R N S INSTITUTE OF TECHNOLOGY

DEPARTMENTOF MATHEMATICS

Channasandra, Bangalore- 560 098



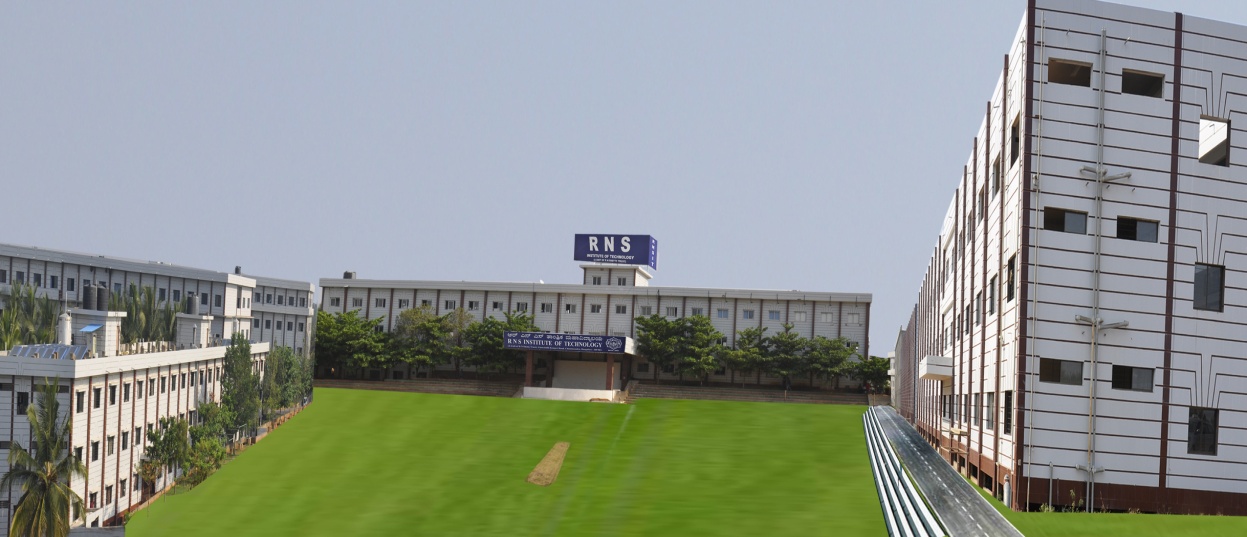
STUDY MATERIALS

FOR

MATHEMATICS

VTU NEW SYLLABUS

MODULE-1



Differential calculus-1

Module-1 Differential calculus-1

Determination of nth order derivatives of Standard functions - Problems. Leibnitz’s theorem (without proof) - problems. Polar Curves - Angle between the radius vector and tangent, Angle between two curves, Pedal equation of polar curves. Derivative of arc length - Cartesian, Parametric and Polar forms (without proof) - problems. Curvature and Radius of Curvature – Cartesian, Parametric, Polar and Pedal forms (without proof) -problems

Synopsis:

* Trigonometric Relation

1. 

2. 

3. 

4. 

5.  6.  8. 

9. 

10.  11. 

* Logarithmic Relation

1.  2. 

3.  4.  5. 

* Leibnitz Theorem



* Polar Curves

1. Relations to find angle between radius vector and the tangent at the point on the polar curve 

2. Relations to find length of the perpendicular from the pole (origin) the tangent to the polar curve is    
 and 

3. Angle between the two polar curves is 

4. For the Cartesian curve the derivative of arc length is 

5. For the Cartesian curve the derivative of arc length is 

6. For the parametric equation of the curve  the derivative of arc length is   
 

7. For the polar equation of the curve  the derivative of arc length is 

8. For the polar equation of the curve  the derivative of arc length is 

9.  10. 

11. The radius of curvature of the curve is 

12. The radius of curvature of the curve in

* Cartesian form is 
* Cartesian form is 
* parametric equation of the curve  is 
* Polar equation is 
* Pedal equation is
* nth Derivatives of some standard functions:

|  |  |  |
| --- | --- | --- |
| Sl. no. | Functions (y) | nth derivative (yn) |
| 1. |  |  |
| 2. |  |  |
| 3. |  |  |
| 4. |  |  |
| 5. |  |  |
| 6. |  |  |
| 7. |  |  |
| 8. |  |  |
| 9. |  |  |
| 10. |  |  |
| 11. |  |  |
| 12. |  |  |
| 13. |  |  |
| 14. |  |  |
| 15. |  |  |

1.0 Introduction: nth Order Derivatives

Differential Calculus is one of the most revered fields of study. Measure the rate of change of a given function. It is used and applied in all the Engineering departments. Higher order derivatives give the characteristics of a function and this function can be any practical relation of a moving body, circuit mechanism, constructional analysis, and many more. It tells the maximum and minimum variations and slope characteristics.

* The first derivative,  denotes velocity at a given time
* The second derivative,  denotes acceleration at a time
* The third derivative,  denotes the [jerk or jolt](http://en.wikipedia.org/wiki/Jerk_%28physics%29) at time , an important quantity in engineering and motion control
* The fourth derivative denotes the [jounce](http://en.wikipedia.org/wiki/Jounce) at time; the jounce is also used in studying motion and in studying the cosmological equation of state.
* Fifth and sixth derivatives of position are also important in some applications/theoretical physics studies, but they have no universally accepted name.
* Fifth derivative and curve fitting are used to do DNA analysis and population matching.
* And higher derivatives are also used for approximating functions using Taylor polynomials, which can be useful when a certain amount of precision is required.
* nth order is used to formulate a generalised function for any order derivative. This makes it easier for calculation

Leibnitz was a German [polymath](https://en.wikipedia.org/wiki/Polymath) and philosopher who occupies a prominent place in

the historyof mathematics and the history of philosophy, having developed 

differential and integral calculus [independently](https://en.wikipedia.org/wiki/Multiple_discovery) of [Isaac Newton](https://en.wikipedia.org/wiki/Isaac_Newton).

[Leibniz's notation](https://en.wikipedia.org/wiki/Leibniz%27s_notation) has been widely used ever since it was published. It was only in the 20th century that his [Law of Continuity](https://en.wikipedia.org/wiki/Law_of_Continuity)  and [Transcendental Law of Homogeneity](https://en.wikipedia.org/wiki/Transcendental_Law_of_Homogeneity)  found mathematical implementation (by means of [non-standard analysis](https://en.wikipedia.org/wiki/Non-standard_analysis)). He became one of the most prolific inventors in the field of [mechanical calculators](https://en.wikipedia.org/wiki/Mechanical_calculator). While working on adding automatic multiplication and division to [Pascal's calculator](https://en.wikipedia.org/wiki/Pascal%27s_calculator), he was the first to describe a [pinwheel calculator](https://en.wikipedia.org/wiki/Pinwheel_calculator) in 1685 and invented the [Leibniz wheel](https://en.wikipedia.org/wiki/Leibniz_wheel), used in the [arithmometer](https://en.wikipedia.org/wiki/Arithmometer), the first mass-produced mechanical calculator. He also refined the [binary number](https://en.wikipedia.org/wiki/Binary_number) system, which is the foundation of virtually all digital computers.

1.1 nth derivative of some standard results:

* Find the nth derivative of .

Solution: Let 

Differentiating w r to x







.

Case(i): If m is a positive integer and m > n. Then, we have,  




In particular, 

Case(ii): If (a positive integer). Then



In particular, 

Case(iii): If m is a positive integer and . 

Then, 

Case(iv): If m is a positive integer and .

Then 

In particular, 

Case(v): If .

Then 

In particular, 

* Find the nth derivative of .

Solution: Let 

Differentiating w r to x

; ;  ;   
 



In particular, 

* Find the nth derivative of .

Solution: Let 

Differentiating w r to x











In particular,   
 

* Find the nth derivative of .

Solution: Let 

Differentiating w r to x











In particular,   
 

* Find the nth derivative of .

Solution: Let 

Differentiating w r to x   
 

Setting  and 

so that , 











Proceeding like this, we get





In particular,   
 

* Find the nth derivative of .

Solution: Let 

Differentiating w r to x



Setting  and 

so that , 









Proceeding like this, we get





In particular,   
 

1.2 Type-1 : In this type, before applying derivatives of standard functions, we convert products of functions of sine and cosine into sums using the following well-known relations.

* Trigonometric Relation

1. 

2. 

3. 

4. 

5.  6.  8. 

9. 

Problems:

1. Find the nth derivative of the following functions:

(*i*)  (VTU., 2006) (*ii*)  (*iii*) 

(*iv*)  (VTU., 2009) (v) 

Solution:

* (i) Let 







Therefore,    
 

* (ii) Let 







* (iii) Let 





* (iv) Let 











* (v) Let 

Therefore, 



1.3 Type-2

In this type, before applying derivatives of standard functions, we convert improper fractions into a sum of a polynomial and a proper functions and use partial fractions.

2. Find the nth derivative of the following functions:

(i)  (VTU 2009) (ii)  (VTU., 2005, 2013) (iii) 

(iv)  (v)  (VTU J2017) (vi)  (vii) 

(viii) 

* Solution: (i) 

Let  (1)

Consider, 

 (2)

putting  in (2), we get 

and putting  in (2) we get .

substituting the values of *A* and *B* in (1),

* (ii) 

Solution: Since the given fraction is an improper fraction.

To convert it into proper fraction divide the numerator polynomial from the denominator polynomial.





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Let  (1)

Consider,  (2)

. (3)

putting  in (3), we get 

and putting  in (3), we get .

Substituting the values of *A* and *B* in (2), 

and the equation (1) gives, 

(iii) 

Solution: Let  (1)

Consider,

. (2)

putting  in (2), we get 

and putting  in (2), we get .

Substituting the values of *A* and *B* in (1),



* (iv) 

Solution: Let  (1)

Consider,

 . (2)

putting  in (2), we get 

and putting  in (2), we get .

Substituting the values of *A* and *B* in (1),



* (v) 

Solution: Let  (1)

Consider 

 (2)

putting  in (2), we get 

and putting  in (2), we get 

Collect the coefficient of  on both sides of equation (2),

we get 



Substituting the values of *A, B* and *C* in (1),







* (vi) 

Solution: Let  (1)

Consider 

. (2)

Taking  in (2), we get 

and taking  in (2), we get .

 Equation (1) gives 

and 





Put  so that  and 

Then,  and 





and 













* (vii) 

Solution: Let  (1)

. (2)

taking , in (2), we get 

and taking  in (2), we get .









Put  so that  and 

Then,  and 

 and 

and 













* (viii) 

Solution: Let  (1)

Consider 

 (2)

By putting  in (2), we get ,

Collecting the coefficient of  on both sides we get  so that ,

Collecting the coefficient of  on both sides we get .

Subzstituting *A, B and C* in (1), we get





 [using the above problem]

1.4 Type-3

In this type we discuss to find nth derivatives of functions whose derivatives can be obtained using the results of previous type.

3. Find the nth derivative of (i)  (ii)  (iii)  (iv) 

* (i) Let 





* (ii) Let 





* (iii) Let 







* (iv) Let 





1.5 Type-4

In this type we discuss to find nth derivatives of logarithmic functions. we use the properties of logarithms

* Logarithmic Relation

1.  2. 

3.  4.  5. 

4. Find the nth derivative of the following:

(i)  (VTU 2010) (ii)  (iii) 

(iv)  (v) .

Solution:

* (i) Let 





* (ii) 







* (iii) 







* (iv) 









* (v) 







1.6 Type-5

* In this type we discuss to find nth derivatives of product of exponential and sine and exponential and cosine functions by converting into sums of exponential and sine functions or exponential and cosine functions, using the formula
* 
* 
* 

5. Find the nth derivative of the following functions:

(i) (VTU 2010S) (ii)  (iii) 

(iv)  (v)  (vi) (VTU D2016)

Solution:

* (i) Let 









* (ii)











* (iii) 

Let 









* (iv) 

Let 







* (v) 

Let













* (vi) 

Let 







Exercise:

1. Find the nth derivative of the following

(i)  Ans: 

(ii)  Ans: 

(iii) Ans: 

(iv) Ans: 

(v)  Ans: 

(vi) Ans: 

2. Find the nth derivative of the following:

(i)  Ans: 

(ii)  Ans: 

3. Find the nth derivative of the following:

(i)  Ans: 

(ii) Ans: 

(iii) Ans: 

(iv) Ans: 

(v) Ans: 

(vi) Ans: 

(vii) Ans: 

(viii)

Ans:

(ix) Ans: 

(x) Ans: 

4. Find the nth derivative of the following:

(i) Ans: 

(ii)  Ans: 

(iii)  Ans: 

(iv)  Ans: 

(v)  Ans: 

(vi)  Ans: 

(vii)  Ans: 

(viii)  Ans: 

(ix)  Ans: 

(x)  Ans: 

(xi)  Ans: 

5. Find the nth derivative of the following:

(i)  Ans: 

(ii)  Ans: 

(iii)  Ans: 

(iv)  Ans: 

(v)  Ans:

(vi)  Ans: 

(vii)  Ans: 

(viii)  Ans: 

(ix)  Ans: 

1.7 Type-6

In this type we discuss to find nth derivatives of given functions use the Leibntiz’s theorem.

1.7.1 LEIBNITZ THEOREM

The invention of calculus is now credited jointly to Isaac Newton (1642 - 1727) and Gottfried W. Leibniz (146 - 1716) the following theorem given by Leibniz helps us to find the nth derivative of the product of two functions.

Statement : If *u* and *v* be two functions of *x* possessing derivatives of the nth order then

 (1)

(where suffixes denote the differentiation of those orders,  and 

1.7.2 Working rule:

* Choose properly  and , since  will become zero for some *n* so that the process   
   terminates at some stage.
* In all the standard type of problems use your discretion for getting the result required.

1.7.3 Problems on Leibnitz’s theorem:

1. Find the nth derivatives of the following functions

(i)  (ii)  (iii)  (iv) 

Solution:

* (i) To find 

Taking and , applying Leibnitz theorem, we have





* (ii) To find 

Taking and applying Leibnitz theorem we have













* (iii) To find 

Let  then



Multiplying both sides by , we have



or  (1)

Differentiating both sides of equation (1)  times using Leibnitz theorem, we have









or 

* (iv) To find 

By Leibnitz theorem, we have







2. Find the nth derivative of .

Solution: Let 

Let 







 etc

Note: Take always *v* is algebraic i.e., 

 By Leibnitz’s theorem, we get







3. Find the nth derivative of 

Solution: Let, 



 etc.

By Leibnitz’s theorem, we get









4. Find the nth derivative of 

Solution: Let 









 etc.

By Leibnitz’s theorem, we get







5. Find the nth derivative of 

Solution: Let 







 etc.

By Leibntiz’s theorem,











6. Find the nth derivative of 

Solution: Let 



 etc.











7. Evaluate 

Solution: Consider 



 etc

By Leibnitz’s theorem, we have











8. If  show that . (VTU 2006)

Solution: Let  or  (1)

Differentiate (1) n times by using Leibnitz’s theorem taking  as the first function, we have

















9. If show that .

Solution: Let 

Differentiate w r t x,









Differentiate every term  times by using Leibnitz’s theorem









 or 

10. If  prove that  and hence show that

 (VTU 2001)

Solution: Let 











 (1)

Putting the values for n = 1, 2, 3, 4, 5, … in (1), we get

n = 1, 

n = 2, 



n = 3, 

 and so on.

Similarly, 

1.8 Type-7

In this type we express y in terms of  and by applying the Libnitz’s theorem to get the expression in terms of .

1. If  then show that

(i) 

(ii)  (VTU 2003)

Solution: Let 

Differentiate with respect to x





Differentiate with respect to x once again





  from given equation

 (1)

Differentiate every terms of equation(1)  times by using Leibnitz’s theorem, we get



I term : .



II term : 

 etc.

III term :  substituting these values in Leibnitz theorem, we get





2. If  or  or  prove that

 (VTU July 2004, D2016)

Solution: Let 



Differentiate with respect to x.





Differentiate every term  times by using Leibnitz’s theorem,









i. e., 

3. If  or  or If , show that

(i) 

(ii) . (VTU : July 2004)

Solution: (i) Consider 

i.e., 

Eliminating  we get 



Differentiate with respect to x,





Squaring on both sides,











Differentiate every term  times by using Leibnitz’s theorem,











4. If  show that (i) ,

(ii)  (VTU 2005)

Solution: (i) Consider 

i.e., 

Eliminating  we get 



Differentiate with respect to x,





Squaring on both sides,











Differentiate every term  times by using Leibnitz’s theorem,











5. If  show that 

or If . Show that  satisfies the equation    
 (VTU Feb2005)

Solution: Let 





Differentiate with respect to x again





Differentiate every term  times by using Leibnitz theorem,













6. If  show that (i) ,

(ii)  (VTU July2006, 2007)

Solution: (i) Let 

Differentiate with respect to x,





Squaring on both sides



Differentiate with respect to x







(ii) Differentiate every term  times by using Leibnitz’s theorem.













7. If  prove that . Also S T (i) 

(ii)  (VTU 2008S, 2011, 2013)

Solution: Given 

 [multiplying throughout by ]







Taking  power on both sides we get



Differentiating with respect to x











Squaring on both sides









or 

(ii) Differentiate every term  times by Leibnitz’s theorem.















Similarly we get for 

8. If  show that (i) 

(ii)  Hence find  (VTU 2009)

Solution: (i) Let 

Differentiate with respect to x,







Squaring on both sides



Differentiate with respect to x







(ii) Differentiate every term  times by using Leibnitz’s theorem.













when *x = 0; *

Also replace n by *n-2*, we get  and 

9. If , prove that . Hence show that   
  (VTU 2009)

Solution: Given 

Differentiate with respect to x,



Squaring on both sides



Differentiate again with respect to x,







Differentiate every term  times by using Leibnitz’s theorem.











 (1)

put x = 0 in (1), 

 (2)

This is a recurrence relation in n

Taking n = 0, 1, 2, ... in (2) we get

n = 0 in (2),  

n = 1 in (2),  

n = 2 in (2), 

n = 3 in (2), 

n = 4 in (2), 

n = 5 in (2), 

n = 6 in (2), 

n = 7 in (2),  and so on.

Hence 

10. If  show that (i) ,

(ii)  at x = 0 (VTU 2009)

Solution: (i) Let 

Differentiate with respect to x,







Squaring on both sides



Differentiate with respect to x





 or 

(ii) Differentiate every term  times by using Leibnitz’s theorem.













when *x = 0; * is the required result.

11. If  show that  (VTU 2010, 2014)

Solution: Let 

Squaring on both sides, we get 

Differentiate with respect to x,









Differentiate again with respect to x,





(ii) Differentiate every term  times by using Leibnitz’s theorem.















12. If  show that

 (VTU 2010S)

Solution: Let 

Differentiate with respect to x,







Squaring on both sides, we get



Differentiate again with respect to x,









Differentiate every term  times by using Leibnitz’s theorem.













13. If  show that (i) 

(ii) . (VTU 2012)

Solution: (i) 

Differentiate with respect to x,



Squaring on both sides



Differentiate once again with respect to x,









(ii)Differentiate every term times by using Leibnitz’s thm, we get,













14. If show that .

Hence determine the value of  when  (VTU 2013)

Solution: (i) 

Differentiate with respect to x,



Squaring on both sides



Differentiate again with respect to x,







(ii)Differentiate every term times by using Leibnitz’s theorem, we get,











 (1)

put x = 0 in (1), we get 

 (2)

when 

put  in (2)

n = 1, 

n = 2, 

n = 3, 

when n is odd

when n is even.

15. If  show that

(i) ,

(ii)  (VTU 2004, 2014, D2016)

Solution: (i) 



Differentiate with respect to x,





Differentiate again with respect to x,







Differentiate every term times by using Leibnitz’s theorem, we get,















16. If show that .

Hence determine the value of all derivatives of with respect to x when 

Solution: Let  (1)

Differentiate with respect to 

 (2)

Differentiate  times by using Leibnitz’s theorem.







 (3)

Putting  in (1), (2) and (3)

 and  (4)

Putting  in (4)

n = 1 in (4), 

n = 2 in (4), 

n = 3 in (4), 

n = 1 in (4), 

Hence when  is odd 

When  is even .

17. If  or  or  show that

(i)  (ii) 

Solution: (i) 





Squaring on both sides



Differentiate with respect to x







(ii) Differentiate every term  times by using Leibnitz’s theorem we get,













18. If  show that

(i)  (ii) 

Solution: (i) Consider 







Differentiate with respect to x,









Once again differentiate with respect to x,











(ii) Differentiate every term  times by using Leibnitz’s theorem,













19. If  show that . Hence deduce that   
 

Solution: Let 

Differentiate with respect to x,





Differentiate with respect to x,









Differentiate  times by Leibnitz theorem, we get











Example 20. If  or  then, Show that

 (VTU J2017)

Solution: Consider the given





i. e., 



Taking  power both sides















Squaring on both sides 

Differentiating w r to x









Differentiate every term  times by using Leibnitz theorem,





I term : 

.

II term : 

...etc

III term : Substituting all these in Leibnitz theorem.







Polar Curves

Polar Curves comprises of Curvature, Radius of Curvature, Arc length and so on.

Curvature is a measure of rate of change of a bentness. Straight line has zero bentness while a circle has constant bentness.

These help was to calculate the orbits of satellites, gravitational theorems, even metallurgical analysis is dependent on these, in automobile design, aerodynamics, and in fluid dynamics.

Polar curves have many applications in engineering fields. Some application of differential calculus to geometry will discuss in this section. Other than Cartesian system of coordinates, there is another system to represent a point and curve analytically in a plane known as the polar coordinate system.

Let us choose a fixed point on the plane and call it as the pole. A fixed line  drawn form the pole is called initial line or polar axis. Then the position of any point P in the plane, is obtaining by joining the points O and P. If the distance OP = r is called radius vector and X, the vectorial angle. The angle  is considered positive when it is measured in the anti-clockwise direction and clearly r  and Then (are called the polar coordinates of the point P.

Let  be the Cartesian coordinates of the point P. Then we have

P(r,θ)

y

x

Y

X

o

 (1)

gives  (2)

The relation (1) finds Cartesian in terms of polar coordinates. Conversely relation (2) gives polar coordinates in terms of Cartesian coordinates. The equation of a curve in polar coordinates will be in the form  A curve specified by a polar equations is referred to as a polar curve.

Angle between Radius Vector and Tangent

1. With usual notation prove that 

Y

o

P(r,θ)

X

θ

ϕθ

where  is the angle between radius vector and tangent.

Let  be any point on a polar curve

 (1)

 and  (2)

Draw a tangent to the curve at , meets the  at the point . Let the angle between radius vector  and the tangent  be  and  be the angle between tangent  and positive .

Let  be the Cartesian coordinates of the point  then we have

, 

 (3)

and  slope of tangent 

 fig

 (4)

 [from (3)]

Dividing numerator and denominator by  we get

 (5)

from (4) and (5), we get, .

Thus .

Angle of intersection of two polar curves

The angle of intersection of two curves is the angle between their tangent at that point. Let the curves

 and  intersect at . Let and  be the angles between the common radius vector OP and tangents PT1 and PT2. Hence the angle between the two tangents is equal to . Therefore the acute angle of intersection of the curves is equal to .

where and  are determined by the formula.

 for 

 for 

Suppose the angle between two curves is given by 

or equivalently , then we say that the curves intersect orthogonally.

Problems:

Find the angle of intersection between the following curves:

1.  and  (VTU D2011)

Solution: Consider the given curves

 (1) and  (2)

Taking log for the both curves

Differentiate with respect to  Differentiate with respect to 

and 

2.  and  VTU 2012 D

Solution: Consider the given curves  (1) and  (2)

Solving the equations 



Differentiate (1) with respect to 







and 

Differentiate (1) with respect to 











At    




3.  and  (VTU 2008S, D2016)

Solution: Consider the given curves  (1)

and  (2)

Taking log for the both curves

Differentiate with respect to  Differentiate with respect to 

therefore the angle between the two curves is given by

 (3)

Solving the equations (1) and (2) 





and 



4.  and  (VTU 2004)

Solution: Consider the given curves (1) and  (2)

Taking log for the both curves

Differentiate with respect to  Differentiate with respect to 

 [dividing Nr and Dr by ] 







5.  and  (VTU F2003, 2006)

Solution: Consider the given curves (1) and  (2)

solving the equations (1) and (2), 



Differentiate with respect to  Differentiate with respect to 

; For all 

when  , 

6. Show that  and  cut orthogonally. (VTU 2003, F06,11S, J15)

Solution: The curves are

For the curve (1) we have







For the curve (2) we have 



But 





 the curves cut orthogonally.

7. Show that  and  cut orthogonally.

Solution : Consider the given curve

Taking log on both sides of given curves, we have

and 

Differentiate with respect to  we get



So that the curves cut 

9. Show that the tangent to the Cardioid  at the points  and   
  are respectively parallel and perpendicular to the initial line.

Solution: Given the curve 

Differentiate with respect to  we get











now we have 

when  

This shows that the tangent and the initial line co insides.

when  

This shows that the tangent and the initial line perpendicular.

10. Find the angle of intersection of curves .

Solution: Let the curves  

Taking log for both curves



Differentiate with respect to 

. 

But,  and 

Solving the given equations, we get





At  





Let  be the angle of intersection of two curves



11. Show that the curves  ,  intersect each other orthogonally. Solution: Consider the given curve  (1)

 (2)

Differentiate (1) with respect to , we get







Consider 

Differentiate with respect to , we get











So the curves intersect orthogonally.

12. Show that the curves  and  cut orthogonally.

Solution: The given curves are

 (1)

 (2)

Differentiate (1) with respect to  we get











For the curve (2), 











So the curves cut orthogonally.

13. Show that the curves and  intersect each other orthogonally.

Solution: The given curves are  (1) and  (2)

For the curve (1)  we have





For the curve (2) we have







Eliminating r between equations (1) and (2), we get

i.e., 

When 

When  

The two points of intersection are  and 

At  

and at  

Hence curves intersect orthogonally.

14. Show that the curves   intersect each other orthogonally.

Solution: The given curves are  (1)  (2)

Taking log both sides for the curves

 and 

Differentiate with respect to , we get

But  



So the curves intersect orthogonally.

15. Show that  and  cut orthogonally. (J2017)

Solution: The curves are

 (1) and  (2)

For the curve (1) we have







For the curve (2) we have 



But 





 the curves cut orthogonally.

Questions Bank 1.3

1. Find the angle between radius victor and the tangent for each of the following curves.

(a)  Ans: 

(b)  Ans : 

(c) . Ans : 

(d)  Ans : 

(e)  Ans : 

(f) 

(g) 

2. Prove that the normal at any pint on the curve makes an angle  with   
 the initial line.

3. Find the slope of the following curves.

(a)  

(b)  

(c) 

(d)  

5. Find the angle of intersection of each of the following pairs of curves.

(a) 

(b) 

(c) 

(d) 

(e) 

(f) and 

6. Show that each of the following pairs of curves intersect orthogonally.

(a) and 

(b)  and 

(c) and 

(d) 

(e)  and 

(f) 

Length of the perpendicular form pole to the Tangent:

Consider a point  on the curve 

Draw  perpendicular form the pole to the tangent at P.

Form Fig.

p

0

X

M

r

 and 

Then we have 

or squaring and inverting, we get



 (1)

Note: This formula can be expressed in another form as

Let  then, 

substituting these in (1), we get



1.4 Pedal Equation

Any equation containing p and r (without ) is called pedal equation or p-r equation and where p is length of perpendicular from the pole to the tangent and r the radius vector.

Note: (i) If  (1)  (2) and  (3)

Eliminating  and  from these equations we get pedal equation.

(ii) 

If  is express in terms of  explicitly then we use (i) otherwise we use (ii).

Example: 1. Find the pedal equation of the curve 

Solution: Consider the given curve 

Differentiate with respect to , we get 







We have, 



Squaring on both sides;



Example: 2 find the pedal equation of the curve  (VTU 2009)

Solution: Consider the given curve 

Differentiate with respect to , we get 







We have, 





Squaring on both sides;



Example: 3. find the pedal equation of the curve 

Solution: Taking log on both sides 

Differentiate with respect to , we get



But 



We have 

 (1)

Consider the given equation  

It can be written as 



 or 

Thus  is the required pedal equation.

Example: 4. Find the pedal equation of the curve 

Solution: The given curve is  (1)

Taking log on both sides for the given curve,



Differentiate with respect to ,









We have 



 (2) 

Substitute (2) in (1), we get



 is the required pedal equation.

Example: 5. Find pedal equation of the curve 

Solution: The given curve 

Differentiate with respect to  we get







Since we are not able to eliminate  in ,

The  equation is obtain by



i.e., 



 [ from (1)  ]

i.e., 

Hence  is the required pedal equation.

Example:6. Find the pedal equation of the curve 

Solution: The given equation is  (1)

Taking log on both sides 

Differentiate with respect to , we get







We have 



or 

Substituting this in (1), we get 

Hence,  is the required pedal equation.

Example : 7. Find the pedal equation of the curve  (VTU 2010, D2016)

Solution: The equation of the given curve  (1)

Taking log on both sides



Differentiate with respect to , we get







We have 







 is the required pedal equation.

Example: 8. Find the pedal equation of the curve,  (VTU F2005, J2015)

Solution: consider the equation 

Taking log on both sides,



Differentiate with respect to 







Consider 

squaring and taking reciprocal we get

 or 









 is the required pedal equation.

Example 9. Find the pedal equation of the curve . (VTU 2010, 2014J, J2017)

Solution: The equation of the curve 

Taking log on both sides,



Differentiate with respect to 







Consider 



 is the required pedal equation.

10. Find the Pedal equation of the curve  (VTU 2004)

Solution: The equation of the curve 

Taking log on both sides,



Differentiate with respect to 







Consider 



 is the required pedal equation.

11. Find the Pedal equation of curve  (VTU 2010)

Solution: The equation of the curve 

Taking log on both sides,



Differentiate with respect to 





Consider 



Squaring on both sides



 is the required pedal equation.

12. Find the Pedal equation of curve  (VTU 2007)

Solution: The equation of the curve 

Taking log on both sides,



Differentiate with respect to 



Consider 

Squaring and taking reciprocal we get

 or 



 is the required pedal equation.

13. find the pedal equation of the curve  (VTU., D2016)

Solution: Taking log on both sides 

Differentiate with respect to , we get



But 



We have 

 (1)

Consider the given equation 

It can be written as 



 or 

Thus  is the required pedal equation.

Question Bank:

1.Find the pedal equation for the following curves

(a) 

(b) 

(c) 

(d) 

(e) : 

(f) 

(g) 

(h) 

2. Show that the pedal equation of the curve  is 

3. Show that the length of the perpendicular from the pole to the tangent at the point  on the curve  is equal to .

Derivative of arc length:

* If the equation of the curve in Cartesian form , then the derivative of arc length is



* If the equation of the curve in Cartesian form , then the derivative of arc length is 
* If the equation of the curve in parametric form  then the derivative of arc length is  where t is a parameter.
* To find , we have 
* For the curve in polar form  we have the derivative of arc length 
* For the curve in polar form  the derivative of arc length is 
* To find , we have 

Problems:

1. Calculate  for the following curves:

(i)  (ii)  (iii) 

Solution: (i) Given  (1)

Differentiating (1) w r to x, we get 

or 



Derivative of arc length 



(ii) 

Solution: Given 

Differentiating w r to x, we get



Derivative of arc length 



(iii) 

Solution: Given 

Differentiating w r to x, we get





Derivative of arc length 



2. Find  for the curve  (VTU 2007)

Solution: For the curve 

Differentiating x and y w r to , we get 

and 

Derivative of arc length 



3. Find the derivative of arc length of  and  (VTU J2017)

Solution: For the curve  and 

Differentiating x and y w r to t, we get 



and 

Derivative of arc length 







4. Show that for the Astroid or ,  at  is .

Solution: Given curve is 

Differentiating x and y w r to , we get



and 

Derivative of arc length 





At the point , 



3. Find  for the following curves:

(i)  (VTU 2004) (ii) 

(ii)  (VTU 2007)

Solution: (i) Given 

Differentiating w r to , we get 

Derivative of arc length is 











(ii) 

Solution: Given 

Taking log on both sides we get 

Differentiating w r to 





Derivative of arc length is 







(iii) 

Solution: Given 

Taking log on both sides we get 

Differentiating w r to 





Derivative of arc length is 







4. Show that for the curve ,  varies inversely as .

Solution: Given 

Taking log on both sides we get ..

Differentiating w r to 





Derivative of arc length is 



   
  [but  ]



  varies inversely as .

5. For the curves  prove that  (VTU 2005)

Solution: Given 

Differentiating w r to r, we get













Derivative of arc length is 









or 

6. With the usual meanings for  for the polar curve  show that

 (VTU 2000)

Solution: we have 

Differentiating w r to s, we get



Multiplying throughout by , we get







or 

Question Bank:

1. Find  and  for the following curves

(i)  (ii)  at (3,3)

Ans: (i)   (ii) 

2. Find  for the following curves

(i)  (ii)  (iii)  (iv) 

Ans: (i)  (ii)  (iii) (iv) 

3. Find  and for the following curves

(i)  (ii)  (iii)  (iv) 

Ans: (i)  (ii)  , (iv)  (iv) 

4. With the usual meanings for  and  for the polar curve , show that 

Curvature:

Let P be any point on a given curve and Q a neighbouring point.

Y

o

P

X

θ

Q

Let arc AP = s and arc . Let the tangents at P and Q

make angle  and  with the x-axis, so that the angle

between the tangents at P and Q is . In moving from the

point P to Q through a distance , the tangents has turned

through the angle . This is called the total bending or total

curvature of the arc PQ.

Therefore the average curvature of arc

The limiting value of average curvature when Q approaches P (i,e., ) is defined as the curvature of the curve at P. Thus curvature 

The unit of measurement of a curvature is radians per unit length.

Radius of curvature: The reciprocal of the curvature of a curve at any point p is called the radius of curvature at p and is denoted by  and defined by .

1. Radius of curvature for Cartesian curve  is given by: 

We know that  or 

Differentiating both sides with respect to x,







Note: 1. The radius of curvature is positive or negative according as  is positive or negative(i.e according as the curve is concave upwards or downwards).

2. Since the radius of curvature is independent of the choice of coordinate axes, therefore interchanging x and y we get   
 this formula is useful when the tangent is perpendicular to the x-axis. i.e  or .

2. Radius of curvature for parametric equation is given by:   


The equation of the curve in parametric form is , then

 where dashes represent the differentiations with respect to t



Substituting the values of  and  in  we get





or 

Problems: (Cartesian form)

1. Find the radius of curvature at the point  of the Folium    
 (VTU2008, J2017)

Solution: Let the given curve is 

Differentiating with respect to x, we get



 (1)

Differentiating (1), 









At  the radius of curvature 

2. Find the radius of curvature at the point  of the curve  (VTU 2010)

Solution: Let the given curve is 

Differentiating with respect to x, we get

 (1)



Differentiate (1) with respect to x, we get





At  the radius of curvature 



3. Show that the radius of curvature at  on the curve  is 

(VTU 2000S, 2014)

Solution: Let the given curve is 

Differentiating with respect to x, we get





 (1)



Differentiating (1) with respect to y, we get





At  the radius of curvature 



4. Show that the radius of curvature at  of the curve  is . (2009S)

Solution: Let the given curve is 

Differentiating with respect to x, we get

 (2)

At , 

Differentiating (1) with respect to x, we get



At , 

At  the radius of curvature 



5. For the curve  show that  . (VTU 2008)

Solution: Let the given curve is 

Differentiate with respect to x, we get





 (1)

 []

Differentiate (1) with respect to x, we get



 []



Therefore the radius of curvature 



Squaring on both sides we get   


Raising the power 1/3 on both sides,   


or 

6. Show that the radius of curvature for the curve  where it crosses the line   
  is .

Solution: Let the given curve is 

Differentiate with respect to x, we get

 (1)

Differentiate (1) with respect to x,







Therefore the radius of curvature 



Solving the two equation  and , we get

 and 

At 

8. Find the radius of curvature at any point  of the parabola 

Solution: Let the given curve is 

Differentiate with respect to x, we get

 (1)

At the point  

Differentiate (1) with respect to x, we get



At the  

Therefore the radius of curvature 



9. Find the radius of curvature at the origin for

(i)  (ii) .

(iii) 

(i) Solution: Let the given curve is 

Differentiate with respect to x, we get

 (1)

At the point  

Differentiate (1) with respect to x, we get



At the point    




Therefore the radius of curvature 

(ii) .

Solution: Let the given curve is 

Differentiate with respect to x, we get



 (1)

At the point  



Rewrite equation (1), 

Differentiate with respect to y, we get



At the point    




Therefore the radius of curvature 

(iii) 

Solution: Let the given curve is 



Differentiate with respect to x, we get



 (1)

At (0, 0) 

Differentiate (1) with respect to x,



At (0, 0)   


Therefore the radius of curvature 

10. Find radius of curvature of  at (1, 1) (VTU D2016)

Solution: Given  (1)

Differentiating (1) w r to x we get





 (2)

At the point (1,1), 

Differentiating (2) w r to x



At the point (1, 1),   


Therefore the radius of curvature 

or 

11. In the Ellipse , show that the radius of curvature at an end point of the major axis is equal to the semi-latus rectum.

Solution: Given the equation of the ellipse  (1)

The length of the latus rectum is .

Differentiating (1) w r to x, we get

 (2)

Differentiating (2) w r to x



 using (2)



Therefore, 

 (3)

At the end point of the major axis, we have 



.

Problems: (Parametric form)

1. Show that the radius of curvature at any point of the cycloid

 is  .

Solution: Let the given curve is 

Differentiate with respect to , we get

 and 

Differentiate again with respect to , we get

 and 

Consider, 





And 





Therefore the radius of curvature 





2. Prove that the radius of curvature at any point of the Astriod  is three   
 times the length of the perpendicular from the origin to the tangent at that point.

Solution: Let the parametric equation of the given curve is

 and 

Differentiate with respect to, we get

 and 

Differentiate again with respect to , we get





and 



Consider, 







And 









Therefore the radius of curvature 





3. Find the radius of curvature at any point on the ellipse: 

Solution: Let the given curve is 

Differentiate with respect to, we get

 and 

Differentiate again with respect to , we get

 and 

Consider, 



And 



Therefore the radius of curvature 



4. Find the radius of curvature at any point on the curve

 .

Solution: Let the given curve is 

Differentiate with respect to t, we get



and 

Differentiate again with respect to t, we get

 and 

Consider, 



And 





Therefore the radius of curvature 

5. Show that the radius of curvature at the point t on the curve  is   
 

Solution: Let the given curve is 

Differentiate with respect to t, we get



and 

Differentiate again with respect to t, we get





and 



Consider, 





And 





Therefore the radius of curvature 

6. Find the radius of curvature for the curve . (VTU J2015)

Solution: Let the given curve is 

Differentiate with respect to t, we get









Differentiate again with respect to , we get





Now, 



Differentiate again with respect to , we get



Consider, 







And 



Therefore the radius of curvature 

3. Radius of curvature for polar curve  is given by: 

Proof: With the usual notations, from the fig.

p

0

X

M

r



Differentiating with respect to s,



 (1)

Also we know that



 where 

 (2)

Also  (3)

Substituting the value from (2) and (3) in (1),



or   


4. Radius of curvature for pedal curve  is given by: .

Solution: Radius of curvature for pedal curve 

From the above figure with usual notation we have 

Differentiating with respect to s,   
 (1)

Also we know that 



 []

 [from equation (1)]

or 

Consequence of the above relation is

Obtain a formula for  in terms of 

Solution: The radius of curvature in the pedal form is  (1)

Rewriting the equation (1) we have





 (2)

And also,  (3)

Squaring and adding (2) and (3), we get

 (4)

Differentiating (4) w r to , we get



or   




Therefore,   


This is the formula for  in terms of  is known as the formula for  in the tangential polar form.

Problems on Polar and pedal form:

1. Show that the radius of curvature at any point  of the cardioids

 varies as  . (VTU2003)

Solution: Given a polar curve 

Differentiating with respect to 



















and 

 the radius of curvature is 



 (1)

 The radius of curvature varies as .

Also squaring on both sides of the equation (1), we get 

.

2.  (VTU D2016)

Solution: Given a polar curve 

Taking log on both sides we get 

Differentiating with respect to 





Differentiating with respect to 















and given the polar curve 

 The radius of curvature is 





 The radius of curvature varies inversely as .

3. Find the radius of curvature for the parabola .

Solution: Given a polar curve 

Taking log on both sides, we get 

Differentiating with respect to 







and 















and  (1)

 The radius of curvature is 



Hence 

 The radius of curvature varies as 

4. Show that  at any point  of the cardioids  is constant. (VTU., 2001)

Solution: Given a polar curve 

Differentiating with respect to 















 []



and 

 The radius of curvature is 







or   


5. If  be the radii of curvature at the extremities of any chord of the Cardioid

 which passes through the pole, show that 

Solution: Let  and  be the two polar chards (chords passing through the pole) of the Cardioid

. Let  be the radii of curvatures at the point P and Q corresponding to the vectorial angle  respectively.

We proved the result for the Cardioid is  in previous problem

 At the point P,  (1)

also at the point Q, ,



 (2)

Adding (1) and (2), we get 





6. For any curve , Prove that .

Solution: For any curve ,

p

0

X

M

r

From the fig. we have  (1)

Differentiating (1) w r to s we get



 (2)

we know that 

The equation (2) 

or 

7. Prove that for the ellipse in pedal form , the radius of curvature at the   
 point  is 

Solution: Given the equation of the ellipse in pedal form is  (1)

Differentiating (1) w r to r, we get  

Therefore, for the given curve 



Question Bank:

1. Show that the radius of curvature for the curve  at the point (a, a) is .

2. Show that the radius of curvature for the curve  at the point (x, y) is .

3. Show that the radius of curvature for the curve  at the point (a, 0) is .

4. Show that the radius of curvature for the curve  at the point t is .

5. Show that the radius of curvature for the curve  at the point t is   
 .

6. Show that the radius of curvature for the curve  at the point is .

7. For the curve , show that  is a constant.

8. Show that the radius of curvature for the curve  at the pole is .