DISTRIBUTIONS

1. X is a random variable that follows a continuous uniform distribution with probability density function

$$f(x) = \begin{cases} 1/12 & (5 \le x \le 17) \\ 0 & else\ where \end{cases}$$

then what is the mean of the distribution?

(a) 11 (b) 15

(c) 16 (d) 12

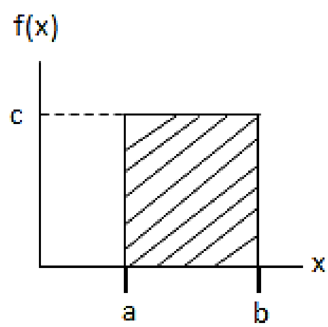
Solution:

Since x ranges from 5 to 17 has uniform probability density $1/_{12}$, the mean E(x) is

$$E(x) = \frac{17+5}{2}$$

$$= 11$$

2.



If x is uniformly distributed random variable that has probability density f(x) in the interval [5, 13]. What is the 'c' in the above diagram?

(a) $1/_{11}$ (b) $1/_{17}$

(c) 11 (d) $1/_{8}$

Solution:

Since the area of the shaded region is 1, the value of c can be obtained as follows:

$$c(b-a) = 1$$

$$c = \frac{1}{b-a}$$

$$= \frac{1}{13-5}$$

$$= \frac{1}{8}$$

3. Suppose that the weight of maple sugar obtained by boiling down a tank of maple-tree sap is uniformly distributed with mean of 10 pounds and a range of 1.8 pounds.

What is the probability that a tank of sap will boil down to between 9 and 10.5 pounds?

(a) 0.633

(b) 0.833

(c) 0.6777

(d) 0.8777

Solution:

The mean, 10 is the center point of a line segment whose length is the range 1.8 pounds.

Dividing the length of the interval by the range gives

$$P(9\ to\ 10.5) = \frac{10.5 - 9}{0.833} = 0.833$$

as the probability that a tank full will boil down to between 9 and 10.5 pounds of sugar.

4. Consider the same scenario as in question 3.

What is the probability that a tankful will boil down to more than 9.5 pounds of sugar?

(a) 0.877

(b) 0.778

(c) 0.633

(d) 0.833

Solution:

We need find

$P(x \geq 9.5)$

It is given that the mean is 10 and range is 1.8.

The line segment extends to $\left[\frac{1.8}{2}\right]$ $0.9\ pound$ to the left and right of 10.

∴ From 9.1 to 10.9 pounds

$\therefore\ P(x \geq 9.5) = \frac{10.9-9.5}{1.8}$ $\qquad\left[P(x \geq p) = \frac{b-p}{b-a}\right]$

$= 0.778$

5. The amount of times, in minutes, that a person must wait for a bus is uniformly distributed between) and 15 minutes inclusive.

What is the probability that a person waits fewer than 12.5 minutes?

(a) 0.633 (b) 0.833

(c) 0.6777 (d) 0.8777

Solution:

Let X = the number of minutes a person must wait for a bus

$$a = 0\ \ b = 15\ \ X \sim \cup\ (0, 15)$$

The probability density function,

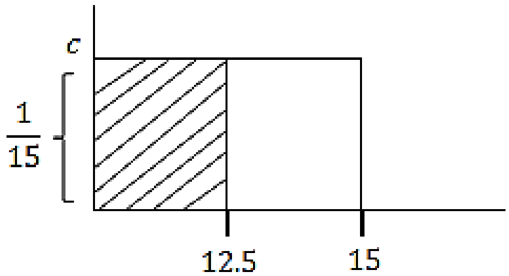
$$f(x) = \frac{1}{b-a}$$

$$= \frac{1}{15-0}$$

$$= \frac{1}{15}\,(0 \leq x \leq 15)$$

Find $P(X < 12.5)$

If you can draw a graph



$\therefore P(X < 12.5) = base \cdot height$

$= (12.5 - 0) \times \frac{1}{15}$

$= 0.8333$

6. Consider the same scenario as in question 5.

90% of the time, the person must wait atmost x minutes. The value of x is ________.

(a) 12.5 (b) 12

(c) 13 (d) 13.5

Solution:

It is given that

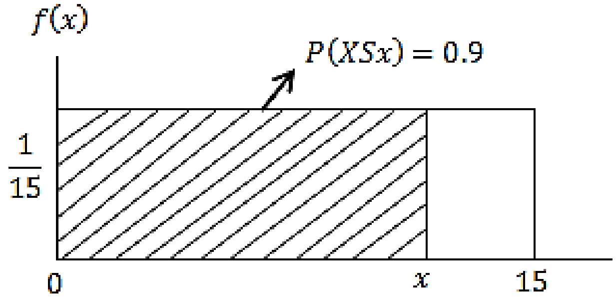
$P(X < x) = 90\%$

If you can draw a graph

$\Longrightarrow$ According to the given data, the probability density function

$$f(x) = \frac{1}{b-a} = \frac{1}{15-0} = \frac{1}{15}$$

The graph looks like this



$\therefore Area\ of\ shaded\ region = 0.9$

$(base) \times height = 0.9$

$(x - 0) \times \dfrac{1}{15} = 0.9$

$x = 0.9 \times 15$

$\boxed{x = 13.5\ minutes}$

7. The average number of donuts a nine-year old child eats per month is uniformly distributed from 0.5 to 4 donuts, inclusive. Let $X =$ the average number of donuts a nine year old child eats per month. Then $X \sim \cup\ (0.5, 4)$

The probability that a different nine year old child eats an average of more than two donuts given that his or her amount is more than 1.5 donuts is ________.

Solution:

Given that $a = 0.5 \quad b = 4$

$\therefore$ The probability density function

$$f(x) = \frac{1}{b-a} = \frac{1}{4-0.5} = \frac{1}{3.5}$$

We need to find

$$P(X > 2/x > 1.5) = \frac{P(X > 2\ \cap x > 1.5}{P(x > 1.5)}$$

$$= \frac{P(X>2)}{P(X>1.5)}$$

$$= \frac{(4-2)\times\frac{1}{3.5}}{(4-1.5)\times\frac{1}{1.5}}$$

$$= \frac{2}{2.5}$$

$$= 0.8$$

$$= \frac{4}{5}$$

8. A full moon occurs every 29 nights. If you choose a night at random and then observe the night sky for 11 successive nights, what is the probability you will see a full moon?

(a) 0.03

(b) 0.11

(c) 0.38

(d) 0.33

Solution:

Given that

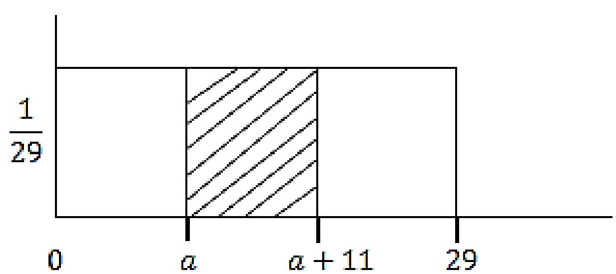
$a = 0$

$b = 29$

The probability density function,

$$f(x) = \frac{1}{29-0} = \frac{1}{29}$$

$\therefore$ *We can draw a graph like*



We need to find

$P(a < X < a + 11) = \text{area of shaded region}$

$$= (a + 11 - a) \times \frac{1}{29}$$

$= \frac{11}{29}$

$= 0.379$

$= 0.38$

9. A arrives at office at 8-10am regularly; B arrives at 9-11 am every day. Probability that one day B arrives before A? [Assume arrival time of both A and B are uniformly distributed]

(a) $^4/_5$ (b) $^7/_8$

(c) $^1/_8$ (d) $^1/_5$

Solution:

v_1 *arrival time of A*

v_2 *arrival time of B*

$$P(v_1 > v_2) = P(u_1 > u_2 | u_1 > 9, u_2 < 10) \times P(u_1 > 9, u_2 < 10)$$

⋆ Uniform conditioned to be in an interval is uniform. v_1, v_2 are independent, so conditioning of one has no effect on the other.

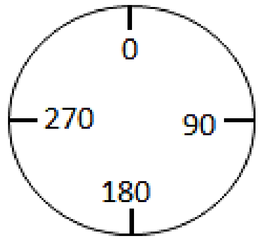
$$\therefore P(v_1 > v_2) = P(u_1 > u_2)\, P(u_1 > 9)\, P(u_2 < 10)$$

$$= \frac{1}{2} \times \frac{10-9}{10-8} \times \frac{11-10}{11-9}$$

$$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$$

$$= \frac{1}{8}$$

10. Suppose that you spin the dial shown in the figure so that it comes to reset at random position, the probability that the dial will land somewhere between 5° and 300°.



(a) 0.877 (b) 0.8194

(c) 0.788 (d) 0.7194

Solution:

Let X be the angle at which the pointer comes to reset.

∴ The interval [0, 360]

∴ The probability density function $= \frac{1}{b-a} = \frac{1}{360-0} = \frac{1}{360}$

$\therefore P(5 < X < 300) = \int_5^{300} f(x)dx = \frac{1}{360}\int_5^{300} dx = \frac{1}{360}[x]_5^{300} = \frac{1}{360}[300-5] = \frac{295}{360} = 0.8194$

11. Which of the following is true for normal distribution?

[Choose most appropriate option]

i) They are always symmetric

ii) They are never fat-failed

iii) They always have a mean of 0

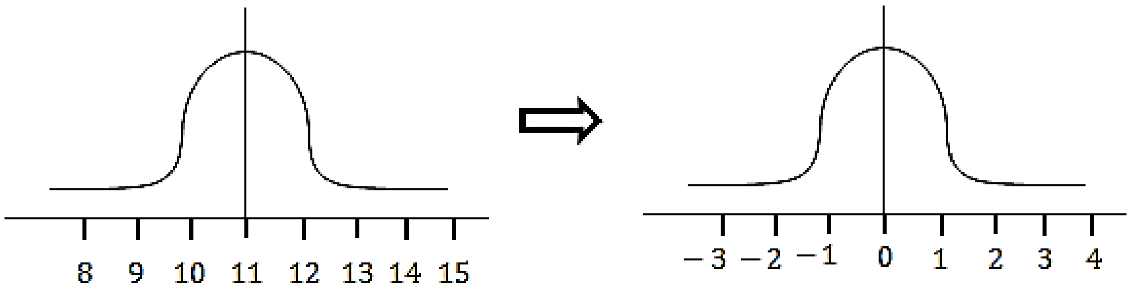
(a) i, ii (b) i, ii, iii

(c) i, iii (d) i

Solution:

⟹ the first two statements are properties of normal distribution

⟹ the normal distribution is no need to have a mean of ‘0’



"Normal Distribution" "Standard Normal Distribution"

So, option 1 is correct.

12. If X is a normally distributed variable with mean $\mu = 11$ and standard deviation $\sigma, S,$ what is the probability$P(X > 21)$.

<u>Note</u>: Use the standard normal distribution table below, where z has $\mu = 0$ and $\sigma = 1$.

z	$P(0 \leq Z \leq z)$
1	0.3413
1.5	0.4332
2.0	0.4772
2.5	0.4938

(a) 0.143

(b) 0.341

(c) 6.68 ∈(–2)

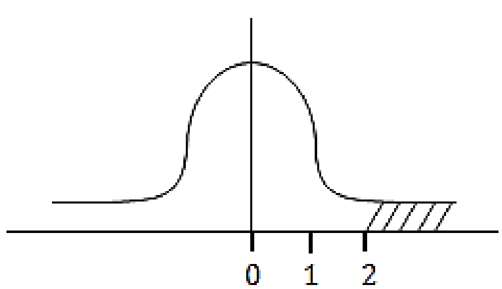
(d) 228 ∈(–2)

Solution:

The Z-value of X = 21

$$Z = \frac{21 - \mu}{\sigma} = \frac{21 - 11}{5} = 2$$

$P(X > 21) = P(Z > 2)$



$\because$ The area under that curve is 1

The area of shaded region $= 0.5 - P(0 \le Z \le 2) = 0.0228$

13. A large group of students took a math test, and their scores obey a normal distribution if the distribution has mean 62 and standard deviation 10, what is the % of these students who scored higher than 72?

<u>Note</u>: $\mu = 0 \quad \sigma = 1$

z	$P(0 \le 2 \le z)$
1	0.3414
1.5	0.4332
2.0	0.4772
2.5	0.4938

(a) 10.34%

(b) 19.14%

(c) 12.60%

(d) 15.87%

Solution:

Let X be the probability distribution of the score

Given $\mu = 62 \quad \sigma = 10$

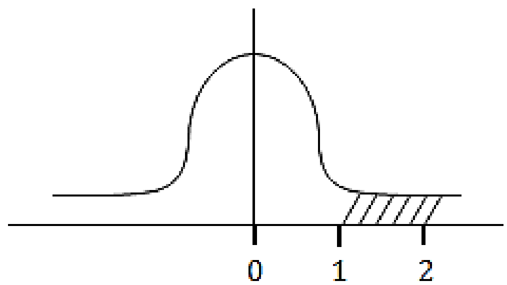
Z-value of X = 72 is

$$Z = \frac{72 - \mu}{\sigma}$$

$$= \frac{72-62}{10}$$

$$= 1$$

$\therefore P(X \ge 72) = P(Z \ge 1)$



$= 0.5 - P(0 \leq 2 \leq 1) = 0.1587$

15. An examination is often regarded as being good (in the sense of determining a valid grade spread for those taking it). If the test scores of those taking the examination can be approximate by a normal density function. The instruction often uses the test scores to estimate the normal parameters μ and σ^2 and then assigns the letter grade A to those whose test score is greater than $\mu + \sigma$, B to those whose score is between μ and $\mu + \sigma$, C to those whose score is between $\mu - \sigma$ and μ , D to those score is between $\mu - 2\sigma$ and $\mu - \sigma$ and F to those getting a score below $\mu - 2\sigma$

i) What % of the class will receive $\frac{\frac{\frac{\frac{A}{B}}{C}}{D}}{F}$ grade ________.

Solution:

a) To find the number of people who gets A

$P(X > \mu + \sigma) = P(X - \mu > \sigma)$

$= P\left(\frac{X-\mu}{\sigma} > 1\right)$

$= P((z > 1)$

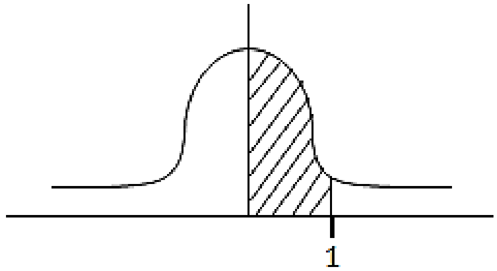
$= 1 - P(Z \leq 1)$

$= 1 - 0.8413$

$= 0.1587$

∴ 16% of the class will receive an A grade.

b) the % of the class who will get a B grade



$\Longrightarrow P(\mu < X < \mu + \sigma)$

$= P(0 < X - \mu < \sigma)$

$= P\left(0 < \dfrac{X - \mu}{\sigma} < 1\right)$

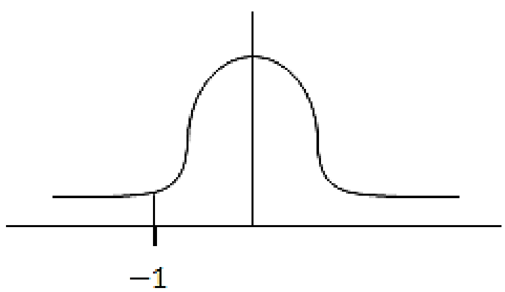
$= P(0 < 2 < 1)$

$= P(Z = 1) - P(z = 0)$

$= 0.8413 - 0.5000$

$= 0.3413$

c) the % of the class who will get a C grade



$\Longrightarrow P(\mu - \sigma < X < \mu)$

$= P(-\sigma < X - \mu < 0)$

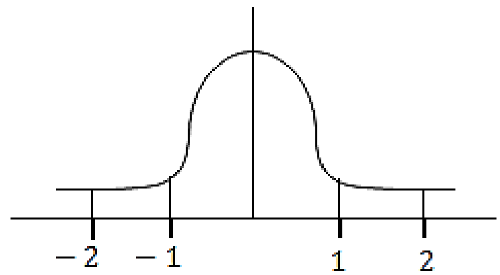
$= P\left(-1 < \dfrac{X - \mu}{\sigma} < 0\right)$

$= P(-1 < Z < 0)$

$= P(Z = 0) - P(Z = -1)$

$= 0.3413$

d) the % of the class who will get a D grade



$P(\mu - 2\sigma < X < \mu - \sigma) = P(-2\sigma < X - \mu < -\sigma)$

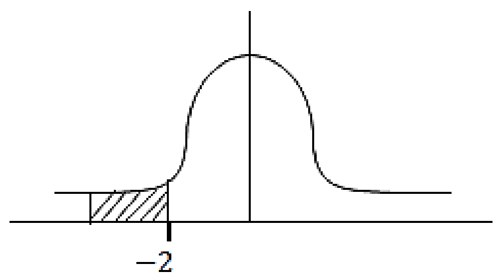
$= P\left(-2 < \frac{X-\mu}{\sigma} < -1\right)$

$= P(-2 < Z < -1)$

$= P(Z = 2) - P(Z = 1)$

$= 0.1359$

e) the % of the class who will get a F grade



$P(X < \mu - 2\sigma) = P(X - \mu < -2\sigma)$

$= P\left(\frac{X-\mu}{\sigma} < -2\right)$

$= P(Z < -2)$

$= 1 - P(X \leq 2)$

$= 0.0228$

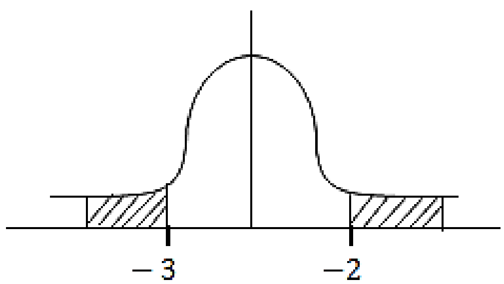
16. An expert witness in a paternity suit testifies that the length (in days) of distributed with parameters $\mu = 270$ and $\sigma^2 = 100$. The defendant in the suit is able to prove that he was out of the country during a period that began 290 days before the birth of the child and ended 240 days before birth.

If the defendant was in fact, the father of the child, what is the probability that the mother could have had the very long or very short pregnancy indicated by the testimony?

Solution:

Let X denotes the length of the pregnancy.

Let's assume that the defendant is the father then the probability that the birth could occur within the indicated period is



$= P(X > 290 \; or \; X < 240)$

$= P(X > 290) + P(X < 240)$

$= P(X - 270 > 20) + P(X - 270 < -30)$

$= P\left(\frac{X - 270}{10} > 2\right) + P\left(\frac{X - 270}{10} < -3\right)$

$= P(Z > 2) + P(Z < -3)$

$= 1 - P(Z \leq 2) + 1 - P(2 \leq 3)$

$= 0.0241$

17. The width of the slot of a duralumin is (in inches) normally distributed with $\mu = 0.900$ and $\sigma = 0.0030$. The specification limits were given as 9000 ± 0.0050.

a) What % of forgings will be defective?

we need to find

$$\Longrightarrow P(0.9 - 0.005 < X < 0.9 + 0.005)$$

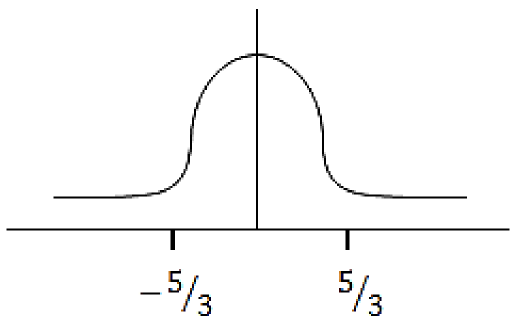
$$= P(-0.005 < X - 0.9 < 0.005)$$

$$= P\left(\frac{-0.005}{0.003} < \frac{X - 0.9}{0.003} < \frac{0.005}{0.003}\right)$$

$$= P\left(-\frac{5}{3} < 2 < \frac{5}{3}\right)$$

$$= 2\phi\left(\frac{5}{3}\right) - 1$$

$$= 0.9050$$



∴ Hence 95% will be defective.

b) What is the maximum allowable value of σ that will permit no more than 1 in 100 defectives when the widths are normally distributed with $\mu = 0.900$ and σ ?

Solution:

We need to find

$$= P(0.9 - 0.005 < X < 0.9 + 0.005)$$

$$= P(-0.005 < X - 0.9 < 0.005)$$

$$= P\left(-\frac{0.005}{\sigma} < \frac{X - 0.9}{\sigma} < \frac{0.005}{\sigma}\right)$$

$$= P\left(-\frac{0.005}{\sigma} < 2 < \frac{0.005}{\sigma}\right)$$

$$= 2\phi\left(-\frac{0.005}{\sigma}\right) - 1 = 0.99$$

$$\phi\left(\frac{0.005}{\sigma}\right) = \frac{1.99}{2}$$

$$\phi\left(\frac{0.005}{\sigma}\right) = 0.995$$

$$\therefore \ \frac{0.005}{\sigma} = 2.575$$

$$\sigma = 0.0019$$

18. The probability is 0.80 that a person of age 20 years will be alive at age 65 years. Suppose that 500 people of age 20 are selected at random, the probability that exactly 390 of them will be alive at 65.

Solution:

we have n = 500 and P = 0.8 therefore

$$nP = 500 \times 0.8 = 400$$

$$n(1 - P) = 500 \times 0.2 = 100$$

we get

$$\mu = nP = 400$$

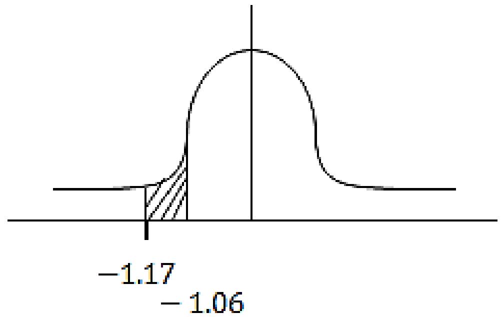
$$\sigma = \sqrt{nP(1 - P)} = \sqrt{500 \times 0.8 \times 0.2} = 8.94$$

To make the correction for continuity, we subtract 0.5 from 390 and 0.5 to 390.

Thus we need to find the area under the normal curve with parameters $\mu = 400$ and $\sigma = 8.94$ that lies between 389.5 and 390.5.

$$Z_1 = \frac{389.5 - 400}{8.94} \approx -1.17$$

$$Z_2 = \frac{390.5 - 400}{8.94} \approx -1.06$$



The Probability $= \phi(1.17) - \phi(1.06) = 0.0235$

∴ with the probability 2.36% that exactly 390 of the 500 people selected will be alive at age 65.

b) the probability that between 375 and 425 of them, inclusive will be alive at age 65 ________.

Solution:

To make the correction for continuity, we subtract 0.5 from 375 and 0.5 to 425.

Thus, we need to determine the area under the normal curve with parameters

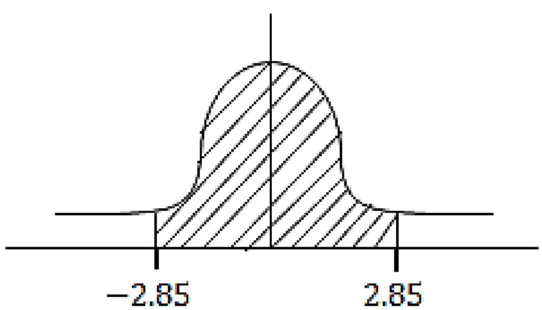
$\mu = 400 \quad \sigma = 8.94$

that lies between 374.5 and 425.5.

We convert to 2-scores

$$Z_1 = \frac{374.5 - 400}{8.94} \approx -2.85$$

$$Z_2 = \frac{425.5 - 400}{8.94} \approx 2.85$$



∴ The area under the curve is $2 \times 0.4978 = 0.9956$

So, $P(375 \leq X \leq 425) = 0.9956$ approximately

19. Suppose that the length of a phone call in minutes is an exponential random variable with parameter $\lambda = \frac{1}{10}$ if someone arrives immediately ahead of you at a public telephone booth, find the probability that you will have to wait between 10 and 20 minutes _________.

Solution:

Letting X denote the length of the call made by the person in the booth

we have that the desired probability is

$P(10 < X < 20) = F(20) - F(10)$

$\because$ X is continuous random variable,

the probability density function is given as

$$f(X) = \begin{cases} \lambda e^{-\lambda x} & if\ X \geq 0 \\ 0 & if\ X < 0 \end{cases}$$

Given $\lambda = \frac{1}{10}$

$$\begin{aligned} F(X = x) &= P\{X \leq x\} \\ &= \int_0^a \lambda e^{-\lambda}\, dx \\ &= 1 - e^{-\lambda x} \end{aligned}$$

$$\begin{aligned} \therefore F(20) - F(10) &= \left(1 - e^{-\frac{1}{20} \times 20}\right) - \left(1 - e^{-\frac{1}{10} \times 10}\right) \\ &= 1 - e^{-2} - 1 + e^{-1} \\ &= e^{-1} - e^{-2} \\ &\approx 0.233 \end{aligned}$$

20. Consider a post office that is staffed by two clerks. Suppose that when Miss. Grangar enters the system, she discovers that Mr. Harry is being served by one of the clerks and Mr. Ron by other.

Suppose also that Miss Grangar is told that her service will begin as soon as either Harry or Ron leaves.

If the amount of time that clerk spends with a customer is exponentially distributed with parameter λ, what is the probability that, of the three customers, Miss. Grangar is the last to leave the post office?

Solution:

⟶ Consider the time at which Miss. Grangar first finds a free clerk.

⟶ At this point either Mr. Harry or Mr. Ron would have just left and the other on would still be in service.

⟹ by the lack of memory of exponential if follows that the additional amount of time that the other person would still have to spend in the post office is exponentially distributed with parameter λ. That is, it is the same as if service for this person were just starting at this point.

Hence, by symmetry, the probability that the remaining person finishes before Miss. Grangar must be equal to $\frac{1}{2}$.