha_0$, $c_{t-1} = \beta + \alpha_1(1 - \eta \operatorname{sgn}(z_{t-1}))|z_{t-1}|$. These two models are special cases of the asymmetric power ARCH (A-PARCH) model of Ding et al. (1993) obtained by allowing $k > 0$ in Eq. (2); for a discussion of its statistical properties see also He and Teräsvirta (1997).
- The volatility switching GARCH(1,1) (VS-GARCH(1,1); Fornari and Mele, 1997) model for $k = 2$ and $g_{t-1} = \alpha_0 + \delta(z_{t-1}^2 - \xi)\operatorname{sgn}(z_{t-1})$, $c_{t-1} = \beta + \alpha_1 z_{t-1}^2 + \zeta z_{t-1}^2 \operatorname{sgn}(z_{t-1})$.
- The threshold GARCH (TGARCH(1,1); Zakoïan, 1994) model for $k = 1$ and $g_{t-1} \equiv \alpha_0$, $c_{t-1} = \beta + (\alpha^+(1 - I(z_{t-1})) + \alpha^- I(z_{t-1}))|z_{t-1}|$.
- The fourth-order nonlinear generalized moving-average conditional heteroskedasticity (4NLGMACH(1,1)) model for $k = 2$ and $g_{t-1} = \alpha_0 + \alpha_1(z_{t-1} - c)^2 + \alpha_2(z_{t-1} - c)^4$, $c_{t-1} = \beta$. This is a further generalization of the family of moving-average conditional heteroskedasticity (MACH) models Yang and Bewley (1995) introduced.
- The generalized quadratic ARCH (GQARCH(1,1); Sentana, 1995) model for $k = 2$ and $g_{t-1} = \alpha_0 + \zeta z_{t-1} h_{t-1}$, $c_{t-1} = \beta + \alpha_1 z_{t-1}^2$. Note that g_{t-1} in this case depends both on z_{t-1} and h_{t-1}, and $\{g_t\}$ is thus not sequence of iid random variables. Yet it is a sequence of uncorrelated random variables, and this property allows us to obtain results that are similar to those obtained in the pure iid case. It is worth mentioning however, that Eq. (3) does not

necessarily hold for the GQARCH(1,1) model without further parameter restrictions.

The most conspicuous exclusions from the family of GARCH(1,1) models defined by Eqs. (1)–(3) are the EGARCH model (Nelson, 1991) and the A-PARCH model. Nonlinear GARCH models such as the smooth transition GARCH (STGARCH; González-Rivera, 1998; Hagerud, 1997) do not fit into this framework either. On the other hand, Hentschel (1995) defined a parametric family of GARCH models that also nests the EGARCH and A-PARCH models but not, say, VS-GARCH, GQARCH, GMACH or STGARCH models. Following Hentschel (1995), let

$$f(z_t) = |z_t - b| - c(z_t - b).$$

By defining $c_t = \alpha_1 \lambda [f(z_t)]^{\delta} + \beta$ and $g_t \equiv \lambda \alpha_0 - \beta + 1$ it is seen that a subset of Hentschel's model family defined by $\lambda = \delta = k$ ($= 1, 2$) is nested in Eqs. (1)–(3). As we are interested in analytic expressions of integer moments of the original GARCH process we do not consider the general choice, $k > 0$ in Eq. (2), here.

3. General moment condition and the fourth moment

In this section we consider moment properties of the GARCH(1,1) model (1)–(3) and begin by introducing notation. Let $v_i = \mathsf{E}|z_t|^i$, $\gamma_{ci} = \mathsf{E}c_t^i$, $\gamma_{gi} = \mathsf{E}g_t^i$ and $\tilde{\gamma}_{gi,cj} = \mathsf{E}g_t^i c_t^j$, where i and j are positive integers. In particular, let $\gamma_c = \gamma_{c1}$, $\gamma_g = \gamma_{g1}$ and $\tilde{\gamma}_{gc} = \tilde{\gamma}_{g1,c1}$. We require $\gamma_g > 0$, which is not a restrictive assumption but is needed for condition (3) to hold. We can now state

Theorem 1. For the general GARCH(1,1) *model (1)–(3) that started at some finite value infinitely many periods ago, the kmth unconditional moment exists if and only if*

$$\gamma_{cm} = \mathsf{E}c_t^m < 1. \tag{4}$$

Under this condition the kmth moment of ε_t *can be expressed recursively as*

$$\mu_{km} = \mathsf{E}|\varepsilon_t|^{km} = \{v_{km}/(1 - \gamma_{cm})\}\left\{\sum_{j=1}^{m} \binom{m}{j} \tilde{\gamma}_{gj,c(m-j)}(\mu_{k(m-j)}/v_{k(m-j)})\right\}. \tag{5}$$

Proof. See the appendix.

Condition (4) is a necessary and sufficient condition for the existence of the kmth unconditional moment of $\{\varepsilon_t\}$. It appeared in Nelson (1990) for the standard GARCH(1, 1) model. Note that the existence of μ_{km} does not depend on g_t whereas the value of μ_{km} does.

Since we are interested in the kurtosis of ε_t and the autocorrelation function of $\{\varepsilon_t^2\}$ we first state

Corollary 1.1. For the general GARCH(1, 1) *model (1)–(3) with* $k = 2$ *and* $\gamma_{c2} < 1$, *the fourth unconditional moment of* ε_t *is given by*

$$\mu_4 = \mathsf{E}\varepsilon_t^4 = \frac{\nu_4\{\gamma_{g2}(1-\gamma_c) + 2\tilde{\gamma}_{gc}\gamma_g\}}{(1-\gamma_c)(1-\gamma_{c2})} \tag{6}$$

and the kurtosis by

$$\kappa_4 = \frac{\kappa_4(z_t)\{\gamma_{g2}(1-\gamma_c) + 2\tilde{\gamma}_{gc}\gamma_g\}(1-\gamma_c)}{\gamma_g^2(1-\gamma_{c2})} \tag{7}$$

where $\kappa_4(z_t) = \nu_4/\nu_2^2$ *is the kurtosis of* z_t.

In particular, for the case $g_t \equiv \alpha_0$ in Eq. (2), expression (7) simplifies to

$$\kappa_4^0 = \kappa_4(z_t)\,(1-\gamma_c^2)/(1-\gamma_{c2}). \tag{8}$$

This corollary gives the fourth moment and the kurtosis for $\{\varepsilon_t\}$ when the conditional variance is a function of past squared values of the process. Note that if $k = 1$, the squares are replaced by absolute values. Since it is our intention to also consider fourth moments of GARCH processes with $k = 1$ we apply Theorem 1 again for obtaining those. We have

Corollary 1.2. For the general GARCH(1, 1) model (1)–(3) with $k = 1$ *and* $\gamma_{c4} < 1$ *the fourth unconditional moment of* ε_t *is given by*

$$\mu_4^* = \nu_4 \Delta_4 \prod_{i=1}^{4} (1-\gamma_{ci})^{-1} \tag{9}$$

and the kurtosis

$$\kappa_4^* = \frac{\kappa_4(z)\Delta_4(1-\gamma_c)(1-\gamma_{c2})}{\Delta_2^2(1-\gamma_{c3})(1-\gamma_{c4})}, \tag{10}$$

where

$$\Delta_2 = \gamma_{g2}(1-\gamma_c) + 2\tilde{\gamma}_{gc}\gamma_g, \tag{11}$$

$$\Delta_3 = (1-\gamma_{c2})[\gamma_{g3}(1-\gamma_c) + 3\tilde{\gamma}_{g2,c1}\gamma_g] + 3\tilde{\gamma}_{g1,c2}\Delta_2, \tag{12}$$

$$\Delta_4 = (1-\gamma_{c2})(1-\gamma_{c3})[\gamma_{g4}(1-\gamma_c) + 4\tilde{\gamma}_{g3,c1}\gamma_g]$$
$$+\ 6\tilde{\gamma}_{g2,c2}(1-\gamma_{c3})\Delta_2 + 4\tilde{\gamma}_{g1,c3}\Delta_3. \tag{13}$$

For the large majority of existing absolute value GARCH models $g_t \equiv \alpha_0$ in Eq. (2). In that case, Eqs. (9) and (10) reduce to

$$\mu_4^* = v_4\alpha_0^4\Delta_4^0\prod_{i=1}^{4}(1-\gamma_{ci})^{-1} \tag{14}$$

and

$$\kappa_4^+ = \frac{\kappa_4(z)\Delta_4^0(1-\gamma_c)(1-\gamma_{c2})}{(1+\gamma_c)^2(1-\gamma_{c3})(1-\gamma_{c4})}, \tag{15}$$

respectively, where

$$\Delta_4^0 = 1 + 3\gamma_c + 5\gamma_{c2} + 3\gamma_{c3} + 3\gamma_c\gamma_{c2} + 5\gamma_c\gamma_{c3} + 3\gamma_{c2}\gamma_{c3} + \gamma_c\gamma_{c2}\gamma_{c3}. \tag{16}$$

Next we consider the case in which $k = 2$ and g_t is also a function of h_t. More precisely, let $g_t = \alpha_0 + f(z_t)h_t$. Defining $f(z_t) = \zeta z_t$ yields the GQARCH model. We have

Corollary 1.3. Assume that $g_{t-1} = \alpha_0 + f(z_{t-1})h_{t-1}$ where $\alpha_0 > 0$, $\gamma_f = \mathsf{E}f(z_{t-1}) = 0$, $\gamma_{f2} = \mathsf{E}f^2(z_{t-1}) > 0$ and $\mathsf{E}f(z_{t-1})c_{t-1} = 0$ in the general GARCH(1,1) model (1) with (2). The fourth unconditional moment of ε_t exists if and only if

$$\gamma_{c2} = \mathsf{E}c_t^2 < 1. \tag{17}$$

Under this condition the fourth moment is given by

$$\mu_4 = \mathsf{E}\varepsilon_t^4 = \frac{v_4\{\alpha_0^2(1+\gamma_c) + \alpha_0\gamma_{f2}\}}{(1-\gamma_{c2})(1-\gamma_c)} \tag{18}$$

and the kurtosis by

$$\kappa_4 = \frac{\kappa_4(z)\{(1+\gamma_c) + (\gamma_{f2}/\alpha_0)\}(1-\gamma_c)}{(1-\gamma_{c2})}. \tag{19}$$

Applying expressions (7), (8), (10), (15) and (19) to the models discussed in Section 2 we automatically obtain the kurtosis of $\{\varepsilon_t\}$. For example, for the

VS-GARCH(1,1) model we have $k = 2$, which leads to Eq. (7) with $\gamma_g \equiv \alpha_0$, $\gamma_{g2} = \alpha_0^2 + \delta^2(v_4 - 2v_2\xi + \xi^2)$, $\gamma_c = \beta + \alpha_1 v_2$, $\gamma_{c2} = \beta^2 + 2\beta\alpha_1 v_2 + (\zeta^2 + \alpha_1^2)v_4$ and $\tilde{\gamma}_{gc} = \alpha_0(\beta + \alpha_1 v_2) + \delta\zeta(v_4 - \xi v_2)$. Setting $v_2 = 1$ and $v_4 = 3$ gives the kurtosis for normal errors; this is the correct version of the corresponding expression in Fornari and Mele (1997).

As an example of the case $k = 1$ we may take the TGARCH (1,1) model. If we assume that z_t has a symmetric density, which is not necessary for the general result (15), we obtain the kurtosis by setting

$$\gamma_c = \beta + (1/2)v_1(\alpha^+ + \alpha^-),$$

$$\gamma_{c2} = \beta^2 + v_1\beta(\alpha^+ + \alpha^-) + (1/2)v_2((\alpha^+)^2 + (\alpha^-)^2),$$

$$\gamma_{c3} = \beta^3 + (3/2)v_1\beta^2(\alpha^+ + \alpha^-) + (3/2)v_2\beta((\alpha^+)^2 + (\alpha^-)^2) + (1/2)v_3((\alpha^+)^3 + (\alpha^-)^3)$$

and

$$\gamma_{c4} = \beta^4 + 2v_1\beta^3(\alpha^+ + \alpha^-) + 3v_2\beta^2((\alpha^+)^2 + (\alpha^-)^2) + 2v_3\beta((\alpha^+)^3 + (\alpha^-)^3) + (1/2)v_4((\alpha^+)^4 + (\alpha^-)^4)$$

in expression (15). As the practitioners often work with symmetric distributions we have made that assumption here. Omitting it would lead to more complicated and less practical expressions.

4. Autocorrelations of squared observations, k=2

In this section we consider the autocorrelation function of $\{\varepsilon_t^2\}$ for our general GARCH(1,1) model with $k = 2$. Let $\bar{\gamma}_g = \mathsf{E}(z_t^2 g_t)$ and $\bar{\gamma}_c = \mathsf{E}(z_t^2 c_t)$. The following result defines the n-th order autocorrelation of $\{\varepsilon_t^2\}$, $\rho_n = \rho_n(\varepsilon_t^2, \varepsilon_{t-n}^2)$, $n \geqslant 1$:

Theorem 2. Assume that $\gamma_{c2} < 1$ in the GARCH(1,1) model (1)–(3) with $k = 2$. Then the autocorrelation function of $\{\varepsilon_t^2\}$ is given by

$$\rho_1 = \frac{v_2\{(1-\gamma_c)(\gamma_g\bar{\gamma}_g(1-\gamma_{c2}) + \bar{\gamma}_c(\gamma_{g2}(1-\gamma_c) + 2\tilde{\gamma}_{gc}\gamma_g)) - v_2\gamma_g^2(1-\gamma_{c2})\}}{v_4\{\gamma_{g2}(1-\gamma_c) + 2\tilde{\gamma}_{gc}\gamma_g\}(1-\gamma_c) - v_2^2\gamma_g^2(1-\gamma_{c2})} \tag{20}$$

and $\rho_n = \rho_1\gamma_c^{n-1}$ for $n > 1$.

Proof. See the appendix.

For $g_t \equiv \alpha_0$, the autocorrelation function of $\{\varepsilon_t^2\}$ has the simplified form

$$\rho_1^0 = \frac{v_2\bar{\gamma}_c(1-\gamma_c^2) - v_2^2\gamma_c(1-\gamma_{c2})}{v_4(1-\gamma_c^2) - v_2^2(1-\gamma_{c2})} \tag{21}$$

and $\rho_n^0 = \rho_1^0\gamma_c^{n-1}$ for $n > 1$.

Similarly, we can derive the autocorrelation function for the case in which $g_{t-1} = \alpha_0 + f(z_{t-1})h_{t-1}$. We have

Theorem 3. Let $g_{t-1} = \alpha_0 + f(z_{t-1})h_{t-1}$ *where* $\alpha_0 > 0$, $\gamma_f = \mathsf{E}f(z_{t-1}) = 0$, $\gamma_{f2} = \mathsf{E}f^2(z_{t-1}) > 0$ *and* $\mathsf{E}f(z_{t-1})c_{t-1} = 0$ *in the GARCH*(1,1) *model* (1)–(3). *If condition* $\gamma_{c2} < 1$ *holds, then the autocorrelation function of* $\{\varepsilon_t^2\}$ *is given by*

$$\rho_1 = \frac{v_2\bar{\gamma}_c(1-\gamma_c)\{\alpha_0(1+\gamma_c)+\gamma_{f2}\} - \alpha_0 v_2^2\gamma_c(1-\gamma_{c2})}{v_4(1-\gamma_c)\{\alpha_0(1+\gamma_c)+\gamma_{f2}\} - \alpha_0 v_2^2(1-\gamma_{c2})} \tag{22}$$

and $\rho_n = \rho_1\gamma_c^{n-1}$ for $n > 1$.

Proof. See the appendix.

The autocorrelation function of $\{\varepsilon_t^2\}$ given in the above theorems is dominated by an exponential decay from the first autocorrelation. From expressions (20)–(22) and retaining the previous definitions we obtain the first-order autocorrelation ρ_1 of $\{\varepsilon_t^2\}$ for any member of our GARCH(1, 1) subfamily defined by $k = 2$. For example, for the GARCH(1, 1) model $\gamma_c = \beta + \alpha_1 v_2$, $\gamma_{c2} = \beta^2 + 2\beta\alpha_1 v_2 + \alpha_1^2 v_4$ and $\bar{\gamma}_c = \beta v_2 + \alpha_1 v_4$ in expression (21). It turns out that in this case, expression (21) simplifies to

$$\rho_1 = \alpha_1 v_2(1-\beta^2-\beta\alpha_1 v_2)/(1-\beta^2-2\beta\alpha_1 v_2). \tag{23}$$

Setting $v_2 = 1$ and $v_4 = 3$ (normality) in expression (23) gives the result in Bollerslev (1988). Note that the existence of the autocorrelation function does depend on the existence of v_4 although expression (23) is not a function of v_4. As an another example, the corresponding first-order autocorrelation for the VS-GARCH(1, 1) model is obtained by setting $\bar{\gamma}_g = \alpha_0 v_2$ and $\bar{\gamma}_c = \beta v_2 + \alpha_1 v_4$ in expression (20).

Note that formula (22) helps to make more precise a statement in Sentana (1995). The GQARCH and GARCH models do not have the same autocorrelation function for $\{\varepsilon_t^2\}$ as Sentana argued, but the decay of the autocorrelation function is the same in both models.

5. Autocorrelations of squared observations, $k=1$

In this section we derive the autocorrelation function of $\{\varepsilon_t^2\}$ for our family of absolute value GARCH models. Set $\bar{\gamma}_{c2} = \mathsf{E}(z_t^2 c_t^2)$, $\bar{\gamma}_{g2} = \mathsf{E}(z_t^2 g_t^2)$ and $\bar{\gamma}_{gc} = \mathsf{E}(z_t^2 g_t c_t)$. This notation allows us to formulate the following result:

Theorem 4. If $\gamma_{c4} < 1$ *in the GARCH*(1, 1) *model* (1)–(3) *with* $k=1$, *then the autocorrelation function of* $\{\varepsilon_t^2\}$ *is defined as follows*:

$$\rho_1^* = \frac{v_2(1-\gamma_c)(1-\gamma_{c2})\{2\bar{\gamma}_{gc}(1-\gamma_{c4})\Delta_3 + \bar{\gamma}_{c2}\Delta_4\}}{\Delta}$$
$$- \frac{v_2\Delta_2(1-\gamma_{c3})(1-\gamma_{c4})\{v_2\Delta_2 - \bar{\gamma}_{g2}(1-\gamma_c)(1-\gamma_{c2})\}}{\Delta}, \tag{24}$$

$$\rho_n^* = \gamma_{c2}\rho_{n-1}^* + \theta\gamma_c^{n-2}, \quad n \geqslant 2, \tag{25}$$

where

$$\Delta = v_4\Delta_4(1-\gamma_c)(1-\gamma_{c2}) - v_2^2\Delta_2^2(1-\gamma_{c3})(1-\gamma_{c4})$$

and

$$\theta = \Delta^{-1}\{2v_2\tilde{\gamma}_{gc}(1-\gamma_{c2})(1-\gamma_{c4})[\Delta_3\bar{\gamma}_c(1-\gamma_c)$$
$$- \Delta_2(1-\gamma_{c3})(v_2\gamma_g - \bar{\gamma}_g(1-\gamma_c))]\}.$$

Proof. See the appendix.

Assume again that $g_t \equiv \alpha_0$. Then formulas (24) and (25) simplify and are given in

Corollary 4.1. If $\gamma_{c4} < 1$ *in the GARCH*(1, 1) *model* (1)–(3) *with* $k=1$ *and* $g_t \equiv \alpha_0$, *then the autocorrelation function of* $\{\varepsilon_t^2\}$ *can be defined as follows*:

$$\rho_1^+ = \frac{v_2(1-\gamma_c)(1-\gamma_{c2})\{2\bar{\gamma}_c(1-\gamma_{c4})\Delta_3^0 + \bar{\gamma}_{c2}\Delta_4^0\}}{\Delta^0}$$
$$- \frac{v_2^2(1+\gamma_c)(1-\gamma_{c3})(1-\gamma_{c4})\{2\gamma_c + \gamma_{c2}(1-\gamma_c)\}}{\Delta^0}, \tag{26}$$

$$\rho_n^+ = \gamma_{c2}\rho_{n-1}^+ + \theta^0\gamma_c^{n-1}, \quad n \geqslant 2, \tag{27}$$

where

$$\Delta^0 = v_4\Delta_4^0(1-\gamma_c)(1-\gamma_{c2}) - v_2^2(1+\gamma_c)^2(1-\gamma_{c3})(1-\gamma_{c4}),$$

$$\theta^0 = (1/\Delta^0)\{2v_2(1-\gamma_{c2})(1-\gamma_{c4})[\Delta_3^0\bar{\gamma}_c(1-\gamma_c) - v_2\gamma_c(1+\gamma_c)(1-\gamma_{c3})]\},$$

$$\Delta_3^0 = 1 + 2\gamma_c + 2\gamma_{c2} + \gamma_c\gamma_{c2}$$

and

$$\Delta_4^0 = 1 + 3\gamma_c + 5\gamma_{c2} + 3\gamma_{c3} + 3\gamma_c\gamma_{c2} + 5\gamma_c\gamma_{c3} + 3\gamma_{c2}\gamma_{c3} + \gamma_c\gamma_{c2}\gamma_{c3}.$$

It follows from Theorem 4 that the autocorrelation function of $\{\varepsilon_t^2\}$ for the absolute value GARCH(1, 1) model ($k = 1$) is radically different from that of the GARCH(1, 1) models covered by Theorem 2. For $k = 2$ the autocorrelation function of $\{\varepsilon_t^2\}$ decays exponentially from the first autocorrelation whereas for $k = 1$ it does not. It is seen from expressions (24) and (26) that the rate of decay is slower than exponential. However, as $\gamma_{c4} \to 1$ then the second term on the right-hand side of expression (24) converges to zero. Thus for $\gamma_{c4} \approx 1$ the decay rate is 'nearly' exponential with the discount factor γ_{c2}. As an example of this theory, the first-order autocorrelation of $\{\varepsilon_t^2\}$ for the TGARCH(1, 1) model is obtained by setting $\bar{\gamma}_c = \beta v_2 + (1/2)v_3(\alpha^+ + \alpha^-)$ and $\bar{\gamma}_{c2} = \beta^2 v_2 + v_3\beta(\alpha^+ + \alpha^-) + (1/2)v_4((\alpha^+)^2 + (\alpha^-)^2)$ in formula (26). This again requires the assumption that z_t has a symmetric density.

6. Autocorrelations of absolute values, $k = 1$

The autocorrelation function of the sequence of absolute values $\{|\varepsilon_t|\}$ for our subfamily of GARCH(1, 1) models with $k = 1$ follows as a byproduct of the results in Section 4. Let $\bar{\gamma}_{1g} = \mathsf{E}(|z_t|g_t)$ and $\bar{\gamma}_{1c} = \mathsf{E}(|z_t|c_t)$. A simple substitution of $\bar{\gamma}_{1g}, \bar{\gamma}_{1c}, v_1$ and v_2 for $\bar{\gamma}_g, \bar{\gamma}_c, v_2$ and v_4, respectively, in (20), leads to

Theorem 5. If $\gamma_{c2} < 1$ in the GARCH(1, 1) model (1)–(3) with $k = 1$, then the autocorrelation function of $\{|\varepsilon_t|\}$ is defined as follows:

$$\rho_1(1) = \frac{v_1\{(1-\gamma_c)[\gamma_g\bar{\gamma}_{1g}(1-\gamma_{2c}) + \bar{\gamma}_{1c}(\gamma_{g2}(1-\gamma_c) + 2\tilde{\gamma}_{gc}\gamma_g)] - v_1\gamma_g^2(1-\gamma_{c2})\}}{v_2\{\gamma_{g2}(1-\gamma_c) + 2\tilde{\gamma}_{gc}\gamma_g\}(1-\gamma_c) - v_1^2\gamma_g^2(1-\gamma_{c2})} \tag{28}$$

and $\rho_n(1) = \rho_1(1)\gamma_c^{n-1}$ for $n > 1$.

For $g_t \equiv \alpha_0$, the autocorrelation function of $\{|\varepsilon_t|\}$ has the form

$$\rho_1^0(1) = \frac{v_1 \bar{\gamma}_{1c}(1-\gamma_c^2) - v_1^2 \gamma_c (1-\gamma_{c2})}{v_2(1-\gamma_c^2) - v_1^2(1-\gamma_{c2})} \tag{29}$$

and $\rho_n^0(1) = \rho_1^0(1)\gamma_c^{n-1}$ for $n > 1$.

7. The Taylor property

In this section we consider the following question. Define $\rho_n(\delta) = \rho(|\varepsilon_t|^\delta, |\varepsilon_{t-n}|^\delta)$. Granger and Ding (1995a) called the empirical relationship

$$\hat{\rho}_n(1) > \hat{\rho}_n(\delta) \quad \text{for any } \delta \neq 1,\ n \geqslant 1$$

the Taylor effect. Taylor (1986, pp. 52–55), by studying a variety of speculative return series, found that very often $\hat{\rho}_1(1) > \hat{\rho}_1(2) = \hat{\rho}_1^*$. Granger and Ding (1995a, b) discovered the Taylor effect in a very large number of high-frequency return series. Ding et al. (1993), who considered the long S&P 500 daily stock return series from 3 January 1928 till 30 April 1991 compiled by William Schwert, also studied it. The results of Sections 5 and 6 enable us to see if GARCH(1, 1) models can generate series with $\hat{\rho}_n(1) > \hat{\rho}_n^*$, or, more precisely, whether or not the theoretical relationship $\rho_n(1) > \rho_n^*$ holds for these models. We focus on the first-order autocorrelations and call this theoretical property the Taylor property. For simplicity we restrict ourselves to the AVGARCH(1, 1) process with normal errors.

Our starting point is that for any fixed β, the first-order autocorrelation of the absolute values, $\rho_1^0(1)$, and that of the squares, ρ_1^+, for the AVGARCH(1, 1) process are functions of α_1 defined on $(0, \alpha)$ where $\alpha > 0$ such $\gamma_{c4} \to 1$ as $\alpha_1 \to \alpha$. We can state the following result:

Theorem 6. Consider the AVGARCH(1, 1) model and assume that $z_t \sim$ nid(0,1). Then for any fixed β there exists a subset $\{\alpha_1 \colon \alpha_1 \in (0, \alpha^*)\}$ such that

$$\rho_1^0(1) < \rho_1^+. \tag{30}$$

Assume $\beta = 0$. Then there exists another subset $\{\alpha_1 \colon \alpha_1 \in (\alpha^{**}, \alpha)\}$ such that

$$\rho_1^0(1) > \rho_1^+ \tag{31}$$

where $\alpha > 0$ such that $\gamma_{c4} = 1$. Furthermore, as $\beta = 0$, $\alpha^* = \alpha^{**}$.

Proof. See the appendix.

When $\beta > 0$ the limiting value α is a solution of a fourth-order equation in α_1 (since for the AVGARCH(1, 1) model $\gamma_{c4} = \beta^4 + 4\beta^3\alpha_1 v_1 + 6\beta^2\alpha_1^2 v_2 + 4\beta\alpha_1^3 v_3 + \alpha_1^4 v_4$), which means that an analytic solution generally does not exist. Numerical calculations indicate that the result also holds in this case, but that is only a conjecture.

If Eq. (31) holds, then there exists a lag $n_0 \geqslant 1$ such that the Taylor property $\rho_n(1) > \rho_n^*$ is valid for $n \in \{1, \dots, n_0\}$.

The situation is illustrated in Fig. 1, Panel (a), for $\beta = 0$, and in Panel (b) for $\beta = 0.9$. As kurtosis is a monotonically increasing function of α_1 we use it as the unit of measurement. It is seen how the Taylor property emerges at high values of the kurtosis. It thus seems that the Taylor property is present only for parameterizations that correspond to very strong leptokurtosis and strong, slowly decaying autocorrelation of squared or absolute-valued observations. But then, these are properties that are regularly observed in financial high-frequency series.

We cannot make corresponding comparisons for the standard GARCH(1, 1) process analytically because an analytic expression of the autocorrelation function of $|\varepsilon_t|$ is not available. We have, however, investigated the situation by simulation. The results appear in Table 1. The simulated processes have a high kurtosis because for the AVGARCH model, the Taylor property is found to be present for parameterizations with that property. The results do not contain any evidence favouring the existence of the Taylor property. The means of estimated

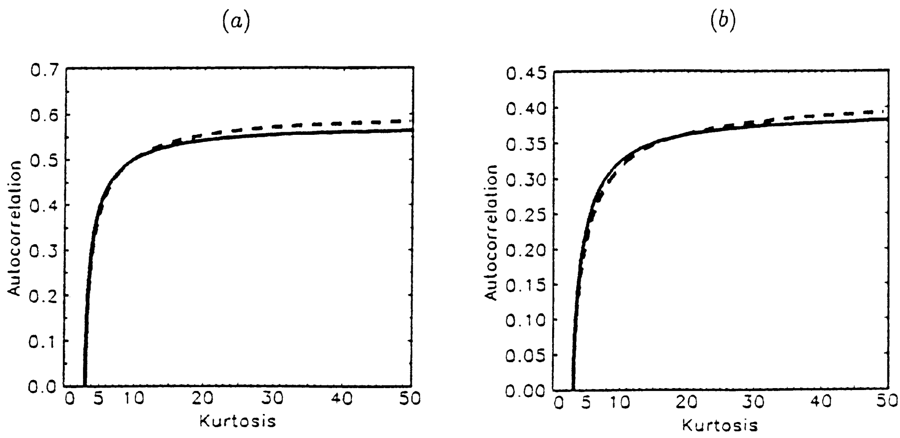

Fig. 1. First-order autocorrelation of absolute-valued (dashed line) and squared (solid line) observations from the AVGARCH(1, 1) model (a) as a function of the kurtosis with $\beta = 0$, (b) as a function of the kurtosis with $\beta = 0.9$.

Table 1
The kurtosis and the first- and fifth-order autocorrelations of squared observations for four GARCH(1, 1) models, the estimates of the autocorrelations of absolute-valued observations, and the number of times (out of 1000) when the value of the estimated first-order autocorrelation of absolute values exceeds that of the squares. For each model, the top sub-row is based on 1000 simulated series of 100,000 observations and the lower one on 1000 series of 1000 observations

α_1	β	κ_4	ρ_1^+	ρ_5^+	$\hat{\rho}_1^+$	$\hat{\rho}_5^+$	$\hat{\rho}_1^0(1)$	$\hat{\rho}_5^0(1)$	Score 1[a]	Score 5[b]
					0.516	0.038	0.457	0.030	88	381
0.575	0	247	0.575	0.063						
					0.452	0.019	0.438	0.021	381	564
					0.386	0.280	0.330	0.243	43	86
0.23	0.70	13.8	0.427	0.319						
					0.315	0.205	0.302	0.211	399	567
					0.406	0.302	0.362	0.276	96	208
0.24	0.70	291	0.471	0.368						
					0.329	0.220	0.324	0.234	471	615
					0.314	0.299	0.300	0.288	248	299
0.0915	0.90	277	0.389	0.376						
					0.199	0.181	0.207	0.192	573	613

[a] Score 1 equals the number of times (out of 1000) in which the simulated value of $\rho_1^0(1)$ (first-order autocorrelation of the absolute values) exceeds that of ρ_1^+ (first-order autocorrelation of the squares).
[b] Score 5 equals the number of times (out of 1000) in which the simulated value of $\rho_5^0(1)$ (fifth-order autocorrelation of the absolute values) exceeds that of ρ_5^+ (fifth-order autocorrelation of the squares).

squared autocorrelations based on simulated series with 100,000 observations (after deleting 50 to discard the initial effects) and 1000 replications always exceed the corresponding means of the autocorrelations of the absolute-valued series. It seems extremely unlikely that the first five autocorrelations of the squared observations would be less than those of the absolute-valued ones. The results are restricted to the standard GARCH(1, 1) process with normal errors.

These results appear to contradict the simulation results in Ding et al. (1993) who found a standard GARCH(1, 1) model capable of generating series with the Taylor effect. But then, the GARCH(1, 1) model they simulated does not have a finite fourth moment so that the results are not directly comparable. Furthermore, sufficiently short series generated by the GARCH(1, 1) model may well display the Taylor effect even if the GARCH process does not have the Taylor property. This is seen from Table 1 which shows what happens when the experiment is repeated with 1000 replications of 1000 instead of 100,000 observations each. Chances of generating a series in which the Taylor effect is present are remarkably larger than in the previous experiment.

There is another detail worth mentioning in Table 1. For the squared observations the theoretical autocorrelation function is known. By comparing its values with the estimated ones it is seen that the autocorrelation estimates are heavily downward biased even for 100,000 observations. The bias is quite dramatic for sequences with 1000 observations. It becomes much less pronounced if the kurtosis is low, but in those cases we do not expect the Taylor property to be present anyway.

8. Conclusions

In this paper we have derived a general existence condition of any integer moment of absolute-valued observation for our family of GARCH processes as well as the moments themselves. The expressions for the kurtosis and autocorrelations of the squared and absolute-valued observations that follow as special cases are of particular importance. They make it possible to see how well any estimated GARCH(1, 1) model reproduces statistical facts observed in the data. Those include high kurtosis and low-starting but persistent autocorrelations of the squared and absolute-valued observations. The investigator may plug estimates of the parameters of a GARCH model into the left-hand side of the existence condition to see what the estimated model implies about the existence of, say, the fourth moment. If the fourth moment condition is satisfied, he or she can do the same for the definitions of the kurtosis and autocorrelations and compare the resulting figures with what is obtained by direct estimation of those statistics from the data. All this can be done for any member of the general GARCH family and for different error distributions. Such comparisons are left for further research.

The only stylized fact investigated to some extent already here is the empirically well-established Taylor effect. The theoretical considerations indicate that some parameterizations of the AVGARCH(1, 1) model of Taylor (1986) with normal errors possess a corresponding theoretical property, so that it is not difficult to generate series with the Taylor effect from these models. On the other hand, our simulation results suggest that for the standard GARCH(1, 1) process with normal errors the first autocorrelations of the squared observations are greater than those of the absolute-valued observations, which contradicts the empirical Taylor effect. Nevertheless, at least if the sample sizes are not unusually high, it is possible to observe the Taylor effect in series generated by this model as well.

All our results concern first-order GARCH processes. It seems straightforward to generalize those about the fourth moment to the GARCH(p, q) case for models for which $g_t \equiv \alpha_0$ and $k = 2$. This would mean obtaining results for, say, the GJR-GARCH and NLGARCH($p, q, 2$) models but not, for example, for the VS-GARCH or QGARCH ones. The basic theory is available in He and Teräsvirta (1999). On the other hand, it would be very hard to generalize the results obtained for models with $k = 1$ (AVGARCH, TGARCH, etc.) with the available techniques. As first-order GARCH models are the ones most commonly used in empirical work, however, we expect our results to be widely applicable in practice.

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Appendix. Proofs

Proof of Theorem 1. Raising Eq. (2) to the power m yields

$$h_t^{km} = c_{t-1}^m h_{t-1}^{km} + \sum_{j=1}^{m} \binom{m}{j} g_{t-1}^j (c_{t-1} h_{t-1}^k)^{m-j}. \tag{A.1}$$

Applying Eq. (2) to the first term on the right-hand side of Eq. (A.1) one obtains

$$h_t^{km} = (c_{t-1} c_{t-2} h_{t-2}^k)^m + \sum_{j=1}^{m} \binom{m}{j} \{ g_{t-1}^j (c_{t-1} h_{t-1}^k)^{m-j} + c_{t-1}^m g_{t-2}^j (c_{t-2} h_{t-2}^k)^{m-j} \}.$$

Further recursion gives

$$h_t^{km} = \left(\prod_{i=1}^{n} c_{t-i}^m \right) h_{t-n}^{km} + \sum_{j=1}^{m} \binom{m}{j} \sum_{i=1}^{n} \left(\prod_{l=1}^{i-1} c_{t-l}^m \right) g_{t-i}^j (c_{t-i} h_{t-i}^k)^{m-j}. \tag{A.2}$$

Assuming that the moments μ_{kp} exist up until $p = m - 1$, the unconditional expectation of expression (A.2) is

$$\mathsf{E} h_t^{km} = \left(\prod_{i=1}^{n} \mathsf{E} c_{t-i}^m \right) \mathsf{E} h_{t-n}^{km}$$

$$+\sum_{j=1}^{m}\binom{m}{j}\sum_{i=1}^{n}\left(\prod_{l=1}^{i-1}\mathsf{E}c_{t-l}^{m}\right)\mathsf{E}(g_{t-i}^{j}c_{t-i}^{m-j})\mathsf{E}h_{t-i}^{k(m-j)}$$

$$=\gamma_{cm}^{n}\mathsf{E}h_{t}^{km}+\sum_{j=1}^{m}\binom{m}{j}\sum_{i=1}^{n}(\gamma_{cm}^{i-1}\tilde{\gamma}_{gj,c(m-j)})\mathsf{E}h_{t}^{k(m-j)}. \tag{A.3}$$

Assume that the process started at some finite value infinitely many periods ago. Then letting $n\to\infty$ in Eq. (A.3) gives

$$\mu_{km}=\mathsf{E}z_{t}^{km}\left\{\sum_{j=1}^{m}\binom{m}{j}\frac{\tilde{\gamma}_{gj,c(m-j)}}{1-\gamma_{cm}}\mathsf{E}h_{t}^{k(m-j)}\right\}$$

$$=(v_{km}/(1-\gamma_{cm}))\left\{\sum_{j=1}^{m}\binom{m}{j}\tilde{\gamma}_{gj,c(m-j)}(\mu_{k(m-j)}/v_{k(m-j)})\right\}$$

if and only if $\gamma_{cm}<1$. □

Proof of Theorem 2. First consider

$$h_{t}^{2}h_{t-1}^{2}=(g_{t-1}+c_{t-1}h_{t-1}^{2})h_{t-1}^{2}. \tag{A.4}$$

Inserting $h_{t-1}^{2}=g_{t-2}+c_{t-2}h_{t-2}^{2}$ to expression (A.4) leads to

$$h_{t}^{2}h_{t-2}^{2}=(g_{t-1}+c_{t-1}g_{t-2}+c_{t-1}c_{t-2}h_{t-2}^{2})h_{t-2}^{2}.$$

Generally, after $n-1$ recursions,

$$h_{t}^{2}h_{t-n}^{2}=\left\{g_{t-1}+\sum_{j=2}^{n}\left(g_{t-j}\prod_{i=1}^{j-1}c_{t-i}\right)+\left(\prod_{i=1}^{n}c_{t-i}\right)h_{t-n}^{2}\right\}h_{t-n}^{2}. \tag{A.5}$$

Multiplying both sides of Eq. (A.5) by $z_{t}^{2}z_{t-n}^{2}$ and taking expectations yields

$$\mathsf{E}(\varepsilon_{t}^{2}\varepsilon_{t-n}^{2})=\left(v_{2}^{2}\gamma_{g}\sum_{j=0}^{n-1}\gamma_{c}^{j}+v_{2}\bar{\gamma}_{g}\gamma_{c}^{n-1}\right)\mathsf{E}h_{t}^{2}+v_{2}\bar{\gamma}_{c}\gamma_{c}^{n-1}\mathsf{E}h_{t}^{4}, \tag{A.6}$$

where $\bar{\gamma}_{g}=\mathsf{E}(z_{t}^{2}g_{t})$ and $\bar{\gamma}_{c}=\mathsf{E}(z_{t}^{2}c_{t})$. From Eq. (A.3) we have $\mathsf{E}h_{t}^{2}=\gamma_{g}/(1-\gamma_{c})$ and $\mathsf{E}h_{t}^{4}=\{\gamma_{g2}(1-\gamma_{c})+2\tilde{\gamma}_{gc}\gamma_{g}\}/(1-\gamma_{c2})(1-\gamma_{c})$. Inserting these into Eq. (A.6) under the assumption $\gamma_{c2}<1$ and applying the definition of $\rho_{n}=\rho_{n}(\varepsilon_{t}^{2},\varepsilon_{t-n}^{2})$ together with some additional manipulation yield (20). □

Proof of Theorem 3. Write

$$h_t^2 = \alpha_0 + f_{t-1}h_{t-1} + c_{t-1}h_{t-1}^2, \tag{A.7}$$

where $f_{t-1} = f(z_{t-1})$. Recursive substitution as in the proof of Theorem 2 leads to

$$h_t^2 h_{t-n}^2 = \left\{\alpha_0\left(1 + \sum_{j=1}^{n-1}\left(\prod_{i=1}^{j} c_{t-i}\right)\right) + \sum_{j=1}^{n}\left(f_{t-j}h_{t-j}\prod_{i=1}^{j-1} c_{t-i}\right) + \left(\prod_{i=1}^{n} c_{t-i}\right)h_{t-n}^2\right\}h_{t-n}^2. \tag{A.8}$$

Multiplying both sides of Eq. (A.8) by $z_t^2 z_{t-n}^2$ and taking expectations yields

$$\mathsf{E}(\varepsilon_t^2\varepsilon_{t-n}^2) = \left(v_2^2\alpha_0\sum_{j=0}^{n-1}\gamma_c^j\right)\mathsf{E}h_t^2 + (v_k\bar{\gamma}_c\gamma_c^{n-1})\mathsf{E}h_t^4, \tag{A.9}$$

where $\bar{\gamma}_c = \mathsf{E}(z_t^2 c_t)$. From Theorem 1, $\mathsf{E}h_t^4 = \{\alpha_0^2(1+\gamma_c) + \alpha_0\gamma_{f2}\}/\{(1-\gamma_{c2})(1-\gamma_c)\}$ and, besides, $\mathsf{E}h_t^2 = \alpha_0/(1-\gamma_c)$. Inserting these into expression (A.9) under the condition $\gamma_{c2} < 1$ and some further manipulation yields Eq. (22). □

Proof of Theorem 4. Setting $k = 1$ in Eq. (2) gives

$$h_t = g_{t-1} + c_{t-1}h_{t-1}. \tag{A.10}$$

Applying Eq. (A.10) to h_t in $h_t^2 h_{t-1}^2$ yields

$$h_t^2 h_{t-1}^2 = (g_{t-1}^2 + 2g_{t-1}c_{t-1}h_{t-1} + c_{t-1}^2 h_{t-1}^2)h_{t-1}^2. \tag{A.11}$$

Multiplying both sides of Eq. (A.11) by $z_t^2 z_{t-1}^2$ and taking expectations we have

$$\mathsf{E}\varepsilon_t^2\varepsilon_{t-1}^2 = v_2\bar{\gamma}_{g2}\mathsf{E}h_t^2 + 2v_2\bar{\gamma}_{gc}\mathsf{E}h_t^3 + v_2\bar{\gamma}_{c2}\mathsf{E}h_t^4. \tag{A.12}$$

Repeating the above steps for $\mathsf{E}\varepsilon_t^2\varepsilon_{t-2}^2$ one obtains

$$\mathsf{E}\varepsilon_t^2\varepsilon_{t-2}^2 = v_2(v_2\gamma_{g2} + \gamma_{c2}\bar{\gamma}_{g2} + 2\tilde{\gamma}_{gc}\bar{\gamma}_g)\mathsf{E}h_t^2 + 2v_2(\tilde{\gamma}_{gc}\bar{\gamma}_c + \bar{\gamma}_{gc}\gamma_{c2})\mathsf{E}h_t^3 + v_2\gamma_{c2}\bar{\gamma}_{c2}\mathsf{E}h_t^4. \tag{A.13}$$

Rewriting the right-hand side of Eq. (A.13) by applying Eq. (A.12) gives

$$\mathsf{E}\varepsilon_t^2\varepsilon_{t-2}^2 = \gamma_{c2}\mathsf{E}\varepsilon_t^2\varepsilon_{t-1}^2 + v_2(v_2\gamma_{g2} + 2\tilde{\gamma}_{gc}\bar{\gamma}_g)\mathsf{E}h_t^2 + 2v_2\tilde{\gamma}_{gc}\bar{\gamma}_c\mathsf{E}h_t^3.$$

In the same fashion, one can write

$$\begin{aligned}\mathsf{E}\varepsilon_t^2\varepsilon_{t-3}^2 = \gamma_{c2}\mathsf{E}\varepsilon_t^2\varepsilon_{t-2}^2 + v_2(v_2\gamma_{g2} + 2v_2\tilde{\gamma}_{gc}\gamma_g \\ + 2\tilde{\gamma}_{gc}\bar{\gamma}_g\gamma_c)\mathsf{E}h_t^2 + 2v_2\tilde{\gamma}_{gc}\bar{\gamma}_c\gamma_c\mathsf{E}h_t^3.\end{aligned}$$

Generally, recursively applying Eq. (A.10) to h_t in $h_t^2h_{t-n}^2$ gives

$$\begin{aligned} h_t^2h_{t-n}^2 = \Bigg\{ & g_{t-1}^2 + \sum_{j=2}^{n}\left(g_{t-j}^2\prod_{i=1}^{j-1}c_{t-i}^2\right) \\ & + 2\sum_{j=2}^{n-1}\left[g_{t-j}\prod_{i=1}^{j-1}c_{t-i}^2\left(\sum_{l=j+1}^{n}g_{t-l}\prod_{k=j}^{l-1}c_{t-k}\right)\right] \\ & + 2g_{t-1}\sum_{j=2}^{n}\left(g_{t-j}\prod_{i=1}^{j-1}c_{t-i}\right) + 2g_{t-1}\prod_{i=1}^{n}c_{t-i}h_{t-n} \\ & + 2\sum_{j=2}^{n}\left(g_{t-j}\prod_{i=1}^{j-1}c_{t-i}\right)\left(\prod_{i=1}^{n}c_{t-i}\right)h_{t-n} + \prod_{i=1}^{n}c_{t-i}^2h_{t-n}^2\Bigg\}h_{t-n}^2. \end{aligned} \tag{A.14}$$

From Eq. (A.14) it follows that

$$\begin{aligned} \mathsf{E}\varepsilon_t^2\varepsilon_{t-n}^2 = v_2\Bigg\{ & v_2\gamma_{g2}\sum_{j=1}^{n-1}\gamma_{c2}^{j-1} + \gamma_{c2}^{n-1}\bar{\gamma}_{g2} \\ & + 2v_2\tilde{\gamma}_{gc}\gamma_g\left(\sum_{j=2}^{n-1}\gamma_{c2}^{j-1}\left(\sum_{l=j+1}^{n-1}\gamma_c^{l-j-1}\right) + \sum_{j=2}^{n-1}\gamma_c^{j-2}\right) \\ & + 2\tilde{\gamma}_{gc}\bar{\gamma}_g\left(\sum_{j=2}^{n-1}\gamma_{c2}^{j-1}\gamma_c^{n-j-1} + \gamma_c^{n-2}\right)\Bigg\}\mathsf{E}h_t^2 \\ & + 2v_2\left\{\bar{\gamma}_{gc}\bar{\gamma}_c\left(\gamma_c^{n-2} + \sum_{j=2}^{n-1}\gamma_{c2}^{j-1}\gamma_c^{n-j-1}\right) + \bar{\gamma}_{gc}\gamma_{c2}^{n-1}\right\}\mathsf{E}h_t^3 \\ & + v_2\gamma_{c2}^{n-1}\bar{\gamma}_{c2}\mathsf{E}h_t^4. \end{aligned} \tag{A.15}$$

On the other hand, applying Eq. (A.15) to $\mathsf{E}\varepsilon_t^2\varepsilon_{t-(n-1)}^2$ we can rewrite $\mathsf{E}\varepsilon_t^2\varepsilon_{t-n}^2$ as

$$\mathsf{E}\varepsilon_t^2\varepsilon_{t-n}^2 = \gamma_{c2}\mathsf{E}\varepsilon_t^2\varepsilon_{t-(n-1)}^2 + v_2\left(v_2\gamma_{g2} + 2v_2\tilde{\gamma}_{gc}\gamma_g\sum_{j=2}^{n-1}\gamma_c^{j-2} + 2\tilde{\gamma}_{gc}\bar{\gamma}_g\gamma_c^{n-2}\right)\mathsf{E}h_t^2$$

$$+ 2v_2\tilde{\gamma}_{gc}\bar{\gamma}_c\gamma_c^{n-2}\mathsf{E}h_t^3. \tag{A.16}$$

Some further manipulations of Eq. (A.16) yield Eqs. (24) and (25). □

Proof of Theorem 6. We assume that $g_t \equiv \alpha_0$ and $c_t = \beta + \alpha_1|z_t|$ and, furthermore, that $z_t \sim \text{nid}(0,1)$. For any fixed β, $\rho_1(1)$ and ρ_1^+ are functions of α_1 defined on $(0,\alpha)$ where $\alpha > 0$ such that $\gamma_{c4} = 1$. Set $\rho_1^0(1,\alpha_1) = \rho_1^0(1)$, $\rho_1^+(\alpha_1) = \rho_1^+$ and, furthermore, $\gamma_{c4}(\alpha_1) = \gamma_{c4}$, respectively to stress this fact. We have

$$\frac{\partial}{\partial\alpha_1}\rho_1^0(1,\alpha_1) = v_1 + 2\beta\alpha_1 v_1^2(1-\beta^2-\beta\alpha_1 v_1)/(1-\beta^2-2\beta\alpha_1 v_1)^2.$$

As α_1 and β are positive, $(\partial/\partial\alpha_1)\rho_1^0(1,\alpha_1) > 0$. This implies that for a given $\beta, \rho_1(1,\alpha_1)$ is a monotonically increasing function of α_1 on $(0,\alpha)$. Next we show that $\rho_1^+(\alpha_1)$ is also a monotonically increasing function of α_1 on $(0,\alpha)$. Choose two arbitrary values $\alpha_{11}, \alpha_{12} \in (0,\alpha)$ such that $\alpha_{11} < \alpha_{12}$. For a fixed β, $\gamma_{c4}(\alpha_{11}) < \gamma_{c4}(\alpha_{12})$. This implies $\rho_1^+(\alpha_{11}) < \rho_1^+(\alpha_{12})$ so that ρ_1^+ is a monotonically increasing function of $\gamma_{c4}(\alpha_1)$.

Second, we show that $\rho_1^0(1,\alpha_1) < \rho_1^+(\alpha_1)$ as $\alpha_1 \to 0$. We have $(\partial/\partial\alpha_1)\rho_1^0(1,\alpha_1)|_{\alpha_1=0} = (\partial/\partial\alpha_1)\rho_1^+(\alpha_1)|_{\alpha_1=0} = v_1 > 0$. Tedious calculation shows that for normal errors, $(\partial^2/\partial\alpha_1^2)\rho_1^+(\alpha_1)|_{\alpha_1=0} > (\partial^2/\partial\alpha_1^2)\rho_1^0(1,\alpha_1)|_{\alpha_1=0} > 0$ for any $\beta > 0$. Thus both autocorrelations are convex functions of α_1, and there exists an interval $(0,\alpha^*)$ such that $\rho_1^0(1,\alpha_1) < \rho_1^+(\alpha_1)$.

Finally, we show that if $\beta = 0$ then $\rho_1^0(1,\alpha) > \rho_1^+(\alpha)$ as $\alpha_1 \to \alpha$. Note that $\alpha_1 \to \alpha$ implies $\kappa_4^+ \to \infty$ and $\gamma_{c4} \to 1$. It is seen from Eq. (26) that $\rho_1^+(\alpha_1) \to v_2/v_4^{1/2}$ as $\alpha_1 \to \alpha$. On the other hand, from Eq. (29) it follows that $\rho_1^0(1,\alpha_1) \to v_1/v_4^{1/4}$ as $\alpha_1 \to \alpha$. Under the normality assumption we have $\rho_1^0(1,\alpha) > \rho_1^+(\alpha)$.

Both $\rho_1^+(\alpha_1)$ and $\rho_1^0(1,\alpha_1)$ are monotonically increasing functions of α_1 on $(0,\alpha)$. It is seen that on $(0,\alpha)$, $(\rho_1^0(1,0), \rho_1^0(1,\alpha)) \supset (\rho_1^+(0), \rho_1^+(\alpha))$. Then for any $\alpha_1 \in (0,\alpha)$, there exists $\alpha^* \in (0,\alpha)$ such that $\rho_1^0(1,\alpha^*) = \rho_1^+(\alpha^*)$. Therefore for any given α_1 we have $\rho_1^0(1,\alpha_1) < \rho_1^+(\alpha_1)$ if $\alpha_1 < \alpha^*$, otherwise $\rho_1^0(1,\alpha_1) > \rho_1^+(\alpha_1)$. □

References

Bollerslev, T., 1986. Generalized autoregressive conditional heteroskedasticity. Journal of Econometrics 31, 307–327.

Bollerslev, T., 1988. On the correlation structure for the generalized autoregressive conditional heteroskedastic process. Journal of Time Series Analysis 9, 121–131.

Bollerslev, T., Engle, R.F., Nelson, D.B., 1994. ARCH models. In: Engle, R.F., McFadden, D.L. (Eds.), Handbook of Econometrics, vol. 4. Elsevier, Amsterdam, pp. 2959–3038.

Ding, Z., Granger, C.W.J., Engle, R.F., 1993. A long memory property of stock market returns and a new model. Journal of Empirical Finance 1, 83–106.

Engle, R.F., 1990. Discussion: Stock market volatility and the crash of 87. Review of Financial Studies 3, 103–106.

Fornari, F., Mele, A., 1997. Sign- and volatility-switching ARCH models: theory and applications to international stock markets. Journal of Applied Econometrics 12, 49–65.

Glosten, L.R., Jagannathan, R., Runkle, D., 1993. On the relation between the expected value and the volatility of the nominal excess return on stocks. Journal of Finance 48, 1779–1801.

González-Rivera, G. 1998. Smooth-transition GARCH models. Studies in Nonlinear Dynamics and Econometrics, 3, 61–78.

Granger, C.W.J., Ding, Z., 1995a. Some properties of absolute returns: an alternative measure of risk. Annales d'économie et de statistique 40, 67–95.

Granger, C.W.J., Ding, Z., 1995b. Stylized facts on the temporal and distributional properties of daily data from speculative markets. Working paper, Department of Economics, University of California, San Diego.

Granger, C.W.J., Ding, Z., 1996. Varieties of long memory models. Journal of Econometrics 73, 61–77.

Hagerud, G.E., 1997. A New Non-linear GARCH Model. EFI Economic Research Institute, Stockholm.

He, C., Teräsvirta, T., 1997. Statistical properties of the Asymmetric Power ARCH process. Working Paper Series in Economics and Finance, No. 199, Stockholm School of Economics.

He, C., Teräsvirta, T., 1999. Fourth moment structure of the GARCH(p, q) process. Econometric Theory, forthcoming.

Hentschel, L., 1995. All in the family. Nesting symmetric and asymmetric GARCH models. Journal of Financial Economics 39, 71–104.

Nelson, D.B., 1990. Stationarity and persistence in the GARCH(1, 1) model. Econometric Theory 6, 318–334.

Nelson, D.B., 1991. Conditional heteroskedasticity in asset returns: a new approach. Econometrica 59, 347–370.

Schwert, G.W., 1989.. Why does stock market volatility change over time? Journal of Finance 45, 1129–1155.

Sentana, E., 1995. Quadratic ARCH models. Review of Economic Studies 62, 639–661.

Taylor, S., 1986. Modelling Financial Time Series. Wiley, New York.

Yang, M., Bewley, R., 1995. Moving average conditional heteroskedastic processes. Economics Letters 49, 367–372.

Zakoïan, J.-M., 1994. Threshold heteroskedastic models. Journal of Economic Dynamics and Control 18, 931–955.