# Intro Real Analysis

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## Contents

# 1 Notions from Set Theory

## 1.1 Sets and Elements. Subsets

## 1.2 Operations on Sets

## 1.3 Functions

## 1.4 Finite and Infinite Sets

# 2 The Real Number System

## 2.1 The Field Properties

**Definition 2.1** (Group). *A Group is a an ordered pair $(G, *)$ where $G$ is some non-empty set along with a closed binary operator $*\colon G \times G \to G$ s.t.:*

1. ***Associative:*** $\forall x, y, z \in G : (x * y) * z = x * (y * z)$.
2. ***Identity Element:*** $\forall x \in G, \exists! i_G \in G : x * i_G = i_G * x = x$.
3. ***Inverse Element:*** $\forall x \in G, \exists! x^{-1} \in G : x * x^{-1} = x^{-1} * x = i_G$.

**Definition 2.2** (Abelian Group). *An Abelian Group is a Commutative Group:*

$$\forall x, y \in G : x * y = y * x$$

**Definition 2.3** (Field). *A Field is an ordered triple $(\mathbb{F}, *, \circ)$ s.t.:*

1. $(\mathbb{F}, *)$ *and* $(\mathbb{F}, \circ)$ *form Abelian Groups with* $0_\mathbb{F} = i_{(\mathbb{F},*)}$ *and* $1_\mathbb{F} = i_{(\mathbb{F},\circ)}$.
2. ***Distributive Property:*** $\forall x, y, z \in \mathbb{F}, x * (y \circ z) = (x * y) \circ (x * z)$.

**Definition 2.4** (Real Numbers). $(\mathbb{R}, +, \cdot)$ *forms a Field where:*

1. ***Addition:*** $+\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $i_\mathbb{R} = 0$.
2. ***Multiplication:*** $\cdot\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $i_\mathbb{R} = 1$, *where 0 has no inverse element.*

Some other common Fields are $(\mathbb{Q}, +, \cdot)$ and $(\mathbb{C}, +, \cdot)$. So far, these have all been infinite Fields, but one can have finite Fields as well. The smallest possible finite Field is $(\{0_\mathbb{F}, 1_\mathbb{F}\}, *, \circ)$. Below are some immediate consequences of our algebraic structures:

1. For any Associative Closed Binary Operation, parantheses can be omitted:
   $\cdot\colon S \times S \to S$ associative $\Rightarrow x \cdot y \cdot z = (x \cdot y) \cdot z = x \cdot (y \cdot z), \forall x, y, z \in S$.
2. For any Abelian Group, the order of elements is immaterial:
   $(G, \cdot)$ Abelian Group $\Rightarrow x \cdot y \cdot z = y \cdot x \cdot z = y \cdot z \cdot x = z \cdot y \cdot x, \forall x, y, z \in G$.
3. For any Group, the equation $x \cdot y = z$ has a unique solution in $x$:
   $(G, \cdot)$ Group $\Rightarrow \forall y, z \in G, \exists! x \in G : x \cdot y = z$.

*Proof.* Let $y, z \in (G, \cdot)$.
$\Rightarrow y^{-1} \in G$.
$\Rightarrow z \cdot y^{-1} \in G$.
$\Rightarrow (z \cdot y^{-1}) \cdot y = z \cdot (y^{-1} \cdot y)$.
$\Rightarrow (z \cdot y^{-1}) \cdot y = z \cdot i_G$.
$\Rightarrow (z \cdot y^{-1}) \cdot y = z$.
$\Rightarrow x = z \cdot y^{-1}$ is a solution to $x \cdot y = z$.
Let $w$ be some other solution to $x \cdot y = z$.
$\Rightarrow w \cdot y = z$.
$\Rightarrow (w \cdot y) \cdot y^{-1} = z \cdot y^{-1}$.
$\Rightarrow w \cdot (y \cdot y^{-1}) = z \cdot y^{-1}$.
$\Rightarrow w \cdot i_G = z \cdot y^{-1}$.
$\Rightarrow w = z \cdot y^{-1}$.
$\Rightarrow w = x$.
Therefore $x = z \cdot y^{-1}$ is a unique solution to $x \cdot y = z$.

□

4. For any Group, the identity element is unique:

$$\forall x \in G, \exists! i_G \in G : i_G \cdot x = x \cdot i_G = x$$

*Proof.* Let $(G, \cdot)$ be a Group with identity element $i_G$.
Let $y, z \in G$ s.t. $y = z$.
$\Rightarrow i_G = y \cdot y^{-1} = z \cdot y^{-1}$.
$\Rightarrow x = i_G$ is a unique solution to $x \cdot y = z$.

□

5. For any Group, inverse elements are unique:

$$\forall x \in G, \exists! x^{-1} \in G : x \cdot x^{-1} = x^{-1} \cdot x = i_G$$

*Proof.* Let $(G, \cdot)$ be a Group with identity element $i_G$.
Let $y, z \in (G, \cdot)$ s.t. $z = i_G$.
$\Rightarrow y^{-1} = i_G \cdot y^{-1} = z \cdot y^{-1}$.
$\Rightarrow x = y^{-1}$ is a unique solution to $x \cdot y = z$. □

6. For any Field, any element multiplied by the Zero element $0_{\mathbb{F}}$ is $0_{\mathbb{F}}$.

*Proof.* Let $x \in (\mathbb{F}, +, \cdot)$.
$\Rightarrow x \cdot 0_{\mathbb{F}} = x \cdot i_{(\mathbb{F},+)} = x \cdot (i_{(\mathbb{F},+)} + i_{(\mathbb{F},+)}) = (x \cdot i_{(\mathbb{F},+)}) + (x \cdot i_{(\mathbb{F},+)})$.
Let $y = x \cdot i_{(\mathbb{F},+)}$.
$\Rightarrow y = y + y$.
$\Rightarrow y + y^{-1} = (y + y) + y^{-1}$.
$\Rightarrow y + y^{-1} = y + (y + y^{-1})$.
$\Rightarrow i_{(\mathbb{F},+)} = y + i_{(\mathbb{F},+)}$.
$\Rightarrow i_{(\mathbb{F},+)} = y$.

$\Rightarrow 0_{\mathbb{F}} = y.$
$\Rightarrow 0_{\mathbb{F}} = x \cdot 0_{\mathbb{F}}.$

$\Rightarrow 0_{\mathbb{F}} \cdot x = i_{(\mathbb{F},+)} \cdot x = (i_{(\mathbb{F},+)} + i_{(\mathbb{F},+)}) \cdot x = (i_{(\mathbb{F},+)} \cdot x) + (i_{(\mathbb{F},+)} \cdot x).$
Let $z = i_{(\mathbb{F},+)} \cdot x.$
$\Rightarrow z = z + z.$
$\Rightarrow z + z^{-1} = (z + z) + z^{-1}.$
$\Rightarrow z + z^{-1} = z + (z + z^{-1}).$
$\Rightarrow i_{(\mathbb{F},+)} = z + i_{(\mathbb{F},+)}.$
$\Rightarrow i_{(\mathbb{F},+)} = z.$
$\Rightarrow 0_{\mathbb{F}} = z.$
$\Rightarrow 0_{\mathbb{F}} = 0_{\mathbb{F}} \cdot x.$

$\Rightarrow x \cdot 0_{\mathbb{F}} = 0_{\mathbb{F}} \cdot x = 0_{\mathbb{F}}.$ □

7. For any Field, if the product of 2 elements is the Zero element, then atleast 1 factor must be the Zero element:

   *Proof.* Let $x, y \in (\mathbb{F}, +, \cdot)$ s.t. $x, y \neq 0_{\mathbb{F}}$.
   Case 1: $x = 0_{\mathbb{F}} \Rightarrow x \cdot y = 0_{\mathbb{F}} \cdot y = 0_{\mathbb{F}}.$
   Case 2: $y = 0_{\mathbb{F}} \Rightarrow x \cdot y = x \cdot 0_{\mathbb{F}} = 0_{\mathbb{F}}.$
   □

8. For any Group, the inverse of the inverse element is the element itself.

   *Proof.* Let $x \in (G, \cdot)$.
   $\Rightarrow \exists! x^{-1} \in G : x \cdot x^{-1} = i_G$
   $\Rightarrow \exists! (x^{-1})^{-1} \in G : (x^{-1})^{-1} \cdot x^{-1} = i_G.$
   $\Rightarrow (x^{-1})^{-1} = x$, since they are both the solution to same equation:

$$x \cdot y = z : y = x^{-1}, z = i_G$$

   □

9. For any Field, we can define negative numbers as $-x = -1_{\mathbb{F}} \cdot x = x \cdot -1_{\mathbb{F}}$, where $-1_{\mathbb{F}}$ is the inverse of $1_{\mathbb{F}}$ under $+$.

   *Proof.* Let $x \in (\mathbb{F}, +, \cdot)$.
   $\Rightarrow \exists! x^{-1} \in (\mathbb{F}, +) : x^{-1} + x = 0_{\mathbb{F}}.$
   Define $x^{-1} = -x$.
   $\Rightarrow -x + x = 0_{\mathbb{F}}.$

   $\Rightarrow -1_{\mathbb{F}} \cdot x + x = -1_{\mathbb{F}} \cdot x + 1_{\mathbb{F}} \cdot x.$
   $\Rightarrow -1_{\mathbb{F}} \cdot x + x = (-1_{\mathbb{F}} \cdot +1_{\mathbb{F}}) \cdot x.$

$\Rightarrow -1_{\mathbb{F}} \cdot x + x = 0_{\mathbb{F}} \cdot x.$
$\Rightarrow -1_{\mathbb{F}} \cdot x + x = 0_{\mathbb{F}}.$

$\Rightarrow x \cdot -1_{\mathbb{F}} + x = x \cdot -1_{\mathbb{F}} + x \cdot 1_{\mathbb{F}}.$
$\Rightarrow x \cdot -1_{\mathbb{F}} + x = x \cdot (-1_{\mathbb{F}} \cdot +1_{\mathbb{F}}).$
$\Rightarrow x \cdot -1_{\mathbb{F}} + x = x \cdot 0_{\mathbb{F}}.$
$\Rightarrow x \cdot -1_{\mathbb{F}} + x = 0_{\mathbb{F}}.$

$\Rightarrow -x = -1_{\mathbb{F}} \cdot x = x \cdot -1_{\mathbb{F}}$, since all three are solutions to same equation:

$$x + y = z : y = x, z = 0_{\mathbb{F}}$$

□

10. For any Field, multiplying 2 negative numbers yields a positive number.

*Proof.* Let $x, y \in (\mathbb{F}, +, \cdot)$.

$$\begin{aligned} -x \cdot -y &= (-1_{\mathbb{F}}) \cdot x \cdot (-1_{\mathbb{F}}) \cdot y \\ &= (-1_{\mathbb{F}}) \cdot (-x \cdot y) \\ &= -(-x \cdot y) \\ &= x \cdot y \end{aligned}$$

□

11. For any Field, the additive inverse of a sum is the sum of the additive inverses.

*Proof.* Let $x, y \in (\mathbb{F}, +, \cdot)$.

$$\begin{aligned} -(x + y) &= (-1_{\mathbb{F}}) \cdot (x + y) \\ &= (-1_{\mathbb{F}}) \cdot x + (-1_{\mathbb{F}}) \cdot y \\ &= (-x) + (-y) \end{aligned}$$

□

12. For any Field, the multiplicative inverse of a product of non-zero elements is the product of the multiplicative inverses reversed.

*Proof.* Let $x, y \in (\mathbb{F}, +, \cdot)$ s.t. $x, y \neq 0_{\mathbb{F}}$.
$\Rightarrow x \cdot y \neq 0_{\mathbb{F}}$
$\Rightarrow \exists! x^{-1}, y^{-1}, (x \cdot y)^{-1} \in (\mathbb{F}, \cdot)$.
$\Rightarrow (x \cdot y)^{-1} \cdot (x \cdot y) = 1_{\mathbb{F}}$.
$\Rightarrow (x \cdot y)^{-1} \cdot (x \cdot y) \cdot y^{-1} = 1_{\mathbb{F}} \cdot y^{-1}$.
$\Rightarrow (x \cdot y)^{-1} \cdot x \cdot (y \cdot y^{-1}) = 1_{\mathbb{F}} \cdot y^{-1}$.
$\Rightarrow (x \cdot y)^{-1} \cdot x \cdot 1_{\mathbb{F}} = 1_{\mathbb{F}} \cdot y^{-1}$.
$\Rightarrow (x \cdot y)^{-1} \cdot x = y^{-1}$.
$\Rightarrow (x \cdot y)^{-1} \cdot x \cdot x^{-1} = y^{-1} \cdot x^{-1}$.
$\Rightarrow (x \cdot y)^{-1} \cdot 1_{\mathbb{F}} = y^{-1} \cdot x^{-1}$.
$\Rightarrow (x \cdot y)^{-1} = y^{-1} \cdot x^{-1}$.

□

### 2.2 Order

### 2.3 The Least Upper Bound Property

### 2.4 The Existence of Square Roots

## 3 Metric Spaces

### 3.1 Definition of Metric Spaces. Examples

**Definition 3.1** (Metric Space)**.** *A Metric Space is an ordered pair $(M, d)$ where $M$ is some set along with a metric function $d \colon M^2 \to \mathbb{R}$ such that $\forall x, y, z \in M$:*

1. ***Symmetry:*** $d(x, y) = d(y, x)$.
2. ***Identity of Indiscernibles:*** $d(x, y) = 0 \Leftrightarrow x = y$.
3. ***Triangle Inequality:*** $d(x, y) \leq d(x, z) + d(z, y)$.

**Theorem 3.1.** *Metric functions are non-negative.*

*Proof.* Let $x, y \in (M, d)$:

$$\begin{aligned}
d(x,y) &= \frac{2 \cdot d(x,y)}{2} \\
&= \frac{d(x,y) + d(x,y)}{2} \\
&= \frac{d(x,y) + d(y,x)}{2} && \text{(Symmetry)} \\
&\geq \frac{d(x,x)}{2} && \text{(Subadditivity)} \\
&= 0 && \text{(Identity of Indiscernibles)}
\end{aligned}$$

□

**Theorem 3.2** (General Triangle Inequality). *For any sequence of points in a metric space:*

$$d(x_1, x_n) \leq \sum_{k=1}^{n-1} d(x_k, x_{k+1})$$

*Proof.* Let $x_1, \ldots, x_n \in (M, d)$.
(Induction)
Base Case: $n = 2$ is trivial and $n = 3$ is satisfied by triangle inequality.
Inductive Hypothesis: Assume our statement holds for all n $\geq 2$
Inductive Step: $(n \longrightarrow n + 1)$

$$\begin{aligned} d(x_1, x_{n+1}) &\leq d(x_1, x_n) + d(x_n, x_{n+1}) \\ &\leq \sum_{k=1}^{n-1} d(x_k, x_{k+1}) + d(x_n, x_{n+1}) \\ &= \sum_{k=1}^{n} d(x_k, x_{k+1}) \end{aligned}$$

□

**Theorem 3.3** (Reverse Triangle Inequality). *For any* $x, y, z \in M$*:*

$$|d(x, z) - d(z, y)| \leq d(x, y)$$

*Proof.* Let $x, y, z \in (M, d)$:

$$\begin{aligned} d(y, z) &\leq d(y, x) + d(x, z) && \text{(Subadditivity)} \\ d(z, y) &\leq d(x, y) + d(x, z) && \text{(Symmetry)} \\ -d(x, y) &\leq d(x, z) - d(z, y) \end{aligned}$$

We also have:

$$\begin{aligned} d(x, z) &\leq d(x, y) + d(y, z) && \text{(Subadditivity)} \\ d(x, z) &\leq d(x, y) + d(z, y) && \text{(Symmetry)} \\ d(x, z) - d(z, y) &\leq d(x, y) \end{aligned}$$

Which gives us:

$$-d(x, y) \leq d(x, z) - d(z, y) \leq d(x, y)$$
$$|d(x, z) - d(z, y)| \leq d(x, y)$$

□

**Definition 3.2** (Dot Product). *The Dot Product of vectors* $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{R}^n$ *is*

$$\boldsymbol{x} \cdot \boldsymbol{y} = \sum_{i=1}^{n} x_i y_i$$

**Definition 3.3** (Euclidean Norm)**.** *The Euclidean norm of* $\boldsymbol{x} \in \mathbf{R}^n$ *is*

$$||\boldsymbol{x}|| = \sqrt{\sum_{i=1}^{n} x_i^2} = \sqrt{\boldsymbol{x} \cdot \boldsymbol{x}}$$

**Theorem 3.4** (Cauchy Schwarz Inequality)**.**

$$|\boldsymbol{x} \cdot \boldsymbol{y}| \leq ||\boldsymbol{x}|| \cdot ||\boldsymbol{y}||, \quad \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$$

*Proof.* Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$. Then,

$$\begin{aligned}
||t \cdot \boldsymbol{y} + \boldsymbol{x}||^2 &\geq 0, \quad \forall t \in \mathbb{R} \\
\sum_{i=1}^{n} (t \cdot y_i + x_i)^2 &\geq 0 \\
\sum_{i=1}^{n} (t^2 \cdot y_i^2 + 2x_i y_i t + x_i^2) &\geq 0 \\
t^2 \sum_{i=1}^{n} y_i^2 + 2t \sum_{i=1}^{n} x_i y_i + \sum_{i=1}^{n} x_i^2 &\geq 0 \\
t^2 ||\boldsymbol{y}||^2 + 2t(\boldsymbol{x} \cdot \boldsymbol{y}) + ||\boldsymbol{x}||^2 &\geq 0 \\
at^2 + bt + c &\geq 0 \\
b^2 - 4ac &\leq 0 \\
(2(\boldsymbol{x} \cdot \boldsymbol{y}))^2 - 4||\boldsymbol{y}||^2 ||\boldsymbol{x}||^2 &\leq 0 \\
4(\boldsymbol{x} \cdot \boldsymbol{y})^2 - 4||\boldsymbol{y}||^2 ||\boldsymbol{x}||^2 &\leq 0 \\
(\boldsymbol{x} \cdot \boldsymbol{y})^2 - ||\boldsymbol{y}||^2 ||\boldsymbol{x}||^2 &\leq 0 \\
(\boldsymbol{x} \cdot \boldsymbol{y})^2 - (||\boldsymbol{x}|| \cdot ||\boldsymbol{y}||)^2 &\leq 0 \\
(\boldsymbol{x} \cdot \boldsymbol{y})^2 &\leq (||\boldsymbol{x}|| \cdot ||\boldsymbol{y}||)^2 \\
|\boldsymbol{x} \cdot \boldsymbol{y}| &\leq ||\boldsymbol{x}|| \cdot ||\boldsymbol{y}||
\end{aligned}$$

□

**Corollary 3.4.1.** *The Euclidean Norm is Subadditive.*

*Proof.* Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{R}^n$. Then,

$$\begin{aligned}
||\boldsymbol{x}+\boldsymbol{y}||^2 &= \sum_{i=1}^{n}(x_i+y_i)^2 \\
&= \sum_{i=1}^{n}(x_i^2+2x_iy_i+y_i^2) \\
&= \sum_{i=1}^{n}x_i^2 + 2\sum_{i=1}^{n}x_iy_i + \sum_{i=1}^{n}y_i^2 \\
&= ||\boldsymbol{x}||^2 + 2(\boldsymbol{x}\cdot\boldsymbol{y}) + ||\boldsymbol{y}||^2 \\
&\leq ||\boldsymbol{x}||^2 + 2||\boldsymbol{x}||\cdot||\boldsymbol{y}| + ||\boldsymbol{y}||^2 && \text{(Cauchy Schwarz Inequality)} \\
&= (||\boldsymbol{x}|| + ||\boldsymbol{y}||)^2 \\
||\boldsymbol{x}+\boldsymbol{y}|| &\leq ||\boldsymbol{x}|| + ||\boldsymbol{y}|| && \square
\end{aligned}$$

**Theorem 3.5** (Euclidean Metric Space)**.** $(\mathbb{R}^n, d)$ *forms a Metric Space where* $d(\boldsymbol{x}, \boldsymbol{y}) = ||\boldsymbol{x}-\boldsymbol{y}||$.

*Proof.* Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ .

**Symmetry:** $d(\boldsymbol{x}, \boldsymbol{y}) = ||\boldsymbol{x}-\boldsymbol{y}|| = ||\boldsymbol{y}-\boldsymbol{x}|| = d(\boldsymbol{y}, \boldsymbol{x})$

**Identity of Indiscernibles:** $d(\boldsymbol{x}, \boldsymbol{x}) = ||\boldsymbol{x}-\boldsymbol{x}|| = ||\mathbf{0}|| = 0$

**Subadditivity:**

$$\begin{aligned}
d(\boldsymbol{x}, \boldsymbol{y}) &= ||\boldsymbol{x}-\boldsymbol{y}|| \\
&= ||(\boldsymbol{x}-\boldsymbol{z}) + (\boldsymbol{z}-\boldsymbol{y})|| \\
&\leq ||\boldsymbol{x}-\boldsymbol{z}|| + ||\boldsymbol{z}-\boldsymbol{y}|| && \text{(Subadditivity of } ||\cdot||) \\
&= d(\boldsymbol{x}, \boldsymbol{z}) + d(\boldsymbol{z}, \boldsymbol{y}) && \square
\end{aligned}$$

**Corollary 3.5.1.** $(\mathbb{R}, d)$ *forms a Metric Space where* $d(x, y) = |x-y|$.

*Proof.* Let n = 1. Then by Theorem 3.5 we obtain a metric space $\square$

**Theorem 3.6** (Taxicab Metric Space)**.** *Any nonempty set* $M$ *with the following metric function forms a Metric Space:*

$$d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

*Proof.* Let $x, y, z \in E$.

**Identity of Indiscernibles:** $d(x, x) = 0$.

**Symmetry:**
Let $x \neq y \Rightarrow d(x, y) = 1 = d(y, x)$.
Let $x = y \Rightarrow d(x, y) = 0 = d(y, x)$.

**Subadditivity:**

$$d(x,z)+d(z,y)=\begin{cases}2 & x\neq z\neq y\\ 0 & x=y=z\\ 1 & else\end{cases}$$

Therefore we have $d(x,y)\leq d(x,z)+d(z,y)$

□

## 3.2 Open and Closed Sets

Let $M$ be some Metric Space, $c\in M$, and $r\in\mathbf{R}^+$.

**Definition 3.4** (Open Ball)**.** *Then the Open Ball in $M$ of center $c$ and radius $r$ is the subset of $M$ given by*

$$B_r(c)=\{x\in M:d(x,c)<r\}$$

**Definition 3.5** (Closed Ball)**.** *Then the Closed Ball in $M$ of center $c$ and radius $r$ is the subset of $M$ given by*

$$\bar{B}_r(c)=\{x\in M:d(x,c)\leq r\}$$

**Theorem 3.7.** *Open/Closed Balls in $\mathbb{R}$ are Open/Closed Intervals.*

*Proof.* Let $c\in\mathbb{R}, r\in\mathbb{R}^+$.

$$\begin{aligned}B_r(c)&=\{x\in M:d(x,c)<r\}\\&=\{x\in\mathbb{R}:|x-c|<r\}\\&=\{x\in\mathbb{R}:c-r<x<c+r\}\\&=(c-r,c+r)\end{aligned}$$

$$\begin{aligned}\bar{B}_r(c)&=\{x\in M:d(x,c)\leq r\}\\&=\{x\in\mathbb{R}:|x-c|\leq r\}\\&=\{x\in\mathbb{R}:c-r\leq x\leq c+r\}\\&=[c-r,c+r]\end{aligned}$$

□

**Definition 3.6** (Open Set)**.** *A subset $S$ of a metric space $M$ is open if every point in $S$ is the center of some open ball contained in $S$:*

$$\forall c\in S,\exists r\in\mathbb{R}^+:B_r(c)\subset S$$

**Theorem 3.8.** *In any metric space, an open ball is an open set.*

*Proof.* Define $S = B_\delta(p)$ s.t. $p \in M, \delta \in \mathbb{R}^+$.
Let $c \in S$.
$\Rightarrow d(c,p) < \delta$.
Choose $r = \delta - d(c,p) \Rightarrow r \in \mathbb{R}^+$.
Suppose $x \in B_r(c)$.
$\Rightarrow d(x,c) < r$.
$\Rightarrow d(x,c) < \delta - d(c,p)$.
$\Rightarrow d(x,c) + d(c,p) < \delta$.
$\Rightarrow d(x,p) < \delta$. (Triangle Inequality)
$\Rightarrow x \in S$.
$\Rightarrow B_r(c) \subset S$. $\square$

**Theorem 3.9.** *For any metric space $(M,d)$:*

1. *$\varnothing$ is open*
2. *$M$ is open*
3. *The union of an arbitrary number of open subsets is open:*
   *$\{V_k\}_{k\in\mathbb{N}}$ open in $M \Rightarrow \bigcup_{k\in\mathbb{N}} V_k$ open in M.*
4. *The intersection of a finite number of open subsets is open:*
   *$\{V_k\}_{k=1}^n$ open in $M \Rightarrow \bigcap_{k=1}^n V_k$ open in M.*

*Proof.* Let $(M,d)$ be a metric space:

1. Let $c \in \varnothing$. But $c \notin \varnothing$. So trivially, $\exists r \in \mathbb{R}^+ : B_r(c) \subset \varnothing$.
2. Let $c \in M$. Suppose $x \in B_r(c) \Rightarrow d(x,c) < r \Rightarrow x \in M \Rightarrow B_r(c) \subset M$.
3. Suppose $\{V_k\}_{k\in\mathbb{N}}$ arbitrary collection of open subsets in M.
   Let $c \in \bigcup_{k\in\mathbb{N}} V_k$.
   $\Rightarrow c \in V_k$ for some $k = 1, \ldots, n$.
   $\Rightarrow \exists r \in \mathbb{R}^+ : B_r(c) \subset V_k$.
   $\Rightarrow \exists r \in \mathbb{R}^+ : B_r(c) \subset \bigcup_{k\in\mathbb{N}} V_k$, since any set is contained in its union.
4. Suppose $\{V_k\}_{k=1}^n$ finite collection of open subsets in M. Let $c \in \bigcap_{k=1}^n V_k$:

$$\Rightarrow c \in V_k, \forall k = 1, \ldots, n.$$
$$\Rightarrow \exists r_k \in \mathbb{R}^+ : B_{r_k}(c) \subset V_k.$$

Let $r = min(r_1, \dots, r_n)$. Suppose $x \in B_r(c)$:

$$\begin{aligned}
&\Rightarrow d(x,c) < r.\\
&\Rightarrow d(x,c) < r_1, \dots, d(x,c) < r_n.\\
&\Rightarrow x \in B_{r_1}(c), \dots, x \in B_{r_n}(c).\\
&\Rightarrow x \in V_1, \dots, x \in V_n.\\
&\Rightarrow x \in \bigcap_{k=1}^{n} V_k.\\
&\Rightarrow B_r(c) \subset \bigcap_{k=1}^{n} V_k.
\end{aligned}$$

□

**Definition 3.7** (Closed Set)**.** *A subset $S$ of a metric space $(M,d)$ is closed if its complement $S^{\complement}$ is open in $M$.*

**Theorem 3.10.** *In any metric space, a closed ball is a closed set.*

*Proof.* Define $S = \bar{B}_\delta(p)$ as some closed ball in $(M,d)$. Let $c \in S^{\complement}$. Then, $d(c,p) > \delta$. Choosing $r = d(c,p) - \delta$, clearly $r > 0$. Suppose $x \in B_r(c)$.

$$\begin{aligned}
&\Rightarrow d(x,c) < r.\\
&\Rightarrow d(x,c) < d(c,p) - \delta.\\
&\Rightarrow d(c,p) - d(x,c) > \delta.\\
&\Rightarrow d(c,x) + d(x,p) - d(x,c) > \delta.\\
&\Rightarrow d(x,p) > \delta.\\
&\Rightarrow x \in S^{\complement}.\\
&\Rightarrow B_r(c) \subset S^{\complement}.
\end{aligned}$$

So $S^{\complement}$ is open, making $S$ closed in $M$. □

**Theorem 3.11.** *For any metric space $(M,d)$:*

1. *$\varnothing$ is closed*
2. *$M$ is closed*
3. *The intersection of an arbitrary number of closed subsets is closed:* $\{V_k\}_{k\in\mathbb{N}}$ *closed in* $M \Rightarrow \bigcap_{k\in\mathbb{N}} V_k$ *closed in M.*
4. *The union of a finite number of closed subsets is closed:* $\{V_k\}_{k=1}^{n}$ *closed in* $M \Rightarrow \bigcup_{k=1}^{n} V_k$ *closed in M.*

*Proof.* Let $(M,d)$ be a metric space:

1. $\varnothing^{\complement} = M \Rightarrow \varnothing^{\complement}$ open $\Rightarrow \varnothing$ closed.

2. $M^{\complement} = \varnothing \Rightarrow M^{\complement}$ open $\Rightarrow M$ closed.

3. Suppose $\{V_k\}_{k\in\mathbb{N}}$ arbitrary collection of closed subsets in M.
$\Rightarrow \{V_k^{\complement}\}_{k\in\mathbb{N}}$ is an abitrary collection of open subsets in M.
$\Rightarrow \bigcup_{k\in\mathbb{N}} V_k^{\complement}$ is open in M.
$\Rightarrow (\bigcap_{k\in\mathbb{N}} V_k)^{\complement}$ is open in M.
$\Rightarrow \bigcap_{k\in\mathbb{N}} V_k$ is closed in M.

4. Suppose $\{V_k\}_{k=1}^n$ finite collection of closed subsets in M.
$\Rightarrow \{V_k^{\complement}\}_{k=1}^n$ is a finite collection of open subsets in M.
$\Rightarrow \bigcap_{k=1}^n V_k^{\complement}$ is open in M.
$\Rightarrow (\bigcup_{k=1}^n V_k)^{\complement}$ is open in M.
$\Rightarrow \bigcup_{k=1}^n V_k$ is closed in M.

□

**Theorem 3.12.** *Any singleton set is closed.*

*Proof.* Define $S = \{p\}$ as the singleton set $\forall p \in (M, d)$.
Let $c \in S^{\complement} = M/\{p\}$.
Then, $c \neq p$.
Choosing $r = d(c, p)$, clearly $r > 0$.
Suppose $x \in B_r(c)$.

$$\begin{aligned}
&\Rightarrow d(x, c) < r.\\
&\Rightarrow d(x, c) < d(c, p).\\
&\Rightarrow d(c, p) - d(x, c) > 0.\\
&\Rightarrow d(p, c) - d(c, x) > 0.\\
&\Rightarrow |d(p, c) - d(c, x)| > 0.\\
&\Rightarrow d(p, x) > 0.\\
&\Rightarrow p \neq x.\\
&\Rightarrow x \in S^{\complement}.\\
&\Rightarrow B_r(c) \subset S^{\complement}.
\end{aligned}$$

So $S^{\complement}$ is open, making $S$ closed in $M$. □

**Corollary 3.12.1.** *Any finite subset of a metric space is closed.*

*Proof.* Let $x_1, \ldots, x_n$ be a finite set of points in $M$.
$\Rightarrow \{x_1\}, \ldots, \{x_n\}$ all closed singleton sets in $M$.
$\Rightarrow \bigcup_{k=1}^n \{x_k\}$ closed in $M$.
$\Rightarrow \{x_1, \ldots, x_n\}$ closed in $M$. □

**Theorem 3.13.** *Any sphere in a metric space is closed.*

*Proof.* Define sphere of radius $r \in \mathbb{R}^+$ centered as:

$$\begin{aligned}S &= \{x \in M : d(x,c) = r\} \\ &= \{x \in M : d(x,c) \leq r\} \cap \{x \in M : d(x,c) \geq r\} \\ &= \{x \in M : d(x,c) \leq r\} \cap \{x \in M : d(x,c) < r\}^{\complement} \\ S &= \bar{B}_r(c) \cap B_r(c)^{\complement}\end{aligned}$$

Note, the closed ball $\bar{B}_r(c)$ is closed, and $B_r(c)^{\complement}$ is closed since its complement $B_r(c)^{\complement^{\complement}} = B_r(c)$ is the open ball, an open set. The finite intersection of closed sets is closed, so $S$ is closed in $M$. □

**Theorem 3.14.** *Any half-interval is neither open nor closed.*

*Proof.* Let $a, b \in \mathbb{R}, a < b$.
Define $S = [a, b)$ as a half-interval on $\mathbb{R}$.
Let $r > 0$.
$\Rightarrow a - r < a$.
$\Rightarrow a - r < min(S)$.
$\Rightarrow a - r \notin S$.
$\Rightarrow \nexists r \in \mathbb{R}^+ : B_r(a) = (a - r, a + r) \subset S$.
$\Rightarrow S$ is not open in $\mathbb{R}$.
$\Rightarrow [b, \infty)$ is not open in $\mathbb{R}$ by same logic.
$\Rightarrow S^{\complement} = (-\infty, a) \cup [b, \infty)$ is not open in $\mathbb{R}$ by same logic.
$\Rightarrow S$ is not closed in $\mathbb{R}$. □

**Theorem 3.15.** *Any subspace of Euclidean space with a strictly bounded component is open. So,* $\forall a \in \mathbb{R}, k \in \{1, \ldots, n\}$*:* $\{x \in \mathbb{R}^n : x_k < a\}$ *and* $\{x \in \mathbb{R}^n : x_k > a\}$ *are open in* $\mathbb{R}^n$*.*

*Proof.* Let $a \in \mathbb{R}$, and $k \in \{1, \ldots, n\}$.
Define $S = \{x \in \mathbb{R}^n : x_k < a\}$ and $S' = \{x \in \mathbb{R}^n : x_k > a\}$.
Let $c \in S$ and $p \in S'$.
$\Rightarrow c_k < a$ and $p_k > a$.
Choose $r = a - c_k$ and $\delta = p_k - a \Rightarrow r, \delta > 0$.
Suppose $x \in B_r(c)$ and $y \in B_\delta(p)$.
$\Rightarrow d(x, c) < r$ and $d(y, p) < \delta$.
$\Rightarrow d(x, c) < a - c_k$ and $d(y, p) < p_k - a$.
$\Rightarrow ||x - c|| < a - c_k$ and $||y - p|| < p_k - a$.
$\Rightarrow \sqrt{\sum_k (x_k - c_k)^2} < a - c_k$ and $\sqrt{\sum_k (y_k - p_k)^2} < p_k - a$.
$\Rightarrow \sqrt{(x_k - c_k)^2} < a - c_k$ and $\sqrt{(y_k - p_k)^2} < p_k - a$.
$\Rightarrow |x_k - c_k| < a - c_k$ and $|y_k - p_k| < p_k - a$.
$\Rightarrow x_k - c_k < a - c_k$ and $p_k - y_k < p_k - a$
$\Rightarrow x_k < a$ and $y_k > a$.
$\Rightarrow x \in S$ and $y \in S'$.
$\Rightarrow B_r(c) \subset S$ and $B_\delta(p) \subset S'$. □

**Theorem 3.16.** *Any subspace of Euclidean space with a weakly bounded component is closed. So, $\forall a \in \mathbb{R}, k \in \{1, \ldots, n\}$: $\{x \in \mathbb{R}^n : x_k \leq a\}$ and $\{x \in \mathbb{R}^n : x_k \geq a\}$ are closed in $\mathbb{R}^n$.*

*Proof.* Let $a \in \mathbb{R}$, and $k \in \{1, \ldots, n\}$.
Define $S = \{x \in \mathbb{R}^n : x_k \leq a\}$ and $S' = \{x \in \mathbb{R}^n : x_k \geq a\}$.
$\Rightarrow S^{\complement} = \{x \in \mathbb{R}^n : x_k > a\}$ and $S'^{\complement} = \{x \in \mathbb{R}^n : x_k < a\}$.
$\Rightarrow S^{\complement}, S'^{\complement}$ are open in $\mathbb{R}^n$.
$\Rightarrow S, S'$ are closed in $\mathbb{R}^n$. $\square$

**Definition 3.8** (Open/Closed Intervals in $\mathbb{R}^n$)**.** *Let $a_1, \ldots, a_n, b_1, \ldots b_n \in \mathbb{R}$. For $a_1 < b_1, \ldots, a_n < b_n$, the open interval in $\mathbb{R}^n$ is:*

$$\{x \in \mathbb{R}^n : a_k < x_k < b_k, \forall k \in \{1, \ldots, n\}\}$$

*For $a_1 \leq b_1, \ldots, a_n \leq b_n$, the closed interval in $\mathbb{R}^n$ is:*

$$\{x \in \mathbb{R}^n : a_k \leq x_k \leq b_k, \forall k \in \{1, \ldots, n\}\}$$

**Theorem 3.17.** *Any open/closed interval in Euclidean space is open/closed set.*

*Proof.* Define $S = \{x \in \mathbb{R}^n : a_k < x_k < b_k, \forall k \in \{1, \ldots, n\}\}$.
$\Rightarrow S = \bigcap_{k=1}^{n}\{x \in \mathbb{R}^n : a_k < x_k < b_k\}$.
$\Rightarrow S = (\bigcap_{k=1}^{n}\{x \in \mathbb{R}^n : x_k > a_k\}) \cap (\bigcap_{k=1}^{n}\{x \in \mathbb{R}^n : x_k < b_k\})$.
$\Rightarrow S$ is open, since it is the finite intersection of $(2n)$ open sets.

Define $S' = \{x \in \mathbb{R}^n : a_k \leq x_k \leq b_k, \forall k \in \{1, \ldots, n\}\}$.
$\Rightarrow S' = \bigcap_{k=1}^{n}\{x \in \mathbb{R}^n : a_k \leq x_k \leq b_k\}$.
$\Rightarrow S' = (\bigcap_{k=1}^{n}\{x \in \mathbb{R}^n : x_k \geq a_k\}) \cap (\bigcap_{k=1}^{n}\{x \in \mathbb{R}^n : x_k \leq b_k\})$.
$\Rightarrow S'$ is closed, since it is the finite intersection of $(2n)$ closed sets. $\square$

**Definition 3.9** (Bounded)**.** *A subset $S$ of a metric space $M$ is bounded if it is contained in some ball.*

**Theorem 3.18.** *Any open/closed interval in Euclidean space is bounded.*

*Proof.* Let $a, b \in \mathbb{R}^n$ s.t. $a \neq b$.
Define $S = \{x \in \mathbb{R}^n : a_k \leq x_k \leq b_k, \forall k \in \{1, \ldots, n\}\}$.
Choose $r = d(b, a) > 0$.
Let $x \in S$.
$\Rightarrow 0 \leq x_k - a_k \leq b_k - a_k$.
$\Rightarrow (x_k - a_k)^2 \leq (b_k - a_k)^2$.
$\Rightarrow \sum\limits_{k=1}^{n}(x_k - a_k)^2 \leq \sum\limits_{k=1}^{n}(b_k - a_k)^2$.
$\Rightarrow \sqrt{\sum\limits_{k=1}^{n}(x_k - a_k)^2} \leq \sqrt{\sum\limits_{k=1}^{n}(b_k - a_k)^2}$.
$\Rightarrow ||x - a|| \leq ||b - a||$.

$\Rightarrow d(x, a) \leq d(b, a)$.
$\Rightarrow d(x, a) \leq r$.
$\Rightarrow x \in \bar{B}_r(a)$.
$\Rightarrow S \subset \bar{B}_r(a)$.

Define $S' = \{x \in \mathbb{R}^n : a_k < x_k < b_k, \forall k \in \{1, \ldots, n\}\}$.
$\Rightarrow S' \subset S$.
$\Rightarrow S' \subset \bar{B}_r(a)$. □

**Theorem 3.19.** *The union of a finite collection of bounded subsets of a metric space is bounded.*

*Proof.* Suppose $V_1, \ldots, V_n$ are bounded subsets in $M$.
Let $c_1 \in V_1, \ldots, c_n \in V_n$.
$\Rightarrow \exists r_1, \ldots, r_n \in \mathbb{R}^+$ s.t. $V_k \subset \bar{B}_{r_k}(c_k), \forall k = 1, \ldots, n$.
Let $x \in \bigcup_{k=1}^n V_k$.
$\Rightarrow x \in V_k$ for some $k = 1, \ldots, n$.
$\Rightarrow x \in \bar{B}_{r_k}(c_k)$.
$\Rightarrow \bigcup_{k=1}^n V_k \subset \bar{B}_{r_k}(c_k)$. □

**Theorem 3.20.** *A nonempty closed subset of $\mathbb{R}$, if it is bounded from above has a greatest element and if it is bounded from below has a least element.*

*Proof.* Let $S \subset \mathbb{R}, S \neq \varnothing$.
Suppose S is closed in $\mathbb{R}$ and bounded above.
Let $c = sup(S)$.
With an eye to contradict, assume $c \notin S$.
$\Rightarrow c \in S^{\complement}$.
$\Rightarrow \exists r \in \mathbb{R}^+$ s.t. $B_r(c) \subset S^{\complement}$, since $S^{\complement}$ open in $\mathbb{R}$.
Then no element in S is greater than $c - r$.
$\Rightarrow c - r$ is an upper bound for $S$.
This must be a contradictions, so $c \in S$. □