Goal: $\displaystyle\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ pointwise on $(-1,1)$.

Setup: Let $\varepsilon > 0,$ and let $x \in (-1,1)$.

Claim: $|x|^n \to 0$ for every $x \in (-1,1)$.

Case 1: $x = 0 \Longrightarrow ||x|^n - 0| = 0 < \varepsilon$.

Case 2: $x \in (-1,0) \cup (0,1),\ 0 < |x| < 1$.

$N_0 = \left\lceil \left|\log_{|x|}(\varepsilon)\right| \right\rceil \Longrightarrow n > N_0 \Longrightarrow |x|^n < \varepsilon.$

Hence $||x|^n - 0| < \varepsilon$.

Choose index: $\exists N \in \mathbb{N} : \forall n > N :\ |x|^n < \varepsilon(1-x)$.

Tail bound:

$$\left|\sum_{k=0}^{n} x^k - \frac{1}{1-x}\right| = \left|\frac{1-x^{n+1}}{1-x} - \frac{1}{1-x}\right|$$

$$= \left|\frac{-x^{n+1}}{1-x}\right| = \frac{|x|^{n+1}}{1-x} < \frac{|x|^n}{1-x} < \frac{\varepsilon(1-x)}{1-x} = \varepsilon. \quad \square$$