

Intro Real Analysis

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1 Notions from Set Theory

1.1 Sets and Elements. Subsets

1.2 Operations on Sets

1.3 Functions

1.4 Finite and Infinite Sets

2 The Real Number System

2.1 The Field Properties

2.2 Order

2.3 The Least Upper Bound Property

2.4 The Existence of Square Roots

3 Metric Spaces

3.1 Definition of Metric Spaces. Examples

Definition 3.1 (Metric Space). *A Metric Space is an ordered pair (E, d) where E is some set along with a metric function $d: E \times E \rightarrow \mathbb{R}$ such that:*

1. **Identity of Indiscernibles:** $d(p, q) = 0 \Leftrightarrow p = q, \quad \forall p, q \in E$
2. **Symmetry:** $d(p, q) = d(q, p), \quad \forall p, q \in E$
3. **Subadditivity:** $d(p, q) \leq d(p, r) + d(r, q), \quad \forall p, q, r \in E$

Theorem 3.1. *Metric functions are non-negative*

Proof. Let d be a metric function with points $p, q \in E$. So,

$$\begin{aligned} 0 &= d(p, p) && \text{(Identity of Indiscernibles)} \\ &= \frac{d(p, p)}{2} \\ &\leq \frac{d(p, q) + d(q, p)}{2} && \text{(Subadditivity)} \\ &= \frac{d(p, q) + d(p, q)}{2} && \text{(Symmetry)} \\ &= 2 \cdot \frac{d(p, q)}{2} \\ &= d(p, q) \end{aligned} \quad \square$$

Theorem 3.2. $d(p_1, p_n) \leq \sum_{k=1}^{n-1} d(p_k, p_{k+1}), \quad \forall p_1, \dots, p_n \in E$

Proof. We show the above statement via Induction. Note the case for $n = 2$ is trivial so we omit it:

Base case ($n = 3$)

$$d(p_1, p_3) \leq d(p_1, p_2) + d(p_2, p_3) = \sum_{k=1}^{3-1} d(p_k, p_{k+1})$$

Inductive Step ($n \rightarrow n + 1$)

$$d(p_1, p_{n+1}) \leq d(p_1, p_n) + d(p_n, p_{n+1}) \leq \sum_{k=1}^{n-1} d(p_k, p_{k+1}) + d(p_n, p_{n+1}) = \sum_{k=1}^{(n+1)-1} d(p_k, p_{k+1})$$

□

Theorem 3.3. $|d(p, r) - d(q, r)| \leq d(p, q), \quad \forall p, q, r \in E$

Proof. Let $p, q, r \in E$. Then:

$$\begin{aligned} d(q, r) &\leq d(q, p) + d(p, r) && \text{(Subadditivity)} \\ d(q, r) &\leq d(p, q) + d(p, r) && \text{(Symmetry)} \\ -d(q, r) &\geq -d(p, q) - d(p, r) \\ d(p, r) - d(q, r) &\geq -d(p, q) \end{aligned}$$

We also have:

$$\begin{aligned} d(p, r) &\leq d(p, q) + d(q, r) && \text{(Subadditivity)} \\ d(p, r) - d(q, r) &\leq d(p, q) \end{aligned}$$

Which gives us:

$$\begin{aligned} -d(p, q) &\leq d(p, r) - d(q, r) \leq d(p, q) \\ |d(p, r) - d(q, r)| &\leq d(p, q) \end{aligned}$$

□

Definition 3.2 (Dot Product). *The Dot Product of vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ is*

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^n x_i y_i$$

Definition 3.3 (Euclidean Norm). *The Euclidean norm of $\mathbf{x} \in \mathbb{R}^n$ is*

$$\|\mathbf{x}\| = \sqrt{\sum_{i=1}^n x_i^2} = \sqrt{\mathbf{x} \cdot \mathbf{x}}$$

Theorem 3.4 (Cauchy Schwarz Inequality). $|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \cdot \|\mathbf{y}\|, \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$

Proof. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Then,

$$\begin{aligned}
& \|t \cdot \mathbf{y} + \mathbf{x}\|^2 \geq 0, \quad \forall t \in \mathbb{R} \\
& \sum_{i=1}^n (t \cdot y_i + x_i)^2 \geq 0 \\
& \sum_{i=1}^n (t^2 \cdot y_i^2 + 2x_i y_i t + x_i^2) \geq 0 \\
& t^2 \sum_{i=1}^n y_i^2 + 2t \sum_{i=1}^n x_i y_i + \sum_{i=1}^n x_i^2 \geq 0 \\
& t^2 \|\mathbf{y}\|^2 + 2t(\mathbf{x} \cdot \mathbf{y}) + \|\mathbf{x}\|^2 \geq 0
\end{aligned}$$

The above is a quadratic in t that is non-negative. Therefore it must have a non-positive discriminant.

$$\begin{aligned}
(2(\mathbf{x} \cdot \mathbf{y}))^2 - 4\|\mathbf{y}\|^2\|\mathbf{x}\|^2 &\leq 0 \\
4(\mathbf{x} \cdot \mathbf{y})^2 - 4\|\mathbf{y}\|^2\|\mathbf{x}\|^2 &\leq 0 \\
(\mathbf{x} \cdot \mathbf{y})^2 - \|\mathbf{y}\|^2\|\mathbf{x}\|^2 &\leq 0 \\
(\mathbf{x} \cdot \mathbf{y})^2 - (\|\mathbf{x}\| \cdot \|\mathbf{y}\|)^2 &\leq 0 \\
(\mathbf{x} \cdot \mathbf{y})^2 &\leq (\|\mathbf{x}\| \cdot \|\mathbf{y}\|)^2 \\
|\mathbf{x} \cdot \mathbf{y}| &\leq \|\mathbf{x}\| \cdot \|\mathbf{y}\|
\end{aligned}$$

□

Corollary 3.4.1. *The Euclidean Norm is Subadditive.*

Proof. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Then,

$$\begin{aligned}
\|\mathbf{x} + \mathbf{y}\|^2 &= \sum_{i=1}^n (x_i + y_i)^2 \\
&= \sum_{i=1}^n (x_i^2 + 2x_i y_i + y_i^2) \\
&= \sum_{i=1}^n x_i^2 + 2 \sum_{i=1}^n x_i y_i + \sum_{i=1}^n y_i^2 \\
&= \|\mathbf{x}\|^2 + 2(\mathbf{x} \cdot \mathbf{y}) + \|\mathbf{y}\|^2 \\
&\leq \|\mathbf{x}\|^2 + 2\|\mathbf{x}\| \cdot \|\mathbf{y}\| + \|\mathbf{y}\|^2 \quad (\text{Cauchy Schwarz Inequality}) \\
&= (\|\mathbf{x}\| + \|\mathbf{y}\|)^2 \\
\|\mathbf{x} + \mathbf{y}\| &\leq \|\mathbf{x}\| + \|\mathbf{y}\|
\end{aligned}$$

□

Theorem 3.5. \mathbb{R}^n along with $d(\mathbf{p}, \mathbf{q}) = \|\mathbf{p} - \mathbf{q}\|$ forms a Metric Space.

Proof. Let $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n$

Identity of Indiscernibles: $d(\mathbf{p}, \mathbf{p}) = \|\mathbf{p} - \mathbf{p}\| = \|\mathbf{0}\| = 0$

Symmetry: $d(\mathbf{p}, \mathbf{q}) = \|\mathbf{p} - \mathbf{q}\| = \|\mathbf{q} - \mathbf{p}\| = d(\mathbf{q}, \mathbf{p})$

Subadditivity:

$$\begin{aligned} d(\mathbf{p}, \mathbf{q}) &= \|\mathbf{p} - \mathbf{q}\| \\ &= \|(\mathbf{p} - \mathbf{r}) + (\mathbf{r} - \mathbf{q})\| \\ &\leq \|(\mathbf{p} - \mathbf{r})\| + \|(\mathbf{r} - \mathbf{q})\| \quad (\text{Subadditivity of } \|\cdot\|) \\ &= d(\mathbf{p}, \mathbf{r}) + d(\mathbf{r}, \mathbf{q}) \quad \square \end{aligned}$$

Corollary 3.5.1. \mathbb{R} along with $d(p, q) = |p - q|$ forms a Metric Space.

Proof. Let $n = 1$. Then by Theorem 3.5 we obtain a metric space \square

Theorem 3.6 (Taxicab metric space). *Let E be some set along with*

$$d(p, q) = \begin{cases} 1 & p \neq q \\ 0 & p = q \end{cases}$$

Then (E, d) forms a Metric Space.

Proof. Let $p, q, r \in E$

Identity of Indiscernibles: $d(p, p) = 0$

Symmetry:

Let $p \neq q$

$$d(p, q) = 1 = d(q, p)$$

Let $p = q$

$$d(p, q) = 0 = d(q, p)$$

Subadditivity:

$$d(p, r) + d(r, q) = \begin{cases} 2 & p \neq r \neq q \\ 0 & p = q = r \\ 1 & \text{else} \end{cases}$$

Therefore we have $d(p, q) \leq d(p, r) + d(r, q)$ \square

3.2 Open and Closed Sets

3.3 Convergent Sequences

3.4 Completeness

3.5 Compactness

3.6 Connectedness

4 Continuous Functions

4.1 Definition of Continuity. Examples

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