Intro Real Analysis

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1 Notions from Set Theory

1.1 Sets and Elements. Subsets

1.2 Operations on Sets

1.3 Functions

1.4 Finite and Infinite Sets

2 The Real Number System

2.1 The Field Properties

Definition 2.1 (Group). *A Group is a an ordered pair* $(G, *)$ *where* G *is some non-empty set along with a closed binary operator* $*\colon G \times G \to G$ *s.t.:*

1. ***Associative:*** $\forall x, y, z \in G : (x * y) * z = x * (y * z)$.
2. ***Identity Element:*** $\forall x \in G, \exists! i_G \in G : x * i_G = i_G * x = x$.
3. ***Inverse Element:*** $\forall x \in G, \exists! x^{-1} \in G : x * x^{-1} = x^{-1} * x = i_G$.

Definition 2.2 (Abelian Group). *An Abelian Group is a Commutative Group:*

$$\forall x, y \in G : x * y = y * x$$

Definition 2.3 (Field). *A Field is an ordered triple* $(\mathbb{F}, *, \circ)$ *s.t.:*

1. $(\mathbb{F}, *)$ *and* $(\mathbb{F}, \circ)$ *form Abelian Groups with* $0_{\mathbb{F}} = i_{(\mathbb{F},*)}$ *and* $1_{\mathbb{F}} = i_{(\mathbb{F},\circ)}$.
2. ***Distributive Property:*** $\forall x, y, z \in \mathbb{F}, x * (y \circ z) = (x * y) \circ (x * z)$.

Definition 2.4 (Real Numbers). $(\mathbb{R}, +, \cdot)$ *forms a Field where:*

1. ***Addition:*** $+\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $i_{\mathbb{R}} = 0$.
2. ***Multiplication:*** $\cdot\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $i_{\mathbb{R}} = 1$, *where 0 has no inverse element.*

Some other common Fields are $(\mathbb{Q}, +, \cdot)$ and $(\mathbb{C}, +, \cdot)$. So far, these have all been infinite Fields, but one can have finite Fields as well. The smallest possible finite Field is $(\{0_{\mathbb{F}}, 1_{\mathbb{F}}\}, *, \circ)$. Below are some immediate consequences of our algebraic structures:

1. For any Associative Closed Binary Operation, parantheses can be omitted:
   $\cdot\colon S \times S \to S$ associative $\Rightarrow x \cdot y \cdot z = (x \cdot y) \cdot z = x \cdot (y \cdot z), \forall x, y, z \in S$.
2. For any Abelian Group, the order of elements is immaterial:
   $(G, \cdot)$ Abelian Group $\Rightarrow x \cdot y \cdot z = y \cdot x \cdot z = y \cdot z \cdot x = z \cdot y \cdot x, \forall x, y, z \in G$.
3. For any Group, the equation $x \cdot y = z$ has a unique solution in x:
   $(G, \cdot)$ Group $\Rightarrow \forall y, z \in G, \exists! x \in G : x \cdot y = z$.

Proof. Let $y, z \in (G, \cdot)$.
$\Rightarrow y^{-1} \in G$.
$\Rightarrow z \cdot y^{-1} \in G$.
$\Rightarrow (z \cdot y^{-1}) \cdot y = z \cdot (y^{-1} \cdot y)$.
$\Rightarrow (z \cdot y^{-1}) \cdot y = z \cdot i_G$.
$\Rightarrow (z \cdot y^{-1}) \cdot y = z$.
$\Rightarrow x = z \cdot y^{-1}$ is a solution to $x \cdot y = z$.
Let w be some other solution to $x \cdot y = z$.
$\Rightarrow w \cdot y = z$.
$\Rightarrow (w \cdot y) \cdot y^{-1} = z \cdot y^{-1}$.
$\Rightarrow w \cdot (y \cdot y^{-1}) = z \cdot y^{-1}$.
$\Rightarrow w \cdot i_G = z \cdot y^{-1}$.
$\Rightarrow w = z \cdot y^{-1}$.
$\Rightarrow w = x$.
Therefore $x = z \cdot y^{-1}$ is a unique solution to $x \cdot y = z$.

□

4. For any Group, the identity element is unique:

$$\forall x \in G, \exists! i_G \in G : i_G \cdot x = x \cdot i_G = x$$

Proof. Let $(G, \cdot)$ be a Group with identity element i_G.
Let $y, z \in G$ s.t. $y = z$.
$\Rightarrow i_G = y \cdot y^{-1} = z \cdot y^{-1}$.
$\Rightarrow x = i_G$ is a unique solution to $x \cdot y = z$.

□

5. For any Group, inverse elements are unique:

$$\forall x \in G, \exists! x^{-1} \in G : x \cdot x^{-1} = x^{-1} \cdot x = i_G$$

Proof. Let $(G, \cdot)$ be a Group with identity element i_G.
Let $y, z \in (G, \cdot)$ s.t. $z = i_G$.
$\Rightarrow y^{-1} = i_G \cdot y^{-1} = z \cdot y^{-1}$.
$\Rightarrow x = y^{-1}$ is a unique solution to $x \cdot y = z$. □

6. For any Field, any element multiplied by the Zero element $0_{\mathbb{F}}$ is $0_{\mathbb{F}}$.

Proof. Let $x \in (\mathbb{F}, +, \cdot)$.
$\Rightarrow x \cdot 0_{\mathbb{F}} = x \cdot i_{(\mathbb{F},+)} = x \cdot (i_{(\mathbb{F},+)} + i_{(\mathbb{F},+)}) = (x \cdot i_{(\mathbb{F},+)}) + (x \cdot i_{(\mathbb{F},+)})$.
Let $y = x \cdot i_{(\mathbb{F},+)}$.
$\Rightarrow y = y + y$.
$\Rightarrow y + y^{-1} = (y + y) + y^{-1}$.
$\Rightarrow y + y^{-1} = y + (y + y^{-1})$.
$\Rightarrow i_{(\mathbb{F},+)} = y + i_{(\mathbb{F},+)}$.
$\Rightarrow i_{(\mathbb{F},+)} = y$.

$\Rightarrow 0_{\mathbb{F}} = y.$
$\Rightarrow 0_{\mathbb{F}} = x \cdot 0_{\mathbb{F}}.$

$\Rightarrow 0_{\mathbb{F}} \cdot x = i_{(\mathbb{F},+)} \cdot x = (i_{(\mathbb{F},+)} + i_{(\mathbb{F},+)}) \cdot x = (i_{(\mathbb{F},+)} \cdot x) + (i_{(\mathbb{F},+)} \cdot x).$
Let $z = i_{(\mathbb{F},+)} \cdot x.$
$\Rightarrow z = z + z.$
$\Rightarrow z + z^{-1} = (z + z) + z^{-1}.$
$\Rightarrow z + z^{-1} = z + (z + z^{-1}).$
$\Rightarrow i_{(\mathbb{F},+)} = z + i_{(\mathbb{F},+)}.$
$\Rightarrow i_{(\mathbb{F},+)} = z.$
$\Rightarrow 0_{\mathbb{F}} = z.$
$\Rightarrow 0_{\mathbb{F}} = 0_{\mathbb{F}} \cdot x.$

$\Rightarrow x \cdot 0_{\mathbb{F}} = 0_{\mathbb{F}} \cdot x = 0_{\mathbb{F}}.$ □

7. For any Field, if the product of 2 elements is the Zero element, then atleast 1 factor must be the Zero element:

   Proof. Let $x, y \in (\mathbb{F}, +, \cdot)$ s.t. $x, y \neq 0_{\mathbb{F}}$.
   Case 1: $x = 0_{\mathbb{F}} \Rightarrow x \cdot y = 0_{\mathbb{F}} \cdot y = 0_{\mathbb{F}}.$
   Case 2: $y = 0_{\mathbb{F}} \Rightarrow x \cdot y = x \cdot 0_{\mathbb{F}} = 0_{\mathbb{F}}.$ □

8. For any Group, the inverse of the inverse element is the element itself.

   Proof. Let $x \in (G, \cdot)$.
   $\Rightarrow \exists! x^{-1} \in G : x \cdot x^{-1} = i_G$
   $\Rightarrow \exists! (x^{-1})^{-1} \in G : (x^{-1})^{-1} \cdot x^{-1} = i_G.$
   $\Rightarrow$ Both $(x^{-1})^{-1}$ and x are solutions since they are both the solution to same equation:
   $$x \cdot y = z : y = x^{-1}, z = i_G$$
   □

9. For any Field, we can define negative numbers as $-x = -1_{\mathbb{F}} \cdot x = x \cdot -1_{\mathbb{F}}$, where $-1_{\mathbb{F}}$ is the inverse of $1_{\mathbb{F}}$ under $+$.

   Proof. Let $x \in (\mathbb{F}, +, \cdot)$.
   $\Rightarrow \exists! x^{-1} \in (\mathbb{F}, +) : x^{-1} + x = 0_{\mathbb{F}}.$
   Define $x^{-1} = -x$.
   $\Rightarrow -x + x = 0_{\mathbb{F}}.$

   $\Rightarrow -1_{\mathbb{F}} \cdot x + x = -1_{\mathbb{F}} \cdot x + 1_{\mathbb{F}} \cdot x.$
   $\Rightarrow -1_{\mathbb{F}} \cdot x + x = (-1_{\mathbb{F}} \cdot +1_{\mathbb{F}}) \cdot x.$
   $\Rightarrow -1_{\mathbb{F}} \cdot x + x = 0_{\mathbb{F}} \cdot x.$

$\Rightarrow -1_{\mathbb{F}} \cdot x + x = 0_{\mathbb{F}}$.

$\Rightarrow x \cdot -1_{\mathbb{F}} + x = x \cdot -1_{\mathbb{F}} + x \cdot 1_{\mathbb{F}}$.
$\Rightarrow x \cdot -1_{\mathbb{F}} + x = x \cdot (-1_{\mathbb{F}} \cdot +1_{\mathbb{F}})$.
$\Rightarrow x \cdot -1_{\mathbb{F}} + x = x \cdot 0_{\mathbb{F}}$.
$\Rightarrow x \cdot -1_{\mathbb{F}} + x = 0_{\mathbb{F}}$.

$\Rightarrow -x = -1_{\mathbb{F}} \cdot x = x \cdot -1_{\mathbb{F}}$, since all three are solutions to same equation:

$$x + y = z : y = x, z = 0_{\mathbb{F}}$$

□

10. For any Field, multiplying 2 negative numbers yields a positive number.

Proof. Let $x, y \in (\mathbb{F}, +, \cdot)$.
$\Rightarrow -x \cdot -y = (-1_{\mathbb{F}}) \cdot x \cdot (-1_{\mathbb{F}}) \cdot y$
$\Rightarrow -x \cdot -y = (-1_{\mathbb{F}}) \cdot (-x \cdot y)$
$\Rightarrow -x \cdot -y = -(-x \cdot y)$
$\Rightarrow -x \cdot -y = x \cdot y$

□

11. For any Field, the additive inverse of a sum is the sum of the additive inverses.

Proof. Let $x, y \in (\mathbb{F}, +, \cdot)$.

$$\begin{aligned} -(x + y) &= (-1_{\mathbb{F}}) \cdot (x + y) \\ &= (-1_{\mathbb{F}}) \cdot x + (-1_{\mathbb{F}}) \cdot y \\ &= (-x) + (-y) \end{aligned}$$

□

12. For any Field, the multiplicative inverse of a product of non-zero elements is the product of the multiplicative inverses reversed.

Proof. Let $x, y \in (\mathbb{F}, +, \cdot)$ s.t. $x, y \neq 0_{\mathbb{F}}$.
$\Rightarrow x \cdot y \neq 0_{\mathbb{F}}$
$\Rightarrow \exists! x^{-1}, y^{-1}, (x \cdot y)^{-1} \in (\mathbb{F}, \cdot)$.
$\Rightarrow (x \cdot y)^{-1} \cdot (x \cdot y) = 1_{\mathbb{F}}$.
$\Rightarrow (x \cdot y)^{-1} \cdot (x \cdot y) \cdot y^{-1} = 1_{\mathbb{F}} \cdot y^{-1}$.
$\Rightarrow (x \cdot y)^{-1} \cdot x \cdot (y \cdot y^{-1}) = 1_{\mathbb{F}} \cdot y^{-1}$.
$\Rightarrow (x \cdot y)^{-1} \cdot x \cdot 1_{\mathbb{F}} = 1_{\mathbb{F}} \cdot y^{-1}$.

$\Rightarrow (x \cdot y)^{-1} \cdot x = y^{-1}$.
$\Rightarrow (x \cdot y)^{-1} \cdot x \cdot x^{-1} = y^{-1} \cdot x^{-1}$.
$\Rightarrow (x \cdot y)^{-1} \cdot 1_{\mathbb{F}} = y^{-1} \cdot x^{-1}$.
$\Rightarrow (x \cdot y)^{-1} = y^{-1} \cdot x^{-1}$.

□

2.2 Order

2.3 The Least Upper Bound Property

2.4 The Existence of Square Roots

3 Metric Spaces

3.1 Definition of Metric Spaces. Examples

Definition 3.1 (Metric Space)**.** *A Metric Space is an ordered pair* (M, d) *where* M *is some set along with a metric function* $d\colon M^2 \to \mathbb{R}$ *s.t.* $\forall x, y, z \in (M, d)$:

1. ***Symmetry:*** $d(x, y) = d(y, x)$
2. ***Identity of Indiscernibles:*** $d(x, y) = 0 \Leftrightarrow x = y$
3. ***Triangle Inequality:*** $d(x, y) \leq d(x, z) + d(z, y)$

Theorem 3.1. *Metric functions are non-negative.*

Proof. Let $x, y \in (M, d)$
$\Rightarrow d(x, y) = \dfrac{2 \cdot d(x, y)}{2}$
$\Rightarrow d(x, y) = \dfrac{d(x, y) + d(x, y)}{2}$
$\Rightarrow d(x, y) = \dfrac{d(x, y) + d(y, x)}{2}$
$\Rightarrow d(x, y) \geq \dfrac{d(x, x)}{2}$
$\Rightarrow d(x, y) \geq 0$

□

Theorem 3.2 (General Triangle Inequality)**.** *For any sequence of points in a metric space:*

$$d(x_1, x_n) \leq \sum_{k=1}^{n-1} d(x_k, x_{k+1})$$

Proof. Let $x_1, \ldots, x_n \in (M, d)$. By induction:

1. Base Case ($n = 3$):
   $\Rightarrow d(x_1, x_3) \leq d(x_1, x_2) + d(x_2, x_3)$

2. Inductive Hypothesis:
   $d(x_1, x_n) \leq \sum_{k=1}^{n-1} d(x_k, x_{k+1})$

3. Inductive Step:
   $\Rightarrow d(x_1, x_n) + d(x_n, x_{n+1}) \leq \sum_{k=1}^{n-1} d(x_k, x_{k+1}) + d(x_n, x_{n+1})$
   $\Rightarrow d(x_1, x_n) + d(x_n, x_{n+1}) \leq \sum_{k=1}^{n} d(x_k, x_{k+1})$
   $\Rightarrow d(x_1, x_{n+1}) \leq \sum_{k=1}^{n} d(x_k, x_{k+1})$

□

Theorem 3.3 (Reverse Triangle Inequality)**.**

$$|d(x, z) - d(z, y)| \leq d(x, y), \quad \forall x, y, z \in (M, d)$$

Proof. Let $x, y, z \in (M, d)$.
$\Rightarrow d(y, z) \leq d(y, x) + d(x, z)$ and $d(x, z) \leq d(x, y) + d(y, z)$.
$\Rightarrow d(z, y) \leq d(x, y) + d(x, z)$ and $d(x, z) \leq d(x, y) + d(z, y)$.
$\Rightarrow -d(x, y) \leq d(x, z) - d(z, y)$ and $d(x, z) - d(z, y) \leq d(x, y)$.
$\Rightarrow -d(x, y) \leq d(x, z) - d(z, y) \leq d(x, y)$.
$\Rightarrow |d(x, z) - d(z, y)| \leq d(x, y)$. □

Definition 3.2 (Dot Product)**.** *The Dot Product of vectors* $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{R}^n$ *is*

$$\boldsymbol{x} \cdot \boldsymbol{y} = \sum_{i=1}^{n} x_i y_i$$

Definition 3.3 (Euclidean Norm)**.** *The Euclidean norm of* $\boldsymbol{x} \in \mathbf{R}^n$ *is*

$$||\boldsymbol{x}|| = \sqrt{\sum_{i=1}^{n} x_i^2} = \sqrt{\boldsymbol{x} \cdot \boldsymbol{x}}$$

Theorem 3.4 (Cauchy Schwarz Inequality)**.**

$$|\boldsymbol{x} \cdot \boldsymbol{y}| \leq ||\boldsymbol{x}|| \cdot ||\boldsymbol{y}||, \quad \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$$

Proof. Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$. Then,

$$
\begin{aligned}
||t \cdot \boldsymbol{y} + \boldsymbol{x}||^2 &\geq 0, \quad \forall t \in \mathbb{R} \\
\sum_{i=1}^{n} (t \cdot y_i + x_i)^2 &\geq 0 \\
\sum_{i=1}^{n} (t^2 \cdot y_i^2 + 2x_i y_i t + x_i^2) &\geq 0 \\
t^2 \sum_{i=1}^{n} y_i^2 + 2t \sum_{i=1}^{n} x_i y_i + \sum_{i=1}^{n} x_i^2 &\geq 0 \\
t^2 ||\boldsymbol{y}||^2 + 2t(\boldsymbol{x} \cdot \boldsymbol{y}) + ||\boldsymbol{x}||^2 &\geq 0 \\
at^2 + bt + c &\geq 0 \\
b^2 - 4ac &\leq 0 \\
(2(\boldsymbol{x} \cdot \boldsymbol{y}))^2 - 4||\boldsymbol{y}||^2 ||\boldsymbol{x}||^2 &\leq 0 \\
4(\boldsymbol{x} \cdot \boldsymbol{y})^2 - 4||\boldsymbol{y}||^2 ||\boldsymbol{x}||^2 &\leq 0 \\
(\boldsymbol{x} \cdot \boldsymbol{y})^2 - ||\boldsymbol{y}||^2 ||\boldsymbol{x}||^2 &\leq 0 \\
(\boldsymbol{x} \cdot \boldsymbol{y})^2 - (||\boldsymbol{x}|| \cdot ||\boldsymbol{y}||)^2 &\leq 0 \\
(\boldsymbol{x} \cdot \boldsymbol{y})^2 &\leq (||\boldsymbol{x}|| \cdot ||\boldsymbol{y}||)^2 \\
|\boldsymbol{x} \cdot \boldsymbol{y}| &\leq ||\boldsymbol{x}|| \cdot ||\boldsymbol{y}||
\end{aligned}
$$

□

Corollary 3.4.1. *The Euclidean Norm is Subadditive.*

Proof. Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{R}^n$. Then,

$$
\begin{aligned}
||\boldsymbol{x} + \boldsymbol{y}||^2 &= \sum_{i=1}^{n} (x_i + y_i)^2 \\
&= \sum_{i=1}^{n} (x_i^2 + 2x_i y_i + y_i^2) \\
&= \sum_{i=1}^{n} x_i^2 + 2\sum_{i=1}^{n} x_i y_i + \sum_{i=1}^{n} y_i^2 \\
&= ||\boldsymbol{x}||^2 + 2(\boldsymbol{x} \cdot \boldsymbol{y}) + ||\boldsymbol{y}||^2 \\
&\leq ||\boldsymbol{x}||^2 + 2||\boldsymbol{x}|| \cdot ||\boldsymbol{y}| + ||\boldsymbol{y}||^2 \qquad \text{(Cauchy Schwarz Inequality)} \\
&= (||\boldsymbol{x}|| + ||\boldsymbol{y}||)^2 \\
||\boldsymbol{x} + \boldsymbol{y}|| &\leq ||\boldsymbol{x}|| + ||\boldsymbol{y}||
\end{aligned}
$$

□

Theorem 3.5 (Euclidean Metric Space)**.** $(\mathbb{R}^n, d)$ *forms a Metric Space where* $d(\boldsymbol{x}, \boldsymbol{y}) = ||\boldsymbol{x} - \boldsymbol{y}||$.

Proof. Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$.

Symmetry: $d(\boldsymbol{x}, \boldsymbol{y}) = ||\boldsymbol{x} - \boldsymbol{y}|| = ||\boldsymbol{y} - \boldsymbol{x}|| = d(\boldsymbol{y}, \boldsymbol{x})$

Identity of Indiscernibles: $d(\boldsymbol{x}, \boldsymbol{x}) = ||\boldsymbol{x} - \boldsymbol{x}|| = ||\mathbf{0}|| = 0$

Subadditivity:

$$\begin{aligned} d(\boldsymbol{x}, \boldsymbol{y}) &= ||\boldsymbol{x} - \boldsymbol{y}|| \\ &= ||(\boldsymbol{x} - \boldsymbol{z}) + (\boldsymbol{z} - \boldsymbol{y})|| \\ &\leq ||\boldsymbol{x} - \boldsymbol{z}|| + ||\boldsymbol{z} - \boldsymbol{y}|| && \text{(Subadditivity of } || \cdot ||) \\ &= d(\boldsymbol{x}, \boldsymbol{z}) + d(\boldsymbol{z}, \boldsymbol{y}) && \square \end{aligned}$$

Corollary 3.5.1. *$(\mathbb{R}, d)$ forms a Metric Space where $d(x, y) = |x - y|$.*

Proof. Let n = 1. Then by Theorem 3.5 we obtain a metric space $\square$

Theorem 3.6 (Taxicab Metric Space)**.** *For any non-empty set M, the Taxicab Metric:*

$$d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

forms a metric space (M, d).

Proof. Let $x, y, z \in E$.

Identity of Indiscernibles: $d(x, x) = 0$.

Symmetry:
Let $x \neq y \Rightarrow d(x, y) = 1 = d(y, x)$.
Let $x = y \Rightarrow d(x, y) = 0 = d(y, x)$.

Subadditivity:

$$d(x, z) + d(z, y) = \begin{cases} 2 & x \neq z \neq y \\ 0 & x = y = z \\ 1 & else \end{cases}$$

Therefore we have $d(x, y) \leq d(x, z) + d(z, y)$

$\square$

3.2 Open and Closed Sets

Let M be some Metric Space, $c \in M$, and $r \in \mathbf{R}^+$.

Definition 3.4 (Open Ball)**.** *Then the Open Ball in M of center c and radius r is the subset of M given by*

$$B_r(c) = \{x \in M : d(x, c) < r\}$$

Definition 3.5 (Closed Ball)**.** *Then the Closed Ball in M of center c and radius r is the subset of M given by*

$$\bar{B}_r(c) = \{x \in M : d(x, c) \leq r\}$$

Theorem 3.7. *Open/Closed Balls in $\mathbb{R}$ are Open/Closed Intervals.*

Proof. Let $c \in \mathbb{R}, r \in \mathbb{R}^+$.

$$\begin{aligned}B_r(c) &= \{x \in M : d(x, c) < r\} \\ &= \{x \in \mathbb{R} : |x - c| < r\} \\ &= \{x \in \mathbb{R} : c - r < x < c + r\} \\ &= (c - r, c + r)\end{aligned}$$

$$\begin{aligned}\bar{B}_r(c) &= \{x \in M : d(x, c) \leq r\} \\ &= \{x \in \mathbb{R} : |x - c| \leq r\} \\ &= \{x \in \mathbb{R} : c - r \leq x \leq c + r\} \\ &= [c - r, c + r]\end{aligned}$$

□

Definition 3.6 (Open Set)**.** *A subset S of a metric space M is open if every point in S is the center of some open ball contained in S:*

$$\forall c \in S, \exists r \in \mathbb{R}^+ : B_r(c) \subset S$$

Theorem 3.8. *In any metric space, an open ball is an open set.*

Proof. Define $S = B_\delta(p)$ s.t. $p \in M, \delta \in \mathbb{R}^+$.
Let $c \in S$.
$\Rightarrow d(c, p) < \delta$.
Choose $r = \delta - d(c, p) \Rightarrow r \in \mathbb{R}^+$.
Suppose $x \in B_r(c)$.
$\Rightarrow d(x, c) < r$.
$\Rightarrow d(x, c) < \delta - d(c, p)$.
$\Rightarrow d(x, c) + d(c, p) < \delta$.
$\Rightarrow d(x, p) < \delta$. (Triangle Inequality)
$\Rightarrow x \in S$.
$\Rightarrow B_r(c) \subset S$. □

Theorem 3.9. *For any metric space (M, d):*

1. *$\varnothing$ is open*
2. *M is open*
3. *The union of an arbitrary number of open subsets is open:*
   $\{V_k\}_{k\in\mathbb{N}}$ *open in M* $\Rightarrow \bigcup_{k\in\mathbb{N}} V_k$ *open in M.*
4. *The intersection of a finite number of open subsets is open:*
   $\{V_k\}_{k=1}^n$ *open in M* $\Rightarrow \bigcap_{k=1}^n V_k$ *open in M.*

Proof. Let (M, d) be a metric space:

1. Let $c \in \varnothing$. But $c \notin \varnothing$. So trivially, $\exists r \in \mathbb{R}^+ : B_r(c) \subset \varnothing$.

2. Let $c \in M$. Suppose $x \in B_r(c) \Rightarrow d(x, c) < r \Rightarrow x \in M \Rightarrow B_r(c) \subset M$.

3. Suppose $\{V_k\}_{k\in\mathbb{N}}$ arbitrary collection of open subsets in M.
   Let $c \in \bigcup_{k\in\mathbb{N}} V_k$.
   $\Rightarrow c \in V_k$ for some $k = 1, \ldots, n$.
   $\Rightarrow \exists r \in \mathbb{R}^+ : B_r(c) \subset V_k$.
   $\Rightarrow \exists r \in \mathbb{R}^+ : B_r(c) \subset \bigcup_{k\in\mathbb{N}} V_k$, since any set is contained in its union.

4. Suppose $\{V_k\}_{k=1}^n$ finite collection of open subsets in M.
   Let $c \in \bigcap_{k=1}^n V_k$: $\Rightarrow c \in V_k, \forall k = 1, \ldots, n$.
   $\Rightarrow \exists r_k \in \mathbb{R}^+ : B_{r_k}(c) \subset V_k$.
   Let $r = min(r_1, \ldots, r_n)$. Suppose $x \in B_r(c)$: $\Rightarrow d(x, c) < r$.
   $\Rightarrow d(x, c) < r_1, \ldots, d(x, c) < r_n$.
   $\Rightarrow x \in B_{r_1}(c), \ldots, x \in B_{r_n}(c)$.
   $\Rightarrow x \in V_1, \ldots, x \in V_n$.
   $\Rightarrow x \in \bigcap_{k=1}^n V_k$.
   $\Rightarrow B_r(c) \subset \bigcap_{k=1}^n V_k$.

□

Definition 3.7 (Closed Set)**.** *A subset S of a metric space (M, d) is closed if its complement $S^{\complement}$ is open in (M, d).*

Theorem 3.10. *In any metric space, a closed ball is a closed set.*

Proof. Define $S = \bar{B}_\delta(p)$ as some closed ball in (M, d).
Let $c \in S^{\complement}$.
Then, $d(c, p) > \delta$.
Choosing $r = d(c, p) - \delta$, clearly $r \in \mathbb{R}^+$.
Suppose $x \in B_r(c)$.
$\Rightarrow d(x, c) < r$.
$\Rightarrow d(x, c) < d(c, p) - \delta$.
$\Rightarrow d(c, p) - d(x, c) > \delta$.
$\Rightarrow d(c, x) + d(x, p) - d(x, c) > \delta$.
$\Rightarrow d(x, p) > \delta$.
$\Rightarrow x \in S^{\complement}$.
$\Rightarrow B_r(c) \subset S^{\complement}$.
$\Rightarrow S^{\complement}$ is open in (M, d).

□

Theorem 3.11. *For any metric space (M, d):*

1. *$\varnothing$ is closed*

2. *M is closed*

3. *The intersection of an arbitrary number of closed subsets is closed:* $\{V_k\}_{k\in\mathbb{N}}$ *closed in* $M \Rightarrow \bigcap_{k\in\mathbb{N}} V_k$ *closed in* M.

4. *The union of a finite number of closed subsets is closed:* $\{V_k\}_{k=1}^{n}$ *closed in* $M \Rightarrow \bigcup_{k=1}^{n} V_k$ *closed in* M.

Proof. Let (M, d) be a metric space:

1. $\varnothing^{\complement} = M \Rightarrow \varnothing^{\complement}$ open $\Rightarrow \varnothing$ closed.

2. $M^{\complement} = \varnothing \Rightarrow M^{\complement}$ open $\Rightarrow M$ closed.

3. Suppose $\{V_k\}_{k\in\mathbb{N}}$ arbitrary collection of closed subsets in M.
   $\Rightarrow \{V_k^{\complement}\}_{k\in\mathbb{N}}$ is an abitrary collection of open subsets in M.
   $\Rightarrow \bigcup_{k\in\mathbb{N}} V_k^{\complement}$ is open in M.
   $\Rightarrow (\bigcap_{k\in\mathbb{N}} V_k)^{\complement}$ is open in M.
   $\Rightarrow \bigcap_{k\in\mathbb{N}} V_k$ is closed in M.

4. Suppose $\{V_k\}_{k=1}^{n}$ finite collection of closed subsets in M.
   $\Rightarrow \{V_k^{\complement}\}_{k=1}^{n}$ is a finite collection of open subsets in M.
   $\Rightarrow \bigcap_{k=1}^{n} V_k^{\complement}$ is open in M.
   $\Rightarrow (\bigcup_{k=1}^{n} V_k)^{\complement}$ is open in M.
   $\Rightarrow \bigcup_{k=1}^{n} V_k$ is closed in M.

$\square$

Theorem 3.12. *Any singleton set is closed.*

Proof. Define $S = \{p\}$ as the singleton set $\forall p \in (M, d)$.
Let $c \in S^{\complement} = M/\{p\}$.
Then, $c \neq p$.
Choose $r = d(c, p) \Rightarrow r \in \mathbb{R}^{+}$.
Suppose $x \in B_r(c)$.
$\Rightarrow d(x, c) < r$.
$\Rightarrow d(x, c) < d(c, p)$.
$\Rightarrow d(c, p) - d(x, c) > 0$.
$\Rightarrow d(p, c) - d(c, x) > 0$.
$\Rightarrow |d(p, c) - d(c, x)| > 0$.
$\Rightarrow d(p, x) > 0$.
$\Rightarrow p \neq x$.
$\Rightarrow x \in S^{\complement}$.
$\Rightarrow B_r(c) \subset S^{\complement}$.
So $S^{\complement}$ is open, making S closed in M. $\square$

Corollary 3.12.1. *Any finite subset of a metric space is closed.*

Proof. Let $x_1, \ldots, x_n \in M$.
$\Rightarrow \{x_1\}, \ldots, \{x_n\}$ all closed in M.
$\Rightarrow \{x_1, \ldots, x_n\} = \bigcup_{k=1}^{n} \{x_k\}$ closed in M.
□

Theorem 3.13. *Any sphere in a metric space is closed.*

Proof. Let $c \in (M, d), r \in \mathbb{R}^+$.
Define r-sphere centered at c as $S = \{x \in (M, d) : d(x, c) = r\}$.
Let $V_1 = \bar{B}_r(c)$.
$\Rightarrow V_1$ closed in (M, d).
Let $V_2 = B_r(c)^{\complement}$.
$\Rightarrow V_2^{\complement} = B_r(c)$.
$\Rightarrow V_2^{\complement}$ open in (M, d).
$\Rightarrow V_2$ closed in (M, d).
$\Rightarrow V_1 \cap V_2$ closed in (M, d).
$\Rightarrow \{x \in (M, d) : d(x, c) \leq r\} \cap \{x \in (M, d) : d(x, c) \geq r\}$ closed in (M, d).
$\Rightarrow \{x \in (M, d) : d(x, c) = r\}$ closed in (M, d).
$\Rightarrow S$ closed in (M, d). □

Theorem 3.14. *Any half-interval is neither open nor closed.*

Proof. Let $a, b \in \mathbb{R}, a < b$.
Define $S = [a, b)$ as a half-interval on $\mathbb{R}$.
Let $r \in \mathbb{R}^+$.
$\Rightarrow a - r < a$.
$\Rightarrow a - r < min(S)$.
$\Rightarrow a - r \in S^{\complement}$.
$\Rightarrow \exists a \in S, \forall r \in \mathbb{R}^+ : B_r(a) = (a - r, a + r) \subset S$.
$\Rightarrow S$ is not open in $\mathbb{R}$.
$\Rightarrow [b, \infty)$ is not open in $\mathbb{R}$ by same logic.
$\Rightarrow S^{\complement} = (-\infty, a) \cup [b, \infty)$ is not open in $\mathbb{R}$ by same logic.
$\Rightarrow S$ is not closed in $\mathbb{R}$. □

Theorem 3.15. *Any subspace of Euclidean space with a strictly bounded component is open. So,* $\forall a \in \mathbb{R}, k \in \{1, \ldots, n\}$*:* $\{x \in \mathbb{R}^n : x_k < a\}$ *and* $\{x \in \mathbb{R}^n : x_k > a\}$ *are open in* $\mathbb{R}^n$*.*

Proof. Let $a \in \mathbb{R}$, and $k \in \{1, \ldots, n\}$.
Define $S = \{x \in \mathbb{R}^n : x_k < a\}$ and $S' = \{x \in \mathbb{R}^n : x_k > a\}$.
Let $c \in S$ and $p \in S'$.
$\Rightarrow c_k < a$ and $p_k > a$.
Choose $r = a - c_k$ and $\delta = p_k - a \Rightarrow r, \delta \in \mathbb{R}^+$.
Suppose $x \in B_r(c)$ and $y \in B_\delta(p)$.
$\Rightarrow d(x, c) < r$ and $d(y, p) < \delta$.
$\Rightarrow d(x, c) < a - c_k$ and $d(y, p) < p_k - a$.
$\Rightarrow ||x - c|| < a - c_k$ and $||y - p|| < p_k - a$.
$\Rightarrow \sqrt{\sum_k (x_k - c_k)^2} < a - c_k$ and $\sqrt{\sum_k (y_k - p_k)^2} < p_k - a$.

$\Rightarrow \sqrt{(x_k - c_k)^2} < a - c_k$ and $\sqrt{(y_k - p_k)^2} < p_k - a$.
$\Rightarrow |x_k - c_k| < a - c_k$ and $|y_k - p_k| < p_k - a$.
$\Rightarrow x_k - c_k < a - c_k$ and $p_k - y_k < p_k - a$.
$\Rightarrow x_k < a$ and $y_k > a$.
$\Rightarrow x \in S$ and $y \in S'$.
$\Rightarrow B_r(c) \subset S$ and $B_\delta(p) \subset S'$. □

Theorem 3.16. *Any subspace of Euclidean space with a weakly bounded component is closed. So, $\forall a \in \mathbb{R}, k \in \{1, \dots, n\}$: $\{x \in \mathbb{R}^n : x_k \leq a\}$ and $\{x \in \mathbb{R}^n : x_k \geq a\}$ are closed in $\mathbb{R}^n$.*

Proof. Let $a \in \mathbb{R}$, and $k \in \{1, \dots, n\}$.
Define $S = \{x \in \mathbb{R}^n : x_k \leq a\}$ and $S' = \{x \in \mathbb{R}^n : x_k \geq a\}$.
$\Rightarrow S^{\complement} = \{x \in \mathbb{R}^n : x_k > a\}$ and $S'^{\complement} = \{x \in \mathbb{R}^n : x_k < a\}$.
$\Rightarrow S^{\complement}, S'^{\complement}$ are open in $\mathbb{R}^n$.
$\Rightarrow S, S'$ are closed in $\mathbb{R}^n$. □

Definition 3.8 (Open/Closed Intervals in $\mathbb{R}^n$)**.** *Let $a_1, \dots, a_n, b_1, \dots b_n \in \mathbb{R}$. For $a_1 < b_1, \dots, a_n < b_n$, the open interval in $\mathbb{R}^n$ is:*

$$\{x \in \mathbb{R}^n : a_k < x_k < b_k, k = 1, \dots, n\}$$

For $a_1 \leq b_1, \dots, a_n \leq b_n$, the closed interval in $\mathbb{R}^n$ is:

$$\{x \in \mathbb{R}^n : a_k \leq x_k \leq b_k, k = 1, \dots, n\}$$

Theorem 3.17. *Any open/closed interval in Euclidean space is open/closed set.*

Proof. Define $S = \{x \in \mathbb{R}^n : a_k < x_k < b_k, k = 1, \dots, n\}$.
$\Rightarrow S = \bigcap_{k=1}^n \{x \in \mathbb{R}^n : a_k < x_k < b_k\}$.
$\Rightarrow S = (\bigcap_{k=1}^n \{x \in \mathbb{R}^n : x_k > a_k\}) \cap (\bigcap_{k=1}^n \{x \in \mathbb{R}^n : x_k < b_k\})$.
$\Rightarrow S$ is open, since it is the finite intersection of $(2n)$ open sets.

Define $S' = \{x \in \mathbb{R}^n : a_k \leq x_k \leq b_k, k = 1, \dots, n\}$.
$\Rightarrow S' = \bigcap_{k=1}^n \{x \in \mathbb{R}^n : a_k \leq x_k \leq b_k\}$.
$\Rightarrow S' = (\bigcap_{k=1}^n \{x \in \mathbb{R}^n : x_k \geq a_k\}) \cap (\bigcap_{k=1}^n \{x \in \mathbb{R}^n : x_k \leq b_k\})$.
$\Rightarrow S'$ is closed, since it is the finite intersection of $(2n)$ closed sets. □

Definition 3.9 (Bounded)**.** *A subset S of a metric space M is bounded if it is contained in some ball.*

Theorem 3.18. *Any open/closed interval in Euclidean space is bounded.*

Proof. Let $a, b \in \mathbb{R}^n$ s.t. $a \neq b$.
Define $S = \{x \in \mathbb{R}^n : a_k \leq x_k \leq b_k, k = 1, \dots, n\}$.
Choose $r = d(b, a) > 0$.
Let $x \in S$.
$\Rightarrow 0 \leq x_k - a_k \leq b_k - a_k$.

$\Rightarrow (x_k - a_k)^2 \leq (b_k - a_k)^2$.
$\Rightarrow \sum_{k=1}^{n} (x_k - a_k)^2 \leq \sum_{k=1}^{n} (b_k - a_k)^2$.
$\Rightarrow \sqrt{\sum_{k=1}^{n} (x_k - a_k)^2} \leq \sqrt{\sum_{k=1}^{n} (b_k - a_k)^2}$.
$\Rightarrow ||x - a|| \leq ||b - a||$.
$\Rightarrow d(x, a) \leq d(b, a)$.
$\Rightarrow d(x, a) \leq r$.
$\Rightarrow x \in \bar{B}_r(a)$.
$\Rightarrow S \subset \bar{B}_r(a)$.

Define $S' = \{x \in \mathbb{R}^n : a_k < x_k < b_k, k = 1, \ldots, n\}$.
$\Rightarrow S' \subset S$.
$\Rightarrow S' \subset \bar{B}_r(a)$. □

Theorem 3.19. *The union of a finite collection of bounded subsets of a metric space is bounded.*

Proof. Suppose $V_1, \ldots, V_n$ are bounded subsets in M.
Let $c_1 \in V_1, \ldots, c_n \in V_n$.
$\Rightarrow \exists r_1, \ldots, r_n \in \mathbb{R}^+$ s.t. $V_k \subset \bar{B}_{r_k}(c_k), \forall k = 1, \ldots, n$.
Let $x \in \bigcup_{k=1}^{n} V_k$.
$\Rightarrow x \in V_k$ for some $k = 1, \ldots, n$.
$\Rightarrow x \in \bar{B}_{r_k}(c_k)$.
$\Rightarrow \bigcup_{k=1}^{n} V_k \subset \bar{B}_{r_k}(c_k)$. □

Theorem 3.20. *A nonempty closed subset of $\mathbb{R}$, if it is bounded from above has a greatest element and if it is bounded from below has a least element.*

Proof. Let $S \subset \mathbb{R}, S \neq \varnothing$.
Suppose S is closed in $\mathbb{R}$ and bounded above.
Let $c = sup(S)$.
With an eye to contradict, assume $c \in S^{\complement}$.
Since $S^{\complement}$ open in $\mathbb{R}$, $\exists r \in \mathbb{R}^+$ s.t. $(c - r, c + r) \subset S^{\complement}$.
Then no element in S is greater than $c - r$.
$\Rightarrow c - r$ is an upper bound for S.
This must be a contradictions, so $c \in S$. □

3.3 Convergent Sequences

Definition 3.10 (Convergent Sequence)**.** *Let* $\{p_n\}_{n \in \mathbb{N}}$

3.4 Completeness

3.5 Compactness

3.6 Connectedness

4 Continuous Functions

4.1 Definition of Continuity. Examples

4.2 Continuity and Limits

4.3 The Continuity of Rational Operations. Functions with values in E^n

4.4 Continuous Functions on a Compact Metric Space

4.5 Continuous Functions on a Connected Metric Space

4.6 Sequences of Functions

5 Differentiation

5.1 Definition of the Derivative

5.2 Rules of Differentiation

5.3 The Mean Value Theorem

5.4 Taylor's Theorem

6 Riemann Integration

6.1 Definition and Examples

6.2 Linearity and Order Properties of the Integral

6.3 Existence of the Integral

6.4 The Fundamental Theorem of Calculus

6.5 The Logarithmic and Exponential Functions

6.6 Definition of Continuity. Examples

7 Interchange of Limit Operations

7.1 Integration and Differentiation of Sequences of Functions

7.2 Infinite Series

7.3 Power Series

7.4 The Trigonometric Functions

7.5 Differentiation under the Integral Sign

8 The Method of Successive Approximations

8.1 The Fixed Point Theorem