

Intro Real Analysis

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1 Notions from Set Theory

1.1 Sets and Elements. Subsets

Set theory supplies the basic language for all that follows, so we fix the most common conventions. We will not attempt to “define” the word set; informally

a set is simply a collection of objects, called its *elements*. Typical sets are the students enrolled at a particular university, the straight lines in the plane, or the real numbers. Elements of a set may themselves be sets: an element of the set of straight lines in the plane is the set of points on that line. We use capital letters for sets, lower case letters for their elements, and the symbol \in for membership. Thus

$$x \in S$$

means that x is a member of the set S , while $x \notin S$ abbreviates “ x is not an element of S ”. A set is completely determined by its elements; explicitly $X = Y$ if and only if every x satisfies $x \in X \Leftrightarrow x \in Y$. The difference between a set and its elements is crucial: the set $\{x\}$ consists of a single element, and must not be confused with x itself.

The notation $\{x_1, x_2, x_3\}$ lists the indicated elements, while set-builder notation

$$\{x \in S : x \text{ satisfies a stated property}\}$$

collects all elements of S with the desired property. For example

$$\{x \in \mathbb{R} : 0 \leq x \leq 1\}, \quad \{n : n \text{ is a positive integer}\}$$

are two subsets of the real numbers \mathbb{R} . The symbol \emptyset denotes the empty set, the unique set with no elements.

Definition 1.1 (Subset). *If X and S are sets we write $X \subseteq S$ when every element of X is also an element of S ; that is*

$$X \subseteq S \iff \forall x (x \in X \Rightarrow x \in S).$$

If $X \subseteq S$ and $X \neq S$ we write $X \subsetneq S$ and call X a proper subset of S .

The relation $X \subseteq S$ is always true when $X = S$, and is vacuously true when $X = \emptyset$ because there are no elements that violate the implication. A frequent source of confusion is mixing up the symbols \subseteq and \in ; for example $\{2, 3\} \subseteq \{1, 2, 3\}$, whereas $2 \in \{1, 2, 3\}$.

1.2 Operations on Sets

The standard operations on sets are intersection, union, and complement.

Definition 1.2 (Intersection and Union). *If X and Y are sets their intersection and union are*

$$X \cap Y = \{x : x \in X \text{ and } x \in Y\}, \quad X \cup Y = \{x : x \in X \text{ or } x \in Y\}.$$

The word “or” is inclusive: x belongs to $X \cup Y$ when it lies in X , or in Y , or in both sets.

Definition 1.3 (Complement and Difference). *If X is a subset of a fixed “ambient” set S , the complement of X in S is*

$$S \setminus X = \{x \in S : x \notin X\},$$

and we often abbreviate this to X^c once S is understood. For any two sets we also write

$$X - Y = X \cap Y^c = \{x \in X : x \notin Y\}$$

and call this the set difference or relative complement of Y in X .

The relations among these operations mirror the familiar algebraic identities. For instance, if $X, Y \subseteq S$ then

$$X^c \cap Y^c = (X \cup Y)^c, \quad X^c \cup Y^c = (X \cap Y)^c,$$

which are the De Morgan laws.

Sets X and Y are *disjoint* if $X \cap Y = \emptyset$; a family of sets is disjoint if every two distinct members are disjoint. Intersections and unions extend to any finite number of sets by iterating the preceding definitions, and to arbitrary families via index sets. If I is an indexing set and $\{X_i\}_{i \in I}$ is a family of subsets of S , then

$$\bigcap_{i \in I} X_i = \{x : x \in X_i \text{ for every } i \in I\}, \quad \bigcup_{i \in I} X_i = \{x : x \in X_i \text{ for at least one } i \in I\}.$$

In particular, if each $X_i \subseteq S$, then $(\bigcap_{i \in I} X_i)^c = \bigcup_{i \in I} X_i^c$.

Definition 1.4 (Ordered Pair and Cartesian Product). *The ordered pair (a, b) records a as the first component and b as the second; $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$. One possible set-theoretic realization is due to Kuratowski:*

$$(a, b) = \{\{a\}, \{a, b\}\}.$$

Given sets X and Y the Cartesian product

$$X \times Y = \{(x, y) : x \in X, y \in Y\}$$

is the set of all ordered pairs whose first coordinate comes from X and second from Y .

1.3 Functions

Definition 1.5 (Function). *If X and Y are sets, a function f from X to Y is a rule that assigns to each $x \in X$ a definite element $f(x) \in Y$. We encode this by writing $f : X \rightarrow Y$, and call X the domain and Y the codomain of f . Two functions $f, g : X \rightarrow Y$ are equal when $f(x) = g(x)$ for every $x \in X$.*

Functions may be described by listing the pairs $(x, f(x))$, by a formula (for example $f(x) = x^2 + 3x - 2$ on \mathbb{R}), or geometrically via their graphs. Any subset of the plane meeting each vertical line in at most one point is the graph of a real-valued function on a subset of \mathbb{R} . The defining rule need not be easy to compute: the function that sends each real number to the digit in its billionth decimal place is a legitimate example.

The graph of $f : X \rightarrow Y$ is the subset $\{(x, f(x)) : x \in X\}$ of $X \times Y$. Conversely, any subset of $X \times Y$ that contains exactly one ordered pair with first coordinate x for each $x \in X$ determines a function $X \rightarrow Y$. In this sense the data of the domain, codomain, and graph together determine the function.

If $f : X \rightarrow Y$ and $X' \subseteq X$, the restriction $f|_{X'} : X' \rightarrow Y$ is defined by $f|_{X'}(x) = f(x)$. If $Y \subseteq Y'$ we may view f as a function into Y' without changing the rule. Given $f : X \rightarrow Y$ and $g : Y \rightarrow Z$, the composition $g \circ f : X \rightarrow Z$ is given by $(g \circ f)(x) = g(f(x))$. The identity map on X is denoted $i_X : X \rightarrow X$, $i_X(x) = x$.

Definition 1.6 (Injective, Surjective, Bijective). *A function $f : X \rightarrow Y$ is injective (or one-to-one) if $f(x_1) = f(x_2)$ implies $x_1 = x_2$. It is surjective (onto) if every $y \in Y$ equals $f(x)$ for some $x \in X$. A bijection is both injective and surjective, and hence establishes a one-to-one correspondence between X and Y .*

If $f : X \rightarrow Y$ is bijective there exists an inverse function $f^{-1} : Y \rightarrow X$ satisfying $f^{-1}(f(x)) = x$ and $f(f^{-1}(y)) = y$. For an arbitrary function f and subsets $X' \subseteq X$, $Y' \subseteq Y$,

$$f(X') = \{f(x) : x \in X'\}, \quad f^{-1}(Y') = \{x \in X : f(x) \in Y'\}$$

are called the image of X' and the inverse image (or preimage) of Y' , respectively. When f is bijective these two uses of f^{-1} are compatible: $f^{-1}(\{y\})$ is the singleton containing $f^{-1}(y)$.

1.4 Finite and Infinite Sets

Let $\mathbb{N} = \{1, 2, 3, \dots\}$ denote the positive integers.

Definition 1.7 (Finite Set). *A set X is finite if either $X = \emptyset$ or there exists $n \in \mathbb{N}$ together with a bijection between X and $\{1, 2, \dots, n\}$. In that case we write $|X| = n$ and call n the number of elements of X .*

Any subset of a finite set is finite; if the subset is proper, then it contains fewer elements than the ambient set. A set is *infinite* when it is not finite: intuitively we can keep choosing distinct elements without ever exhausting the set. The following characterization is often useful.

Theorem 1.1. *A set X is infinite if and only if it can be put into a one-to-one correspondence with a proper subset of itself.*

Proof. If X is finite, any proper subset has fewer elements, so no bijection can exist. Conversely, suppose X is infinite. Choose distinct elements x_1, x_2, x_3, \dots in X , and let $Y = X \setminus \{x_1, x_2, x_3, \dots\}$. Define $f : X \rightarrow \{x_2, x_3, \dots\} \cup Y$ by $f(x_k) = x_{k+1}$ for $k \geq 1$ and $f(y) = y$ for $y \in Y$. This f is a bijection from X onto a proper subset of X . \square

The natural numbers also provide a convenient definition of sequences. An n -tuple of elements of a set X is a function from $\{1, 2, \dots, n\}$ to X ; we usually write it as (x_1, x_2, \dots, x_n) . An infinite sequence (or simply a sequence) of elements of X is a function from \mathbb{N} to X , often denoted (x_1, x_2, x_3, \dots) .

2 The Real Number System

2.1 The Field Properties

Analysis rests on the exact algebraic and order structure of the real numbers. We begin by recording the axioms they satisfy.

Definition 2.1 (Group). A group is a nonempty set G equipped with a binary operation \cdot such that:

1. $x \cdot y \in G$ for all $x, y \in G$ (closure).
2. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all $x, y, z \in G$ (associativity).
3. There exists $e \in G$ with $e \cdot x = x \cdot e = x$ for every $x \in G$ (identity).
4. Each $x \in G$ has $x^{-1} \in G$ satisfying $x \cdot x^{-1} = x^{-1} \cdot x = e$ (inverse).

The group is Abelian if additionally $x \cdot y = y \cdot x$ for all x, y .

Definition 2.2 (Field). A field is a set \mathbb{F} together with two binary operations, addition and multiplication, such that

1. $(\mathbb{F}, +)$ is an Abelian group with identity $0_{\mathbb{F}}$.
2. $(\mathbb{F} \setminus \{0_{\mathbb{F}}\}, \cdot)$ is an Abelian group with identity $1_{\mathbb{F}}$.
3. Multiplication distributes over addition:

$$x \cdot (y + z) = x \cdot y + x \cdot z, \quad (x + y) \cdot z = x \cdot z + y \cdot z.$$

Definition 2.3 (Real Numbers). The real number system \mathbb{R} is the unique (up to isomorphism) complete ordered field. Practically we view \mathbb{R} as a set endowed with addition, multiplication, and an order relation satisfying the axioms listed in this section.

Besides \mathbb{R} , the rationals \mathbb{Q} and complexes \mathbb{C} are important fields. Finite examples also exist (for instance $\mathbb{Z}/p\mathbb{Z}$ when p is prime), but they lack the order and completeness required for analysis.

Field axioms. Rosenlicht enumerates seven explicit properties; we collect them in two groups.

- (F1) **Commutativity.** $a + b = b + a$ and $ab = ba$.
- (F2) **Associativity.** $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$.
- (F3) **Identities.** There exist 0 and 1 with $a + 0 = a$ and $a \cdot 1 = a$.
- (F4) **Inverses.** Every a has $-a$ with $a + (-a) = 0$; each $a \neq 0$ has a^{-1} with $aa^{-1} = 1$.
- (F5) **Distributivity.** $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$.

These axioms imply all of the familiar algebraic rules. We spell out the basic consequences because they are used constantly.

Lemma 2.1 (Cancellation). *If (G, \cdot) is a group and $xy = xz$, then $y = z$; likewise $yx = zx$ implies $y = z$.*

Proof. Multiply $xy = xz$ on the left by x^{-1} to obtain $y = z$. \square

Corollary 2.0.1. *The identity element of a group is unique, inverses are unique, and $(x^{-1})^{-1} = x$.*

Proposition 2.1 (Algebraic identities). *Let $(\mathbb{F}, +, \cdot)$ be a field. For any $x, y \in \mathbb{F}$ the following hold.*

1. $0 \cdot x = x \cdot 0 = 0$ and $(-1)x = x(-1) = -x$.
2. $xy = 0$ implies $x = 0$ or $y = 0$.
3. $(-x)(-y) = xy$ and $-(x + y) = (-x) + (-y)$.
4. If $x, y \neq 0$ then $(xy)^{-1} = y^{-1}x^{-1}$.

Proof. Each statement is a short application of the axioms and Lemma 2.1. For instance,

$$x \cdot 0 = x \cdot (0 + 0) = x \cdot 0 + x \cdot 0,$$

so subtracting $x \cdot 0$ from both sides shows $x \cdot 0 = 0$. The other items are similar. \square

2.2 Order

To do analysis, the field must also be ordered.

Definition 2.4 (Ordered field). *An ordered field is a field endowed with a relation $<$ such that for all x, y, z :*

1. Exactly one of $x < y$, $x = y$, or $x > y$ holds (trichotomy).
2. If $x < y$, then $x + z < y + z$.

3. If $0 < z$ and $x < y$, then $xz < yz$.

We write $x \leq y$ when $x < y$ or $x = y$ and denote the set of positive elements by $\mathbb{F}_{>0} = \{x : x > 0\}$. The order interacts with addition and multiplication exactly as intuition dictates.

Lemma 2.2 (Order manipulations). *If $x > 0$ and $y > 0$, then $x + y > 0$ and $xy > 0$. If $x < y$ and $z < w$, then $x + z < y + w$. Multiplying by a negative element reverses inequalities.*

In an ordered field we define intervals, rays, and the absolute value $|x| = \max\{x, -x\}$. These capture the geometric idea of distance on the number line.

2.3 Completeness and Bounds

The property that distinguishes \mathbb{R} from \mathbb{Q} is completeness.

Definition 2.5 (Bounds). *Let $S \subseteq \mathbb{R}$. A number u is an upper bound if $x \leq u$ for all $x \in S$. If u is an upper bound and $u \leq v$ for every upper bound v , then u is the supremum $\sup S$. Lower bounds and the infimum $\inf S$ are defined similarly.*

Theorem 2.1 (Least Upper Bound Axiom). *Every nonempty subset of \mathbb{R} that is bounded above has a supremum in \mathbb{R} ; every nonempty subset bounded below has an infimum.*

Theorem 2.2 (Monotone convergence). *If $\{a_n\}$ is increasing and bounded above, then $a_n \rightarrow \sup\{a_n\}$.*

Proof. Let $s = \sup\{a_n\}$. Given $\varepsilon > 0$, $s - \varepsilon$ is not an upper bound, so some N satisfies $a_N > s - \varepsilon$. For $n \geq N$ we have $s - \varepsilon < a_n \leq s$, proving $a_n \rightarrow s$. \square

Theorem 2.3 (Archimedean property). *For every $x > 0$ there exists $n \in \mathbb{N}$ with $n > x$.*

Proof. Suppose no such n exists. Then the set $\mathcal{A} = \{n : n \in \mathbb{N}\}$ is bounded above by x , so $s = \sup \mathcal{A}$ exists. But $s - 1$ is not an upper bound, so some $m \in \mathbb{N}$ satisfies $m > s - 1$, i.e., $m + 1 > s$. Since $m + 1 \in \mathbb{N}$, we contradict the definition of s . \square

Theorem 2.4 (Density of \mathbb{Q}). *If $a < b$ in \mathbb{R} , then there exists $q \in \mathbb{Q}$ with $a < q < b$.*

Proof. Consider $(b - a) > 0$. By the Archimedean property choose n with $n(b - a) > 1$. There exists an integer m such that $na < m < nb$, and then $m/n \in \mathbb{Q}$ satisfies $a < m/n < b$. \square

2.4 The Existence of Square Roots

Completeness quickly yields the fundamental fact that positive numbers possess square roots.

Theorem 2.5. *For every $a > 0$ there exists a unique $b > 0$ such that $b^2 = a$.*

Proof. Consider the set $S = \{x \in \mathbb{R} : x \geq 0 \text{ and } x^2 \leq a\}$. It is nonempty (since $0 \in S$) and bounded above (for example by $a + 1$). Let $b = \sup S$. We first show $b^2 \leq a$. Otherwise $b^2 > a$ and putting $\varepsilon = b^2 - a > 0$ we may take $\eta = \varepsilon/(2b)$; note that $b > 0$ because $a > 0$. Then $(b - \eta)^2 = b^2 - 2b\eta + \eta^2 = b^2 - \varepsilon + \eta^2 > a$, so every $x \in S$ satisfies $x < b - \eta$. That means $b - \eta$ is an upper bound smaller than b , contradicting the definition of b . Hence $b^2 \leq a$.

If $b^2 < a$, set $\varepsilon = a - b^2 > 0$ and choose $\eta = \min\{1, \varepsilon/(2b + 1)\}$. Then

$$(b + \eta)^2 = b^2 + 2b\eta + \eta^2 \leq b^2 + \eta(2b + 1) \leq b^2 + \varepsilon = a,$$

so $b + \eta \in S$. This contradicts b being an upper bound. Therefore $b^2 = a$.

Uniqueness follows from the order axioms: if $c > 0$ also satisfies $c^2 = a$, then $(b - c)(b + c) = 0$. Proposition 2.1 implies either $b = c$ or $b = -c$. The latter is impossible because $b, c > 0$, so $b = c$. \square

The existence of square roots supplies the positive branch of the absolute value and, together with the field axioms, completes the algebraic description of \mathbb{R} .

3 Metric Spaces

3.1 Definition and Examples

Metric spaces generalize familiar geometric notions of distance.

Definition 3.1 (Metric Space). *A metric space is a pair (M, d) consisting of a set M and a function $d: M \times M \rightarrow \mathbb{R}$ satisfying for all $x, y, z \in M$:*

1. $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$.
2. $d(x, y) = d(y, x)$.
3. $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

Lemma 3.1. *Nonnegativity is automatic: $d(x, y) = \frac{1}{2}(d(x, y) + d(y, x)) \geq \frac{1}{2}d(x, x) = 0$.*

Theorem 3.1 (General Triangle Inequality). *Given points x_1, \dots, x_n in (M, d) ,*

$$d(x_1, x_n) \leq \sum_{k=1}^{n-1} d(x_k, x_{k+1}).$$

Corollary 3.1.1 (Reverse Triangle Inequality). *For all $x, y, z \in M$, $|d(x, z) - d(z, y)| \leq d(x, y)$.*

These inequalities turn the metric into a powerful bookkeeping device for estimates.

Examples.

1. On \mathbb{R}^n the Euclidean metric $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$ arises from the dot product.
The Cauchy–Schwarz inequality

$$|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \|\mathbf{y}\|$$

implies $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$, which provides the triangle inequality for d .

2. The *taxicab* (or ℓ^1) metric on \mathbb{R}^n is $d_1(\mathbf{x}, \mathbf{y}) = \sum |x_i - y_i|$; each coordinate satisfies the one-dimensional triangle inequality, so the sum does as well.
3. The *discrete* metric on an arbitrary set M is $d(x, y) = 0$ when $x = y$ and $d(x, y) = 1$ otherwise.

Subspaces. If $E \subseteq M$, the restriction of d to $E \times E$ makes E into a metric space; this is the *subspace metric*.

3.2 Open Sets, Closed Sets, and Neighborhoods

For $p \in M$ and $r > 0$ define the open ball $B_r(p) = \{x : d(x, p) < r\}$ and the closed ball $\overline{B}_r(p) = \{x : d(x, p) \leq r\}$.

Definition 3.2. A subset $U \subseteq M$ is open if for every $x \in U$ there exists $r > 0$ with $B_r(x) \subseteq U$. A subset $F \subseteq M$ is closed if its complement is open. The empty set and M are both open and closed.

Open sets collect points together with a “cushion” around them. Closed sets are precisely those that contain all limit points of sequences within them (see Proposition 3.2).

Proposition 3.1. Open balls are open, closed balls are closed, and finite intersections of open (resp. closed) balls are open (resp. closed).

Proof. If $x \in B_r(p)$ set $\rho = r - d(x, p) > 0$. Whenever $y \in B_\rho(x)$, the triangle inequality gives $d(y, p) \leq d(y, x) + d(x, p) < \rho + d(x, p) = r$, so $y \in B_r(p)$. The closed case follows by considering complements. \square

Theorem 3.2. Let $\{U_\alpha\}_{\alpha \in A}$ be open subsets of M and F_1, \dots, F_n closed subsets.

1. $\bigcup_{\alpha \in A} U_\alpha$ is open.
2. $\bigcap_{k=1}^n U_k$ is open for any finite subcollection.
3. $\bigcap_{\alpha \in A} F_\alpha$ is closed and $\bigcup_{k=1}^n F_k$ is closed.

Limit points, interior, and closure.

Definition 3.3. A point x is a limit point of $E \subseteq M$ if every ball about x intersects $E \setminus \{x\}$. The set of all limit points is denoted E' . The closure $\overline{E} = E \cup E'$ is the smallest closed set containing E , and the interior $\text{int}(E)$ is the largest open set contained in E .

Proposition 3.2. E is closed if and only if it contains its limit points; equivalently $E = \overline{E}$.

Proof. If E is closed and $x \in E'$ then every ball around x meets E , hence x cannot lie in the open complement. Conversely, if E contains its limit points, E^c is open: given $x \in E^c$, there exists $r > 0$ with $B_r(x) \cap E = \emptyset$, so $B_r(x) \subseteq E^c$. \square

Proposition 3.3. In \mathbb{R} the open balls are intervals (a, b) , while closed balls are $[a, b]$. In \mathbb{R}^n rectangular boxes $\prod_{i=1}^n (a_i, b_i)$ are open, their closures $\prod [a_i, b_i]$ are closed, and both are bounded.

3.3 Boundedness and Extremal Points

Definition 3.4 (Bounded Set). A subset S is bounded if $S \subseteq \overline{B}_r(p)$ for some p and r .

Balls witness boundedness, so finite unions of bounded sets are bounded. In \mathbb{R}^n closed rectangles are bounded because each coordinate difference is bounded.

Theorem 3.3. A nonempty closed subset of \mathbb{R} that is bounded above has a maximum; one bounded below has a minimum.

3.4 Convergent Sequences

Definition 3.5 (Convergence). A sequence $\{p_n\}$ converges to p if for every $\varepsilon > 0$ there exists N with $d(p_n, p) < \varepsilon$ for $n \geq N$.

Limits are unique. A sequence converges to p if and only if every subsequence has a further subsequence converging to p . Open and closed sets can be characterized via sequences:

Proposition 3.4. 1. U is open if and only if no sequence entirely outside U converges to a point of U .
 2. F is closed if and only if whenever $p_n \in F$ and $p_n \rightarrow p$, then $p \in F$.

3.5 Completeness

Definition 3.6 (Cauchy Sequence). A sequence $\{p_n\}$ is Cauchy if for every $\varepsilon > 0$ there exists N with $d(p_n, p_m) < \varepsilon$ for all $m, n \geq N$. A metric space is complete if every Cauchy sequence converges.

Proposition 3.5. \mathbb{R}^n with the Euclidean metric is complete.

Proof. Each coordinate of a Cauchy sequence is Cauchy in \mathbb{R} , hence convergent. The coordinatewise limit lies in \mathbb{R}^n and is the limit of the vector sequence. \square

Theorem 3.4 (Completion criterion). *A subspace $E \subseteq M$ is complete if and only if it is closed in M (when M itself is complete).*

3.6 Compactness

Definition 3.7 (Compactness). *A subset $K \subseteq M$ is compact if every open cover $\{U_\alpha\}$ of K admits a finite subcover. Equivalently, K is compact if every sequence in K has a convergent subsequence with limit in K (sequential compactness).*

Theorem 3.5 (Heine–Borel). *A subset of \mathbb{R}^n is compact if and only if it is closed and bounded.*

Proposition 3.6. *Closed subsets of compact sets are compact, and continuous images of compact sets are compact.*

3.7 Connectedness

Definition 3.8 (Connected Set). *A metric space M is connected if it cannot be written as the union of two disjoint nonempty open sets. A subset $E \subseteq M$ is connected if it is connected with the subspace metric.*

Proposition 3.7. *Intervals in \mathbb{R} are connected, and any connected subset of \mathbb{R} is an interval. Products of connected sets are connected. Continuous images of connected sets are connected.*

Connectedness ensures that intermediate values cannot be skipped; this feeds directly into the Intermediate Value Theorem in the next chapter.

4 Continuous Functions

Continuous functions will carry the weight of the topological results developed in Chapter 3. Rosenlicht treats functions between arbitrary metric spaces; the real- and vector-valued cases fall out as specializations.

4.1 Definition of Continuity. Examples

Let (E, d) and (E', d') be metric spaces.

Definition 4.1 (Continuity at a point). *A function $f: E \rightarrow E'$ is continuous at $p_0 \in E$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d(p, p_0) < \delta$ implies $d'(f(p), f(p_0)) < \varepsilon$. It is continuous if it is continuous at every point.*

The definition admits two equivalent reformulations.

Proposition 4.1 (Neighborhood formulation). *The following are equivalent for $f: E \rightarrow E'$ and $p_0 \in E$.*

1. *f is continuous at p_0 .*
2. *For every open set $U \subseteq E'$ with $f(p_0) \in U$, there exists an open set $V \subseteq E$ containing p_0 such that $f(V) \subseteq U$.*
3. *For every sequence $p_n \rightarrow p_0$ in E , we have $f(p_n) \rightarrow f(p_0)$ in E' .*

Corollary 4.0.1. *f is continuous if and only if the preimage of every open set is open (equivalently, the preimage of every closed set is closed).*

These criteria prove continuity for the functions encountered in calculus: polynomials are continuous everywhere, rational functions are continuous on their domain, absolute value and maximum/minimum are continuous combinations of basic operations, and so on.

4.2 Continuity and Limits

Continuity aligns perfectly with function limits.

Theorem 4.1 (Sequential characterization). *If $f: E \rightarrow E'$ and $p \in E$, then f is continuous at p if and only if for every sequence $p_n \rightarrow p$ we have $f(p_n) \rightarrow f(p)$.*

Theorem 4.2 (Composition and restriction). *If $f: E \rightarrow E'$ is continuous at $p \in E$ and $g: E' \rightarrow E''$ is continuous at $f(p)$, then $g \circ f$ is continuous at p . If $A \subseteq E$ carries the subspace metric and f is continuous on E , then the restriction $f|_A$ is continuous on A .*

Proposition 4.2 (Gluing lemma). *If $E = F \cup G$ with F, G closed subsets and $f: E \rightarrow E'$ agrees with continuous maps on F and on G , then f is continuous.*

Continuity also respects monotone bijections between intervals: if $U, V \subseteq \mathbb{R}$ are intervals and $f: U \rightarrow V$ is strictly increasing and onto, then both f and f^{-1} are continuous.

4.3 Rational Operations and Vector-Valued Functions

The algebra of continuous real-valued functions mimics the algebra of real numbers.

Theorem 4.3 (Algebra of continuous functions). *Let $f, g: E \rightarrow \mathbb{R}$ be continuous and let $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ be continuous. Then the functions $f + g$, $f - g$, fg , and $\varphi \circ f$ are continuous. If $g(p) \neq 0$ for all p , then f/g is continuous.*

Corollary 4.3.1. *Polynomials and rational functions (on the domain where the denominator is nonzero) are continuous. If $\mathbf{f}: E \rightarrow \mathbb{R}^n$ is given by $\mathbf{f}(p) = (f_1(p), \dots, f_n(p))$, then \mathbf{f} is continuous if and only if each coordinate function f_k is continuous.*

4.4 Continuous Functions on Compact Metric Spaces

Compactness translates topological information into quantitative statements.

Theorem 4.4 (Continuous images of compact sets). *If $K \subseteq E$ is compact and $f: E \rightarrow E'$ is continuous, then $f(K) \subseteq E'$ is compact.*

Specializing to real-valued functions produces the classical extreme value and boundedness results.

Corollary 4.4.1. *If K is compact and $f: K \rightarrow \mathbb{R}$ is continuous, then f is bounded and attains both its maximum and minimum on K .*

Theorem 4.5 (Heine–Cantor). *Continuous functions on compact metric spaces are uniformly continuous.*

Proof. Suppose f is not uniformly continuous. Then there exists $\varepsilon_0 > 0$ and points $p_n, q_n \in K$ with $d(p_n, q_n) \rightarrow 0$ but $|f(p_n) - f(q_n)| \geq \varepsilon_0$. Compactness provides a convergent subsequence $p_{n_k} \rightarrow p$; the paired points q_{n_k} also converge to p . Continuity contradicts the fixed gap ε_0 . \square

Uniform continuity guarantees that Cauchy sequences of function values behave well. In particular, the sup metric

$$d_\infty(f, g) = \sup_{p \in K} d'(f(p), g(p))$$

turns $C(K, E')$ (continuous maps $K \rightarrow E'$) into a metric space. When K is compact and E' is complete, $C(K, E')$ is complete under d_∞ : a uniformly Cauchy sequence of continuous functions converges uniformly to a continuous function.

4.5 Continuous Functions on Connected Metric Spaces

Connectedness rules out “jumps”.

Theorem 4.6 (Image of a connected set). *If E is connected and $f: E \rightarrow E'$ is continuous, then $f(E)$ is connected.*

In \mathbb{R} the connected subsets are intervals; applying Theorem 4.6 yields the Intermediate Value Theorem.

Corollary 4.6.1 (Intermediate Value Theorem). *If $f: [a, b] \rightarrow \mathbb{R}$ is continuous and y lies between $f(a)$ and $f(b)$, then there exists $c \in [a, b]$ with $f(c) = y$.*

Further consequences include: a continuous function on an interval that never changes sign is either strictly positive, strictly negative, or identically zero; a continuous bijection from a compact interval to an interval is monotone with a continuous inverse.

4.6 Sequences of Functions

Finally we examine sequences of continuous functions. Let (f_n) be functions $E \rightarrow E'$.

Definition 4.2 (Uniform convergence). f_n converges uniformly to f if for every $\varepsilon > 0$ there exists N with $d'(f_n(p), f(p)) < \varepsilon$ for all $p \in E$ whenever $n \geq N$.

Theorem 4.7 (Uniform limits preserve continuity). If each f_n is continuous and $f_n \rightarrow f$ uniformly on E , then f is continuous.

Proof. Given $\varepsilon > 0$, choose N such that $d'(f_n(p), f(p)) < \varepsilon/3$ for all p and $n \geq N$. Continuity of f_N provides $\delta > 0$ so that $d(p, q) < \delta$ implies $d'(f_N(p), f_N(q)) < \varepsilon/3$. For $d(p, q) < \delta$,

$$d'(f(p), f(q)) \leq d'(f(p), f_N(p)) + d'(f_N(p), f_N(q)) + d'(f_N(q), f(q)) < \varepsilon,$$

so f is continuous. \square

Uniform convergence is naturally encoded by the sup metric on $C(K, E')$ discussed earlier: $f_n \rightarrow f$ uniformly if and only if $d_\infty(f_n, f) \rightarrow 0$. When K is compact and E' complete, $C(K, E')$ is itself complete under d_∞ .

Examples include Fourier-type partial sums converging uniformly on compact subsets, or piecewise linear approximations to continuous paths. This perspective will be revisited in the later chapters on series and approximations.

5 Differentiation

- 5.1 Definition of the Derivative**
- 5.2 Rules of Differentiation**
- 5.3 The Mean Value Theorem**
- 5.4 Taylor's Theorem**

6 Riemann Integration

- 6.1 Definition and Examples**
- 6.2 Linearity and Order Properties of the Integral**
- 6.3 Existence of the Integral**
- 6.4 The Fundamental Theorem of Calculus**
- 6.5 The Logarithmic and Exponential Functions**
- 6.6 Definition of Continuity. Examples**

7 Interchange of Limit Operations

- 7.1 Integration and Differentiation of Sequences of Functions**
- 7.2 Infinite Series**
- 7.3 Power Series**
- 7.4 The Trigonometric Functions**
- 7.5 Differentiation under the Integral Sign**

8 The Method of Successive Approximations

- 8.1 The Fixed Point Theorem**
- 8.2 The Simplest Case of the Implicit Function Theorem**
- 8.3 Existence and Uniqueness Theorems for Ordinary Differential Equations**

9 Partial Differentiation

- 9.1 Definitions and Basic Properties**
- 9.2 Higher Derivatives**
- 9.3 The Implicit Function Theorem**

10 Multiple Integrals

- 10.1 Riemann Integration on Closed Intervals of E^n . Examples and Basic Properties**
- 10.2 Existence of the Integral. Integration on Arbitrary**