

# Intro Real Analysis

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## Contents

<b>1</b>	<b>Notions from Set Theory</b>	<b>3</b>
1.1	Sets and Elements. Subsets . . . . .	3
1.2	Operations on Sets . . . . .	3
1.3	Functions . . . . .	3
1.4	Finite and Infinite Sets . . . . .	3
<b>2</b>	<b>The Real Number System</b>	<b>3</b>
2.1	The Field Properties . . . . .	3
2.2	Order . . . . .	3
2.3	The Least Upper Bound Property . . . . .	3
2.4	The existence of Square Roots . . . . .	3
<b>3</b>	<b>Metric Spaces</b>	<b>3</b>
3.1	Definition of Metric Spaces. Examples . . . . .	3
3.2	Open and Closed Sets . . . . .	4
3.3	Convergent Sequences . . . . .	4
3.4	Completeness . . . . .	4
3.5	Compactness . . . . .	4
3.6	Connectedness . . . . .	4
<b>4</b>	<b>Continuous Functions</b>	<b>4</b>
4.1	Definition of Continuity. Examples . . . . .	4
4.2	Continuity and Limits . . . . .	4
4.3	The Continuity of Rational Operations. Functions with values in $E^n$ . . . . .	4
4.4	Continuous Functions on a Compact Metric Space . . . . .	4
4.5	Continuous Functions on a Connected Metric Space . . . . .	4
4.6	Sequences of Functions . . . . .	4

<b>5</b>	<b>Differentiation</b>	<b>4</b>
5.1	Definition of the Derivative . . . . .	4
5.2	Rules of Differentiation . . . . .	4
5.3	The Mean Value Theorem . . . . .	4
5.4	Taylor's Theorem . . . . .	4
<b>6</b>	<b>Riemann Integration</b>	<b>4</b>
6.1	Definition and Examples . . . . .	4
6.2	Linearity and Order Properties of the Integral . . . . .	4
6.3	Existence of the Integral . . . . .	4
6.4	The Fundamental Theorem of Calculus . . . . .	4
6.5	The Logarithmic and Exponential Functions . . . . .	4
6.6	Definition of Continuity. Examples . . . . .	4
<b>7</b>	<b>Interchange of Limit Operations</b>	<b>4</b>
7.1	Integration and Differentiation of Sequences of Functions . . . . .	4
7.2	Infinite Series . . . . .	4
7.3	Power Series . . . . .	4
7.4	The Trigonometric Functions . . . . .	4
7.5	Differentiation under the Integral Sign . . . . .	4
<b>8</b>	<b>The Method of Successive Approximations</b>	<b>4</b>
8.1	The Fixed Point Theorem . . . . .	4
8.2	The Simplest Case of the Implicit Function Theorem . . . . .	4
8.3	Existence and Uniqueness Theorems for Ordinary Differential Equations . . . . .	4
<b>9</b>	<b>Partial Differentiation</b>	<b>4</b>
9.1	Definitions and Basic Properties . . . . .	4
9.2	Higher Derivatives . . . . .	4
9.3	The Implicit Function Theorem . . . . .	4
<b>10</b>	<b>Multiple Integrals</b>	<b>4</b>
10.1	Riemann Integration on Closed Intervals of $E^n$ . Examples and Basic Properties . . . . .	4
10.2	Existence of the Integral. Integration on Arbitrary Subsets of $E^n$ . Volume . . . . .	4
10.3	Iterated Integrals . . . . .	4
10.4	Change of Variable . . . . .	4

# 1 Notions from Set Theory

## 1.1 Sets and Elements. Subsets

## 1.2 Operations on Sets

## 1.3 Functions

## 1.4 Finite and Infinite Sets

# 2 The Real Number System

## 2.1 The Field Properties

## 2.2 Order

## 2.3 The Least Upper Bound Property

## 2.4 The existence of Square Roots

# 3 Metric Spaces

## 3.1 Definition of Metric Spaces. Examples

**Definition 3.1** (Metric Space). A Metric Space is an ordered pair  $(E, d)$  where  $E$  is some set along with a metric function  $d: E \times E \rightarrow \mathbb{R}$  such that:

1. **Identity of Indiscernibles:**  $d(p, q) = 0 \Leftrightarrow p = q, \forall p, q \in E$
2. **Symmetry:**  $d(p, q) = d(q, p), \forall p, q \in E$
3. **Subadditivity:**  $d(p, q) \leq d(p, r) + d(r, q), \forall p, q, r \in E$

**Theorem 3.1.** *Metric functions are non-negative*

*Proof.* Let  $d$  be a metric function with points  $p, q \in E$ . So,

$$d(p, p) \leq d(p, q) + d(q, p). \quad (\text{Subadditivity})$$

$$d(p, p) \leq d(p, q) + d(p, q). \quad (\text{Symmetry})$$

$$d(p, p) \leq 2 \cdot d(p, q).$$

$$0 \leq 2 \cdot d(p, q). \quad (\text{Identity of Indiscernibles})$$

$$0 \leq d(p, q). \quad \square$$

- 3.2 Open and Closed Sets
- 3.3 Convergent Sequences
- 3.4 Completeness
- 3.5 Compactness
- 3.6 Connectedness
- 4 Continuous Functions
  - 4.1 Definition of Continuity. Examples
  - 4.2 Continuity and Limits
  - 4.3 The Continuity of Rational Operations. Functions with values in  $E^n$
  - 4.4 Continuous Functions on a Compact Metric Space
  - 4.5 Continuous Functions on a Connected Metric Space
  - 4.6 Sequences of Functions
- 5 Differentiation
  - 5.1 Definition of the Derivative
  - 5.2 Rules of Differentiation
  - 5.3 The Mean Value Theorem
  - 5.4 Taylor's Theorem
- 6 Riemann Integration
  - 6.1 Definition and Examples
  - 6.2 Linearity and Order Properties of the Integral
  - 6.3 Existence of the Integral
  - 6.4 The Fundamental Theorem of Calculus
  - 6.5 The Logarithmic and Exponential Functions
  - 6.6 Definition of Continuity. Examples
- 7 Interchange of Limit Operations
  - 7.1 Integration and Differentiation of Sequences of Functions
  - 7.2 Infinite Series
  - 7.3 Power Series
  - 7.4 The Trigonometric Functions
  - 7.5 Differentiation under the Integral Sign