Intro Real Analysis

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3 Metric Spaces

3.1 Definition of Metric Spaces. Examples

Definition 3.1 (Metric Space)**.** *A Metric Space is an ordered pair* (E, d) *where* E *is some set along with a metric function* $d\colon E \times E \to \mathbb{R}$ *such that:*

1. ***Identity of Indiscernibles:*** $d(p, q) = 0 \Leftrightarrow p = q, \ \ \forall p, q \in E$
2. ***Symmetry:*** $d(p, q) = d(q, p), \ \ \forall p, q \in E$
3. ***Subadditivity:*** $d(p, q) \leq d(p, r) + d(r, q), \ \ \forall p, q, r \in E$

Theorem 3.1. *Metric functions are non-negative*

Proof. Let d be a a metric function with points $p, q \in E$. So,

$$\begin{aligned}
0 &= d(p,p) && \text{(Identity of Indiscernibles)} \\
&= \frac{d(p,p)}{2} \\
&\leq \frac{d(p,q) + d(q,p)}{2} && \text{(Subadditivity)} \\
&= \frac{d(p,q) + d(p,q)}{2} && \text{(Symmetry)} \\
&= 2 \cdot \frac{d(p,q)}{2} \\
&= d(p,q)
\end{aligned}$$

□

Theorem 3.2. $d(p_1, p_n) \leq \sum_{k=1}^{n-1} d(p_k, p_{k+1}), \ \ \forall p_1, \ldots, p_n \in E$

Proof. We show the above statement via Induction. Note the case for $n = 2$ is trivial so we omit it:
Base case ($n = 3$)

$$d(p_1, p_3) \leq d(p_1, p_2) + d(p_2, p_3) = \sum_{k=1}^{3-1} d(p_k, p_{k+1})$$

Inductive Step ($n \longrightarrow n + 1$)

$$d(p_1, p_{n+1}) \leq d(p_1, p_n) + d(p_n, p_{n+1}) \leq \sum_{k=1}^{n-1} d(p_k, p_{k+1}) + d(p_n, p_{n+1}) = \sum_{k=1}^{(n+1)-1} d(p_k, p_{k+1})$$

□

Theorem 3.3. $|d(p, r) - d(q, r)| \leq d(p, q), \quad \forall p, q, r \in E$

Proof. Let $p, q, r \in E$. Then:

$$\begin{aligned} d(q, r) &\leq d(q, p) + d(p, r) && \text{(Subadditivity)} \\ d(q, r) &\leq d(p, q) + d(p, r) && \text{(Symmetry)} \\ -d(q, r) &\geq -d(p, q) - d(p, r) \\ d(p, r) - d(q, r) &\geq -d(p, q) \end{aligned}$$

We also have:

$$\begin{aligned} d(p, r) &\leq d(p, q) + d(q, r) && \text{(Subadditivity)} \\ d(p, r) - d(q, r) &\leq d(p, q) \end{aligned}$$

Which gives us:

$$-d(p, q) \leq d(p, r) - d(q, r) \leq d(p, q)$$

$$|d(p, r) - d(q, r)| \leq d(p, q)$$

□

Definition 3.2 (Dot Product)**.** *The Dot Product of vectors* $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{R}^n$ *is*

$$\boldsymbol{x} \cdot \boldsymbol{y} = \sum_{i=1}^{n} x_i y_i$$

Definition 3.3 (Euclidean Norm)**.** *The Euclidean norm of* $\boldsymbol{x} \in \mathbf{R}^n$ *is*

$$||\boldsymbol{x}|| = \sqrt{\sum_{i=1}^{n} x_i^2} = \sqrt{\boldsymbol{x} \cdot \boldsymbol{x}}$$

Theorem 3.4 (Cauchy Schwarz Inequality)**.** $|\boldsymbol{x} \cdot \boldsymbol{y}| \leq ||\boldsymbol{x}|| \cdot ||\boldsymbol{y}||, \quad \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$

Proof. Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$. Then,

$$\begin{aligned}
||t \cdot \boldsymbol{y} + \boldsymbol{x}||^2 &\geq 0, \quad \forall t \in \mathbb{R}^n \\
\sum_{i=1}^{n} (t \cdot y_i + x_i)^2 &\geq 0 \\
\sum_{i=1}^{n} (t^2 \cdot y_i^2 + 2x_i y_i t + x_i^2) &\geq 0 \\
t^2 \sum_{i=1}^{n} y_i^2 + 2t \sum_{i=1}^{n} x_i y_i + \sum_{i=1}^{n} x_i^2 &\geq 0 \\
t^2 ||\boldsymbol{y}||^2 + 2t(\boldsymbol{x} \cdot \boldsymbol{y}) + ||\boldsymbol{x}||^2 &\geq 0
\end{aligned}$$

The above is a quadratic in t that is non-negative. Therefore it must have a non-positive discriminant.

$$\begin{aligned}
(2(\boldsymbol{x} \cdot \boldsymbol{y}))^2 - 4||\boldsymbol{y}||^2 ||\boldsymbol{x}||^2 &\leq 0 \\
4(\boldsymbol{x} \cdot \boldsymbol{y})^2 - 4||\boldsymbol{y}||^2 ||\boldsymbol{x}||^2 &\leq 0 \\
(\boldsymbol{x} \cdot \boldsymbol{y})^2 - ||\boldsymbol{y}||^2 ||\boldsymbol{x}||^2 &\leq 0 \\
(\boldsymbol{x} \cdot \boldsymbol{y})^2 - (||\boldsymbol{x}|| \cdot ||\boldsymbol{y}||)^2 &\leq 0 \\
(\boldsymbol{x} \cdot \boldsymbol{y})^2 &\leq (||\boldsymbol{x}|| \cdot ||\boldsymbol{y}||)^2 \\
|\boldsymbol{x} \cdot \boldsymbol{y}| &\leq ||\boldsymbol{x}|| \cdot ||\boldsymbol{y}||
\end{aligned}$$

□

Corollary 3.4.1. *The Euclidean Norm is Subadditive.*

Proof. Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{R}^n$. Then,

$$\begin{aligned}
||\boldsymbol{x} + \boldsymbol{y}||^2 &= \sum_{i=1}^{n} (x_i + y_i)^2 \\
&= \sum_{i=1}^{n} (x_i^2 + 2x_i y_i + y_i^2) \\
&= \sum_{i=1}^{n} x_i^2 + 2\sum_{i=1}^{n} x_i y_i + \sum_{i=1}^{n} y_i^2 \\
&= ||\boldsymbol{x}||^2 + 2(\boldsymbol{x} \cdot \boldsymbol{y}) + ||\boldsymbol{y}||^2 \\
&\leq ||\boldsymbol{x}||^2 + 2||\boldsymbol{x}|| \cdot ||\boldsymbol{y}| + ||\boldsymbol{y}||^2 && \text{(Cauchy Schwarz Inequality)} \\
&= (||\boldsymbol{x}|| + ||\boldsymbol{y}||)^2 \\
||\boldsymbol{x} + \boldsymbol{y}|| &\leq ||\boldsymbol{x}|| + ||\boldsymbol{y}||
\end{aligned}$$

□

Theorem 3.5. $\mathbb{R}^n$ *along with* $d(\boldsymbol{p}, \boldsymbol{q}) = ||\boldsymbol{p} - \boldsymbol{q}||$ *forms a Metric Space.*

Proof. Let $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{R}^n$

Identity of Indiscernibles: $d(\boldsymbol{p},\boldsymbol{p}) = ||\boldsymbol{p}-\boldsymbol{p}|| = ||\boldsymbol{0}|| = 0$

Symmetry: $d(\boldsymbol{p},\boldsymbol{q}) = ||\boldsymbol{p}-\boldsymbol{q}|| = ||\boldsymbol{q}-\boldsymbol{p}|| = d(\boldsymbol{q},\boldsymbol{p})$

Subadditivity:

$$\begin{aligned} d(\boldsymbol{p},\boldsymbol{q}) &= ||\boldsymbol{p}-\boldsymbol{q}|| \\ &= ||(\boldsymbol{p}-\boldsymbol{r}) + (\boldsymbol{r}-\boldsymbol{q})|| \\ &\leq ||(\boldsymbol{p}-\boldsymbol{r})|| + ||(\boldsymbol{r}-\boldsymbol{q})|| && \text{(Subadditivity of } ||\cdot||) \\ &= d(\boldsymbol{p},\boldsymbol{r}) + d(\boldsymbol{r},\boldsymbol{q}) && \square \end{aligned}$$

Corollary 3.5.1. $\mathbb{R}$ *along with* $d(p,q) = |p-q|$ *forms a Metric Space.*

Proof. Let n = 1. Then by Theorem 3.5 we obtain a metric space □

Theorem 3.6 (Taxicab metric space)**.** *Let* E *be some set along with*

$$d(p,q) = \begin{cases} 1 & p \neq q \\ 0 & p = q \end{cases}$$

Then (E,d) *forms a Metric Space.*

Proof. Let $p,q,r \in E$

Identity of Indiscernibles: $d(p,p) = 0$

Symmetry:
Let $p \neq q$

$$d(p,q) = 1 = d(q,p)$$

Let $p = q$

$$d(p,q) = 0 = d(q,p)$$

Subadditivity:

$$d(p,r) + d(r,q) = \begin{cases} 2 & p \neq r \neq q \\ 0 & p = q = r \\ 1 & else \end{cases}$$

Therefore we have $d(p,q) \leq d(p,r) + d(r,q)$

□