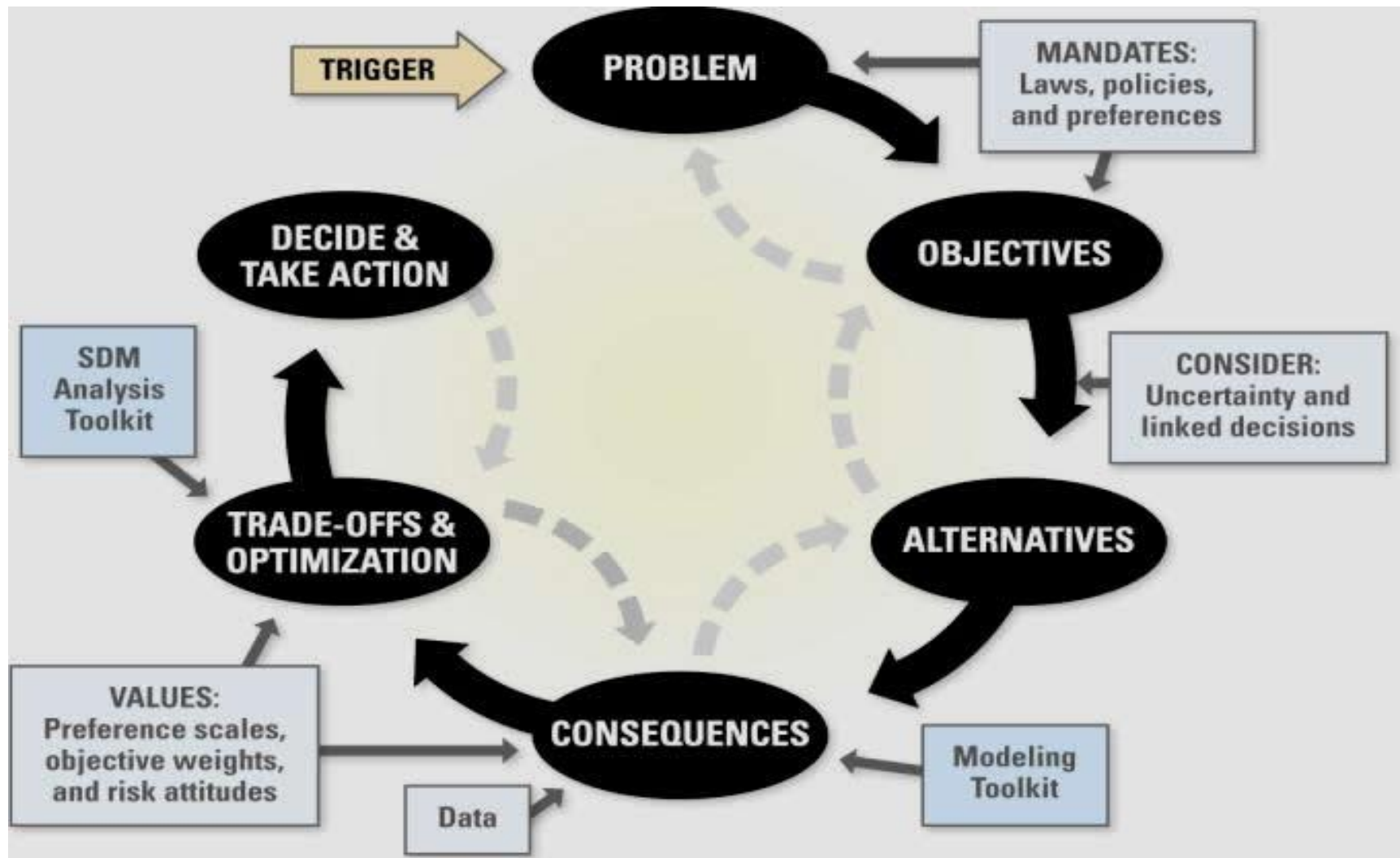
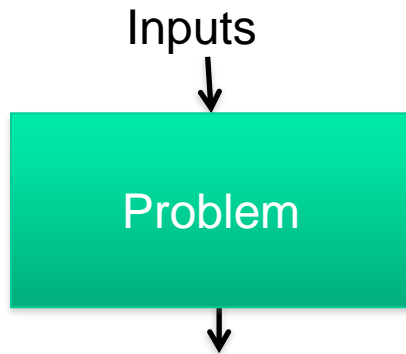




Strategic Decision Making (SDM)



Computational Problems



Decision Problems

(e.g., Yes or No decision: Can it be done and how?)

or

Optimization Problems

(e.g., What's the optimal solution and how?)

Decision Problem (Graph Coloring)

Can we color a map with 4 colors?



Decision Problem (Set Covering)

Can we protect the forest with 10 forest rangers?



Optimization Problem (Linear Programming)

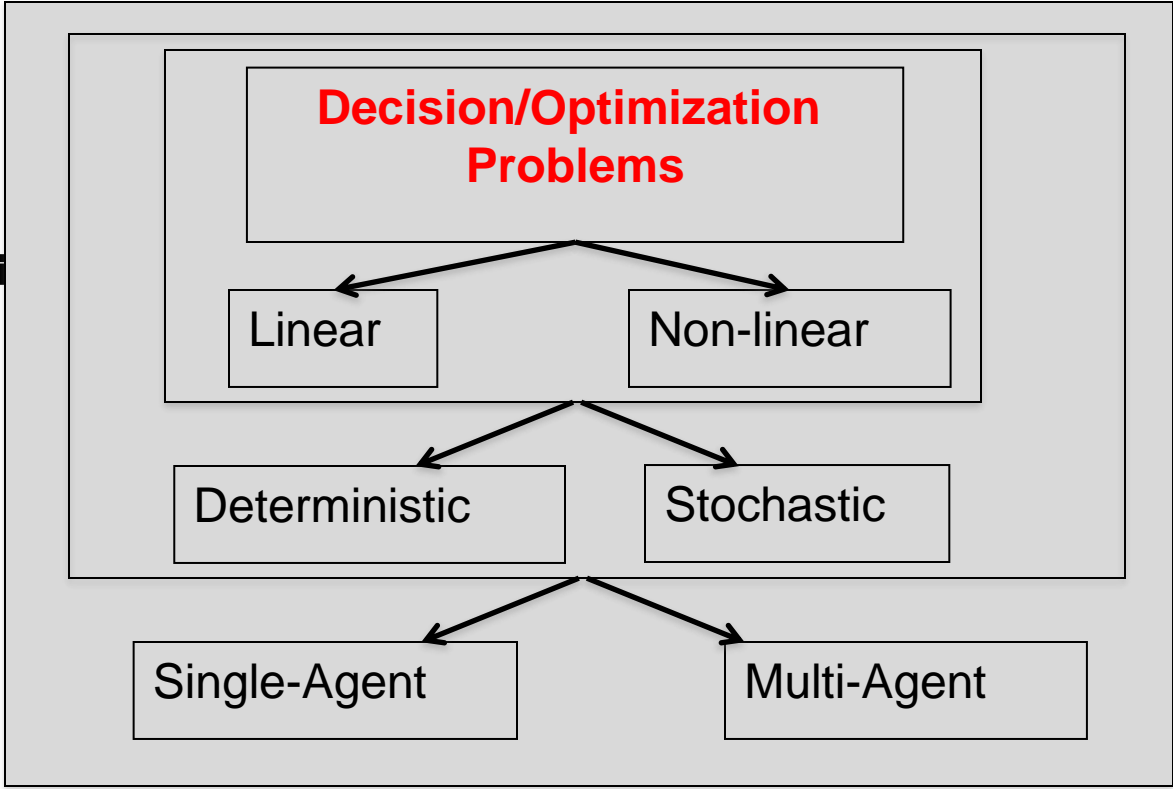
How to minimize Pulp mill pollution?



Optimization Problem (Knapsack Problem)

What Renewable Energy Investments should the Green Power Company pick to maximize profit?





Individual
Perspective

Social
Perspective

Solution Methods

Exact or Complete

Linear/Mixed/
Integer/Quadratic
Programming
Dynamic Programming
Stochastic Programming
Network Flow
(e.g., Shortest Path)
Non-Linear Programming
Branch and Bound
Others...

Incomplete, Sampling, Simulation

Approximation Algorithms
Simulated Annealing
Local Search
Genetic Algorithms
Sampling Based Methods
Particle Swarm Optimization
Markov Chain
Monte Carlo Methods
Agent Based Simulation
Computable General Equilibrium
Meta-heuristic Methods
Others...

Optimization: Mathematical Program

- Optimization Problem in which the objective and constraints are given as mathematical functions and functional relationships.

Minimize $f(x_1, x_2, \dots, x_n)$

Subject to (s.t.):

$$g_1(x_1, x_2, \dots, x_n) =, \geq, \leq b_1$$

$$g_2(x_1, x_2, \dots, x_n) =, \geq, \leq b_2$$

...

$$g_m(x_1, x_2, \dots, x_n) =, \geq, \leq b_m$$

Linear Programming

Linear Programming (LP)

- Linear – All the functions are Linear

$$\text{Ex: } f(x_1, x_2, \dots, x_n) = c_1 x_1 + c_2 x_2 + \dots c_n x_n$$

- Programming – doesn't refer to Computer Programming, but refers to “planning” of activity to find an optimal result i.e., reaches the specified goal (the best solution) (according to the mathematical model) among all feasible alternatives.

- Linear Programming (LP)

- LP has gained widespread industrial acceptance since 1950 for on-line optimization, blending etc.

Components of a Linear Programming Model

- A Linear Programming (LP) Model Consists of:
 - A Set of Decision Variables
 - A (Linear) Objective Function
 - A Set of (Linear) Constraints

Linear Programming (LP) Problems

Both objective function and constraints are linear.

Solutions are highly structured and can be rapidly obtained.

- Linear Constraints can arise due to:
 1. Production Limitations for example: Equipment Limitations, Storage Limits, Market Constraints.
 2. Raw Material Limitation
 3. Safety Restrictions, for example: Allowable Operating Ranges for Temperature and Pressures.
 4. Physical Property Specifications for example Product Quality Constraints when a blend property can be calculated as an average of pure component properties:
$$\bar{P} = \sum_{i=1}^n y_i P_i \leq \alpha$$
 5. Material and Energy Balances
 - Tend to yield equality constraints.
 - Constraints can change frequently (daily or hourly).

Linear Programming (LP) Problems

- Effect of Inequality Constraints

- Consider Linear and Quadratic Objective Functions on the next slide.
- For the LP problem, the optimum must lie on one or more constraints.

- Generic Statement of the LP Problem:

$$\max f = \sum_{i=1}^n c_i x_i$$

subject to (s. t.): $x_i \geq 0 \quad i = 1, 2, \dots, n$

$$\sum_{j=1}^n a_{ij} x_j \leq b_i \quad i = 1, 2, \dots, n$$

- Solution of LP Problems

- Simplex Method (Dantzig, 1947)
- Examine only constraint boundaries
- Very efficient, even for large problems

Steps in Setting up a LP

1. Determine and Label the *Decision Variables*.
2. Determine the Objective and Use the Decision Variables to write an expression for the *Objective Function*.
3. Determine the Constraints - *Feasible Region*.
 1. Determine the *Explicit Constraints* and write a functional expression for each of them.
 2. Determine the *Implicit Constraints (Nonnegativity Constraints)*.

Nature Connection: Recreational Sites

Nature Connection is planning two new public recreational sites: **The forested wilderness area** and a **sightseeing and hiking park**. They own **80 hectares of forested wilderness area** and **20 hectares suitable for the sightseeing and hiking park** but they don't have enough resources to make the entire areas available to the public. They have an overall **budget of \$120K** per year. They estimate a yearly management and maintenance cost of **\$1K per hectare** for the forested wilderness area, and **\$4K per hectare** for the sightseeing and hiking park. The expected average number of visiting hours a day per hectare are: **10 for the forest** and **20 for the sightseeing and hiking park**.

Question: How many hectares should Nature Connection allocate to the public sightseeing and hiking park and to the public forested wilderness area, in order to maximize the amount of recreation, (in average number of visiting hours a day for the total area to be open to the public, for both sites) given their budget constraint?

Problem Formulation as a LP

1. Decision Variables

x_1 – # hectares to allocate to the public forested wilderness area

x_2 – # hectares to allocate the public sightseeing and hiking park

2. Objective Function

$$\text{Max } \{Z = 10 x_1 + 20 x_2\}$$

3. Constraints

$$x_1 \leq 80$$

$$x_2 \leq 20$$

$$x_1 + 4 x_2 \leq 120$$

$x_1 \geq 0; x_2 \geq 0$ Non-negativity constraints

Formulation of the Problem as a Linear Program

Nature Connection is planning two new public recreational sites: a forested wilderness area and a sightseeing and hiking park. They own 80 hectares of forested wilderness area and 20 hectares suitable for the sightseeing and hiking park but they don't have enough resources to make the entire areas available to the public. They have an annual budget of \$120K so that they can estimate the annual management and maintenance cost of \$1K per hectare, for the forested wilderness area, and \$4K per hectare for the sightseeing and hiking park. The expected average number of visiting hours a day per hectare are: 10 for the forest and 20 for the sightseeing and hiking park.

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x_1 – # hectares to allocate to the public forested wilderness area

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2. Objective Function

$$\text{Max } \{Z = 10 x_1 + 20 x_2\}$$

3. Constraints

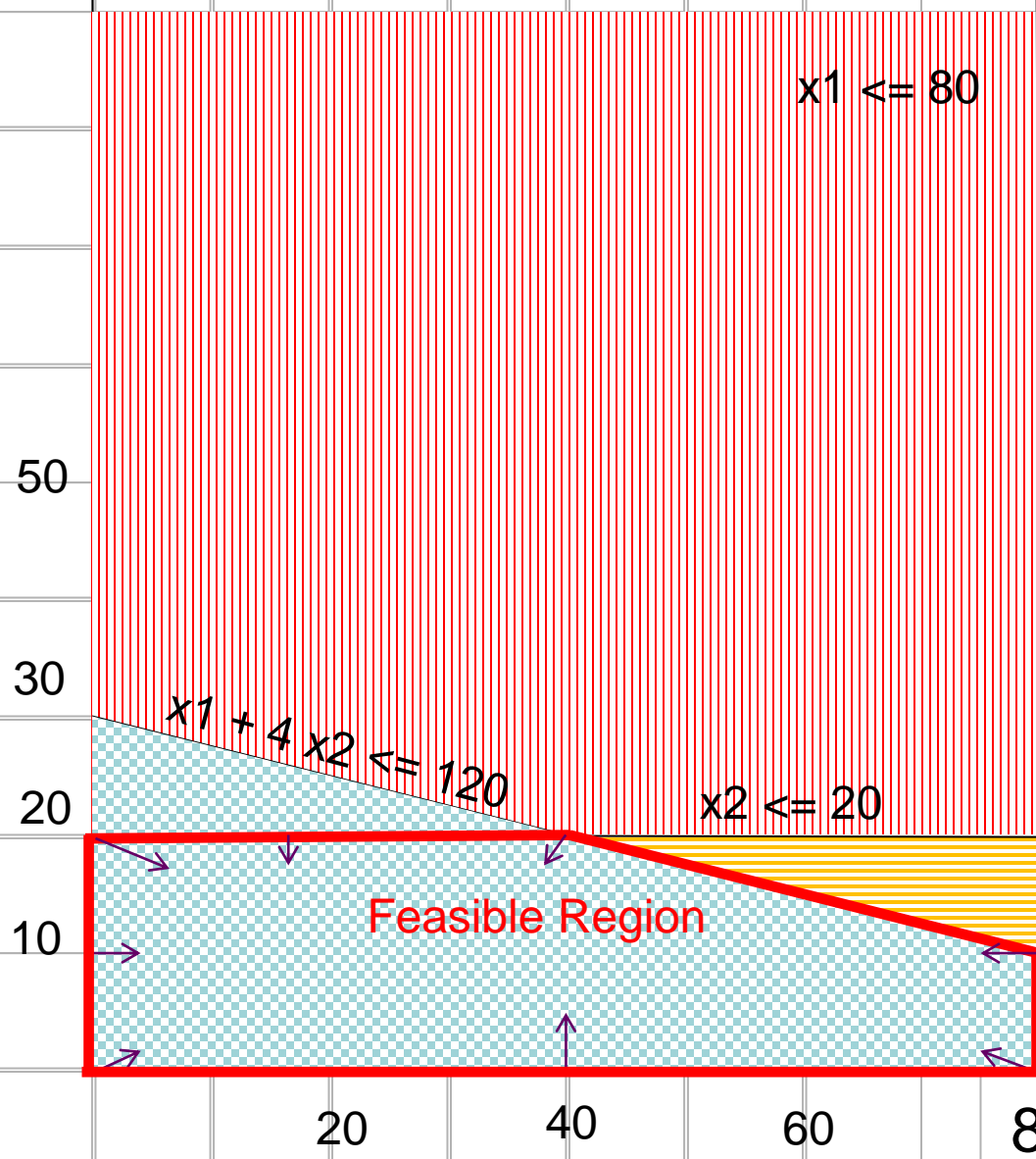
$$x_1 \leq 80 \quad \text{Land for forest}$$

$$x_2 \leq 20 \quad \text{Land for Park}$$

$$x_1 + 4 x_2 \leq 120 \quad \text{Budget}$$

$$x_1 \geq 0; x_2 \geq 0 \quad \text{Non-negativity constraints}$$

x2-Park



$$x_1 \leq 80$$

$$\text{Max } \{Z = 10x_1 + 20x_2\}$$

$$x_1 \leq 80$$

$$x_2 \leq 20$$

$$x_1 + 4x_2 \leq 120$$

$$x_1 \geq 0; x_2 \geq 0$$

$$x_2 \leq 20$$

$$x_1 + 4x_2 \leq 120$$

Feasible Region

120

100

80

60

40

20

x1-Forest

x2-Park

The vector representing the gradient of the objective function is given by:

10
20

$$\text{Max } \{Z = 10x_1 + 20x_2\}$$

$$x_1 \leq 80$$

$$x_2 \leq 20$$

$$x_1 + 4x_2 \leq 120$$

$$x_1 \geq 0; x_2 \geq 0$$

50

30

20

10

20

40

60

80

100

120

x1-Forest

Feasible Region

x2-Park

The vector representing the gradient of the objective function is given by:

10
20

$$\text{Max } \{Z = 10x_1 + 20x_2\}$$

$$x_1 \leq 80$$

$$x_2 \leq 20$$

$$x_1 + 4x_2 \leq 120$$

$$x_1 \geq 0; x_2 \geq 0$$

Isobenefit lines (for different forest and park areas)

$$10x_1 + 20x_2 = Z$$
$$x_2 = -\frac{1}{2}x_1 + \frac{Z}{20}$$

$$Z = 600$$

x1-Forest

x2-Park

The vector representing the gradient of the objective function is given by:

10
20

Isobenefit lines (for different forest and park areas)

$$\text{Max } \{Z = 10x_1 + 20x_2\}$$

$$x_1 \leq 80$$

$$x_2 \leq 20$$

$$x_1 + 4x_2 \leq 120$$

$$x_1 \geq 0; x_2 \geq 0$$

$$x_2 = -\frac{1}{2}x_1 + Z^*$$

$$Z^* = 10(80) + 20(10) = 1000$$

$$Z = 200$$

$$Z = 600$$

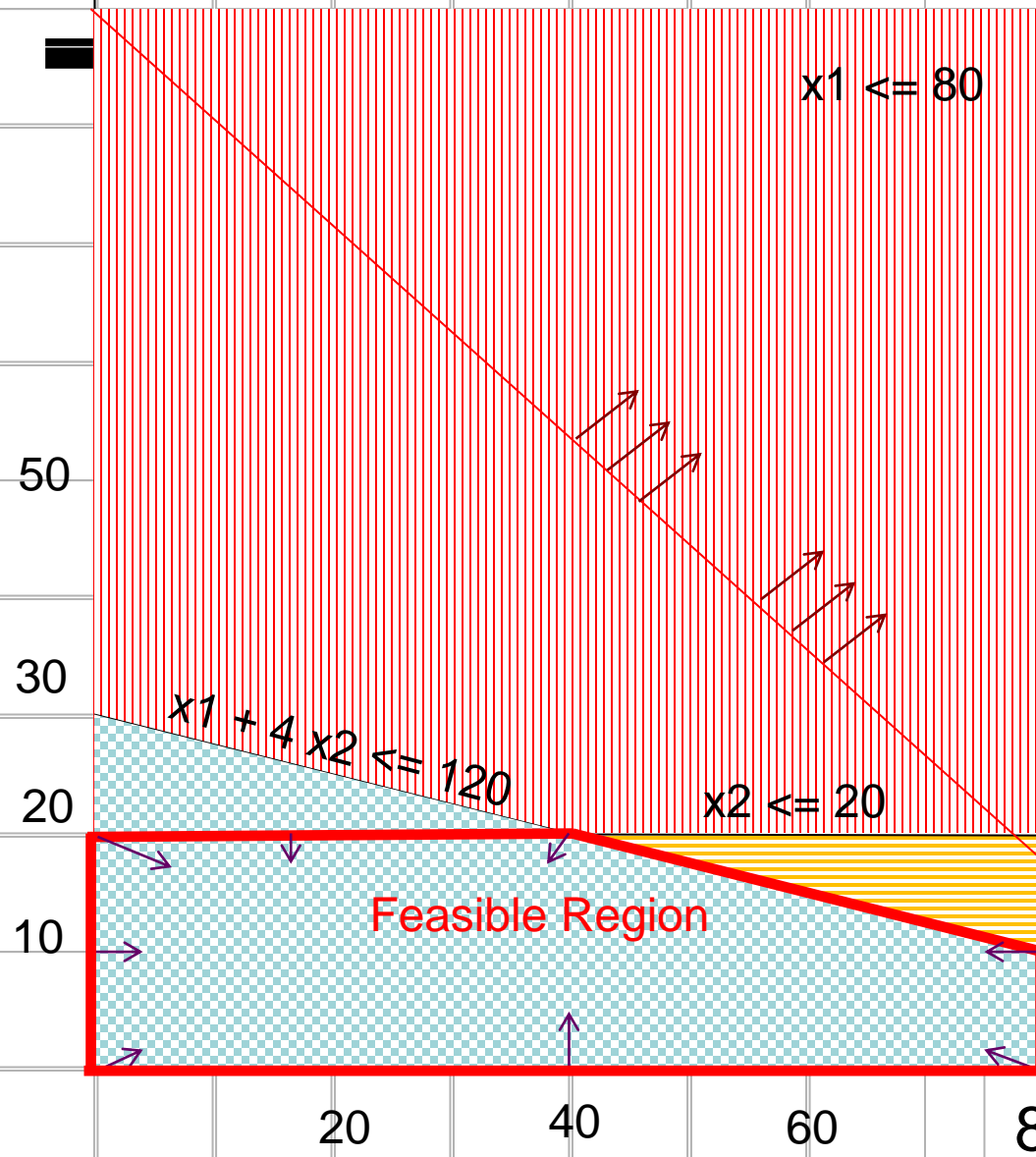
x1-Forest

Summary of the Graphical Method

- Draw the constraint boundary line for each constraint. Use the origin (or any point not on the line) to find which side of line is permitted by the constraint.
- Find the feasible region by determining where all the constraints are satisfied simultaneously.
- Determine the slope of one objective function line (which is perpendicular to its gradient vector). All other objective function lines will have same slope.
- Move a straight edge with this slope through the feasible region in the direction of improving values of the objective function (which is the direction of the gradient). Stop at the last instant that the straight edge still passes through a point in the feasible region. This is optimal objective function line.
- A feasible point on the optimal objective function line is an optimal solution.

No Feasible Solutions – Why?

x2-Park



$$\text{Max } \{Z = 10x_1 + 20x_2\}$$

$$x_1 \leq 80$$

$$x_2 \leq 20$$

$$x_1 + 4x_2 \leq 120$$

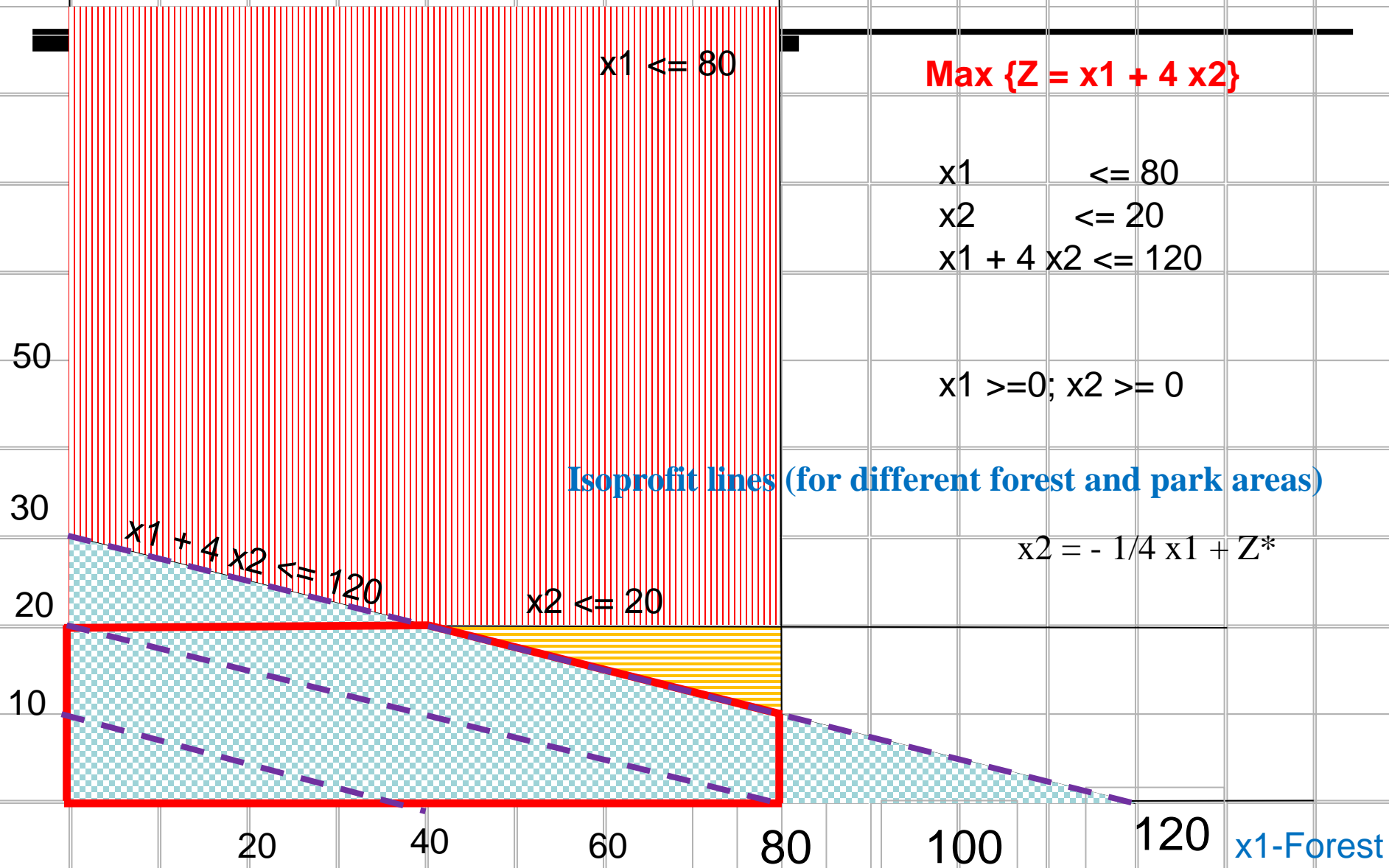
$$x_1 + x_2 \geq 100$$

$$x_1 \geq 0; x_2 \geq 0$$

x1-Forest

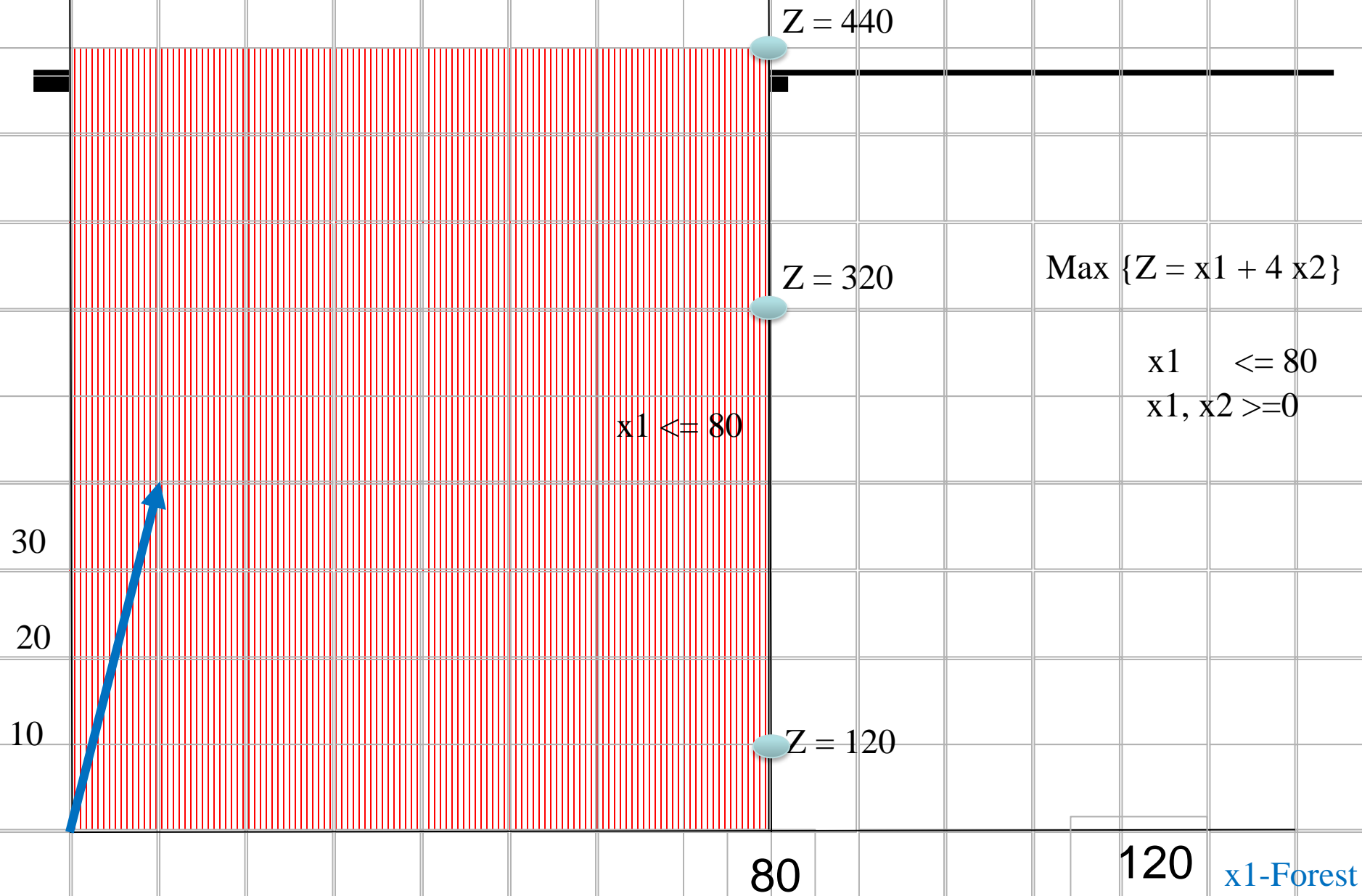
Multiple Optimal Solutions – Why

x2-Park



Unbounded – Why?

x2-Park



Key Categories of LP Problems

- Resource-Allocation Problems
- Cost-benefit-trade-off Problems
- Distribution-Network Problems
- Mixed Problems

Second Example: Keeping the River Clean

Cost-benefit-trade-off problems

Choose the mix of levels of various activities to
Achieve the minimum acceptable levels for
various benefits at a minimum cost.

Second Example: Keeping the River Clean

A pulp mill in Maine makes mechanical and chemical pulp, polluting the river in which it spills its spent waters. This has created several problems, leading to a change in management.

The previous owners felt that it would be too expensive to reduce pollution, so they decided to sell the pulp mill. The mill has been bought back by the employees and local businesses, who now own the mill as a cooperative. The new owners have several objectives:

- (i) – to keep **at least 300 people employed at the mill** (300 workers a day);
- (ii) – to generate at least \$40,000 of revenue per day

They estimate that this will be enough to pay operating expenses and yield a return that will keep the mill competitive in the long run. Within these limits, everything possible should be done to **minimize the pollution**.

Both the chemical and mechanical pulp units require the labor (workforce) of one worker for 1 day (1 workday, wd) per ton produced;

Mechanical pulp sells at \$100 per ton; Chemical pulp sells at \$200 per ton;

Pollution is measured by the Biological Oxygen Demand (BOD). One ton of mechanical pulp produces 1 unit of BOD; One ton of chemical pulp produces 1.5 units of BOD.

The maximum capacity of the pulp mill is to make the mechanical pulp is 300 tons per day, and for chemical pulp is 200 tons per day. The two manufacturing processes are statistically independent (mechanical pulp line cannot be used to make chemical pulp and vice versa).

- The Pollution, Employment, and the Revenues result from the Production of both types of Pulp (Chemical and Mechanical). So a natural choice for the variables is:

Decision Variables

- x_1 amount of mechanical pulp produced (in tons per day, or t/d) and
- x_2 amount of chemical pulp produce (in tons per day, or t/d)

$$\text{Minimize } Z = 1 x_1 + 1.5 x_2$$

(BOD/day) (BOD/t) (t/d) (BOD/t) (t/d)

Subject to (s.t.):

$$\begin{array}{llll} 1 x_1 & + & 1 x_2 & \geq 300 \text{ Workers/day} \\ (\text{wd/t}) (\text{t/d}) & & (\text{wd/t})(\text{t/d}) & \\ 100 x_1 & + & 200 x_2 & \geq 40,000 \text{ revenue/day} \\ (\$/\text{t}) (\text{t/d}) & + & (\$/\text{t})(\text{t/d}) & \$/\text{d} \\ x_1 & & & \leq 300 \text{ (mechanical pulp)} \\ (\text{t/d}) & & & (\text{t/d}) \\ & & x_2 & \leq 200 \text{ (chemical pulp)} \\ & & (\text{t/d}) & (\text{t/d}) \end{array}$$

$$x_1 \geq 0; x_2 \geq 0$$

Distribution Network Problems

Distribution-Network Problem

- The International Hospital Share Organization is a non-profit organization that refurbishes a variety of used equipment for hospitals of developing countries at two international factories (F1 and F2). One of its products is a large X-Ray machine.
- Orders have been received from three large communities for the X-Ray machines.

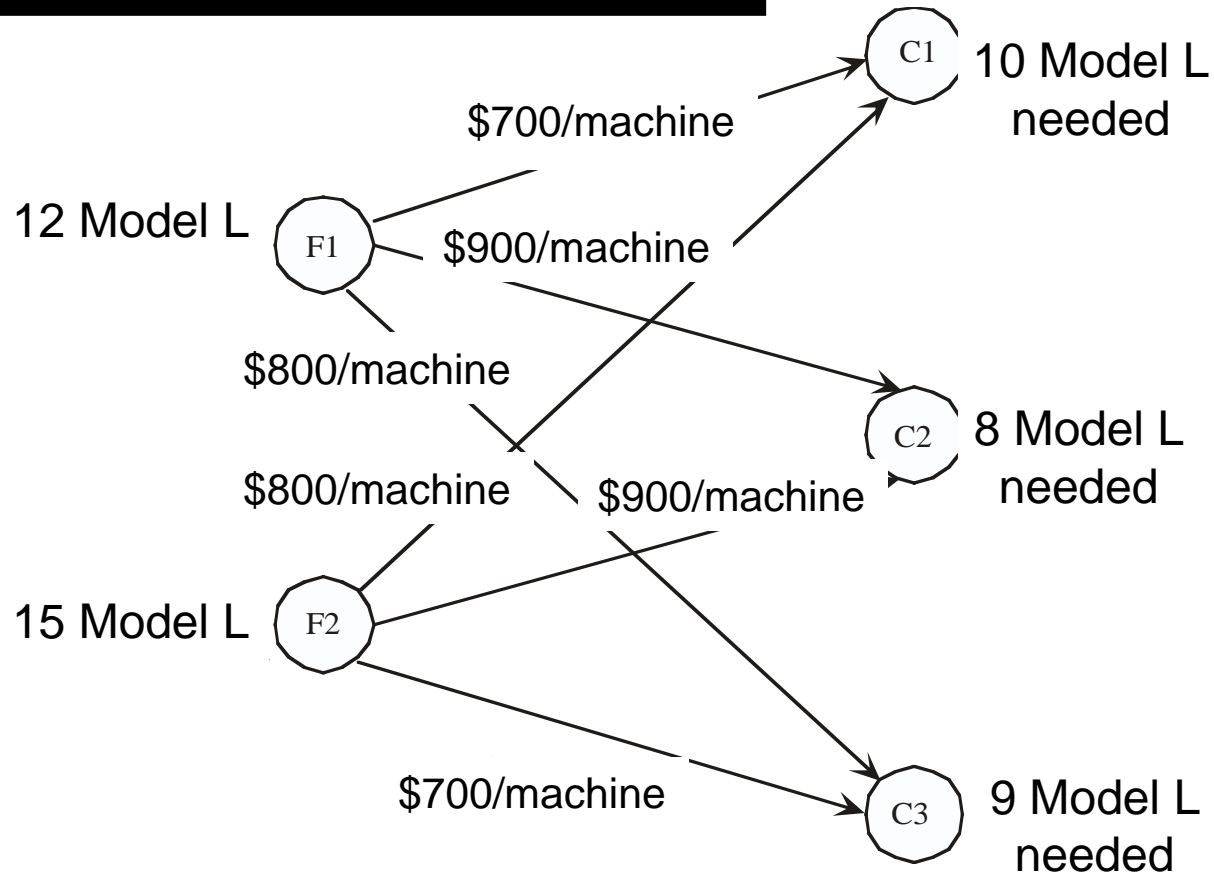
Distribution-Network Problem

Shipping Cost for Each Machine (Model L)

From	To	Community 1	Community 2	Community 3	Output
Factory 1		\$700	\$900	\$800	12 X-ray machines
Factory 2		800	900	700	15 X-Ray machines
Order Size		10 X-ray machines	8 X-Ray machines	9 X-Ray machines	

Question: How many X-Ray machines (model L) should be shipped from each factory to each hospital so that shipping costs are minimized?

The Distribution Network



Question: How many machines (model L) should be shipped from each factory to each customer so that shipping costs are minimized?

-
- **Activities – shipping lanes** (not the level of production which has already been defined)
 - Level of each activity – number of machines of model L shipped through the corresponding shipping lane.
 - Best mix of shipping amounts

Example:

Requirement 1: Factory 1 must ship 12 machines

Requirement 2: Factory 2 must ship 15 machines

Requirement 3: Customer 1 must receive 10 machines

Requirement 4: Customer 2 must receive 8 machines

Requirement 5: Customer 3 must receive 9 machines

Algebraic Formulation

Let S_{ij} = Number of machines to ship from i to j ($i = F1, F2$; $j = C1, C2, C3$).

$$\begin{aligned}\text{Minimize Cost} = & \$700S_{F1-C1} + \$900S_{F1-C2} + \$800S_{F1-C3} \\ & + \$800S_{F2-C1} + \$900S_{F2-C2} + \$700S_{F2-C3}\end{aligned}$$

subject to

$$S_{F1-C1} + S_{F1-C2} + S_{F1-C3} = 12$$

$$S_{F2-C1} + S_{F2-C2} + S_{F2-C3} = 15$$

$$S_{F1-C1} + S_{F2-C1} = 10$$

$$S_{F1-C2} + S_{F2-C2} = 8$$

$$S_{F1-C3} + S_{F2-C3} = 9$$

and

$$S_{ij} \geq 0 \quad (i = F1, F2; j = C1, C2, C3).$$

Algebraic Formulation

Let S_{ij} = Number of machines to ship from i to j ($i = F1, F2$; $j = C1, C2, C3$).

$$\begin{aligned}\text{Minimize Cost} = & \$700S_{F1-C1} + \$900S_{F1-C2} + \$800S_{F1-C3} \\ & + \$800S_{F2-C1} + \$900S_{F2-C2} + \$700S_{F2-C3}\end{aligned}$$

subject to

$$\text{Factory 1: } S_{F1-C1} + S_{F1-C2} + S_{F1-C3} = 12$$

$$\text{Factory 2: } S_{F2-C1} + S_{F2-C2} + S_{F2-C3} = 15$$

$$\text{Customer 1: } S_{F1-C1} + S_{F2-C1} = 10$$

$$\text{Customer 2: } S_{F1-C2} + S_{F2-C2} = 8$$

$$\text{Customer 3: } S_{F1-C3} + S_{F2-C3} = 9$$

and

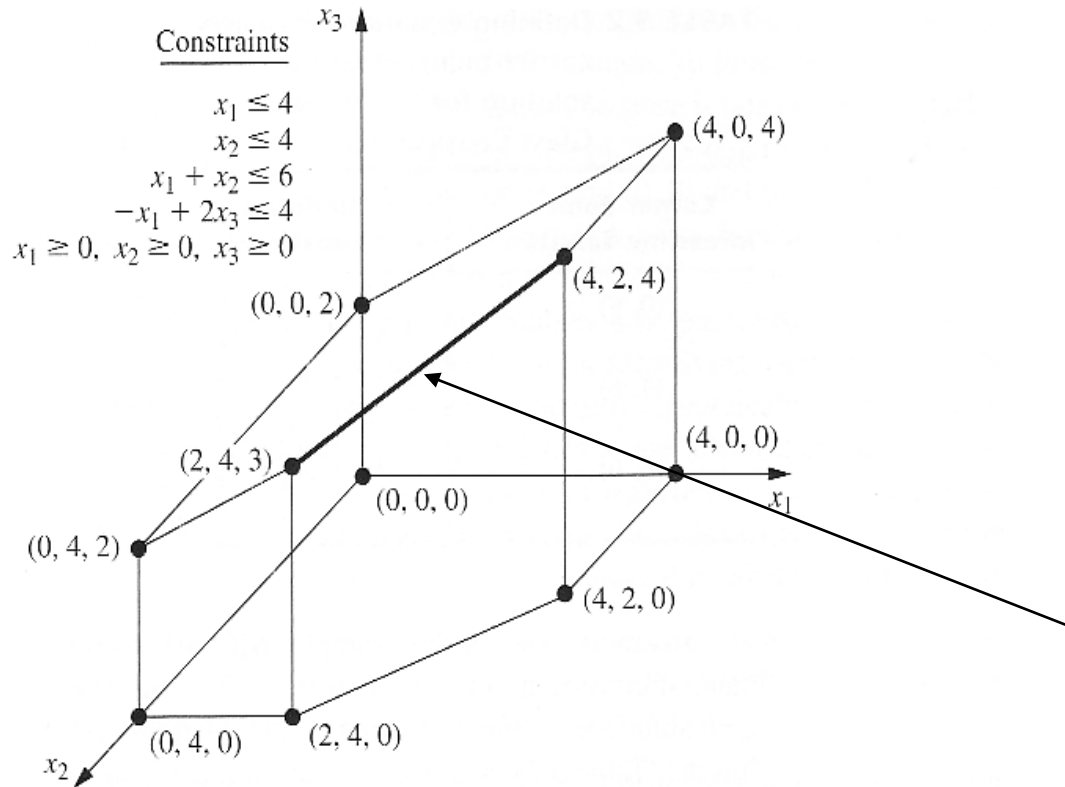
$$S_{ij} \geq 0 \text{ (} i = F1, F2; j = C1, C2, C3\text{)}.$$

Terminology and Notations

Terminology of Solutions in LP Model

- Solution – not necessarily the final answer to the problem!!!
- Feasible Solution – Solution that satisfies all the constraints
- Infeasible Solution – Solution for which at least one of the constraints is violated
- Feasible Region – Set of all points that satisfies all constraints (possible to have a problem without any feasible solutions)
- Binding Constraint – The left-hand side (LHS) and the right-hand side (RHS) of the constraint are equal, i.e., constraint is satisfied in equality. Otherwise the constraint is nonbinding.
- Optimal Solution – Feasible Solution that has the Best Value of the Objective Function.
 - Largest Value → Maximization Problems
 - Smallest Value → Minimization Problems
- Multiple Optimal Solutions, no optimal solutions, unbounded Z

3D Feasible Region

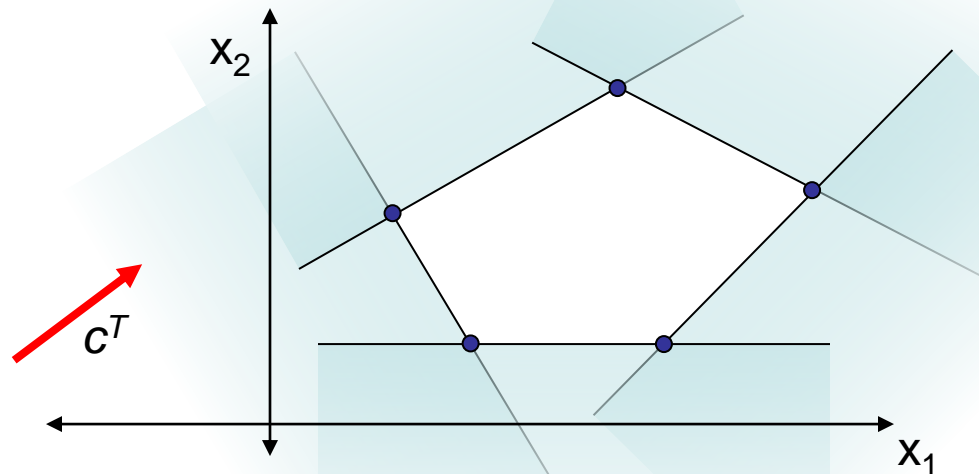


Solving Linear Programs

Solution Methods for Linear Programs

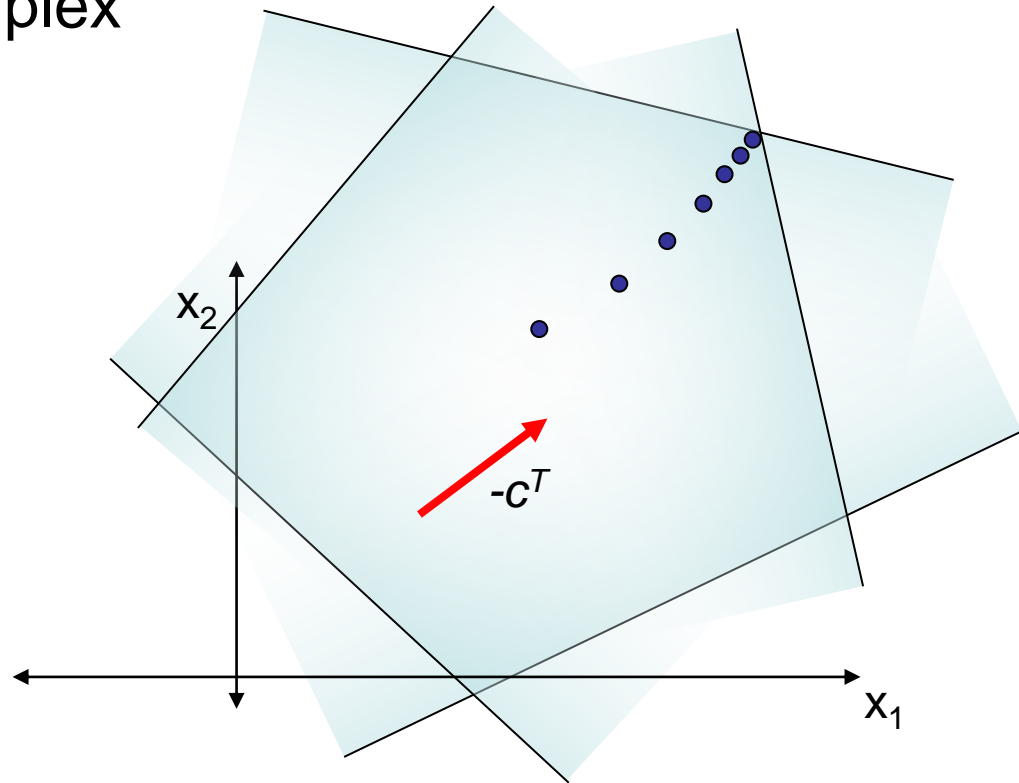
- Simplex Method

- Optimum must be at the intersection of constraints
- Intersections are easy to find, change inequalities to equalities



Solution Methods for Linear Programs

- Interior Point Methods
- Benefits
 - Scales Better than Simplex



Standard Form of the LP Model

$$\text{minimize } Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n,$$

subject to the restrictions

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m,$$

$$x_i \geq 0, (i = 1, 2, \dots, n)$$

Other Variants:

Maximize Z (instead of minimizing Z ; but $\text{Min } Z = -\text{Max } -Z$)

Some constraints have other signs ($=$; and \geq)

Some variables have unrestricted sign, i.e., they are not subject to (s.t.) the non-negativity constraints.

Solving Linear Programs

- To solve LPs, typically need to put them in standard form:

$$\begin{aligned} & \underset{z}{\text{minimize}} && c^T z \\ & \text{subject to} && Az \leq b \end{aligned}$$

- $z \in \mathbb{R}^n$, $A \in \mathbb{R}^{N_i \times n}$, $b \in \mathbb{R}^{N_i}$
- For absolute loss LP

$$z = \begin{bmatrix} \theta \\ \nu \end{bmatrix}, \quad c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad A = \begin{bmatrix} \Phi & -I \\ -\Phi & -I \end{bmatrix}, \quad b = \begin{bmatrix} y \\ -y \end{bmatrix}$$

$$\underset{\theta}{\text{minimize}} \quad \sum_{i=1}^m |\theta^T \phi(x_i) - y_i| \quad \longleftrightarrow \quad \begin{aligned} & \underset{\theta, \nu}{\text{minimize}} && \sum_{i=1}^m \nu_i \\ & \text{subject to} && -\nu_i \leq \theta^T \phi(x_i) - y_i \leq \nu_i \end{aligned}$$

Violates Divisibility Assumption of LP

- **Divisibility Assumption of Linear Programming:**
Decision variables in a Linear Programming (LP) model are allowed to have *any* values, including the *fractional (noninteger)* values, that satisfy all the functional and the nonnegativity constraints i.e., activities can be run at the *fractional levels*.

When the divisibility assumption is violated then we need to apply **Integer Linear Programming!!!**