Intro Real Analysis

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2.1 The Field Properties

Definition 2.1 (Group). *A Group is a an ordered pair $(G, *)$ where G is some non-empty set along with a closed binary operator $*: G \times G \to G$ s.t.:*

1. ***Associative:*** $\forall x, y, z \in G : (x * y) * z = x * (y * z)$.
2. ***Identity Element:*** $\forall x \in G, \exists! i_G \in G : x * i_G = i_G * x = x$.
3. ***Inverse Element:*** $\forall x \in G, \exists! x^{-1} \in G : x * x^{-1} = x^{-1} * x = i_G$.

Definition 2.2 (Abelian Group). *An Abelian Group is a Commutative Group:*

$$\forall x, y \in G : x * y = y * x$$

Definition 2.3 (Field). *A Field is an ordered triple $(\mathbb{F}, *, \circ)$ s.t.:*

1. $(\mathbb{F}, *)$ *and* $(\mathbb{F}, \circ)$ *form Abelian Groups with* $0_{\mathbb{F}} = i_{(\mathbb{F},*)}$ *and* $1_{\mathbb{F}} = i_{(\mathbb{F},\circ)}$.
2. ***Distributive Property:*** $\forall x, y, z \in \mathbb{F}, x * (y \circ z) = (x * y) \circ (x * z)$.

Definition 2.4 (Real Numbers). $(\mathbb{R}, +, \cdot)$ *forms a Field where:*

1. ***Addition:*** $+: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $i_{\mathbb{R}} = 0$.
2. ***Multiplication:*** $\cdot: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $i_{\mathbb{R}} = 1$, *where 0 has no inverse element.*

Some other common Fields are $(\mathbb{Q}, +, \cdot)$ and $(\mathbb{C}, +, \cdot)$. So far, these have all been infinite Fields, but one can have finite Fields as well. The smallest possible finite Field is $(\{0_{\mathbb{F}}, 1_{\mathbb{F}}\}, *, \circ)$. Below are some immediate consequences of our algebraic structures:

1. For any Associative Closed Binary Operation, parantheses can be omitted:
   $\cdot: S \times S \to S$ associative $\Rightarrow x \cdot y \cdot z = (x \cdot y) \cdot z = x \cdot (y \cdot z), \forall x, y, z \in S$.
2. For any Abelian Group, the order of elements is immaterial:
   $(G, \cdot)$ Abelian Group $\Rightarrow x \cdot y \cdot z = y \cdot x \cdot z = y \cdot z \cdot x = z \cdot y \cdot x, \forall x, y, z \in G$.
3. For any Group, the equation $x \cdot y = z$ has a unique solution in x:
   $(G, \cdot)$ Group $\Rightarrow \forall y, z \in G, \exists! x \in G : x \cdot y = z$.

Proof. Let $y, z \in (G, \cdot)$.
$\Rightarrow y^{-1} \in G$.
$\Rightarrow z \cdot y^{-1} \in G$.
$\Rightarrow (z \cdot y^{-1}) \cdot y = z \cdot (y^{-1} \cdot y)$.
$\Rightarrow (z \cdot y^{-1}) \cdot y = z \cdot i_G$.
$\Rightarrow (z \cdot y^{-1}) \cdot y = z$.
$\Rightarrow x = z \cdot y^{-1}$ is a solution to $x \cdot y = z$.
Let w be some other solution to $x \cdot y = z$.
$\Rightarrow w \cdot y = z$.
$\Rightarrow (w \cdot y) \cdot y^{-1} = z \cdot y^{-1}$.
$\Rightarrow w \cdot (y \cdot y^{-1}) = z \cdot y^{-1}$.
$\Rightarrow w \cdot i_G = z \cdot y^{-1}$.
$\Rightarrow w = z \cdot y^{-1}$.
$\Rightarrow w = x$.
Therefore $x = z \cdot y^{-1}$ is a unique solution to $x \cdot y = z$.

□

4. For any Group, the identity element is unique:

$$\forall x \in G, \exists! i_G \in G : i_G \cdot x = x \cdot i_G = x$$

Proof. Let $(G, \cdot)$ be a Group with identity element i_G.
Let $y, z \in G$ s.t. $y = z$.
$\Rightarrow i_G = y \cdot y^{-1} = z \cdot y^{-1}$.
$\Rightarrow x = i_G$ is a unique solution to $x \cdot y = z$.

□

5. For any Group, inverse elements are unique:

$$\forall x \in G, \exists! x^{-1} \in G : x \cdot x^{-1} = x^{-1} \cdot x = i_G$$

Proof. Let $(G, \cdot)$ be a Group with identity element i_G.
Let $y, z \in (G, \cdot)$ s.t. $z = i_G$.
$\Rightarrow y^{-1} = i_G \cdot y^{-1} = z \cdot y^{-1}$.
$\Rightarrow x = y^{-1}$ is a unique solution to $x \cdot y = z$. □

6. For any Field, any element multiplied by the Additive identity element yields the Additive identity element.

Proof. Let $x \in (\mathbb{F}, +, \cdot)$.
$\Rightarrow x \cdot 0_{\mathbb{F}} = x \cdot i_{(\mathbb{F},+)} = x \cdot (i_{(\mathbb{F},+)} + i_{(\mathbb{F},+)}) = (x \cdot i_{(\mathbb{F},+)}) + (x \cdot i_{(\mathbb{F},+)})$.
Let $y = x \cdot i_{(\mathbb{F},+)}$.
$\Rightarrow y = y + y$.
$\Rightarrow y + y^{-1} = (y + y) + y^{-1}$.
$\Rightarrow y + y^{-1} = y + (y + y^{-1})$.
$\Rightarrow i_{(\mathbb{F},+)} = y + i_{(\mathbb{F},+)}$.

$\Rightarrow i_{(\mathbb{F},+)} = y$.
$\Rightarrow 0_{\mathbb{F}} = y$.
$\Rightarrow 0_{\mathbb{F}} = x \cdot 0_{\mathbb{F}}$. $\Rightarrow 0_{\mathbb{F}} \cdot x = i_{(\mathbb{F},+)} \cdot x = (i_{(\mathbb{F},+)} + i_{(\mathbb{F},+)}) \cdot x = (i_{(\mathbb{F},+)} \cdot x) + (i_{(\mathbb{F},+)} \cdot x)$.
Let $z = i_{(\mathbb{F},+)} \cdot x$.
$\Rightarrow z = z + z$.
$\Rightarrow z + z^{-1} = (z + z) + z^{-1}$.
$\Rightarrow z + z^{-1} = z + (z + z^{-1})$.
$\Rightarrow i_{(\mathbb{F},+)} = z + i_{(\mathbb{F},+)}$.
$\Rightarrow i_{(\mathbb{F},+)} = z$.
$\Rightarrow 0_{\mathbb{F}} = z$.
$\Rightarrow 0_{\mathbb{F}} = 0_{\mathbb{F}} \cdot x$.
$\Rightarrow x \cdot 0_{\mathbb{F}} = 0_{\mathbb{F}} \cdot x = 0_{\mathbb{F}}$. □

7. For any Field, if the product of 2 elements is the Zero element, then atleast 1 factor must be the Zero element:

   Proof. Let $x, y \in (\mathbb{F}, +, \cdot)$ s.t. $x, y \neq 0_{\mathbb{F}}$.
   Case 1: $x = 0_{\mathbb{F}} \Rightarrow x \cdot y = 0_{\mathbb{F}} \cdot y = 0_{\mathbb{F}}$.
   Case 2: $y = 0_{\mathbb{F}} \Rightarrow x \cdot y = x \cdot 0_{\mathbb{F}} = 0_{\mathbb{F}}$. □

8. For any Group, the inverse of the inverse element is the element itself.

   Proof. Let $x \in (G, \cdot)$.
   $\Rightarrow \exists! x^{-1} \in G : x \cdot x^{-1} = i_G$
   $\Rightarrow \exists! (x^{-1})^{-1} \in G : (x^{-1})^{-1} \cdot x^{-1} = i_G$.
   $\Rightarrow$ Both $(x^{-1})^{-1}$ and x are solutions since they are both the solution to same equation:
   $$x \cdot y = z : y = x^{-1}, z = i_G$$
   □

9. For any Field, we can define negative numbers as $-x = -1_{\mathbb{F}} \cdot x = x \cdot -1_{\mathbb{F}}$, where $-1_{\mathbb{F}}$ is the inverse of $1_{\mathbb{F}}$ under $+$.

   Proof. Let $x \in (\mathbb{F}, +, \cdot)$.
   $\Rightarrow \exists! x^{-1} \in (\mathbb{F}, +) : x^{-1} + x = 0_{\mathbb{F}}$.
   Define $x^{-1} = -x$.
   $\Rightarrow -x + x = 0_{\mathbb{F}}$.
   $\Rightarrow -1_{\mathbb{F}} \cdot x + x = -1_{\mathbb{F}} \cdot x + 1_{\mathbb{F}} \cdot x$.
   $\Rightarrow -1_{\mathbb{F}} \cdot x + x = (-1_{\mathbb{F}} \cdot +1_{\mathbb{F}}) \cdot x$.
   $\Rightarrow -1_{\mathbb{F}} \cdot x + x = 0_{\mathbb{F}} \cdot x$.
   $\Rightarrow -1_{\mathbb{F}} \cdot x + x = 0_{\mathbb{F}}$.
   Also $x \cdot -1_{\mathbb{F}} + x = x \cdot -1_{\mathbb{F}} + x \cdot 1_{\mathbb{F}}$.
   $\Rightarrow x \cdot -1_{\mathbb{F}} + x = x \cdot (-1_{\mathbb{F}} \cdot +1_{\mathbb{F}})$.
   $\Rightarrow x \cdot -1_{\mathbb{F}} + x = x \cdot 0_{\mathbb{F}}$.

$\Rightarrow x \cdot -1_{\mathbb{F}} + x = 0_{\mathbb{F}}$.
$\Rightarrow -x = -1_{\mathbb{F}} \cdot x = x \cdot -1_{\mathbb{F}}$, since all three are solutions to same equation:

$$x + y = z : y = x, z = 0_{\mathbb{F}}$$

$\square$

10. For any Field, multiplying 2 negative numbers yields a positive number.

Proof. Let $x, y \in (\mathbb{F}, +, \cdot)$.
$\Rightarrow -x \cdot -y = (-1_{\mathbb{F}}) \cdot x \cdot (-1_{\mathbb{F}}) \cdot y$
$\Rightarrow -x \cdot -y = (-1_{\mathbb{F}}) \cdot (-x \cdot y)$
$\Rightarrow -x \cdot -y = -(-x \cdot y)$
$\Rightarrow -x \cdot -y = x \cdot y$

$\square$

11. For any Field, the additive inverse of a sum is the sum of the additive inverses.

Proof. Let $x, y \in (\mathbb{F}, +, \cdot)$
$\Rightarrow -(x + y) = (-1_{\mathbb{F}}) \cdot (x + y)$
$\Rightarrow -(x + y) = (-1_{\mathbb{F}}) \cdot x + (-1_{\mathbb{F}}) \cdot y$
$\Rightarrow -(x + y) = (-x) + (-y)$ $\square$

12. For any Field, the multiplicative inverse of a product of non-zero elements is the product of the multiplicative inverses reversed.

Proof. Let $x, y \in (\mathbb{F}, +, \cdot)$ s.t. $x, y \neq 0_{\mathbb{F}}$.
$\Rightarrow x \cdot y \neq 0_{\mathbb{F}}$
$\Rightarrow \exists! x^{-1}, y^{-1}, (x \cdot y)^{-1} \in (\mathbb{F}, \cdot)$.
$\Rightarrow (x \cdot y)^{-1} \cdot (x \cdot y) = 1_{\mathbb{F}}$.
$\Rightarrow (x \cdot y)^{-1} \cdot (x \cdot y) \cdot y^{-1} = 1_{\mathbb{F}} \cdot y^{-1}$.
$\Rightarrow (x \cdot y)^{-1} \cdot x \cdot (y \cdot y^{-1}) = 1_{\mathbb{F}} \cdot y^{-1}$.
$\Rightarrow (x \cdot y)^{-1} \cdot x \cdot 1_{\mathbb{F}} = 1_{\mathbb{F}} \cdot y^{-1}$.
$\Rightarrow (x \cdot y)^{-1} \cdot x = y^{-1}$.
$\Rightarrow (x \cdot y)^{-1} \cdot x \cdot x^{-1} = y^{-1} \cdot x^{-1}$.
$\Rightarrow (x \cdot y)^{-1} \cdot 1_{\mathbb{F}} = y^{-1} \cdot x^{-1}$.
$\Rightarrow (x \cdot y)^{-1} = y^{-1} \cdot x^{-1}$.

$\square$

2.2 Order

2.3 The Least Upper Bound Property

2.4 The Existence of Square Roots

3 Metric Spaces

3.1 Definition of Metric Spaces. Examples

Definition 3.1 (Metric Space)**.** *A Metric Space is an ordered pair* (M, d) *where* M *is some set along with a metric function* $d\colon M^2 \to \mathbb{R}$ *s.t.* $\forall x, y, z \in (M, d)$:

1. ***Symmetry:*** $d(x, y) = d(y, x)$
2. ***Identity of Indiscernibles:*** $d(x, y) = 0 \Leftrightarrow x = y$
3. ***Triangle Inequality:*** $d(x, y) \leq d(x, z) + d(z, y)$

Theorem 3.1. *Metric functions are non-negative.*

Proof. Let $x, y \in (M, d)$
$\Rightarrow d(x, y) = \dfrac{2 \cdot d(x, y)}{2}$
$\Rightarrow d(x, y) = \dfrac{d(x, y) + d(x, y)}{2}$
$\Rightarrow d(x, y) = \dfrac{d(x, y) + d(y, x)}{2}$
$\Rightarrow d(x, y) \geq \dfrac{d(x, x)}{2}$
$\Rightarrow d(x, y) \geq 0$

□

Theorem 3.2 (General Triangle Inequality)**.** *For any sequence of points in a metric space:*

$$d(x_1, x_n) \leq \sum_{k=1}^{n-1} d(x_k, x_{k+1})$$

Proof. Let $x_1, \ldots, x_n \in (M, d)$. By induction:

1. Base Case ($n = 3$):
   $\Rightarrow d(x_1, x_3) \leq d(x_1, x_2) + d(x_2, x_3)$
2. Inductive Hypothesis:
   $d(x_1, x_n) \leq \sum_{k=1}^{n-1} d(x_k, x_{k+1})$
3. Inductive Step:
   $\Rightarrow d(x_1, x_n) + d(x_n, x_{n+1}) \leq \sum_{k=1}^{n-1} d(x_k, x_{k+1}) + d(x_n, x_{n+1})$

$\Rightarrow d(x_1,x_n)+d(x_n,x_{n+1}) \leq \sum\limits_{k=1}^{n} d(x_k,x_{k+1})$

$\Rightarrow d(x_1,x_{n+1}) \leq \sum\limits_{k=1}^{n} d(x_k,x_{k+1})$

$\square$

Theorem 3.3 (Reverse Triangle Inequality)**.**

$$|d(x,z)-d(z,y)| \leq d(x,y), \;\; \forall x,y,z \in (M,d)$$

Proof. Let $x,y,z \in (M,d)$.
$\Rightarrow d(y,z) \leq d(y,x)+d(x,z)$ and $d(x,z) \leq d(x,y)+d(y,z)$.
$\Rightarrow d(z,y) \leq d(x,y)+d(x,z)$ and $d(x,z) \leq d(x,y)+d(z,y)$.
$\Rightarrow -d(x,y) \leq d(x,z)-d(z,y)$ and $d(x,z)-d(z,y) \leq d(x,y)$.
$\Rightarrow -d(x,y) \leq d(x,z)-d(z,y) \leq d(x,y)$.
$\Rightarrow |d(x,z)-d(z,y)| \leq d(x,y)$. $\square$

Definition 3.2 (Dot Product)**.** *The Dot Product of vectors* $\boldsymbol{x},\boldsymbol{y} \in \mathbf{R}^n$ *is*

$$\boldsymbol{x}\cdot\boldsymbol{y} = \sum_{i=1}^{n} x_i y_i$$

Definition 3.3 (Euclidean Norm)**.** *The Euclidean norm of* $\boldsymbol{x} \in \mathbf{R}^n$ *is*

$$||\boldsymbol{x}|| = \sqrt{\sum_{i=1}^{n} x_i^2} = \sqrt{\boldsymbol{x}\cdot\boldsymbol{x}}$$

Theorem 3.4 (Cauchy Schwarz Inequality)**.**

$$|\boldsymbol{x}\cdot\boldsymbol{y}| \leq ||\boldsymbol{x}||\cdot||\boldsymbol{y}||, \;\; \forall \boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^n$$

Proof. Let $\boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^n, t \in \mathbb{R}$
$\Rightarrow ||t\cdot\boldsymbol{y}+\boldsymbol{x}||^2 \geq 0$
$\Rightarrow \sum\limits_{i=1}^{n} (t\cdot y_i + x_i)^2 \geq 0$
$\Rightarrow \sum\limits_{i=1}^{n} (t^2\cdot y_i^2 + 2x_i y_i t + x_i^2) \geq 0$
$\Rightarrow t^2 \sum\limits_{i=1}^{n} y_i^2 + 2t\sum\limits_{i=1}^{n} x_i y_i + \sum\limits_{i=1}^{n} x_i^2 \geq 0$
$\Rightarrow t^2||\boldsymbol{y}||^2 + 2t(\boldsymbol{x}\cdot\boldsymbol{y}) + ||\boldsymbol{x}||^2 \geq 0$
$\Rightarrow at^2+bt+c \geq 0$
$\Rightarrow b^2-4ac \leq 0$
$\Rightarrow (2(\boldsymbol{x}\cdot\boldsymbol{y}))^2 - 4||\boldsymbol{y}||^2||\boldsymbol{x}||^2 \leq 0$
$\Rightarrow 4(\boldsymbol{x}\cdot\boldsymbol{y})^2 - 4||\boldsymbol{y}||^2||\boldsymbol{x}||^2 \leq 0$
$\Rightarrow (\boldsymbol{x}\cdot\boldsymbol{y})^2 - ||\boldsymbol{y}||^2||\boldsymbol{x}||^2 \leq 0$
$\Rightarrow (\boldsymbol{x}\cdot\boldsymbol{y})^2 - (||\boldsymbol{x}||\cdot||\boldsymbol{y}||)^2 \leq 0$
$\Rightarrow (\boldsymbol{x}\cdot\boldsymbol{y})^2 \leq (||\boldsymbol{x}||\cdot||\boldsymbol{y}||)^2$
$\Rightarrow |\boldsymbol{x}\cdot\boldsymbol{y}| \leq ||\boldsymbol{x}||\cdot||\boldsymbol{y}||$ $\square$

Corollary 3.4.1. *The Euclidean Norm is Subadditive.*

Proof. Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{R}^n$. Then,

$$\begin{aligned}
||\boldsymbol{x}+\boldsymbol{y}||^2 &= \sum_{i=1}^{n}(x_i+y_i)^2 \\
&= \sum_{i=1}^{n}(x_i^2+2x_iy_i+y_i^2) \\
&= \sum_{i=1}^{n}x_i^2+2\sum_{i=1}^{n}x_iy_i+\sum_{i=1}^{n}y_i^2 \\
&= ||\boldsymbol{x}||^2+2(\boldsymbol{x}\cdot\boldsymbol{y})+||\boldsymbol{y}||^2 \\
&\leq ||\boldsymbol{x}||^2+2||\boldsymbol{x}||\cdot||\boldsymbol{y}|+||\boldsymbol{y}||^2 && \text{(Cauchy Schwarz Inequality)} \\
&= (||\boldsymbol{x}||+||\boldsymbol{y}||)^2 \\
||\boldsymbol{x}+\boldsymbol{y}|| &\leq ||\boldsymbol{x}||+||\boldsymbol{y}|| && \square
\end{aligned}$$

Theorem 3.5 (Euclidean Metric Space)**.** $(\mathbb{R}^n, d)$ *forms a Metric Space where* $d(\boldsymbol{x}, \boldsymbol{y}) = ||\boldsymbol{x}-\boldsymbol{y}||$.

Proof. Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$.

Symmetry: $d(\boldsymbol{x},\boldsymbol{y}) = ||\boldsymbol{x}-\boldsymbol{y}|| = ||\boldsymbol{y}-\boldsymbol{x}|| = d(\boldsymbol{y},\boldsymbol{x})$

Identity of Indiscernibles: $d(\boldsymbol{x},\boldsymbol{x}) = ||\boldsymbol{x}-\boldsymbol{x}|| = ||\mathbf{0}|| = 0$

Subadditivity:

$$\begin{aligned}
d(\boldsymbol{x},\boldsymbol{y}) &= ||\boldsymbol{x}-\boldsymbol{y}|| \\
&= ||(\boldsymbol{x}-\boldsymbol{z})+(\boldsymbol{z}-\boldsymbol{y})|| \\
&\leq ||\boldsymbol{x}-\boldsymbol{z}||+||\boldsymbol{z}-\boldsymbol{y}|| && \text{(Subadditivity of } ||\cdot||) \\
&= d(\boldsymbol{x},\boldsymbol{z})+d(\boldsymbol{z},\boldsymbol{y}) && \square
\end{aligned}$$

Corollary 3.5.1. $(\mathbb{R}, d)$ *forms a Metric Space where* $d(x, y) = |x-y|$.

Proof. Let n = 1. Then by Theorem 3.5 we obtain a metric space $\square$

Theorem 3.6 (Taxicab Metric Space)**.** *For any non-empty set* M*, the Taxicab Metric:*

$$d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

forms a metric space (M, d).

Proof. Let $x, y, z \in E$.

Identity of Indiscernibles: $d(x,x) = 0$.

Symmetry:
Let $x \neq y \Rightarrow d(x,y) = 1 = d(y,x)$.
Let $x = y \Rightarrow d(x,y) = 0 = d(y,x)$.

Subadditivity:

$$d(x,z) + d(z,y) = \begin{cases} 2 & x \neq z \neq y \\ 0 & x = y = z \\ 1 & else \end{cases}$$

Therefore we have $d(x,y) \leq d(x,z) + d(z,y)$

□

3.2 Open and Closed Sets

Let M be some Metric Space, $c \in M$, and $r \in \mathbf{R}^+$.

Definition 3.4 (Open Ball)**.** *Then the Open Ball in M of center c and radius r is the subset of M given by*

$$B_r(c) = \{x \in M : d(x,c) < r\}$$

Definition 3.5 (Closed Ball)**.** *Then the Closed Ball in M of center c and radius r is the subset of M given by*

$$\bar{B}_r(c) = \{x \in M : d(x,c) \leq r\}$$

Theorem 3.7. *Open/Closed Balls in $\mathbb{R}$ are Open/Closed Intervals.*

Proof. Let $c \in \mathbb{R}, r \in \mathbb{R}^+$.

$$\begin{aligned} B_r(c) &= \{x \in M : d(x,c) < r\} \\ &= \{x \in \mathbb{R} : |x - c| < r\} \\ &= \{x \in \mathbb{R} : c - r < x < c + r\} \\ &= (c - r, c + r) \end{aligned}$$

$$\begin{aligned} \bar{B}_r(c) &= \{x \in M : d(x,c) \leq r\} \\ &= \{x \in \mathbb{R} : |x - c| \leq r\} \\ &= \{x \in \mathbb{R} : c - r \leq x \leq c + r\} \\ &= [c - r, c + r] \end{aligned}$$

□

Definition 3.6 (Open Set)**.** *A subset S of a metric space M is open if every point in S is the center of some open ball contained in S:*

$$\forall c \in S, \exists r \in \mathbb{R}^+ : B_r(c) \subset S$$

Theorem 3.8. *In any metric space, an open ball is an open set.*

Proof. Define $S = B_\delta(p)$ s.t. $p \in M, \delta \in \mathbb{R}^+$.
Let $c \in S$.
$\Rightarrow d(c,p) < \delta$.
Choose $r = \delta - d(c,p) \Rightarrow r \in \mathbb{R}^+$.
Suppose $x \in B_r(c)$.
$\Rightarrow d(x,c) < r$.
$\Rightarrow d(x,c) < \delta - d(c,p)$.
$\Rightarrow d(x,c) + d(c,p) < \delta$.
$\Rightarrow d(x,p) < \delta$. (Triangle Inequality)
$\Rightarrow x \in S$.
$\Rightarrow B_r(c) \subset S$. □

Theorem 3.9. *For any metric space (M,d):*

1. *$\varnothing$ is open*
2. *M is open*
3. *The union of an arbitrary number of open subsets is open:*
   $\{V_k\}_{k\in\mathbb{N}}$ open in $M \Rightarrow \bigcup_{k\in\mathbb{N}} V_k$ open in M.
4. *The intersection of a finite number of open subsets is open:*
   $\{V_k\}_{k=1}^n$ open in $M \Rightarrow \bigcap_{k=1}^n V_k$ open in M.

Proof. Let (M,d) be a metric space:

1. Let $c \in \varnothing$. But $c \notin \varnothing$. So trivially, $\exists r \in \mathbb{R}^+ : B_r(c) \subset \varnothing$.
2. Let $c \in M$. Suppose $x \in B_r(c) \Rightarrow d(x,c) < r \Rightarrow x \in M \Rightarrow B_r(c) \subset M$.
3. Suppose $\{V_k\}_{k\in\mathbb{N}}$ arbitrary collection of open subsets in M.
   Let $c \in \bigcup_{k\in\mathbb{N}} V_k$.
   $\Rightarrow c \in V_k$ for some $k = 1,\dots,n$.
   $\Rightarrow \exists r \in \mathbb{R}^+ : B_r(c) \subset V_k$.
   $\Rightarrow \exists r \in \mathbb{R}^+ : B_r(c) \subset \bigcup_{k\in\mathbb{N}} V_k$, since any set is contained in its union.
4. Suppose $\{V_k\}_{k=1}^n$ finite collection of open subsets in M.
   Let $c \in \bigcap_{k=1}^n V_k$: $\Rightarrow c \in V_k, \forall k = 1,\dots,n$.
   $\Rightarrow \exists r_k \in \mathbb{R}^+ : B_{r_k}(c) \subset V_k$.
   Let $r = min(r_1,\dots,r_n)$. Suppose $x \in B_r(c)$: $\Rightarrow d(x,c) < r$.
   $\Rightarrow d(x,c) < r_1,\dots,d(x,c) < r_n$.
   $\Rightarrow x \in B_{r_1}(c),\dots,x \in B_{r_n}(c)$.
   $\Rightarrow x \in V_1,\dots,x \in V_n$.
   $\Rightarrow x \in \bigcap_{k=1}^n V_k$.
   $\Rightarrow B_r(c) \subset \bigcap_{k=1}^n V_k$.

□

Definition 3.7 (Closed Set)**.** *A subset S of a metric space (M,d) is closed if its complement $S^{\complement}$ is open in (M,d).*

Theorem 3.10. *In any metric space, a closed ball is a closed set.*

Proof. Define $S = \bar{B}_\delta(p)$ as some closed ball in (M, d).
Let $c \in S^\complement$.
Then, $d(c, p) > \delta$.
Choosing $r = d(c, p) - \delta$, clearly $r \in \mathbb{R}^+$.
Suppose $x \in B_r(c)$.
$\Rightarrow d(x, c) < r$.
$\Rightarrow d(x, c) < d(c, p) - \delta$.
$\Rightarrow d(c, p) - d(x, c) > \delta$.
$\Rightarrow d(c, x) + d(x, p) - d(x, c) > \delta$.
$\Rightarrow d(x, p) > \delta$.
$\Rightarrow x \in S^\complement$.
$\Rightarrow B_r(c) \subset S^\complement$.
$\Rightarrow S^\complement$ is open in (M, d).

□

Theorem 3.11. *For any metric space (M, d):*

1. *$\varnothing$ is closed*

2. *M is closed*

3. *The intersection of an arbitrary number of closed subsets is closed:*
   $\{V_k\}_{k\in\mathbb{N}}$ *closed in* $(M, d) \Rightarrow \bigcap_{k\in\mathbb{N}} V_k$ *closed in* (M, d)

4. *The union of a finite number of closed subsets is closed:*
   $\{V_k\}_{k=1}^n$ *closed in* $(M, d) \Rightarrow \bigcup_{k=1}^n V_k$ *closed in* (M, d)

Proof. Let (M, d) be a metric space:

1. $\varnothing^\complement = M \Rightarrow \varnothing^\complement$ open $\Rightarrow \varnothing$ closed.

2. $M^\complement = \varnothing \Rightarrow M^\complement$ open $\Rightarrow M$ closed.

3. Suppose $\{V_k\}_{k\in\mathbb{N}}$ arbitrary collection of closed subsets in (M, d)
   $\Rightarrow \{V_k^\complement\}_{k\in\mathbb{N}}$ is an abitrary collection of open subsets in (M, d)
   $\Rightarrow \bigcup_{k\in\mathbb{N}} V_k^\complement$ is open in (M, d)
   $\Rightarrow (\bigcap_{k\in\mathbb{N}} V_k)^\complement$ is open in (M, d)
   $\Rightarrow \bigcap_{k\in\mathbb{N}} V_k$ is closed in (M, d)

4. Suppose $\{V_k\}_{k=1}^n$ finite collection of closed subsets in (M, d)
   $\Rightarrow \{V_k^\complement\}_{k=1}^n$ is a finite collection of open subsets in (M, d)
   $\Rightarrow \bigcap_{k=1}^n V_k^\complement$ is open in (M, d)
   $\Rightarrow (\bigcup_{k=1}^n V_k)^\complement$ is open in (M, d)
   $\Rightarrow \bigcup_{k=1}^n V_k$ is closed in (M, d)

□

Theorem 3.12. *Any singleton set is closed.*

Proof. Define $S = \{p\}$ as the singleton set $\forall p \in (M, d)$.
Let $c \in S^{\complement} = M/\{p\}$.
Then, $c \neq p$.
Choose $r = d(c, p) \Rightarrow r \in \mathbb{R}^+$.
Suppose $x \in B_r(c)$.
$\Rightarrow d(x, c) < r$.
$\Rightarrow d(x, c) < d(c, p)$.
$\Rightarrow d(c, p) - d(x, c) > 0$.
$\Rightarrow d(p, c) - d(c, x) > 0$.
$\Rightarrow |d(p, c) - d(c, x)| > 0$.
$\Rightarrow d(p, x) > 0$.
$\Rightarrow p \neq x$.
$\Rightarrow x \in S^{\complement}$.
$\Rightarrow B_r(c) \subset S^{\complement}$.
So $S^{\complement}$ is open, making S closed in M. □

Corollary 3.12.1. *Any finite subset of a metric space is closed.*

Proof. Let $x_1, \ldots, x_n \in M$.
$\Rightarrow \{x_1\}, \ldots, \{x_n\}$ all closed in M.
$\Rightarrow \{x_1, \ldots, x_n\} = \bigcup_{k=1}^{n} \{x_k\}$ closed in M.
□

Theorem 3.13. *Any sphere in a metric space is closed.*

Proof. Let $c \in (M, d), r \in \mathbb{R}^+$.
Define r-sphere centered at c as $S = \{x \in (M, d) : d(x, c) = r\}$.
Let $V_1 = \bar{B}_r(c)$.
$\Rightarrow V_1$ closed in (M, d).
Let $V_2 = B_r(c)^{\complement}$.
$\Rightarrow V_2^{\complement} = B_r(c)$.
$\Rightarrow V_2^{\complement}$ open in (M, d).
$\Rightarrow V_2$ closed in (M, d).
$\Rightarrow V_1 \cap V_2$ closed in (M, d).
$\Rightarrow \{x \in (M, d) : d(x, c) \leq r\} \cap \{x \in (M, d) : d(x, c) \geq r\}$ closed in (M, d).
$\Rightarrow \{x \in (M, d) : d(x, c) = r\}$ closed in (M, d).
$\Rightarrow S$ closed in (M, d). □

Theorem 3.14. *Any half-interval is neither open nor closed.*

Proof. Let $a, b \in \mathbb{R}, a < b$.
Define $S = [a, b)$ as a half-interval on $\mathbb{R}$.
Let $r \in \mathbb{R}^+$.
$\Rightarrow a - r < a$.
$\Rightarrow a - r < min(S)$.
$\Rightarrow a - r \in S^{\complement}$.
$\Rightarrow \exists a \in S, \forall r \in \mathbb{R}^+ : B_r(a) = (a - r, a + r) \subset S$.
$\Rightarrow S$ is not open in $\mathbb{R}$.

$\Rightarrow [b, \infty)$ is not open in $\mathbb{R}$ by same logic.
$\Rightarrow S^{\complement} = (-\infty, a) \cup [b, \infty)$ is not open in $\mathbb{R}$ by same logic.
$\Rightarrow S$ is not closed in $\mathbb{R}$. $\square$

Theorem 3.15. *Any subspace of Euclidean space with a strictly bounded component is open. So, $\forall a \in \mathbb{R}, k \in \{1, \ldots, n\}$: $\{x \in \mathbb{R}^n : x_k < a\}$ and $\{x \in \mathbb{R}^n : x_k > a\}$ are open in $\mathbb{R}^n$.*

Proof. Let $a \in \mathbb{R}$, and $k \in \{1, \ldots, n\}$.
Define $S = \{x \in \mathbb{R}^n : x_k < a\}$ and $S' = \{x \in \mathbb{R}^n : x_k > a\}$.
Let $c \in S$ and $p \in S'$.
$\Rightarrow c_k < a$ and $p_k > a$.
Choose $r = a - c_k$ and $\delta = p_k - a \Rightarrow r, \delta \in \mathbb{R}^+$.
Suppose $x \in B_r(c)$ and $y \in B_\delta(p)$.
$\Rightarrow d(x, c) < r$ and $d(y, p) < \delta$.
$\Rightarrow d(x, c) < a - c_k$ and $d(y, p) < p_k - a$.
$\Rightarrow ||x - c|| < a - c_k$ and $||y - p|| < p_k - a$.
$\Rightarrow \sqrt{\sum_k (x_k - c_k)^2} < a - c_k$ and $\sqrt{\sum_k (y_k - p_k)^2} < p_k - a$.
$\Rightarrow \sqrt{(x_k - c_k)^2} < a - c_k$ and $\sqrt{(y_k - p_k)^2} < p_k - a$.
$\Rightarrow |x_k - c_k| < a - c_k$ and $|y_k - p_k| < p_k - a$.
$\Rightarrow x_k - c_k < a - c_k$ and $p_k - y_k < p_k - a$.
$\Rightarrow x_k < a$ and $y_k > a$.
$\Rightarrow x \in S$ and $y \in S'$.
$\Rightarrow B_r(c) \subset S$ and $B_\delta(p) \subset S'$. $\square$

Theorem 3.16. *Any subspace of Euclidean space with a weakly bounded component is closed. So, $\forall a \in \mathbb{R}, k \in \{1, \ldots, n\}$: $\{x \in \mathbb{R}^n : x_k \leq a\}$ and $\{x \in \mathbb{R}^n : x_k \geq a\}$ are closed in $\mathbb{R}^n$.*

Proof. Let $a \in \mathbb{R}$, and $k \in \{1, \ldots, n\}$.
Define $S = \{x \in \mathbb{R}^n : x_k \leq a\}$ and $S' = \{x \in \mathbb{R}^n : x_k \geq a\}$.
$\Rightarrow S^{\complement} = \{x \in \mathbb{R}^n : x_k > a\}$ and $S'^{\complement} = \{x \in \mathbb{R}^n : x_k < a\}$.
$\Rightarrow S^{\complement}, S'^{\complement}$ are open in $\mathbb{R}^n$.
$\Rightarrow S, S'$ are closed in $\mathbb{R}^n$. $\square$

Definition 3.8 (Open/Closed Intervals in $\mathbb{R}^n$)**.** *Let $a_1, \ldots, a_n, b_1, \ldots b_n \in \mathbb{R}$. For $a_1 < b_1, \ldots, a_n < b_n$, the open interval in $\mathbb{R}^n$ is:*

$$\{x \in \mathbb{R}^n : a_k < x_k < b_k, k = 1, \ldots, n\}$$

For $a_1 \leq b_1, \ldots, a_n \leq b_n$, the closed interval in $\mathbb{R}^n$ is:

$$\{x \in \mathbb{R}^n : a_k \leq x_k \leq b_k, k = 1, \ldots, n\}$$

Theorem 3.17. *Any open/closed interval in Euclidean space is open/closed set.*

Proof. Let $S = \{x \in \mathbb{R}^n : a_k < x_k < b_k, k = 1, \ldots, n\}$
$\Rightarrow S = \bigcap_{k=1}^{n} \{x \in \mathbb{R}^n : a_k < x_k < b_k\}$.
$\Rightarrow S = (\bigcap_{k=1}^{n} \{x \in \mathbb{R}^n : x_k > a_k\}) \cap (\bigcap_{k=1}^{n} \{x \in \mathbb{R}^n : x_k < b_k\})$.
$\Rightarrow S$ is open, since it is the finite intersection of $(2n)$ open sets.
Define $S' = \{x \in \mathbb{R}^n : a_k \leq x_k \leq b_k, k = 1, \ldots, n\}$.
$\Rightarrow S' = \bigcap_{k=1}^{n} \{x \in \mathbb{R}^n : a_k \leq x_k \leq b_k\}$.
$\Rightarrow S' = (\bigcap_{k=1}^{n} \{x \in \mathbb{R}^n : x_k \geq a_k\}) \cap (\bigcap_{k=1}^{n} \{x \in \mathbb{R}^n : x_k \leq b_k\})$.
$\Rightarrow S'$ is closed, since it is the finite intersection of $(2n)$ closed sets. $\square$

Definition 3.9 (Bounded)**.** *A subset S of a metric space M is bounded if it is contained in some ball.*

Theorem 3.18. *Any open/closed interval in Euclidean space is bounded.*

Proof. Let $a, b \in \mathbb{R}^n$ s.t. $a \neq b$
Define $S = \{x \in \mathbb{R}^n : a_k \leq x_k \leq b_k, k = 1, \ldots, n\}$
Choose $r = d(b, a) > 0$
Let $x \in S$
$\Rightarrow 0 \leq x_k - a_k \leq b_k - a_k$
$\Rightarrow (x_k - a_k)^2 \leq (b_k - a_k)^2$
$\Rightarrow \sum\limits_{k=1}^{n} (x_k - a_k)^2 \leq \sum\limits_{k=1}^{n} (b_k - a_k)^2$
$\Rightarrow \sqrt{\sum\limits_{k=1}^{n} (x_k - a_k)^2} \leq \sqrt{\sum\limits_{k=1}^{n} (b_k - a_k)^2}$
$\Rightarrow ||x - a|| \leq ||b - a||$
$\Rightarrow d(x, a) \leq d(b, a)$
$\Rightarrow d(x, a) \leq r$
$\Rightarrow x \in \bar{B}_r(a)$
$\Rightarrow S \subset \bar{B}_r(a)$
Let $S' = \{x \in \mathbb{R}^n : a_k < x_k < b_k, k = 1, \ldots, n\}$
$\Rightarrow S' \subset S$
$\Rightarrow S' \subset \bar{B}_r(a)$ $\square$

Theorem 3.19. *The union of a finite collection of bounded subsets of a metric space is bounded.*

Proof. Suppose $V_1, \ldots, V_n$ are bounded subsets in M.
Let $c_1 \in V_1, \ldots, c_n \in V_n$.
$\Rightarrow \exists r_1, \ldots, r_n \in \mathbb{R}^+$ s.t. $V_k \subset \bar{B}_{r_k}(c_k), \forall k = 1, \ldots, n$.
Let $x \in \bigcup_{k=1}^{n} V_k$.
$\Rightarrow x \in V_k$ for some $k = 1, \ldots, n$.
$\Rightarrow x \in \bar{B}_{r_k}(c_k)$.
$\Rightarrow \bigcup_{k=1}^{n} V_k \subset \bar{B}_{r_k}(c_k)$. $\square$

Theorem 3.20. *A nonempty closed subset of $\mathbb{R}$, if it is bounded from above has a greatest element and if it is bounded from below has a least element.*

Proof. Let $S \subset \mathbb{R}, S \neq \varnothing$.
Suppose S is closed in $\mathbb{R}$ and bounded above.
Let $c = sup(S)$.
With an eye to contradict, assume $c \in S^{\complement}$.
Since $S^{\complement}$ open in $\mathbb{R}$, $\exists r \in \mathbb{R}^{+}$ s.t. $(c - r, c + r) \subset S^{\complement}$.
Then no element in S is greater than $c - r$.
$\Rightarrow c - r$ is an upper bound for S.
This must be a contradictions, so $c \in S$. $\square$

3.3 Convergent Sequences

Definition 3.10 (Convergent Sequence). *Let* $\{p_n\}_{n \in \mathbb{N}}$