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Euclidean wormholes with minimally coupled scalar fields

Soumendranath Ruz[1], **Subhra Debnath**[2], **Abhik Kumar Sanyal**[2] **and Bijan Modak**[1]

[1] Department of Physics, University of Kalyani, West Bengal-741235, India
[2] Department of Physics, Jangipur College, Murshidabad, West Bengal-742213, India

E-mail: ruzfromju@gmail.com, subhra_dbnth@yahoo.com, sanyal_ak@yahoo.com and bijanmodak@yahoo.co.in

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Abstract

A detailed study of quantum and semiclassical Euclidean wormholes for Einstein's theory with a minimally coupled scalar field has been performed for a class of potentials. Massless, constant, massive (quadratic in the scalar field) and inverse (linear) potentials admit the Hawking and Page wormhole boundary condition both in the classically forbidden and allowed regions. An inverse quartic potential has been found to exhibit a semiclassical wormhole configuration. Classical wormholes under a suitable back-reaction leading to a finite radius of the throat, where the strong energy condition is satisfied, have been found for the zero, constant, quadratic and exponential potentials. Treating such classical Euclidean wormholes as an initial condition, a late stage of cosmological evolution has been found to remain unaltered from standard Friedmann cosmology, except for the constant potential which under the back-reaction produces a term like a negative cosmological constant.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Lot of research was initiated in wormhole physics only after Giddings and Strominger [1] for the first time presented a wormhole solution taking into account a third-rank antisymmetric tensor field as the matter source, and the rest of the world came to know that gravitational instantons could really exist [2]. Wormholes, by then, were considered as gravitational instantons, which are the saddle points of the Euclidean path integrals and as such are the solutions of the Euclidean field equations. Thereafter, research in wormhole physics was oriented (apart from traversable and non-traversable Lorentzian wormholes [3]) by and large in two directions. On the one hand, people were motivated to find wormhole solutions for different types of matter fields [4–7] and on the other, the physical consequences of wormhole dynamics were studied extensively [8–15]. In a nutshell, the outcome of all these works is as follows. First, it was learnt that quantum coherence is not really lost by the fact that wormholes connect two asymptotically flat or de-Sitter regions by a throat of radius of the order of Planck length. Next, microscopic wormholes might provide us with the mechanism that would solve the cosmological constant problem, while macroscopic wormholes might be responsible for the final stage of evaporation and complete disappearance of black hole. While these compelling results demand further investigation, the final outcome is disastrous, which is, not all types of matter fields admit wormhole solutions. If wormholes are considered seriously to be responsible for regularizing important physical parameters as stated above, then not only all types of matter fields but also pure gravity should admit wormhole solutions. This led Hawking and Page [16] to interpret wormholes in a different manner. They proposed that instead of considering wormholes as solutions to the classical field equations, they should be treated as the solutions of the Wheeler–DeWitt (W-D) equation under the boundary conditions that the wavefunctional ψ should be exponentially damped for large 3-geometry and it should be regular in some suitable way when the 3-geometry degenerates. The boundary condition mentioned above might not hold for the whole super-space; however, if it is satisfied in minisuperspace models, then wormholes are supposed to exist. Let us try to understand in brief how our usual conception on wormholes tallies with such a boundary condition. The W-D equation is independent of the lapse function and as such holds for both the Euclidean and Lorentzian geometry. Solutions to the W-D equation are obtained both for classically allowed and forbidden regions depending on the signature of the potential. The solutions in the classically allowed regions are oscillatory and the corresponding states are not normalizable since the motion of the gravitational field is unbounded. For the Robertson–Walker (R-W) minisuperspace model, these solutions represent Friedmann solutions with unavoidable singularities. The solutions in the classically forbidden region are exponential in nature and represent Euclidean solutions. These solutions corresponding to the classically forbidden region might represent wormholes if the wavefunctional, as already stated, is damped exponentially for large 3-geometry ensuring asymptotic flat regions which represents Euclidean space. Further, if it is regular as the 3-geometry degenerates, it ensures a non-evolving throat of radius of the order of Planck length, instead of a singularity. Hence, it is clear that the proposal of Hawking and Page [16] is perfectly in tune with the preoccupied conception of wormholes.

Along with their proposal, Hawking and Page [16] also presented a couple of such wormhole solutions corresponding to the massless and massive scalar fields. For the massless case, under suitable transformation of variables, they [16] obtained a solution to the W-D equation as the product of two noninteracting harmonic oscillator wavefunctions with opposite energies that satisfies the boundary condition. The authors [16], however, did not indicate how much separation is required between the two harmonic oscillators to make such an approximation to the W-D equation. For the massive case, their method of finding a wormhole solution is so cumbersome that it really does not make much appeal. Further, they have not indicated how to find wormhole solutions for pure gravity. Later, Garay [17] had explored the wormhole wavefunctional for a conformally coupled massless scalar field in the path integral method. Thereafter, Coule [18], without going for the solutions of the W-D equation, had shown that wormhole solutions might exist for a nonminimally coupled scalar field, just by studying the form of the potential. However, the main issue has not yet been settled, i.e., whether all forms of matter fields really admit a wormhole solution, and a complete study in this regard does not exist in the literature. In this context, the motivation of this paper is to take into account the Einstein–Hilbert action with a minimally coupled scalar field and to explore the quantum and semiclassical wormhole solutions for different forms of the potentials, in the background of the R-W minisuperspace model. It is important to mention that the radii of the throats of the wormholes which exist under the semiclassical approximation are post-Planckian. It has been observed that wormhole boundary conditions [16] are satisfied in the semiclassical limit only for a limited form of potentials and for a limited class of operator ordering indices admitting the back-reaction. Although it appears unrealistic, Kim [19] had suggested that if the wavefunctional behaved well for some operator ordering indices although divergence appears for different choice, it should still be considered to satisfy the Hawking–Page boundary condition for wormholes.

In the following section, we write the action for minimally coupled scalar theory of gravity and discuss our primary motivation in connection with some essential aspects of classical wormhole solutions which lead us to go for the semiclassical approximation of the W-D equation. In section 3, we make the WKB approximation to the W-D equation by expanding the phase of the wavefunctional in the power series of the gravitational constant G, or equivalently, ${m_p}^2$, where m_p is the reduced Planck's mass [20, 21], instead of $\hbar$. In the process, one obtains the Hamilton–Jacobi (H-J) equation for the source free gravity to the leading order of approximation. The H-J equation in turn reduces to the classical vacuum Einstein's equation under a suitable choice of the time parameter. To the next order of approximation, one obtains the Tomonaga–Schwinger equation, which is essentially the functional Schrödinger equation for the matter field propagation in the background of classical curved space. The underlying motivation is that, with this type of approximation, the dynamics of the matter field is determined by the quantum field theory in curved spacetime and when this equation is combined with the vacuum Einstein's equation, a possible back-reaction might arise. If one chooses the Hartle–Hawking type of wavefunction [22] for pure gravity, it is not difficult to see that the wavefunctional exponentially decays for large 3-geometry. Further, as the 3-geometry degenerates, the divergence due to the presence of the Van-Vleck determinant can be regularized if the back-reaction exists. In section 4, we take a massless scalar field and a constant potential. In section 5, we take up power law potentials in the form of the massive scalar field ($V_0\phi^2$), quartic potential ($\lambda\phi^4$) and inverse potentials ($\frac{V_0}{\phi^n}$, $n = 1, 3, 4$). In section 6, we take up an exponential potential. In all the cases, we first try to find quantum wormhole solutions by solving the W-D equation directly and then we explore the semiclasssical wormhole solution following the technique mentioned above. Next, we present the classical wormhole equation and calculate the throat from the back-reaction in the cases for which the semiclassical wavefunction admits the back-reaction. Finally, we check if a viable classical cosmological evolution is admissible taking into account such a back-reaction as an initial condition.

2. Action, motivation and the Wheeler–DeWitt equation

The gravitational action with a minimally coupled scalar field is

$$S_c = \int d^4x\sqrt{-g}\left[\frac{R}{16\pi G} - \frac{1}{2\pi^2}\left(\frac{1}{2}\phi_{,\alpha}\phi^{,\alpha} + V(\phi)\right)\right] + \frac{1}{8\pi G}\int_{\sum} d^3x\sqrt{h}K. \tag{1}$$

Here, $V(\phi)$ is an arbitrary potential and the surface term includes the determinant of the induced metric h_{ij} along with the trace of the extrinsic curvature K. The trace energy tensor for the matter field under consideration is given by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_\phi}{\delta g^{\mu\nu}} = \phi_{,\mu}\phi_{,\nu} - g_{\mu\nu}\left[\frac{1}{2}\phi_{,\alpha}\phi^{,\alpha} + V(\phi)\right], \tag{2}$$

with S_ϕ being the action for the matter field. The field equation for such an action may be computed as

$$R_{\mu\nu} = 8\pi G[\phi_{,\mu}\phi_{,\nu} + g_{\mu\nu}V(\phi)]. \tag{3}$$

It has been conjectured [23] that the necessary (but not the sufficient) condition for a classical wormhole to exist is that the eigenvalues of the Ricci tensor must be negative somewhere on the manifold. This conjecture ensures a real throat and so it is a necessary condition but since it does not ensure an asymptotic flat or de-Sitter space, therefore, it is not sufficient. Now, since $R_{\mu\nu}$ does not have a negative eigenvalue for $V(\phi) > 0$, so clearly a wormhole does not exist for a real scalar in general in the Euclidean regime. For this reason, Hawking and Page [16] conjectured the wormhole as a solution to the W-D equation under an appropriate boundary condition, mentioned in the introduction. Such a boundary condition has been found to be satisfied for the real massless and massive scalar fields [16]. We shall review these cases to show that the negative eigenvalue of $R_{\mu\nu}$ exists under the back-reaction following a semiclassical approximation. In the following we elaborate this fact. In the R-W minisuperspace model,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\ \mathrm{d}\phi^2)\right]. \tag{4}$$

Action (1) reduces to

$$S_c = M\int\left[-\frac{1}{2}a\dot{a}^2 + \frac{ka}{2} + \frac{1}{M}\left(\frac{1}{2}\dot{\phi}^2 - V(\phi)\right)a^3\right]\mathrm{d}t. \tag{5}$$

Here, $M = \frac{3\pi}{2G} = \frac{3\pi m_p^2}{2}$, with m_p being the reduced Planck mass and the curvature parameter $k = 0, \pm1$, which stands for the flat, closed and open models, respectively. The field equations are

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{2}{M}\rho_\phi = \frac{2}{M}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right]. \tag{6}$$

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + V_{,\phi} = 0. \tag{7}$$

Note that the other equation, namely (1 1) = (2 2) = (3 3) is not relevant since it is not an independent one. Under Wick rotation ($t = \mathrm{i}\tau$), the field equations (6) and (7) take the form

$$\frac{a_{,\tau}^2}{a^2} = \frac{k}{a^2} + \frac{2}{M}\left[\frac{1}{2}\phi_{,\tau}^2 - V(\phi)\right] \tag{8}$$

$$\phi_{,\tau\tau} + 3\frac{a_{,\tau}}{a}\phi_{,\tau} = V_{,\phi}. \tag{9}$$

Classical Euclidean wormholes, as mentioned, require two asymptotically flat spaces connected by a non-evolving throat, where $a_{,\tau} = 0$ and is typically described by the form

$$a_{,\tau}^2 = 1 - \frac{l^2}{a^n}, \quad n > 0, \tag{10}$$

where l and n are constants. This form guarantees a real throat together with the condition $a_{,\tau}^2 > 0$ asymptotically, ensuring an asymptotic Euclidean (flat) space. For a better understanding, let us take $n = 2$ to obtain a solution in the form

$$a^2 = l^2 + (\tau - \tau_0)^2. \tag{11}$$

It is now clear that at $\tau = \tau_0$, a real throat $a_0 = l$ is found, while as $\tau \to \pm\infty$, $a \to \infty$ and the wormhole boundary condition is satisfied. Cotsakis *et al* [24] had chosen an ansatz in the form $\phi_{,\tau} = \frac{l}{a^n}$ and found a wormhole solution for the real scalar field in the Euclidean domain which appeared due to some errors in sign being pointed out later by Coule [25]. As a result, (8) and (9) admit the above form provided the scalar field is imaginary in the Euclidean regime, i.e., letting $\phi \to \mathrm{i}\phi$, or $l \to \mathrm{i}l$ which are well known. Instead, a real scalar field (in the Lorentzian regime) being described by the following ansatz,

$$\dot{\phi} = \frac{l}{a^q}, \tag{12}$$

where q is a constant, is more natural since similar form is found as the Noether conserved current $I = l = \dot{\phi} a^3$ $(q = 3)$ for a cyclic scalar field ϕ (Note that under the assumption $\phi_{,\tau} = \frac{l}{a^q}$, one obtains $i\dot{\phi} a^3 =$ conserved, which is unphysical). In view of the ansatz (12), equation (7) or (9) may then be solved to obtain the form of the potential as

$$V = \frac{l^2(3-q)}{2qa^{2q}}, \tag{13}$$

which restricts the potential to be positive definite for $q \leqslant 3$. In view of the above form of the potential (13), equation (8) may be expressed as

$$a_{,\tau}^2 = k - \frac{3l^2}{Mqa^{2(q-1)}}. \tag{14}$$

It is apparent that equation (14) is in the form of equation (10) for $k = 1$ and so the necessary condition for the wormhole to exist is satisfied. To understand the importance of this equation, let us consider pure radiation instead of the scalar field. The Friedmann equation (6) then reads

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8\pi G}{3}\frac{\rho_{r0}}{a^4}, \tag{15}$$

with ρ_{r0} being a constant. Under Wick rotation $(t = i\tau)$, the above equation reads

$$a_{,\tau}^2 = 1 - \frac{8\pi G}{3}\frac{\rho_{r0}}{a^2} \tag{16}$$

setting $k = +1$. Equation (16) is clearly in the same form as (14) under the choice $q = 2$. For pressureless dust again, the Friedmann equation is

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8\pi G}{3}\frac{\rho_{m0}}{a^3}, \tag{17}$$

with ρ_{m0} being a constant. Under Wick rotation $(t = i\tau)$, the above equation reads

$$a_{,\tau}^2 = 1 - \frac{8\pi G}{3}\frac{\rho_{m0}}{a} \tag{18}$$

setting $k = +1$ and the equation is again in the same form as (14) under the choice $q = 3/2$. Therefore, both radiation and pressureless dust admit the Euclidean wormhole boundary condition. Nevertheless, instead of radiation or pressureless dust, the early universe is vacuum dominated and so it is required to consider higher order curvature invariant terms or at least a scalar field. For a scalar field, which is our present consideration, the ansatz (12) however under Wick rotation becomes $\phi_{,\tau} = i\frac{l}{a^q}$ and so the scalar field ϕ again becomes imaginary in the Euclidean domain. But then, we are not going to choose such an ansatz (12), rather we show that a more general form of the ansatz (12) is obtainable under a semiclassical approximation as a back-reaction phenomenon, if the Hawking–Page wormhole boundary condition is satisfied. More clearly, one can calculate $\langle\rho_\phi\rangle$ as a function of the scale factor (a) under the back-reaction appearing in the semiclassical approximation of the W-D equation. In the process, for the real scalar field it is possible to obtain an equation having a more general form than equation (14) without making a Wick rotation to the scalar field. Note that the only scalar field which exists in the form of a Higgs boson has recently been observed at LHC in the ATLAS [26] and CMS [27] detectors applying $\sqrt{s} = 8$ TeV, which probes 10^{-18} cm. Thus, the energy scale of the Higgs particle is much smaller than the scale of gravity and so it does not affect the geometry of the spacetime. Hence, if a scalar is assumed to exist in the Planckian epoch, then we show that it might only leave its trace on the cosmological evolution through the back-reaction. With this motivation, we proceed to make a semiclassical approximation of the W-D equation in the following section. The Hamilton constraint equation for the system under consideration is

$$-\frac{1}{2M}\frac{P_a^2}{a} + \frac{P_\phi^2}{2a^3} - \frac{M}{2}ka + a^3V(\phi) = 0, \tag{19}$$

where P_a and P_ϕ are the corresponding momenta that canonically conjugate to a and ϕ and so the W-D equation reads

$$\left[\frac{\hbar^2}{2M}\left(\frac{\partial^2}{\partial a^2}+\frac{p}{a}\frac{\partial}{\partial a}\right)-\frac{M}{2}ka^2-\frac{\hbar^2}{2a^2}\frac{\partial^2}{\partial\phi^2}+a^4V(\phi)\right]|\Psi\rangle=0, \tag{20}$$

where p removes some of the operator ordering ambiguities. Let us recall that the W-D equation is independent of the lapse function and therefore its form is the same both in the Lorentzian and Euclidean spacetime. In the following section, we shall turn our attention to the semiclassical approximation of the W-D equation (20).

3. Semiclassical approximation

For the semiclassical approximation of the W-D equation (20), we choose $\Psi(a,\phi) = \exp[\frac{\mathrm{i}}{\hbar}S(a,\phi)]$ and substitute it along with its derivatives in the W-D equation (20) to obtain

$$\left[\frac{\mathrm{i}\hbar}{2M}S_{,aa}-\frac{1}{2M}S_{,a}^2+\frac{\mathrm{i}\hbar}{2M}p\frac{S_{,a}}{a}-\frac{M}{2}ka^2-\frac{\mathrm{i}\hbar}{2a^2}S_{,\phi\phi}+\frac{1}{2a^2}S_{,\phi}^2+a^4V\right]\psi=0. \tag{21}$$

Now, let us expand the functional $S(a,\phi)$ in the power series of M^{-1} (instead of $\hbar$) as, $S = MS_0 + S_1 + M^{-1}S_2\cdots$ etc and after substituting it (mentally) in the above equation (21), equate the coefficients of different orders of M to zero. To the highest, namely, M^2 order, we obtain

$$\frac{\partial S_0}{\partial\phi}=0, \tag{22}$$

which implies that S_0 is purely a functional of the gravitational field i.e., $S_0 = S_0(a)$. The next, i.e., the M^1 order term, gives the following source free Einstein–Hamilton–Jacobi (EHJ) equation:

$$\left(\frac{\partial S_0}{\partial a}\right)^2+ka^2=0. \tag{23}$$

If one now identifies the derivative of the phase S_0 with classical momenta as $M\frac{\partial S_0}{\partial a} = P_a = -Ma\dot{a}$, then the EHJ equation (23) reduces to the vacuum Einstein's equation, namely

$$\dot{a}^2+k=0, \tag{24}$$

or equivalently,

$$\frac{1}{2M}P_a^2+\frac{M}{2}ka^2=0, \tag{25}$$

under the following choice of the time parameter,

$$\frac{\partial}{\partial t}=-\frac{1}{a}\frac{\partial S_0}{\partial a}\frac{\partial}{\partial a}. \tag{26}$$

At this stage, we note that the curvature parameter $k = 0$ leads to a static model and so we leave it. Next, since neither equation (23) nor (24) admits a real solution of S_0 for $k = +1$, therefore, apparently it is required to switch over to the Euclidean time following the transformation $t = \mathrm{i}\tau$. In the process, equation (24) reduces to $a_\tau^2 - k = 0$ and so a real solution now exists for $k = +1$ in Euclidean section. If one now requires to obtain the time parameter (26) from the action principle, then one has to rotate the EHJ function S_0 to the Euclidean plane following the transformation $S_0 = \mathrm{i}I_{E_0}$, where I_{E_0} is the Euclidean H-J functional. Thus, equation (23) reduces to

$$\left(\frac{\partial I_{E_0}}{\partial a}\right)^2-ka^2=0, \tag{27}$$

and as mentioned, real solutions are now admissible for $k = +1$. Finally, if there exists a lower limit a_0 to a, then the analytic structure of the Coleman–Hawking wormhole arises quite naturally. Thus, we observe that the problem associated with the type of semiclassical approximation under consideration for a closed ($k = +1$) FRW model is resolved if one invokes a wormhole configuration. It now remains to be shown how this finite resolution limit of the order of Planck's length in the scale factor a appears. The validity of the WKB approximation under consideration requires $|\frac{d\lambda_a}{da}| \ll 1$, where λ_a is the de-Broglie wavelength for pure gravity. Identifying the derivative of the phase factor S_0 with the canonical momenta, the above statement reduces to $|\frac{d}{da}(\frac{\hbar}{MS_{0,a}})| \ll 1$. In view of equation (23), it implies that $a \gg \sqrt{\hbar/M}$ or $a \gg 7.45 \times 10^{-34}$ cm. The validity of the WKB approximation under consideration therefore requires any Planckian or post-Planckian value of the scale factor and so both the microscopic and macroscopic wormholes will possibly exist. Hence, this method of semiclassical approximation is well posed to explain both the problems of vanishing of the cosmological constant and the final stage of evaporation and complete disappearance of a black hole. However, these are not our present concerns, rather we shall also find that the open Friedmann model ($k = -1$) does not satisfy the Hawking–Page wormhole boundary condition. Now, the next (M^0) order of approximation yields

$$\frac{i\hbar}{2}\left[S_{0,aa} + \frac{p}{a}S_{0,a}\right] - S_{0,a}S_{1,a} - \frac{i\hbar}{2a^2}S_{1,\phi\phi} + \frac{1}{2a^2}S_{1,\phi}^2 + a^4V(\phi) = 0 \tag{28}$$

which, using equation (23) and the time parameter defined in equation (26), may be rearranged to obtain the following functional Schrödinger equation, also known as the Tomonaga–Schwinger equation, propagating in the background of curved spacetime, namely

$$\begin{aligned} -\frac{i\hbar}{a}\left(\frac{\partial S_0}{\partial a}\right)\frac{\partial f(a,\phi)}{\partial a} = i\hbar\frac{\partial f(a,\phi)}{\partial t} \Rightarrow \left[-\frac{\hbar^2}{2a^3}\frac{\partial^2}{\partial\phi^2} + a^3V(\phi)\right]f(a,\phi) \\ = \pm\,\hbar\sqrt{k}\frac{\partial f(a,\phi)}{\partial a}, \end{aligned} \tag{29}$$

provided,

$$\frac{\partial S_0}{\partial a}\left(\frac{\partial \mathcal{D}(a)}{\partial a}\right) - \frac{1}{2}\left(\frac{\partial^2 S_0}{\partial a^2} + \frac{p}{a}\frac{\partial S_0}{\partial a}\right)\mathcal{D}(a) = 0, \tag{30}$$

where $f(a,\phi)$ and $\mathcal{D}(a)$ are related by

$$f(a,\phi) = \mathcal{D}(a)\exp\left(\frac{iS_1}{\hbar}\right). \tag{31}$$

Here, $\mathcal{D}(a)$ plays the role of the Van–Vleck determinant and can be solved exactly in view of equation (30) to yield

$$\mathcal{D}(a) = \mu^{-1}a^{\frac{p+1}{2}}, \tag{32}$$

where μ is the constant of integration. Upto this (M^0) order of approximation, the wavefunctional

$$\Psi(a,\phi) = \exp\left[\frac{i}{\hbar}(MS_0 + S_1)\right] \tag{33}$$

takes the following form:

$$\Psi(a,\phi) = \mu a^{-\frac{p+1}{2}}\exp\left[-\frac{M}{2\hbar}\sqrt{k}a^2\right]f(a,\phi). \tag{34}$$

To obtain the wormhole wavefunctional, we have chosen negative sign in the exponent of equation (34), which is the Hartle–Hawking [22] choice, as mentioned earlier. The exponent part is well behaved both for $a \to 0$ and $a \to \infty$. However, for $p + 1 > 0$, the determinant diverges as $a \to 0$. If the solution of $f(a, \phi)$ can somehow control this divergence as $a \to 0$, then only one can expect the wormhole configuration for other values of operator ordering index p. In the following sections, we shall attempt to find quantum wormholes as solutions to the W-D equation (20) and also try to find semiclassical wormholes in view of the wavefunction (34) by solving $f(a, \phi)$ explicitly in view of equation (29) for a class of potentials $V(\phi)$. Throat of the wormhole is then found for potentials which admit the back-reaction and finally, classical cosmological evolution will be studied considering the back-reaction as an initial condition.

4. Wormholes for zero and constant potentials

4.1. Case 1, $V(\phi) = 0$

As mentioned in the introduction, the wormhole solution for a vanishing potential was first found for an axion coupled to gravity in Euclidean spacetime [1]. These are classical charged wormholes. The same type of classical wormhole solutions were found later, where gravity is coupled to a positive energy massless complex scalar field [28]. Hawking and Page [16] represented wormholes in a more general manner in the quantum domain as the solution of W-D equation with an appropriate boundary condition. They [16] also showed that a wormhole solution exists for both the massless and massive scalar fields in this new approach. In the case of the massless scalar field, the solution of the W-D equation apparently does not admit the Hawking–Page boundary condition in general. However, under change of variables $x = a \sinh\phi$ and $y = a \cosh\phi$, the W-D equation reduces to two harmonic oscillators with opposite signs of energy. These solutions are regular at the origin and damped at infinity for $p = 1$. Here, we first review the case and then show that the wormhole boundary condition is obeyed under a semiclassical approximation in a straightforward manner for an arbitrary factor ordering index p.

4.1.1. Quantum wormhole. For the massless scalar field (vanishing potential), the W-D equation (20) may be re-expressed in the following form:

$$\left[\frac{\hbar^2}{2M}\left(a^2\frac{\partial^2}{\partial a^2} + pa\frac{\partial}{\partial a}\right) - \frac{M}{2}ka^4\right]\Psi(a, \phi) = -\frac{\hbar^2}{2}\frac{\partial^2}{\partial \phi^2}\Psi(a, \phi). \tag{35}$$

Under separation of variable $\Psi(a, \phi) = A(a)B(\phi)$, the above equation takes the form

$$\frac{1}{A}\left[\frac{\hbar^2}{2M}\left(a^2\frac{\partial^2 A}{\partial a^2} + pa\frac{\partial A}{\partial a}\right) - \frac{M}{2}ka^4 A\right] = -\frac{\hbar^2}{2B}\frac{\partial^2 B}{\partial \phi^2} = \omega^2, \tag{36}$$

where ω^2 is the separation constant. Explicit solution of the wavefunction $\Psi(a, \phi)$ is therefore,

$$\Psi(a, \phi) = (-1)^b \left(\frac{a^2 M\sqrt{k}}{4\hbar}\right)^{\frac{1-p}{4}} \left[C_1 I_{-c}\left(\frac{a^2 M\sqrt{k}}{2\hbar}\right)\Gamma(1-c) + C_2(-1)^c\, I_c\left(\frac{a^2 M\sqrt{k}}{2\hbar}\right)\Gamma(1+c)\right]B(\phi) \tag{37}$$

Here, $I_\alpha(x)$ is the modified Bessel function of first kind (see the appendix for its properties), while C_1 and C_2 are constants of integration and the constants b and c are given by

$$b = \frac{1-p}{8} - \frac{c}{2} \qquad \text{and} \qquad c = \frac{\sqrt{\hbar^2(p-1)^2 + 8M\omega^2}}{4\hbar}. \tag{38}$$

Since $B(\phi)$ is a regular oscillatory function, therefore, it appears that the wavefunction (37) does not satisfy the wormhole boundary condition. Nonetheless, instead of assuming separation of variables, the W-D equation for the massless scalar field with $p = k = 1$, under the change of variable $x = a \sin h\phi$ and $y = a \cos h\phi$, was expressed by Hawking and Page [16] in the following form (in the unit $\hbar = M = 1$):

$$\left(\frac{\partial^2}{\partial y^2} - y^2 - \frac{\partial^2}{\partial x^2} + x^2\right)\psi(x, y) = 0. \tag{39}$$

The above equation represents two harmonic oscillators with opposite energy and so is well behaved at both ends, confirming the existence of the wormhole for the massless scalar field. This is possible because neither $A(a)$ nor $B(\phi)$ is regular but the combined solution $\Psi(a, \phi)$ turns out to be regular.

4.1.2. Semiclassical wormhole. Having obtained the quantum wormhole solution for the vanishing potential $V(\phi) = 0$ for the operator ordering index $p = 1$, let us now proceed to find its fate under the semiclassical approximation. Equation (29) under separation of variables $f(a, \phi) = A(a)\Phi(\phi)$ takes the following form:

$$\pm 2\hbar\sqrt{k}a^3 \frac{1}{A}\frac{\partial A}{\partial a} = -\hbar^2 \frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2} = 4C^2, \tag{40}$$

where C^2 is a separation constant. $A(a)$ and $\Phi(\phi)$ may now be solved immediately to obtain

$$f(a, \phi) = \exp\left(\mp\frac{C^2}{\hbar\sqrt{k}a^2}\right)\Phi(\phi), \tag{41}$$

where Φ is a regular oscillatory function. If we now restrict ourselves to the negative sign in the exponent of $f(a, \phi)$ (since for the other sign in $f(a, \phi)$ wormhole does not exists), then the wavefunctional Ψ given in equation (34) takes the following form:

$$\Psi(a, \phi) = \mu a^{-\frac{p+1}{2}} \exp\left[-\frac{1}{\hbar}\left(\frac{M}{2}\sqrt{k}a^2 + \frac{C^2}{\sqrt{k}a^2}\right)\right]\Phi(\phi). \tag{42}$$

Clearly, the wavefunctional is exponentially damped at large 3-geometry and regular at small 3-geometry in the case of a closed ($k = +1$) model for an arbitrary factor ordering index p. The exponent of the matter wavefunctional $f(a, \phi)$ not only controls the Van–Vleck determinant $\mathcal{D}(a)$ given in equation (32), but also does the same to any arbitrary but finite oscillation of the scalar field ϕ in both ways $a \to 0$ and $a \to \infty$. Thus, the massless scalar field admits the Hawking–Page wormhole boundary condition in the semiclassical limit for an arbitrary operator ordering index p, while quantum wormholes are realized only for $p = 1$. Note that for $k = -1$, the wavefunction becomes oscillatory leading to an inhabitable singularity instead of a wormhole.

4.1.3. Back-reaction and the throat. Let us now go a bit further. In order to show that the scalar field indeed induces a back-reaction on pure gravity and to calculate the throat, it is required to express the exponent of the solution (42) as $\exp[\frac{i}{\hbar}S_t]$, fixing the curvature parameter to $k = +1$, since the semiclassical wormhole for the massless scalar field exists for a closed model only. Therefore, we need to define the quantity S_t as

$$S_t = \mathrm{i}\left[\frac{M}{2}a^2 + \frac{C^2}{a^2}\right], \tag{43}$$

Now, taking derivative with respect to a and upon squaring, equation (43) reduces to

$$S_{t,a}^2 = -M^2a^2 + \frac{4C^2M}{a^2} - \frac{4C^4}{a^6}. \tag{44}$$

Now, treating $S_{t,a} = -Ma\dot{a}$ as the classical momentum, the above equation (44) may be viewed simply as the Einstein equation along with the back-reaction term, namely

$$\frac{\dot{a}^2}{a^2} + \frac{1}{a^2} = \frac{2}{M}\langle\rho_\phi\rangle = \frac{2}{M}\left\langle\frac{1}{2}\dot{\phi}^2\right\rangle = \frac{2}{M}\left[\frac{2C^2}{a^6} - \frac{2C^4}{Ma^{10}}\right], \tag{45}$$

which under Wick rotation takes the form

$$a_{,\tau}^2 = 1 - \frac{2}{M}\left[\frac{2C^2}{a^4} - \frac{2C^4}{Ma^8}\right], \tag{46}$$

which is in a much general form than equation (10) and clearly admits the classical wormhole boundary condition. The throat of the wormhole can now be found setting $a_{,\tau} = 0$ and is given by

$$a_0 = \left(\frac{2C^2}{M}\right)^{\frac{1}{4}}. \tag{47}$$

Further, since $a_{,\tau}^2 > 0$, therefore, asymptotically flat Euclidean space is ensured. In the case under consideration, $p_\phi = \rho_\phi$ and so an ultraviolet cutoff $a \geqslant (\frac{C^2}{M})^{1/4}$ is required to satisfy the strong energy conditions ($\rho_\phi + 3p_\phi \geqslant 0$). The size of the throat clearly shows the existence of an automatic ultraviolet cutoff and the strong energy condition is obeyed. Depending on the value of the only parameter $C \gg \sqrt{\hbar/2}$, the size of the throat may be anything of the order of the post-Planckian length, which is the limit of the semiclassical approximation as demonstrated in section 3. For example, if the constant C is chosen of the order 1, then the throat is of the order of micrometre. It is also possible to obtain the so-called classical wormhole boundary condition, namely equation (10), if one sets the last term in equation (45) to vanish which is only possible when the scale factor is sufficiently large i.e., in the classical limit. In that case, from the right-hand side of equation (45), one obtains $a^3\dot{\phi} = 2C$, which is true at the classical level since ϕ is cyclic. Condition (12) thus results in quite trivially.

4.1.4. Late time cosmic evolution. Our analysis reveals that under the semiclassical approximation, the massless scalar field admits the wormhole boundary condition for an arbitrary factor ordering index (unlike quantum wormhole) corresponding to which a back-reaction exists leading to the gravitational potential

$$U(a) = \frac{1}{a^2} - \frac{4C^2}{Ma^6} + \frac{4C^4}{M^2a^{10}}. \tag{48}$$

The throat of the wormhole ensures that the scale factor a is now bounded from below and so the gravitational potential $U(a)$ does not suffer from short distance instability, i.e., there exists an ultraviolet cutoff. Likewise, it is also clearly evident from the above equation (48) that $U(a)$ is free from long distance instability. The asymptotic de-Sitter/flat space on the other hand ensures an early inflationary epoch. Inflation must have ended by $10^{-32\pm 6}$ s to give way to hot big bang before $T \sim 100$ GeV, for free quarks to exist restoring an electroweak phase transition and to validate standard nucleosynthesis. Thus, inflation is purely a quantum phenomenon and so classical field equations might not always exhibit such behaviour. It is now important to see how such an initial wormhole boundary condition which sets up a throat affects cosmological evolution. For the purpose, it is required to add some contributions from radiation immediately after the asymptotic flat/de-Sitter space ($k = 0$) has been arrived at and also some contribution from matter at the late stage. Hence, the classical field equations, which we need to solve, are

$$\frac{\dot{a}^2}{a^2} = \frac{4}{M}\left[\frac{C^2}{a^6} - \frac{C^4}{Ma^{10}} + \pi^2\left(\frac{\rho_{ro}}{a^4} + \frac{\rho_{mo}}{a^3}\right)\right] \tag{49}$$

$$a^3\dot{\phi} = 2C, \tag{50}$$

where the wormhole boundary condition has been incorporated. In the above equation, ρ_{ro} and ρ_{mo} are the amount of radiation and the matter available at the present epoch, respectively. Clearly, as Universe expands, the contributions from the first and second terms are negligible and the Universe evolves like the usual Friedmann model with $a \propto t^{1/2}$ in the radiation era and $a \propto t^{2/3}$ in the matter dominated era. As a result, baryogenesis, nucleosynthesis, LSS (large scale structure) along with WMAP redshift data on the matter-radiation equality and decoupling remain unaltered. However, late time cosmic acceleration definitely requires some form of dark energy, which has not been considered here. Thus, the massless scalar field turns out to be a good candidate to explain the cosmological evolution.

4.2. $V(\phi) = V_0$, *where* V_0 *is a constant*

Lee [29] had shown that the quantum theory of a complex scalar field with a constant potential admits the wormhole boundary condition provided V_0 has a minima, indicating that the size of the wormhole must be less than the horizon length. Here, we study the case for a real scalar field.

4.2.1. Quantum wormhole. The W-D equation (20) may now be expressed in the form,

$$\left[\frac{\hbar^2}{2M}\left(a^2\frac{\partial^2}{\partial a^2} + pa\frac{\partial}{\partial a}\right) - \frac{M}{2}ka^4 + a^6V_0\right]\Psi(a,\phi) = -\frac{\hbar^2}{2}\frac{\partial^2}{\partial\phi^2}\Psi(a,\phi). \tag{51}$$

Under separation of variables $\Psi(a,\phi) = A(a)B(\phi)$, the above equation reads

$$\frac{1}{A}\left[\frac{\hbar^2}{2M}\left(a^2\frac{\partial^2 A}{\partial a^2} + pa\frac{\partial A}{\partial a}\right) - \frac{M}{2}ka^4A + a^6V_0A\right] = -\frac{\hbar^2}{2B}\frac{\partial^2 B}{\partial\phi^2} = \omega^2, \tag{52}$$

with ω^2 being the separation constant. The gravitational part of the equation is solvable only for $\omega^2 = 0$ and for some particular value of p. For $p = -1$ or $p = 3$, we can write solutions in the following form:

$$\Psi = Ai\left(\frac{2^{\frac{2}{3}}\left(\frac{kM^2}{4\hbar^2} - \frac{a^2MV_0}{2\hbar^2}\right)}{\left(\frac{-MV_0}{\hbar^2}\right)^{\frac{2}{3}}}\right)C_3B(\phi) + Bi\left(\frac{2^{\frac{2}{3}}\left(\frac{kM^2}{4\hbar^2} - \frac{a^2MV_0}{2\hbar^2}\right)}{\left(\frac{-MV_0}{\hbar^2}\right)^{\frac{2}{3}}}\right)C_4B(\phi) \tag{53}$$

and

$$\Psi = \frac{1}{a^2}Ai\left(\frac{2^{\frac{2}{3}}\left(\frac{kM^2}{4\hbar^2} - \frac{a^2MV_0}{2\hbar^2}\right)}{\left(\frac{-MV_0}{\hbar^2}\right)^{\frac{2}{3}}}\right)C_5B(\phi) + \frac{1}{a^2}Bi\left(\frac{2^{\frac{2}{3}}\left(\frac{kM^2}{4\hbar^2} - \frac{a^2MV_0}{2\hbar^2}\right)}{\left(\frac{-MV_0}{\hbar^2}\right)^{\frac{2}{3}}}\right)C_6B(\phi), \tag{54}$$

where $Ai(x)$ and $Bi(x)$ are Airy functions of the first and second kind, respectively, while C_3, C_4, C_5 and C_6 are constants. Airy functions with negative argument are highly oscillatory (see the appendix) leading to a Lorentzian regime with unavoidable singularity and the wormhole boundary condition therefore is not satisfied. Nevertheless, instead of attempting solution under the assumption of separation of variables, the W-D equation (51) under the transformation $x = a\sin h\phi$, $y = a\cos h\phi$ and for $p = k = 1$ reduces to

$$\left[\frac{\partial^2}{\partial y^2} - y^2 - \frac{\partial^2}{\partial x^2} + x^2 + 2(y^2 - x^2)^2V_0\right]\Psi = 0, \tag{55}$$

which represents a coupled harmonic oscillator with opposite energies and is well behaved at both ends. Thus, here again the wormhole boundary condition is satisfied for $p = 1$.

4.2.2. Semiclassical wormhole. Equation (29) can again be solved for $f(a,\phi)$ using the method of separation of variables and for this particular case it can be written in the following form:

$$\pm\frac{1}{A(a)}\frac{\partial A(a)}{\partial a}=\frac{2C^2}{\hbar\sqrt{k}}\frac{1}{a^3}+\frac{V_0}{\hbar\sqrt{k}}a^3 \qquad \text{and} \qquad -\hbar^2\frac{1}{\Phi(\phi)}\frac{\partial^2\Phi(\phi)}{\partial\phi^2}=4C^2. \tag{56}$$

We have chosen the same separation constant $4C^2$ as in the previous case to make a comparison of the throat of the wormhole. Equations (56) may be solved and the explicit form of the wavefunctional in view of equation (34) is expressed as

$$\Psi(a,\phi)=\mu a^{-\frac{p+1}{2}}\exp\left[-\frac{1}{\hbar}\left(\frac{M}{2}\sqrt{k}a^2+\frac{C^2}{\sqrt{k}a^2}+\frac{V_0a^4}{4\sqrt{k}}\right)\right]\Phi(\phi), \tag{57}$$

where $\Phi(\phi)$ admits the same type of oscillatory solution as before. The above form of the wavefunction is again well behaved i.e., regular at $a\to 0$ and damped out exponentially as $a\to\infty$ for $k=+1$ and for all values of the operator ordering index p. The exponent here again can control both the Van–Vleck determinant and arbitrary oscillations appearing in $\Phi(\phi)$. Thus, $\Psi(a,\phi)$ obtained in (57) admits the wormhole boundary condition.

4.2.3. Back-reaction and the throat. Proceeding as in the earlier case, i.e., expressing the exponent of the solution (57) as $\exp[\frac{i}{\hbar}S_t]$, we obtain

$$S_{t,a}^2=-M^2a^2-\frac{4C^4}{a^6}+\frac{4MC^2}{a^2}+4V_0C^2-2MV_0a^4, \tag{58}$$

neglecting the higher order term in the scale factor (namely a^6) since such term does not contribute to the throat. Hence, Einstein's equation with the back-reaction now takes the form

$$\frac{\dot{a}^2}{a^2}+\frac{1}{a^2}=\frac{2}{M}\langle\rho_\phi\rangle=\frac{2}{M}\left[\frac{2C^2}{a^6}-\frac{2C^4}{Ma^{10}}+\frac{2V_0C^2}{Ma^4}-V_0\right], \tag{59}$$

which under Wick rotation reads

$$a_{,\tau}^2=1-\frac{2}{M}\left[\frac{2C^2}{a^4}-\frac{2C^4}{Ma^8}+\frac{2V_0C^2}{Ma^2}-V_0a^2\right]. \tag{60}$$

It is apparent that equation (60) ensures an asymptotic flat Euclidean regime as $a_{,\tau}^2>0$ and admits a throat ($a_{,\tau}=0$) of size

$$a_0=\left(\frac{2C^2}{M}\right)^{\frac{1}{4}}, \tag{61}$$

which is the same as obtained in the massless case. One can observe that the weak energy condition ($\rho_\phi\geqslant 0$ and $\rho_\phi+p_\phi\geqslant 0$) holds, provided

$$V_0\leqslant\frac{C^2}{a^6}. \tag{62}$$

On the other hand, the strong energy condition (which additionally requires $\rho_\phi+3p_\phi\geqslant 0$) holds, provided

$$V_0\leqslant\frac{4C^2}{a^6}\left(\frac{Ma^4-C^2}{5Ma^4-4C^2}\right). \tag{63}$$

At the radius of the throat (61), the above conditions (62) and (63) imply

$$V_0\leqslant\frac{C^2}{a_0^6}; \qquad \text{and} \qquad V_0\leqslant\frac{2C^2}{3a_0^6}, \tag{64}$$

giving an upper limit to V_0. Hence, a viable classical wormhole solution also exists for a constant potential.

4.2.4. Late time cosmic evolution. The gravitational potential along with the back-reaction term given by

$$U_1(a) = \frac{1}{a^2} - \frac{4C^2}{Ma^6} + \frac{4C^4}{M^2a^{10}} - \frac{4C^2V_0}{M^2a^4} + \frac{2V_0}{M} \tag{65}$$

clearly does not suffer from either short distance or long distance instabilities. Further, at the late stage when radiation and matter in the form of pressureless dust are incorporated, the $\binom{0}{0}$ equation of Einstein in the flat space ($k = 0$) under the wormhole boundary condition reads

$$\begin{aligned}\frac{\dot{a}^2}{a^2} &= \frac{4}{M}\left[\frac{C^2}{a^6} - \frac{C^4}{Ma^{10}} + \frac{C^2V_0}{Ma^4} + \pi^2\left(\frac{\rho_{ro}}{a^4} + \frac{\rho_{mo}}{a^3}\right) - \frac{V_0}{2}\right] \\ &\approx \frac{4}{M}\left[\left(\frac{C^2V_0 + M\pi^2\rho_{ro}}{Ma^4}\right) + \pi^2\frac{\rho_{mo}}{a^3} - \frac{V_0}{2}\right],\end{aligned} \tag{66}$$

where we have neglected the first two terms in the last equation as they should be at the late stage of the cosmic evolution. Nevertheless, the constant potential we have started with behaves as a negative effective cosmological constant under the back-reaction. As a result, viable late time cosmic evolution is not possible. Thus, a scalar field with a constant potential is not suitable for late time cosmological evolution.

5. Wormholes for power law potentials

Regular, bounded and well-behaved quantum wormhole in the power series approximation and semiclassical wormhole in the WKB approximation had been demonstrated by Hawking and Page [16] for the massive scalar field $V = V_0\phi^2$ in the R-W minisuperspace model with a curvature parameter $k = +1$. For this purpose, they made the approximation $\phi^2 \ll 1$ and considered a large scale factor a. Later, Kim [19] also made a detailed analysis in this connection in the R-W metric, for the conformally and minimally coupled scalar fields both for the power law potential of the type $\frac{\lambda_{2p}}{2p}\phi^{2p}$ suitable for a chaotic inflationary model and a polynomial potential of the type $\frac{\lambda_{2p}}{2p}\phi^{2p} + 2V_0\phi^2$ suitable for a new inflationary model, where p is an integer. It was pointed out that operator ordering plays an important role for wavefunctions to follow Hawking–Page boundary conditions. The solutions were obtained by the product integral formulation of wavefunctions and it was found that half of wavefunctions were exponentially damped, whereas the other half were diverging out at large 3-geometry. Kim [19] interpreted the former as tunnelling out wavefunction into and the latter as tunnelling in wavefunction from different universes with the same or different topology. This motivated him to suggest that it is the modulus of wavefunction, instead of wavefunction itself, that should be regular up to some negative power of the 3-geometry as the 3-geometry collapses and should be damped at large 3-geometry. With the help of the Liouville–Green transformation, Kim and Page [30] had also shown that a wormhole solution for the minimally coupled power law scalar field potential exists under the condition that the cosmological constant should vanish. Twamley and Page [31] also found a wormhole solution for the minimally coupled imaginary scalar field, taking potential in the forms $V = \frac{1}{4}\lambda\phi^4$ and $V = V_0\phi^2 + \frac{1}{4}\lambda\phi^4$ following the Runge–Kutta method of iteration. The solution differs from those obtained by others in the respect that it does not possess conserved charge. It also dispels a conjecture made by Halliwell and Hartle regarding the behaviour of the real part of the action for wormholes possessing complex geometries. It can also overcome the problem with a macroscopic wormhole in connection with its stability as argued by Fischler and Susskind [11].

The inverse power law was first introduced by Peebles and Ratra [32]. In recent years, potentials with the inverse power law had played an important role in explaining late time cosmic acceleration [33]. The inverse power law potential is also linked to particle physics models [34]. Therefore, in this section we take up the power law potential in the form

$$V(\phi) = V_0\phi^{-\alpha}, \tag{67}$$

where α may be both positive and negative, so that power law potentials can be handled in the same frame. To explore the possibility of obtaining quantum wormholes, the following transformation relation:

$$\eta = a^m\phi^n \tag{68}$$

is useful, in view of which the W-D equation (20) is expressed as

$$\frac{\hbar^2}{2M}(a^2\Psi_{aa} + ap\Psi_a) - \frac{M}{2}ka^4\Psi = \frac{\eta^{\frac{2n-2}{n}}}{a^{-\frac{2m}{n}}}\left[\frac{\hbar^2}{2}\left(n^2\Psi_{\eta\eta} + \frac{n(n-1)}{\eta}\Psi_\eta\right)\right.$$
$$\left. -V_0 a^{\left(6+\frac{m(\alpha-2)}{n}\right)}\eta^{\left(\frac{2-2n-\alpha}{n}\right)}\Psi\right]. \tag{69}$$

Separation of variable in the form $\Psi(a,\eta) = A(a)B(\eta)$ is possible, provided

$$m = \frac{6n}{2-\alpha} \qquad \text{and} \qquad \alpha \neq 2 \tag{70}$$

for a finite value of m and for a meaningful form of the wavefunction as well. However, condition (70) implies that inverse square law potentials ($V(\phi) = V_0\phi^{-2}$) cannot be treated in the same frame. Thus, we have

$$\frac{a^{-\frac{2m}{n}}}{A}\left[\frac{\hbar^2}{2M}(a^2A_{aa} + apA_a) - \frac{M}{2}ka^4A\right]$$
$$= \frac{\eta^{\frac{2n-2}{n}}}{B}\left[\frac{\hbar^2}{2}\left(n^2B_{\eta\eta} + \frac{n(n-1)}{\eta}B_\eta\right) - V_0\eta^{\frac{2(3-m)}{m}}B\right] = \omega^2, \tag{71}$$

where ω^2 is the separation constant.

5.1. Massive scalar field $V = V_0\phi^2$

5.1.1. Quantum wormhole. This case corresponds to $\alpha = -2$ in view of (67). Now, if we choose $n = 1$, then equation (70) yields $m = 3/2$, implying $\eta = a^{3/2}\phi$ in view of (68). In the process, we find that the solution of (71) exists only under the choice for the separation constant $\omega^2 = 0$. The solutions are

$$A(a) = a^{\frac{1-p}{2}}\left(\frac{2^4\hbar^2}{kM^2}\right)^{\frac{p-1}{8}}\left[C_7(-1)^{\frac{1-p}{4}}I_{\left(\frac{1-p}{4}\right)}\left(\frac{a^2\sqrt{k}M}{2\hbar}\right)\Gamma\left(\frac{5-p}{4}\right)\right.$$
$$\left. + C_8 I_{-\left(\frac{1-p}{4}\right)}\left(\frac{a^2\sqrt{k}M}{2\hbar}\right)\Gamma\left(\frac{3+p}{4}\right)\right] \tag{72}$$

and

$$B(\eta) = C_9 D_{\left(-\frac{1}{2}\right)}\left(\frac{(8V_0)^{\frac{1}{4}}\eta}{\sqrt{\hbar}}\right) + C_{10}D_{\left(-\frac{1}{2}\right)}\left(\mathrm{i}\frac{(8V_0)^{\frac{1}{4}}\eta}{\sqrt{\hbar}}\right). \tag{73}$$

Here, $I_\alpha(x)$, $D_\nu(x)$ are the modified Bessel function of first kind and parabolic cylinder function, respectively, while C_7, C_8, C_9 and C_{10} are integration constants. The real part of the wavefunction $\Psi(a,\eta) = A(a)B(\eta)$ exhibits ultraviolet divergence for $p > 1$. Nevertheless, for $p \leqslant 1$, the parabolic cylinder function controls the divergence (see the appendix) appearing in the modified Bessel function and hence $\Psi(a,\eta)$ becomes regular at both ends. Thus, the wormhole exists for $p \leqslant 1$, which has been exhibited in figure 1, setting $C_7 = C_8 = C_9 = \hbar = M = k = p = 1$.

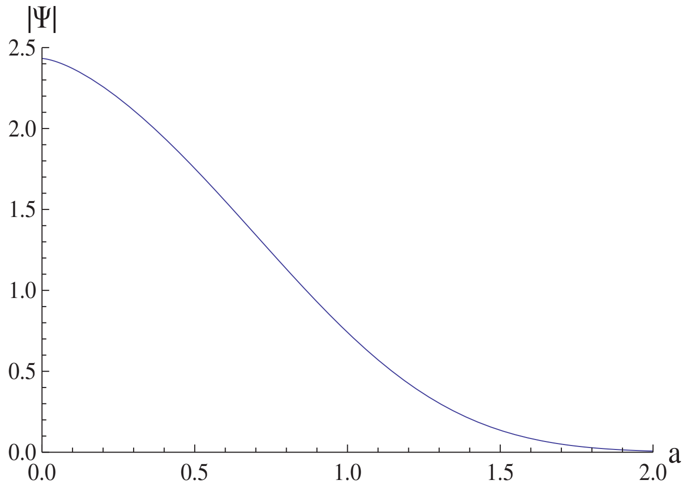

Figure 1. The figure depicts that $|\Psi|$ satisfies the Hawking–Page wormhole boundary condition in the case of $V = V_0\phi^2$ for $p \leqslant 1$ and $\phi > 0$. The figure has been plotted setting the constants $C_7 = C_8 = C_9 = \hbar = M = k = p = \phi = 1$ after testing a wide number of cases with different values of the constants viz., C_7, C_8, C_9 and p for which the nature of $|\Psi|$ remains unaltered.

5.1.2. Semiclassical wormhole. Equation (29) now takes the form

$$\pm 2\hbar\sqrt{k}\frac{\partial f}{\partial a} = -\frac{\hbar^2}{a^3}\frac{\partial^2 f}{\partial\phi^2} + 2V_0 a^3\phi^2 f. \tag{74}$$

Here, again let us choose the same variable, $\eta = a^{\frac{3}{2}}\phi$, to reduce the above equation to

$$\pm 2\hbar\sqrt{k}\frac{1}{f}\left(\frac{\partial f}{\partial a}\right) = -\hbar^2\frac{1}{f}\left(\frac{\partial^2 f}{\partial\eta^2}\right) + 2V_0\eta^2. \tag{75}$$

Separating the variables as $f(a,\eta) = A(a)B(\eta)$, we have

$$\pm 2\hbar\sqrt{k}\frac{1}{A}\left(\frac{\partial A}{\partial a}\right) = -\hbar^2\frac{1}{B}\left(\frac{\partial^2 B}{\partial\eta^2}\right) + 2V_0\eta^2 = \omega^2. \tag{76}$$

where ω^2 is the separation constant. Solving the above equations for a and η, we obtain

$$f(a,\eta) = A_0\exp\left(\pm\frac{\omega_0^2\hbar a}{2\sqrt{k}}\right)\times\left[C_{11}D_{\frac{\omega_0^2-\omega_1}{2\omega_1}}(\sqrt{2\omega_1}\eta) + C_{12}D_{\left(-\frac{\omega_0^2+\omega_1}{2\omega_1}\right)}(\mathrm{i}\sqrt{2\omega_1}\eta)\right], \tag{77}$$

where

$$\omega_0^2 = \frac{\omega^2}{\hbar^2} \qquad \text{and} \qquad \omega_1^2 = 2\frac{V_0}{\hbar^2}. \tag{78}$$

Here, again $D_\nu(x)$ denotes the parabolic cylinder function, while C_{11} and C_{12} are constants. Now, taking only the real part of the above equation and in view of (34), we obtain the following form of the wavefunction, namely

$$\Psi(a,\phi) = (A_0C_{11}\mu)a^{-\frac{p+1}{2}}\exp\left[-\frac{1}{\hbar}\left(\frac{M}{2}\sqrt{k}a^2 - \frac{\omega_0^2\hbar^2 a}{2\sqrt{k}}\right)\right]\times D_{\frac{\omega_0^2-\omega_1}{2\omega_1}}(\sqrt{2\omega_1}\eta). \tag{79}$$

Since the parabolic cylinder function (see the appendix) is well behaved at both ends ($a \to 0$ and $a \to \infty$), therefore, Ψ satisfies the wormhole boundary condition for $p \leqslant -1$ for both the positive and negative signs in the exponential for $k = +1$. Note that the semiclassical wormhole is much restrictive than the quantum one since the quantum wormhole is admissible for $p \leqslant 1$. Here, we would like to mention that Hawking and Page set $\phi^2 \ll 1$ at $a \to 0$ to satisfy the W-H boundary condition for all values of p [16].

5.1.3. Back-reaction and the throat. To find the wormhole throat, we proceed as before (i.e., express the exponent of the solution (79) as $\exp[\frac{i}{\hbar}S_t]$) to obtain the Einstein $\binom{0}{0}$ equation with the back-reaction term as

$$\frac{\dot{a}^2}{a^2}+\frac{1}{a^2}=\frac{2}{M}<\left(\frac{1}{2}\dot{\phi}^2+V_0\phi^2\right)>=\frac{2}{M}\langle\rho_\phi\rangle=\frac{2}{M}\left(\frac{\omega_0^2\hbar^2}{2a^3}-\frac{\omega_0^4\hbar^4}{8Ma^4}\right), \tag{80}$$

which under Wick rotation reads

$$a_{,\tau}^2=1-\frac{2}{M}\left(\frac{\omega_0^2\hbar^2}{2a}-\frac{\omega_0^4\hbar^4}{8Ma^2}\right). \tag{81}$$

$a_{,\tau}^2$ is clearly positive ensuring asymptotic flat Euclidean universe and the radius of the throat of the wormhole is

$$a_0=\frac{\omega_0^2\hbar^2}{2M}, \tag{82}$$

Now, the strong energy condition requires

$$\rho_\phi+p_\phi=\frac{\omega_0^2\hbar^2}{a^3}-\frac{\omega_0^4\hbar^4}{4Ma^4}-2V_0\phi^2>0. \tag{83}$$

$$\rho_\phi+3p_\phi=\frac{2\omega_0^2\hbar^2}{a^3}-\frac{\omega_0^4\hbar^4}{2Ma^4}-6V_0\phi^2>0. \tag{84}$$

Therefore, at the throat, the strong energy condition is satisfied under the condition $V(\phi)\leqslant\frac{4M^3}{3{\omega_0}^4\hbar^4}$. The condition that the potential should not be too large at any stage is of course justifiable. Hence, a well-behaved classical wormhole is obtainable from the semiclassical one, under the back-reaction.

5.1.4. Late time cosmic evolution. The gravitational potential

$$U_2(a)=\frac{1}{a^2}-\frac{\omega_0^2\hbar^2}{Ma^3}+\frac{\omega_0^4\hbar^4}{4M^2a^4} \tag{85}$$

here again does not suffer from any short or long distance instability. As discussed earlier, if we add some contribution from radiation and matter in the form of dust as well (for $k=0$), the $\binom{0}{0}$ component of Einstein's equation together with the wormhole boundary condition reads

$$\frac{\dot{a}^2}{a^2}=\frac{2}{M}\left[\frac{1}{a^4}\left(2\pi^2\rho_{r0}-\frac{\omega_0^4\hbar^4}{8M}\right)+\frac{1}{a^3}\left(2\pi^2\rho_{m0}+\frac{\omega_0^2\hbar^2}{2}\right)\right]. \tag{86}$$

Thus, the back-reaction terms only reduces the radiation density and increases the matter density slightly and hence keeps the Friedmann solutions ($a\propto\sqrt{t}$ in radiation era and $a\propto t^{\frac{2}{3}}$ in the matter dominated era) along with all cosmological observations unaltered. This is of course a wonderful feature of the massive scalar field since late time cosmological evolution remains unaltered from the standard Friedmann cosmology. Thus, the massive scalar field $V=V_0\phi^2$ is the best candidate to describe the history of cosmological evolution.

5.2. Quartic potential, $V(\phi)=\lambda\phi^4$

5.2.1. Quantum wormhole. This case in view of equation (67) corresponds to $\alpha=-4$. Now, the choice $m=n=1$ clearly satisfies equation (70) which results in $\eta=a\phi$ in view of equation (68). Equation (71) may now be solved again only for $\omega^2=0$. The solution of $A(a)$

is therefore the same modified Bessel function of first kind presented in (72), while $B(\eta)$ may be solved as

$$B(\eta) = \left(\frac{\lambda}{18\hbar^2}\right)^{\frac{1}{12}} \sqrt{\eta}\left[C_{13}(-1)^{\frac{1}{6}} I_{\frac{1}{6}}\left(\frac{\sqrt{2\lambda}\eta^3}{3\hbar}\right)\Gamma\left(\frac{7}{6}\right) + C_{14} I_{-\frac{1}{6}}\left(\frac{\sqrt{2\lambda}\eta^3}{3\hbar}\right)\Gamma\left(\frac{5}{6}\right)\right], \tag{87}$$

where C_{13} and C_{14} are integration constants. The wavefunction, $\Psi(a, \eta) = A(a)B(\eta)$ is a product of two modified Bessel functions $[I_\alpha(x)]$ and so suffers from both infrared and ultraviolet divergences for $p > 1$, while it exhibits infrared divergence for $p \leqslant 1$. Thus, the quantum wormhole does not exist for the quartic potential $V = \lambda\phi^4$.

5.2.2. Semiclassical wormhole. In this case, equation (29) takes the form

$$\pm 2\hbar\sqrt{k}\frac{\partial f}{\partial a} = -\frac{\hbar^2}{a^3}\frac{\partial^2 f}{\partial\phi^2} + 2\lambda a^3\phi^4 f. \tag{88}$$

Choosing a new variable $\eta = a\phi$ as in the quantum case, the above equation can be rewritten as

$$\pm 2\hbar\sqrt{k}\frac{a}{f}\left(\frac{\partial f}{\partial a}\right) = -\hbar^2\frac{1}{f}\left(\frac{\partial^2 f}{\partial\eta^2}\right) + 2\lambda\eta^4. \tag{89}$$

Now, using the method of separation of variables by taking $f(a, \eta) = A(a)B(\eta)$, we have

$$\pm 2\hbar\sqrt{k}\frac{a}{A}\left(\frac{\partial A}{\partial a}\right) = -\hbar^2\frac{1}{B}\left(\frac{\partial^2 B}{\partial\eta^2}\right) + 2\lambda\eta^4 = \omega^2, \tag{90}$$

where ω^2 is the separation constants. The solutions of the above equation exists here again, only for $\omega^2 = 0$, which are

$$A(a) = \text{constant} = A_0 \tag{91}$$

$$B(\eta) = \sqrt{\eta}\left(\frac{\lambda}{18\hbar^2}\right)^{\frac{1}{12}}\left[C_{15}(-1)^{\frac{1}{6}} I_{\frac{1}{6}}\left(\frac{\sqrt{2\lambda}\eta^3}{3\hbar}\right)\Gamma\left(\frac{7}{6}\right) + C_{16} I_{-\frac{1}{6}}\left(\frac{\sqrt{2\lambda}\eta^3}{3\hbar}\right)\Gamma\left(\frac{5}{6}\right)\right] \tag{92}$$

where, as already mentioned, $I_\alpha(x)$ is the modified Bessel function of first kind and C_{15} and C_{16} are integration constants. Now in view of equation (34), the wavefunction takes the following form:

$$\Psi(a, \eta) = A_0\mu a^{-\frac{p+1}{2}} \exp\left(-\frac{M}{2\hbar}\sqrt{k}a^2\right)B(\eta). \tag{93}$$

This wavefunction clearly does not satisfy the wormhole boundary condition and so the semiclassical wormhole also does not exist for the quartic potential.

5.2.3. Series solution. No one has obtained the wormhole configuration for the quartic potential in a straightforward manner, as discussed at the beginning of this section. We therefore make yet another attempt to find a series solution for equation (29) under the choice $f = f(\eta)$, where, $\eta = a\phi$, as before. Equation (29) may then be expressed as

$$\frac{\mathrm{d}^2 f}{\mathrm{d}\eta^2} \pm \left(\frac{2\eta\sqrt{k}}{\hbar}\right)\frac{\mathrm{d}f}{\mathrm{d}\eta} - \left(\frac{2\lambda\eta^4}{\hbar^2}\right)f = 0. \tag{94}$$

Further, under the choice

$$f = g(\eta)\exp\left(-\frac{b\eta^2}{4}\right), \qquad \text{where,} \qquad b = \mp\frac{\sqrt{k}}{\hbar}, \tag{95}$$

the above equation (94) may be re-expressed as

$$\frac{d^2g}{d\eta^2} + \left(\mp\frac{\sqrt{k}}{\hbar} + \frac{k}{\hbar^2}\eta^2 - \frac{2\lambda}{\hbar^2}\eta^4\right) g = 0. \tag{96}$$

Now taking,

$$g = \sum_{n=0}^{\infty} g_n \eta^n \tag{97}$$

equation (96) reads

$$-\frac{2\lambda}{\hbar^2}g_n + \frac{k}{\hbar^2}g_{n+2} \mp \frac{\sqrt{k}}{\hbar}g_{n+4} + (n+5)(n+6)g_{n+6} = 0. \tag{98}$$

All the coefficients from g_2 to g_6 can now be found in terms of g_0 and g_1 which remain arbitrary, as follows:

$$\begin{aligned}
g_2 &= \pm\frac{\sqrt{k}}{2\hbar}g_0\\
g_3 &= \pm\frac{\sqrt{k}}{2.3\hbar}g_1\\
g_4 &= -\frac{6g_0}{2.3.4} - \frac{k}{2\hbar^2}\frac{g_3}{2.3.4}\\
g_5 &= \pm\frac{\sqrt{k}}{(2.3.4.5)\hbar}g_3 - \frac{k}{(2.3.4.5)\hbar^2}g_1\\
g_6 &= \pm\frac{\sqrt{k}}{(2.3.4.5.6)\hbar}g_4 - \frac{k}{(2.3.4.5.6)\hbar^2}g_2 + \frac{2\lambda}{(2.3.4.5.6)\hbar^2}g_0.
\end{aligned} \tag{99}$$

Thus, the solution to $f(a,\phi)$ is

$$f(a,\phi) = \sum_{n=0}^{\infty} g_n (a\phi)^n \exp\left(\pm\frac{\sqrt{k}}{\hbar}\times\frac{a^2\phi^2}{4}\right). \tag{100}$$

So in view of (34), the wavefunction finally takes the following form:

$$\Psi(a,\phi) = \mu a^{-\frac{p+1}{2}} \times \left(\sum_{n=0}^{\infty} g_n(a\phi)^n\right) \times \exp\left(-\frac{\sqrt{k}a^2}{2\hbar}\left(M \mp \frac{1}{2}\right)\right) \times \exp\left(\pm\frac{\sqrt{k}}{4\hbar}\phi^2\right). \tag{101}$$

Note that both the terms in the exponent now have the same form and it appears that for $k = +1$ and $p \leqslant -1$ the wavefunction is well behaved at both ends. Nevertheless, at small 3-volume, the wavefunction vanishes irrespective of the signature (instead of being finite) as has been depicted in figure 2. As a result, the formation of baby universe with a finite throat is not possible.

5.3. $V = \frac{V_0}{\phi}$

5.3.1. Quantum wormhole. Inverse potentials, as mentioned have been found proved its importance in the context of late time acceleration. So, to study the effect of such potentials in the early universe, at first let us take it as $V(\phi) = \frac{V_0}{\phi}$, which in comparison with equation (67) requires $\alpha = 1$. Further choosing $n = 1$, $m = 6$, so that $\eta = \phi a^6$ in view of equation (68), the equation for $A(a)$ reads

$$\frac{\hbar^2}{2M}\left(A_{aa} + \frac{p}{a}A_a\right) - \frac{M}{2}ka^2A - \omega^2 a^{10}A = 0 \tag{102}$$

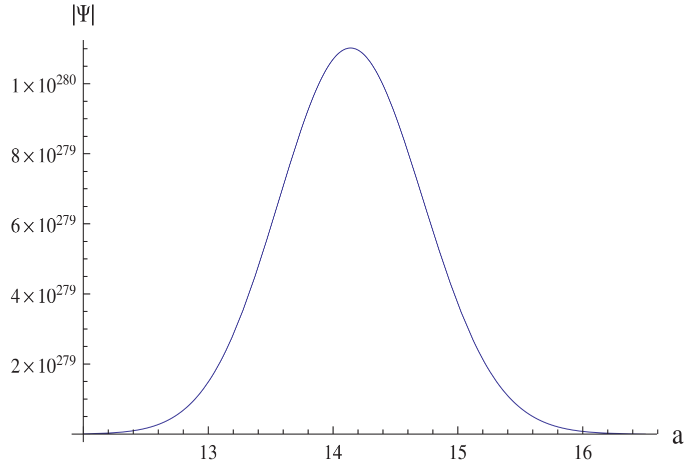

Figure 2. Interestingly enough, the series solution for $V = \lambda\phi^4$ shows Gaussian nature for large but finite n irrespective of the sign appearing in the exponent. Since the wavefunction vanishes as $a \to 0$, so formation of baby universe with a finite throat remains obscure even for $p \leqslant -1$. This graph has been plotted with $k = \hbar = M = 1$, $p = -1$ and taking the first $n = 300$ terms in the series solution. For larger and larger n, the maxima increases and the graph shifts to the right.

Equation (102) can be solved under the choice $\omega^2 = 0$, resulting in the same solution as in (72). In general, for arbitrary ω^2, a series solution of equation (102) can be found which shows a regular singularity of pole of order 1 and one can find indicial roots for it. Further, the $B(\eta)$ equation

$$\frac{\hbar^2}{2}B_{\eta\eta} - \frac{V_0}{\eta}B - \omega^2 B = 0, \tag{103}$$

under the same choice $\omega^2 = 0$, yields the solution

$$B(\eta) = \frac{\sqrt{2\eta V_0}}{\hbar}\left[-C_{17}I_1\left(\frac{\sqrt{8\eta V_0}}{\hbar}\right) + 2C_{18}K_1\left(\frac{\sqrt{8\eta V_0}}{\hbar}\right)\right] \tag{104}$$

where $I_\alpha(x)$ and $K_\alpha(x)$ are the modified Bessel functions of first and second kind, respectively, while C_{17} and C_{18} are arbitrary constants. The wavefunction $[\Psi(a,\eta) = A(a)B(\eta)]$ shows both UV and IR divergences for $p > 1$ and IR divergence for $p \leqslant 1$. However, if we consider $K_\alpha(x)$ to be the particular solution of $B(\eta)$ as

$$B(\eta) = \frac{\sqrt{8\eta V_0}}{\hbar}K_1\left(\frac{\sqrt{8\eta V_0}}{\hbar}\right), \tag{105}$$

then the wormhole boundary condition is satisfied for $p \leqslant 1$ (see the appendix). The plot of such a regular wavefunction $|\Psi| = |A(a)B(\eta)|$ has been presented in figure 3, setting $k = M = C_{17} = C_{18} = \phi = V_0 = \hbar = p = 1$.

5.3.2. Semiclassical wormhole. Under the same above choice, namely $\eta = \phi a^6$ with $n = 1$ and $m = 6$, equation (29) can be expressed as

$$\pm 2\frac{\hbar\sqrt{k}}{a^9}\frac{1}{f}\left(\frac{\partial f}{\partial a}\right) = -\hbar^2\frac{1}{f}\left(\frac{\partial^2 f}{\partial \eta^2}\right) + \frac{2V_0}{\eta}. \tag{106}$$

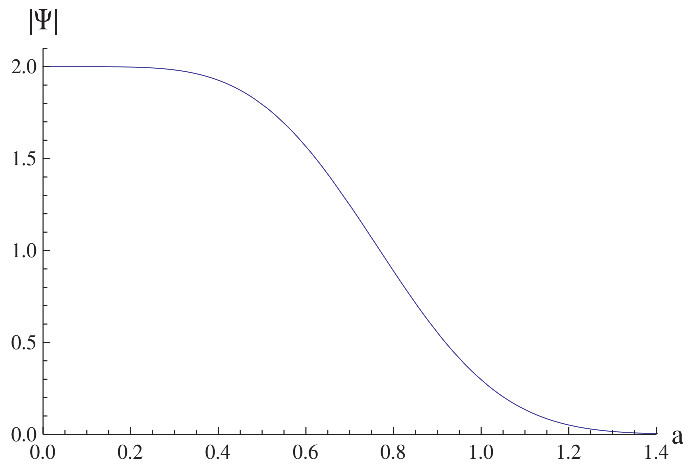

Figure 3. This is the plot of quantum solution of $|\Psi|$ for the case $V(\phi) = \frac{V_0}{\phi}$, where we have set $k = M = C_{17} = C_{18} = \phi = V_0 = \hbar = p = 1$. It is important to mention that other choices of constants do not alter the shape of the graph.

Now, separation of variables in the form $f(a, \eta) = A(a)B(\eta)$ leads to

$$\pm 2\frac{\hbar\sqrt{k}}{a^9}\frac{1}{A}\left(\frac{\partial A}{\partial a}\right) = -\hbar^2\frac{1}{B}\left(\frac{\partial^2 B}{\partial \eta^2}\right) + \frac{2V_0}{\eta} = -\omega^2. \tag{107}$$

Equation (107) may now be solved to yield

$$A(a) = A_0 \exp\left(\mp\frac{\omega^2 a^{10}}{20\hbar\sqrt{k}}\right) \tag{108}$$

and

$$B(\eta) = \frac{1}{\hbar^2}\eta\, \mathrm{e}^{-\omega_0\eta}\left[C_{19}\ {}_1F_1\left(1 + \frac{\omega_1^2}{2\omega_0}, 2; 2\omega_0\eta\right) + C_{20}\ U\left(1 + \frac{\omega_1^2}{2\omega_0}, 2; 2\omega_0\eta\right)\right], \tag{109}$$

where ω_0 and ω_1 are the same as given in equation (78), while A_0, C_{19} and C_{20} are integration constants. ${}_1F_1(a, b; x)$ and $U(a, b; x)$ are confluent hypergeometric functions of first and second kinds, respectively. So in view of equation (34), the wavefunction for this particular case is found as

$$\Psi(a, \eta) = \mu a^{-\frac{p+1}{2}} \exp\left[-\frac{1}{\hbar}\left(\frac{M}{2}\sqrt{k}a^2 + \frac{\omega^2 a^{10}}{20\sqrt{k}}\right)\right]B(\eta). \tag{110}$$

Under the choice $C_{19} = 0$, the wavefunction $\Psi(a, \phi)$ admits the wormhole boundary condition for $p \leqslant -1$ and $k = +1$ (see the appendix).

5.3.3. Back-reaction and throat. As before, setting $k = +1$, the Einstein $\binom{0}{0}$ equation with back-reaction terms takes the form

$$\frac{\dot{a}^2}{a^2} + \frac{1}{a^2} = -\frac{2}{M}\left(\frac{\omega^2 a^6}{2} + \frac{\omega^4 a^{14}}{8M}\right), \tag{111}$$

which under Wick rotation reads

$$a_{,\tau}^2 = 1 + \frac{2}{M}\left(\frac{\omega^2 a^8}{2} + \frac{\omega^4 a^{16}}{8M}\right). \tag{112}$$

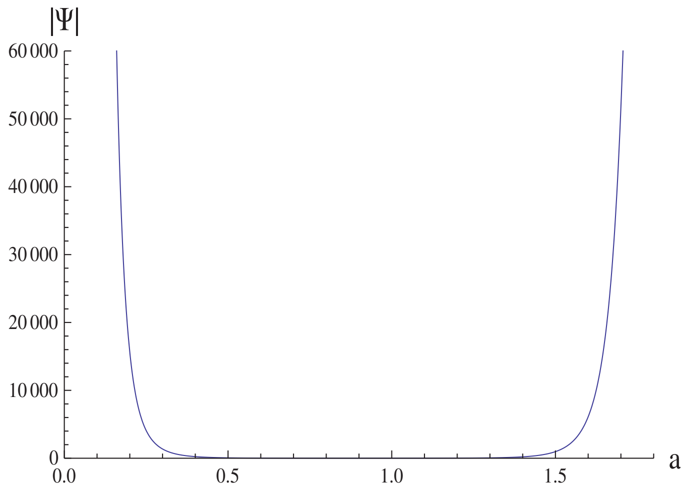

Figure 4. Quantum wormhole wavefunction $|\Psi|$ shows both UV and IR divergence for all values of p in the case of inverse potential in the form $\frac{V_0}{\phi^3}$. The present plot has been presented setting $C_{21} = C_{22} = M = V_0 = \phi = \hbar = p = k = 1$.

Although $a^2_{,\tau} > 0$, the throat clearly becomes imaginary. Thus, the semiclassical wormhole although became apparent from the wavefunction (110), its classical counterpart does not yield a viable wormhole solution. Further, the effective gravitational potential shows infrared divergence which implies that inflation never halts. Since the wormhole exists in the quantum domain, therefore, such inverse form of potential is nice to explain very early universe but is not suitable to explain late time cosmic evolution.

5.4. $V = \frac{V_0}{\phi^3}$

5.4.1. Quantum wormhole. In comparison with equations (67), (68) and (70), the present case is found to correspond with $\alpha = 3, n = 1$ and $m = -6$, so that $\eta = \frac{\phi}{a^6}$. As before, the solution is found in a straightforward manner choosing the separation constant $\omega^2 = 0$. Obviously, the solution $A(a)$ is the same as in equation (72) and the solution for $B(\eta)$ is given below

$$B(\eta) = \hbar\sqrt{\frac{\eta}{2V_0}}\left[C_{21}I_1\left(\frac{2}{\hbar}\sqrt{\frac{2V_0}{\eta}}\right) + 2C_{22}K_1\left(\frac{2}{\hbar}\sqrt{\frac{2V_0}{\eta}}\right)\right], \tag{113}$$

where $I_\alpha(x)$ and $K_\alpha(x)$ are modified Bessel functions as already mentioned, while C_{21} and C_{22} are constants of integration. In this case, the wavefunction $\Psi = A(a)B(\eta)$ exhibits both UV and IR divergences for all p as depicted in figure 4 and so the quantum wormhole does not exist. The wormhole boundary condition is not satisfied even under the choice $C_{21} = 0$, due to the appearance of a multiplicative factor $\sqrt{\eta} = \frac{\phi}{a^3}$ (see the appendix).

5.4.2. Semiclassical wormhole. In this case, equation (29) takes the form

$$\pm 2\hbar\sqrt{k}\frac{\partial f}{\partial a} = -\frac{\hbar^2}{a^3}\frac{\partial^2 f}{\partial \phi^2} + 2V_0\frac{a^3}{\phi^3}f. \tag{114}$$

Choosing the same new variable $\eta = \frac{\phi}{a^6}$ as in the quantum case, the above equation can be rewritten as

$$\pm 2\hbar\sqrt{k}\frac{a^{15}}{f}\left(\frac{\partial f}{\partial a}\right) = -\hbar^2\frac{1}{f}\left(\frac{\partial^2 f}{\partial \eta^2}\right) + 2\frac{V_0}{\eta^3}. \tag{115}$$

Now using the method of separation of variables by taking $f(a,\eta) = A(a)B(\eta)$, we have

$$\pm 2\hbar\sqrt{k}\frac{a^{15}}{A}\left(\frac{\partial A}{\partial a}\right) = -\hbar^2\frac{1}{B}\left(\frac{\partial^2 B}{\partial \eta^2}\right) + 2\frac{V_0}{\eta^3} = \omega^2, \tag{116}$$

where ω^2 is the separation constants. The solutions of the above equation here again exists only for $\omega^2 = 0$. The solutions are $A(a) = \text{constant} = A_0$, while $B(\eta)$ is the same modified Bessel function as presented in equation (113). Now, in view of equation (34), the wavefunction takes the following form:

$$\Psi(a,\eta) = A_0\mu a^{-\frac{p+1}{2}}\exp\left(-\frac{M}{2\hbar}\sqrt{k}a^2\right)B(\eta). \tag{117}$$

Since the modified Bessel function $B(\eta)$ itself shows divergence in its behaviour, so the Van–Vleck determinant ($\mathcal{D}$) also remains unregulated and the wavefunction does not satisfy the wormhole boundary condition. Thus, the semiclassical wormhole remains absent also for $V = \frac{V_0}{\phi^3}$.

5.5. $V = \frac{V_0}{\phi^4}$

5.5.1. Quantum wormhole. This case corresponds to, $\alpha = 4, n = 1, m = -3$ with $\eta = \frac{\phi}{a^3}$ as can be seen while comparing with equations (67), (68) and (70). As before, the solution is found in a straightforward manner only under the choice $\omega^2 = 0$. Obviously, the solution for $A(a)$ is again the same as in equation (72) and the solution $B(\eta)$ is

$$B(\eta) = \eta\left(C_{23}\,\mathrm{e}^{\frac{\sqrt{2V_0}}{\eta\hbar}} + C_{24}\frac{\hbar}{\sqrt{8V_0}}\,\mathrm{e}^{-\frac{\sqrt{2V_0}}{\eta\hbar}}\right), \tag{118}$$

where C_{23} and C_{24} are the constants of integration. Clearly, the first term shows divergence in its behaviour, while the second term is well behaved at both ends. Therefore, the solution of $B(\eta)$ in no way can control the diverging behaviour of the solution for $A(a)$ presented in (72). Hence, the wavefunction Ψ exhibits both UV and IR divergences for all values of p in the similar fashion as depicted in figure 4.

5.5.2. Semiclassical wormhole. In this case, equation (29) takes the form

$$\pm 2\hbar\sqrt{k}\frac{\partial f}{\partial a} = -\frac{\hbar^2}{a^3}\frac{\partial^2 f}{\partial \phi^2} + 2V_0\frac{a^3}{\phi^4}f. \tag{119}$$

Choosing the same new variable $\eta = \frac{\phi}{a^3}$ as taken in the quantum case, the above equation can be rewritten as

$$\pm 2\hbar\sqrt{k}\frac{a^9}{f}\left(\frac{\partial f}{\partial a}\right) = -\hbar^2\frac{1}{f}\left(\frac{\partial^2 f}{\partial \eta^2}\right) + 2\frac{V_0}{\eta^4}. \tag{120}$$

Now, using the method of separation of variables under the choice $f(a,\eta) = A(a)B(\eta)$, we have

$$\pm 2\hbar\sqrt{k}\frac{a^9}{A}\left(\frac{\partial A}{\partial a}\right) = -\hbar^2\frac{1}{B}\left(\frac{\partial^2 B}{\partial \eta^2}\right) + 2\frac{V_0}{\eta^4} = \omega^2, \tag{121}$$

where ω^2 is the separation constants. The above differential equation for $B(\eta)$ admits solution in closed form only when $\omega^2 = 0$, which is already present in equation (118), whence

$A(a) = \text{constant} = A_0$. Thus, in view of equation (34), the wavefunction takes the following form:

$$\Psi(a,\eta) = \frac{C_{24}A_0\mu\hbar}{\sqrt{8V_0}} a^{-\frac{p+7}{2}} \exp\left[-\frac{1}{\hbar}\left(\frac{M}{2}\sqrt{k}a^2 + \frac{\sqrt{2V_0}}{\phi}a^3\right)\right]\phi. \tag{122}$$

Here, we have taken only the second part of the solution (118) setting $C_{23} = 0$. The wavefunction (122) is found to satisfy the wormhole boundary condition for $p \leqslant -7$, $k = +1$ and for regular functional behaviour of ϕ.

5.5.3. Back-reaction and the throat. To find the back-reaction term, we choose

$$\frac{\mathrm{i}}{\hbar}S_t = -\frac{M}{2\hbar}\sqrt{k}a^2 - \frac{2V_0}{\hbar\phi}a^3 \tag{123}$$

and identify $S_{t,a}$ with the classical momentum as before, to obtain

$$\frac{\dot{a}^2}{a^2} + \frac{1}{a^2} = -\frac{2}{M}\left(\frac{18V_0^2}{M\phi^2} + \frac{6V_0}{\phi a}\right), \tag{124}$$

which under Wick rotation reads

$$a_{,\tau}^2 = 1 + \frac{2}{M}\left(\frac{18V_0^2a^2}{M\phi^2} + \frac{6V_0a}{\phi}\right). \tag{125}$$

Here, again we encounter the same situation as occurred in the case of inverse potential in the form V_0/ϕ; i.e., although $a_{,\tau}^2 > 0$, the throat does not have a real root. Thus, the semiclassical wormhole (122) does not yield a classical counterpart. Nevertheless, unlike the previous situation referred, the effective gravitational potential

$$U(a) = \frac{1}{a^2} + \frac{2}{M}\left(\frac{18V_0^2}{M\phi^2} + \frac{6V_0}{\phi a}\right) \tag{126}$$

does not show infrared divergence.

6. Quantum and semiclassical wormholes for exponential potential

6.1. Exponential potential in the form $V = V_0 e^{-\phi/\lambda}$

6.1.1. Quantum wormhole. The exponential potential in the said form was first introduced by Ratra and Peebles [35]. Later it was found to play a significant role to explain late time cosmic acceleration [36]. However, here we consider both the signs of λ. To solve the W-D equation (20) for the case under consideration, let us make a change of variable as $y = a^6 e^{-\phi/\lambda}$. Now separating the wavefunction as $\Psi = A(a)B(y)$ and taking ω^2 as the separation constant, the W-D equation can be expressed as

$$\frac{\hbar^2}{2M}(a^2A_{aa} + paA_a) - \left(\frac{M}{2}ka^4 + \omega^2\right)A = 0, \tag{127}$$

and

$$\frac{\hbar^2}{2\lambda^2}(y^2B_{yy} + yB_y) - (V_0y + \omega^2)B = 0. \tag{128}$$

It is to be noted that λ^2 appears in equation (128) and so the solution is independent of the choice of the signature of λ. The solutions of equation (127) are

$$A(a) = (-1)^{\frac{1-p-4x}{8}}\left(\frac{2^4\hbar^2}{a^4kM^2}\right)^{\frac{p-1}{8}}\left[C_{25}(-1)^xI_x\left(\frac{a^2\sqrt{k}M}{2\hbar}\right)\Gamma(1+x)\right.$$
$$\left. + C_{26}I_{-x}\left(\frac{a^2\sqrt{k}M}{2\hbar}\right)\Gamma(1-x)\right], \tag{129}$$

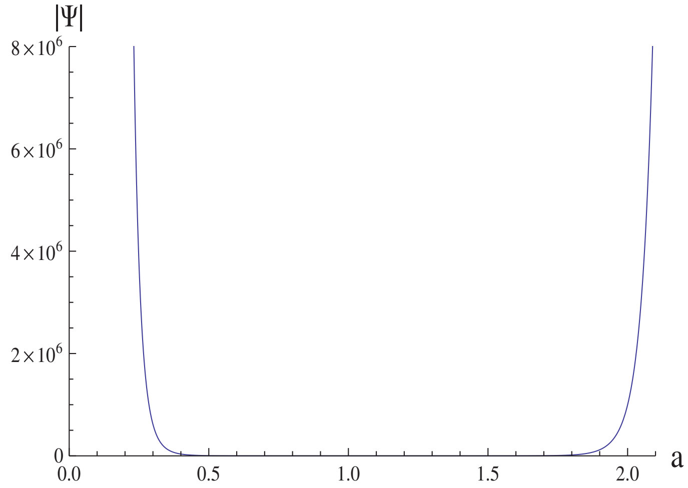

Figure 5. $|\Psi|$ shows both UV and IR divergence behaviour for all p in case of exponential potential. In the present plot, we have chosen $C_{25} = C_{26} = C_{27} = C_{28} = \hbar = V_0 = M = p = k = \omega = \lambda = \phi = 1$.

where C_{25} and C_{26} are constants of integration and

$$x = \frac{\sqrt{\hbar^2(p-1)^2 + 8M\omega^2}}{4\hbar}. \tag{130}$$

The solution of equation (128) can be written in the following form:

$$B(y) = (-1)^{\frac{x_1}{2}}[C_{27}I_{x_1}(x_2)\Gamma(1+x_1) + C_{28}I_{-x_1}(x_2)\Gamma(1-x_1)] \tag{131}$$

with C_{27} and C_{28} being the constants of integration, while $I_\alpha(x)$ is the modified Bessel function of the first kind, with $x_1 = \frac{2\sqrt{2}\omega\lambda}{\hbar}$ and $x_2 = \frac{\sqrt{8yV_0}\lambda}{\hbar}$. Here again, the wavefunction Ψ shows both the UV and IR divergences for all values of p as presented in figure 5.

6.1.2. Semiclassical wormhole. Expressing equation (29) in the following form,

$$\pm\hbar\sqrt{k}\frac{a^3}{f}\frac{\partial f}{\partial a} = -\frac{\hbar^2}{2}\frac{1}{f}\frac{\partial^2 f}{\partial\phi^2} + a^6 \mathrm{e}^{-\frac{\phi}{\lambda}}, \tag{132}$$

it is apparent that the above equation is not separable in its present form. So let us consider a change of variable,

$$\alpha = \ln\left(a^6 \mathrm{e}^{-\frac{\phi}{\lambda}}\right) = 6\ln a - \frac{\phi}{\lambda} \tag{133}$$

and equation (132) may now be expressed as

$$\pm\hbar\sqrt{k}\frac{a^3}{f}\frac{\partial f}{\partial a} = -\frac{\hbar^2}{2\lambda^2}\frac{1}{f}\frac{\partial^2 f}{\partial\alpha^2} + \mathrm{e}^{\alpha} = \omega^2, \tag{134}$$

which is now separable, with ω^2 being the separation constant. Now choosing $f(a,\alpha) = A(a)B(\alpha)$, the above equation reads

$$\begin{cases} \frac{\mathrm{d}^2B}{\mathrm{d}\alpha^2} + \frac{2\lambda^2}{\hbar^2}(\omega^2 - e^{\alpha})B(\alpha) = 0 \\ \\ \frac{1}{A(a)}\frac{\mathrm{d}A(a)}{\mathrm{d}a} = \pm\frac{\omega^2}{\hbar\sqrt{k}}\frac{1}{a^3}. \end{cases} \tag{135}$$

Solving the first equation of (135), we obtain the following expression:

$$B(\alpha) = C_{29}(-1)^{\frac{i\sqrt{2}\omega\lambda}{\hbar}} I_x(y)\Gamma(1+x) + C_{30}(-1)^{\frac{-i\lambda\sqrt{2}\omega}{\hbar}} I_{-x}(y)\Gamma(1-x), \tag{136}$$

where C_{29} and C_{30} are constants of integration, $I_\alpha(x)$ is the modified Bessel function of first kind, while

$$x = \frac{2\lambda\omega i\sqrt{2}}{\hbar} \qquad \text{and} \qquad y = \frac{2\lambda\sqrt{2e^\alpha}}{\hbar}. \tag{137}$$

As in the quantum case here again it is important to note that the sign of λ does not affect the solution $B(\alpha)$. Solution of the second equation of (135) is

$$A(a) = A_0\, e^{\mp\frac{\omega^2}{2\hbar\sqrt{k}a^2}}, \tag{138}$$

where A_0 is the integration constant. So the corresponding wavefunction takes the form

$$\Psi(a,\alpha) = A_0\mu a^{-\frac{p+1}{2}} \exp\left[-\frac{1}{\hbar}\left(\frac{M}{2}\sqrt{k}a^2 + \frac{\omega^2}{2\sqrt{k}a^2}\right)\right]B(\alpha). \tag{139}$$

The exponential part of the wavefunction is similar to one obtained for the massless scalar field ($V_0 = 0$) and is well behaved; nonetheless, the factor $B(\alpha)$ kills the Hawking–Page boundary condition for all values of p. Thus, the semiclassical wormhole also does not exist for the exponential potential.

6.1.3. Back-reaction and the wormhole throat. Although neither quantum nor the semiclassical wormhole exists for the exponential potential under consideration, nevertheless, the semiclassical wavefunction leaves a nice exponential part suitable to find the back-reaction term. Einstein's equation with such a back-reaction term here takes the similar form as massless scalar field, namely

$$\frac{\dot{a}^2}{a^2} + \frac{1}{a^2} = \frac{2}{M}\langle\rho_\phi\rangle = \frac{2}{M}\left(\frac{\omega^2}{a^6} - \frac{\omega^4}{2Ma^{10}}\right), \tag{140}$$

which under Wick rotation reads

$$a_{,\tau}^2 = 1 - \frac{2}{M}\left(\frac{\omega^2}{a^4} - \frac{\omega^4}{2Ma^8}\right) \tag{141}$$

and it is apparent that $a_\tau^2 > 0$ ensuring the asymptotic flat Euclidean regime. The radius of the throat is given by

$$a_0 = \left(\frac{\omega^2}{M}\right)^{\frac{1}{4}}. \tag{142}$$

The strong energy condition is satisfied, provided

$$V_0 \leqslant \left(\frac{M\sqrt{M}}{3\omega}\right) e^{\frac{\phi}{\lambda}}, \tag{143}$$

which is a reasonable condition for both the signatures of λ. Thus, even in the absence of a quantum or semiclassical wormhole solution, the back-reaction under the semiclassical approximation leads to a well-behaved classical wormhole. The late time cosmic evolution with such a wormhole initial condition is determined by the similar set of equations (49) and (50) presented in the case of the massless scalar field and so the wormhole initial condition does not alter Friedmann solutions.

7. Concluding remarks

Euclidean wormholes, which connect two asymptotically flat/de-Sitter spaces with a throat, are the solutions of Einstein's field equations under Wick rotation. Such macroscopic wormholes are realized for radiation dominated and matter (pressureless dust) dominated era as demonstrated in equations (16) and (18), but not for the real scalar field. For microscopic wormholes, one has to probe the very early universe, which is vacuum dominated or might contain a scalar field. The Hawking–Page boundary condition is useful to find such microscopic wormholes. We have explored the possibility of existence of such microscopic wormholes by solving the W-D equation corresponding to the Einstein–Hilbert action being minimally coupled to a scalar field with varied types of potentials. We have also shown that a scalar can only leave behind a trace which might affect classical cosmological evolution in the form of back-reaction obtained under the semiclassical approximation. In the process, $\langle\rho_\phi\rangle$ becomes a function of the scale factor and possibility of obtaining classical Euclidean wormhole solutions for a real scalar field emerge. The findings are as follows.

(1) All types of potentials do not admit the Hawking–Page wormhole boundary condition.

(2) Potentials in the form $V(\phi) = 0, V_0, V_0\phi^2$ and $V_0\phi^{-1}$ have been found to exhibit both quantum and semiclassical wormhole configurations. Further, classical wormholes under the back-reaction exist for all except the inverse potential ($V_0\phi^{-1}$). Additionally, classical cosmological evolution under the wormhole initial condition (put up by the semiclassical back-reaction) has been found to remain unaltered from the Friedmann solution in the radiation and matter dominated era, for the massless $V(\phi) = 0$, massive $V(\phi) = V_0\phi^2$ and exponential $V(\phi) = V_0\,\mathrm{e}^{-\frac{\phi}{\lambda}}$ scalar fields. The constant potential case leaves a negative cosmological constant under the back-reaction and so classical cosmological evolution is unrealistic.

(3) For the potential in the form $V = V_0\phi^{-4}$, quantum wormhole does not exist. Although it admits semiclassical wormhole for $p \leqslant -7$, the back-reaction does not give a real throat. Hence, the scalar in this case does not leave behind a trace for classical wormhole to exist.

(4) Neither quantum nor semiclassical wormhole exists for exponential potential. Nevertheless, the back-reaction term leads to a well-behaved classical wormhole solution. Taking into account such a wormhole boundary condition, the late stage of cosmological evolution has been found to remain unchanged from the Friedmann solutions in the radiation and matter dominated era.

(5) Potentials in the form $V = V_0\phi^4$ and $V = V_0\phi^{-3}$ do not admit any of the wormhole configurations.

(6) Wormhole fixes the curvature parameter $k = +1$ in the early universe.

(7) Semiclassical wormholes for zero ($V(\phi) = 0$) and constant ($V(\phi) = V_0$) scalar fields are found for arbitrary operator ordering index p. Nevertheless, in general, wormholes if exist, require the operator ordering index $p \leqslant 1$. Quantum wormhole obtained for the massless and constant scalar fields require $p = 1$. Thus, the wormhole boundary condition also fixes the factor ordering parameter and $p = 1$ may be chosen in general.

A renormalized theory of gravity and string effective action under the weak field approximation require higher order curvature invariant terms in the gravitational action, which have not been considered here. The wormhole configuration may change considerably if such quantum corrections are incorporated. This we pose in future.

Appendix. A brief account of the role of special functions used in the literature

We have obtained solutions in the literature in terms of some special functions, namely modified Bessel, parabolic cylinder, Airy and confluent hypergeometric functions. Not all the readers are conversant with the properties of these special functions, although these are available in any standard text books on mathematical methods. To make this work self-consistent, we briefly underlying properties of these special functions.

A.1. Modified Bessel functions

Modified Bessel functions of first kind I_α and second kind K_α are linearly independent solutions to the differential equation

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+x\frac{\mathrm{d}y}{\mathrm{d}x}-(x^2+\alpha^2)y=0 \tag{A.1}$$

and are defined as

$$I_\alpha(x)=i^{-\alpha}J_\alpha(\mathrm{i}x)=\sum_{m=0}^{\infty}\frac{1}{m!\Gamma(m+\alpha+1)}\left(\frac{x}{2}\right)^{2m+\alpha} \quad \text{and} \quad K_\alpha(x)=\frac{\pi}{2}\frac{I_{-\alpha}(x)-I_\alpha(x)}{\sin(\alpha\pi)}, \tag{A.2}$$

where $J_\alpha(\mathrm{i}x)$ is the ordinary Bessel functions of imaginary argument. Therefore, unlike ordinary Bessel functions which are oscillating as functions of a real argument, modified Bessel functions are functions of imaginary argument and so, $I_\alpha(x)$ and $K_\alpha(x)$ are exponentially growing and decaying functions, respectively. $I_\alpha(x)$ vanishes at the origin $x=0$ for $\alpha>0$, but remains finite for $\alpha=0$, and then grows exponentially with x. Analogously, $K_\alpha(x)$ diverges at $x=0$ and exponentially decays as x increases.

Now, let us turn our attention to equation (36) which under the choice $x=\frac{a^2M\sqrt{k}}{2\hbar}$, becomes

$$x^2\frac{\mathrm{d}^2A}{\mathrm{d}x^2}+\left(\frac{p+1}{2}\right)x\frac{\mathrm{d}A}{\mathrm{d}x}-\left(x^2+\frac{M\omega^2}{2\hbar^2}\right)A=0. \tag{A.3}$$

For $p=1$, equation (A.3) takes exactly the same form as equation (A.1). But for arbitrary p, the solution presented in equation (37) is a combination of $I_\alpha(x)$ and gamma function. Due to the diverging behaviour of $I_\alpha(x)$, the Hawking–Page boundary condition is not satisfied for the solutions of Ψ, presented in (37), (72), (87), (92), (129), (131) and (136). But as the behaviour of $K_\alpha(x)$ is just opposite to that of $I_\alpha(x)$, so in equation (105), the term $K_1(\frac{\sqrt{8\eta V_0}}{\hbar})$ being multiplied with $\sqrt{\eta}=a^3\sqrt{\phi}$ is finite at $\eta=0$, i.e, $a=0$ and kills the IR divergence of equation (72). So, the wormhole boundary condition is satisfied for this case. Although similar term appears in (113), but due to the presence of $\sqrt{\eta}=\frac{\sqrt{\phi}}{a^3}$ term, unavoidable UV divergence appears at $a=0$.

A.2. Airy function

Airy functions $Ai(x)$ and $Bi(x)$, named after the British astronomer George Biddell Airy, are solutions to the Airy differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-xy=0. \tag{A.4}$$

For real values of x, the Airy function can be defined by the improper Riemann integral

$$Ai(x)=\frac{1}{\pi}\int_0^\infty\cos\left(\frac{t^3}{3}+xt\right)\mathrm{d}t. \tag{A.5}$$

The Airy function of the second kind, denoted $Bi(x)$, is defined as the solution with the same amplitude of oscillation as $Ai(x)$ as $x \to \infty$ which differs in phase by $\pi/2$, namely

$$Bi(x) = \frac{1}{\pi}\int_0^\infty \left[\exp\left(-\frac{t^3}{3}+xt\right)+\sin\left(\frac{t^3}{3}+xt\right)\right]\mathrm{d}t. \tag{A.6}$$

When x is positive, $Ai(x)$ is positive, convex and decreasing exponentially to zero, while $Bi(x)$ is positive, convex and increasing exponentially. When x is negative, $Ai(x)$ and $Bi(x)$ oscillate around zero with ever-increasing frequency and slowly ever-decreasing amplitude which is never damped.

Now under the choice $x = \frac{2^{\frac{2}{3}}\left(\frac{kM^2}{4\hbar^2}-\frac{a^2MV_0}{2\hbar^2}\right)}{\left(\frac{-MV_0}{\hbar^2}\right)^{\frac{2}{3}}}$, equation (52) becomes exactly the same as equation (A.4), whose solution is given in equation (53). However, x is clearly negative, due to the presence of a^2 in the second term and so the function is highly oscillatory.

A.3. Parabolic cylinder function

Parabolic cylinder functions $[D_\nu(x)]$ and $[D_{-\nu-1}(\mathrm{i}x])$ are two independent solutions to the Weber differential equation

$$y''(x) + \left(\nu + \tfrac{1}{2} - \tfrac{1}{4}x^2\right)y(x) = 0. \tag{A.7}$$

The two independent solutions are given by $y = D_\nu(x)$ and $y = D_{-\nu-1}(\mathrm{i}x)$, where

$$\begin{aligned} D_\nu(x) &= 2^{\nu/2+1/4}x^{-1/2}W_{\nu/2+1/4,-1/4}\left(\frac{1}{2}x^2\right) \\ &= \frac{2^{\nu/2}\,\mathrm{e}^{x^2/4}(-\mathrm{i}x)^{1/4}(\mathrm{i}x)^{1/4}}{\sqrt{x}}U\left(-\frac{1}{2}\nu,\frac{1}{2},\frac{1}{2}x^2\right), \end{aligned} \tag{A.8}$$

where $W_{km}(x)$ is the Whittaker function and $U(a,b,x)$ is a confluent hypergeometric function of the first kind. This function is implemented in Mathematica as *ParabolicCylinderD*[ν, z] and it is a regular function. For ν, a non-negative integer n, the solution D_n reduces to

$$D_n(x) = 2^{-n/2}\mathrm{e}^{x^2/4}H_n\left(-\frac{x}{\sqrt{2}}\right), \tag{A.9}$$

where $H_n(x)$ is a Hermite polynomial. For positive ν, $D_\nu(x)$ is oscillatory but converges, while for negative ν, $D_\nu(x)$ starts from some finite value and converges rapidly.

Now for the choice $n = 1$, $m = \frac{3}{2}$, we have from equation (71)

$$B_{,\eta\eta} - \frac{2V_0}{\hbar^2}\eta^2 B - \frac{2\omega^2}{\hbar^2}B = 0, \tag{A.10}$$

which under the transformation $x = \frac{(8V_0)^{\frac{1}{4}}}{\sqrt{\hbar}}$ becomes

$$B_{,xx} + \left[\left(-\frac{1}{2} - \frac{\omega^2}{\hbar\sqrt{2V_0}}\right) + \frac{1}{2} - \frac{x^2}{4}\right]B = 0, \tag{A.11}$$

whose solution is

$$B(\eta) = C_{31}D_{\left(-\frac{1}{2}-\frac{\omega^2}{\hbar\sqrt{2V_0}}\right)}\left(\frac{(8V_0)^{\frac{1}{4}}\eta}{\sqrt{\hbar}}\right) + C_{32}D_{\left(-\frac{1}{2}-\frac{\omega^2}{\hbar\sqrt{2V_0}}\right)}\left(\mathrm{i}\frac{(8V_0)^{\frac{1}{4}}\eta}{\sqrt{\hbar}}\right). \tag{A.12}$$

But the solution for A part is possible only for $\omega^2 = 0$, given in equation (72). So under the choice $\omega^2 = 0$, equation (A.12) is exactly the same as equation (73). Since $\nu = -\frac{1}{2}$ here, the parabolic cylinder function controls the IR divergence appearing in the modified Bessel function in equation (72). Thus, the wormhole exists for a massive scalar field.

A.4. Confluent hypergeometric functions (Kummers function)

The confluent hypergeometric equation,

$$xy'' + (b - x)y' - ay = 0, \tag{A.13}$$

is obtained from the hypergeometric equation by merging two of its singularities. It has a regular singularity at $x = 0$ and one irregular singularity at $x = \infty$. The independent solutions of the above equation are called the confluent hypergeometric functions of first $[M(a, b; x) = {}_1F_1(a, b; x)]$ and second $[U(a, b; x)]$ kinds. In terms of the Pochhammer symbols, these are expressed as

$$M(a, b; x) = {}_1F_1(a, b; x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{x^n}{n!} \tag{A.14}$$

and

$$U(a, b; x) = \frac{\Gamma(1-b)}{\Gamma(a-b+1)} M(a, b; x) + \frac{\Gamma(b-1)}{\Gamma(a)} M(a-b+1, 2-b; x). \tag{A.15}$$

$M(a, b; x)$ becomes singular if b becomes negative, otherwise it is a fast increasing function while $U(a, b; x)$ is a fast decreasing function.

Now, solution (109) of the differential equation (107) for $B(\eta)$ contains the product of a linear term in η, an exponent ($\mathrm{e}^{-\omega_0\eta}$), and the confluent hypergeometric function. Under the choice $C_{19} = 0$, the IR divergence of ${}_1F_1(a, b; x)$ disappears, while with $\omega_0 > 0$ the UV divergence of $U(a, b; x)$ is controlled by the $\eta = \phi a^6$ term. In the process, the wormhole boundary condition is realized for $p \leqslant -1$.

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