Linear Algebra

Rank → number of linear indep rows or colums.

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 4 & 8 & 12 \end{bmatrix} \rightarrow \text{Rank} = 1.$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \text{Rank} = 3.$$

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$ → Row echleon form

- ↳ All zero rows of A occors below non zero rows of A.
- ↳ first non zero element in any row of mat A occurs in jth column of A, then all ele in jth column of A below non zero elements of row i are zero
- ↳ The first non zero entry in ith row should lie in the left of first non zero entry of (i+1) th row.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 0 & 5 \end{bmatrix}$$ find Rank,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & -6 & -4 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 2R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & 0 & 0 \end{bmatrix} \text{Rank} = 2.$$

Scanned with ACE Scanner

Adjoint of matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Cov of $A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

$A_{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

Adj = Transpose of Cov.

$= \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

Inverse $= \frac{Adj(A)}{|A|}$

$A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}$ Cov $A = \begin{bmatrix} 2 & 21 & -18 \\ 6 & -7 & 6 \\ 4 & -8 & 4 \end{bmatrix}$

Adj $= \begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}$

$|A| = 1(-28+30) - 1(-18) = 18+2 = 20$

Inv $= \frac{\begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ 18 & 6 & 4 \end{bmatrix}}{20}$

→ $2x + y + 2z = 0$

$2x - y + z = 10$

$x + 3y - z = 5$.

$\begin{bmatrix} 2 & 1 & 2 \\ 2 & -1 & 1 \\ 1 & 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ 5 \end{bmatrix}$

$\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 \\ +2 & -1 & 1 \\ 1 & 3 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 10 \\ 5 \end{bmatrix}$

$$\begin{bmatrix} 2 & 1 & 2 \\ 2 & -1 & 1 \\ 1 & 3 & -1 \end{bmatrix}^{-1} = adj\left(\begin{bmatrix} -2 & 3 & 7 \\ 7 & -4 & -5 \\ 3 & \cancel{-3}\ +2 & -4 \end{bmatrix}\right) = \frac{\begin{bmatrix} -2 & 7 & 3 \\ 3 & -4 & +2 \\ 7 & -5 & -4 \end{bmatrix}}{13}$$

$$|A| = 2(1-3) - 1(-2-1) + 2(6+1)$$
$$= -4 + 3 + 14$$
$$= 13$$

$$\frac{1}{13}\begin{bmatrix} -2 & 7 & 3 \\ 3 & -4 & +2 \\ 7 & -5 & -4 \end{bmatrix}\begin{bmatrix} 0 \\ 10 \\ 5 \end{bmatrix} = \frac{1}{13}\begin{bmatrix} 85 \\ -30 \\ -70 \end{bmatrix}$$

$$x = \frac{85}{13}, \quad y = \frac{-30}{13}, \quad z = \frac{-70}{13}.$$

→ You have 10,000$ to invest, you want to invest money in stock mutual fund, Bond MF, MM MF. Expected returns are 10%, 7%, 5% respectively. You want your investment to obtain overall 8%. A financial planner recommends that u invest same amount in stocks is money m, bonds combined. How much should you invest in each fund.

$$10\% x + 7\% y + 5\% z = 8\% \Rightarrow 10x + 7y + 5z = 8$$

$$7y + 5z = 10(y+z)$$

$$\cancel{3y =}$$

$$\cancel{y = x} + z$$

$$x + y + z = 10,000$$
$$10x + 7y + 5z = 8 \times 10,000$$
$$x = y + z.$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 10 & 7 & 5 \\ 1 & -1 & -1 \end{pmatrix}^{-1}\begin{pmatrix} 10,000 \\ 80,000 \\ 0 \end{pmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 10 & 7 & 5 \\ 1 & -1 & -1 \end{bmatrix}^{-1} = \frac{\begin{bmatrix} -2 & 15 & -21 \\ 0 & -2 & 2 \\ -2 & 5 & -3 \end{bmatrix}}{-4}$$

$1(-7+5)-1(-10-5)+1(-10-7)$

$-2+15-17$

-4

$$\frac{1}{-4}\begin{bmatrix} -2 & 0 & -2 \\ 15 & -2 & 5 \\ -21 & 2 & -3 \end{bmatrix}\begin{bmatrix} 10{,}000 \\ 80{,}000 \\ 0 \end{bmatrix} = \begin{bmatrix} -20{,}000 \\ -10{,}000 \\ -10{,}000 \end{bmatrix} \div -4$$

$$= \begin{bmatrix} 5000 \\ \cancel{2500} \; 2500 \\ 2500 \end{bmatrix}$$

→ Eigen vectors and eigen values:-

$$\underset{\substack{\downarrow \\ \text{Matrix}}}{A}\underset{\substack{\downarrow \\ \text{Eigen vector}}}{x} = \overset{\text{Eigen value}}{\lambda} x$$

Eigen value → It is scaler which is used to transform the Eigen vector.

Eigen vector → Vectors (non zero) that donot change direction when any linear transformation is applied.

$$Ax = \lambda I x$$

$$Ax - \lambda I x = 0$$

if x is non zero vector, $|A - \lambda I| = 0$.

$A = \begin{bmatrix} 6 & 3 \\ 4 & 5 \end{bmatrix}$ find Eigen vector & Eigen value.

$|A - \lambda I| = 0$

$\left| \begin{bmatrix} 6 & 3 \\ 4 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$.

$\begin{vmatrix} 6-\lambda & 3 \\ 4 & 5-\lambda \end{vmatrix} = 0$

$(6-\lambda)(5-\lambda) - 12 = 0$ $\quad (\lambda - 5)(\lambda - 6) = 12$

$\lambda^2 - 11\lambda + 18 = 0$

$\lambda = 9, 2$.

$\begin{bmatrix} 6 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 9x_1 \\ 9x_2 \end{bmatrix}$

$6x_1 + 3x_2 = 9x_1$ $\qquad x_1 = x_2$

$4x_1 + 5x_2 = 9x_2$

for $A = \begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix}$.

$|A - \lambda I| = 0$

$\left| \begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$ $\Rightarrow (-6-\lambda)(5-\lambda) = 12$

$\Rightarrow (\lambda + 6)(5 - \lambda) = -12$

$\Rightarrow -\lambda^2 + 30 - \lambda = 12$

$\Rightarrow \lambda^2 + \lambda - 42 = 0$

$\Rightarrow \cancel{\lambda^2 + 6\lambda - 5\lambda - 42 = 0}$

$\cancel{\lambda = -7, 6}$.

$\lambda^2 + \lambda - 42 = 0$.

$\lambda^2 + 7\lambda - 6\lambda - 42 = 0$

$\lambda = -7, 6$.

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if $\lambda = -7$, $\begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -7x \\ -7y \end{bmatrix}$.

$-6x + 3y = -7x$. $\quad 4x + 5y = -7y$.

$x + 3y = 0$ $\quad 4x + 12y = 0$

$x + 3y = 0$.

$Ex \to x = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

$\begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \cancel{\begin{bmatrix} 6+9 \\ -4+15 \end{bmatrix} = \begin{bmatrix} 15 \\ 11 \end{bmatrix}}$

$= \begin{bmatrix} 21 \\ -7 \end{bmatrix} = -7 \begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

$\rightarrow A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 5 \\ 0 & 4 & 3 \end{bmatrix}$

$\left| \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 5 \\ 0 & 4 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \right| = 0$.

$\left| \begin{pmatrix} 2-\lambda & 0 & 0 \\ 0 & 4-\lambda & 5 \\ 0 & 4 & 3-\lambda \end{pmatrix} \right| = 0$.

$(2-\lambda)\left((4-\lambda)(3-\lambda) - 20 \right) = 0$.

$\lambda = 2$ or $\quad (\lambda-3)(\lambda-4) = 20$.

$\lambda^2 - 7\lambda - 8 = 0$

$\lambda^2 - 8\lambda + \lambda - 8 = 0$

$\lambda = -1, 8$.

if $\lambda = 2$, $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 5 \\ 0 & 4 & 3 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{bmatrix}$

$\Rightarrow \begin{pmatrix} 2x_1 \\ 4x_2 + 5x_3 \\ 4x_2 + 3x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{pmatrix}$

$\Rightarrow 4x_2 + 5x_3 = 2x_2$ $\qquad$ $4x_2 + 3x_3 = 2x_3$

$2x_2 + 5x_3 = 0$ $\qquad$ $4x_2 + x_3 = 0.$

$x_2 = x_3 = 0. \; x_1 \neq 0.$

$\lambda = -1,$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 5 \\ 0 & 4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_1 \\ -x_2 \\ -x_3 \end{bmatrix}.$$

$$\begin{bmatrix} 2x_1 \\ 4x_2 + 5x_3 \\ 4x_2 + 3x_3 \end{bmatrix} = \begin{bmatrix} -x_1 \\ -x_2 \\ -x_3 \end{bmatrix}.$$

$5x_2 + 5x_3 = 0.$ $\qquad$ $x_2 + x_3 = 0$

$4x_2 + 4x_3 = 0$

$x_1 = 0, \; x_2 = 1, \; x_3 = -1.$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 5 \\ 0 & 4 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \; \lambda = -1 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

$\lambda = 8,$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 5 \\ 0 & 4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8x_1 \\ 8x_2 \\ 8x_3 \end{bmatrix}$$

$2x_1 = 8x_1$

$4x_2 + 5x_3 = 8x_2$ $\qquad$ $4x_2 + 3x_3 = 8x_3$

$5x_3 = 4x_2$ $\qquad$ $4x_2 = 5x_3.$

$x_1 = 0$

$x_2 = 5$

$x_3 = 4.$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 5 \\ 0 & 4 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 40 \\ 32 \end{bmatrix}$$

Equal.

$$8 \cdot \begin{bmatrix} 0 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 40 \\ 32 \end{bmatrix}.$$

Principal component

$$SD = \sqrt{\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n-1}}$$

$$\text{Variance}(x) = \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n-1}$$

$$Cov(x,y) = \sum_{i=1}^{n}\frac{(x_i-\bar{x})(y_i-\bar{y})}{n-1}$$

$Cov(x,y) < 0 \Rightarrow$ negative correlated

$Cov(x,y) > 0 \Rightarrow$ positive correlated

$Cov(x,y) = 0 \Rightarrow$ No linear correlation.

$$Cov(x,y,z) = \begin{bmatrix} cov(x,x) & cov(x,y) & cov(x,z) \\ cov(y,x) & cov(y,y) & cov(y,z) \\ cov(z,x) & cov(z,y) & cov(z,z) \end{bmatrix}$$

Steps in PCA:

1) Collect the data.

2) Subtract mean from data.

3) Calculate Covariance matrix

4) Calculate Eigen vectors & Eigen values of Co-variance matrix.

5) Forming a feature vector.

orders the Eigen vectors by Eigen values (highest to lowest)

Feature vector = Eigen vector & Eigen values

PC1 → The Eigen value with the largest absolute value indicates that the data have largest variance.

x	y	$x - \bar{x}$ mean	$y - \bar{y}$ mean
2.5	2.4	0.69	0.49
0.5	0.7	-1.31	-1.21
2.2	2.9	0.39	0.99
1.9	2.2	0.09	0.29
3.1	3.0	1.29	1.09
2.3	2.7	0.49	0.79
2	1.6	0.19	-0.31
1	1.1	-0.81	-0.81
1.5	1.6	-0.31	-0.31
1.1	1.9	0.71	-1.01
1.81	2.01		

Covariance Matrix

$$\mathrm{Cov}(x, y) = \sum_{i=1}^{n} \frac{(x_i - \bar{x})(y_i - \bar{y})^T}{n-1}$$

$$\cancel{(x_i - \bar{x})} = \begin{bmatrix} 0.69 \\ 0.49 \end{bmatrix}$$

$$\cancel{(x_i - \bar{x})^T} = [0.69 \quad 0.49]$$

$$\cancel{(x_i - \bar{x})(x_i - \bar{x})^T} = [$$

$$\mathrm{Cov}(x, y) = \sum_{i=1}^{n} \frac{(x_i - \bar{x})(y_i - \bar{y})}{n-1}$$

$$\mathrm{Cov}(x, y) = \begin{bmatrix} \mathrm{Cov}(x, x) & \mathrm{Cov}(x, y) \\ \mathrm{Cov}(y, x) & \mathrm{Cov}(y, y) \end{bmatrix}$$

$$\mathrm{Cov}(x, x) = \sum_{i=1}^{n} \frac{(x_i - \bar{x})^2}{n-1}$$

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Calculate new data, (X)

$$\begin{pmatrix} -0.6778 \\ -0.7351 \end{pmatrix}$$
$\downarrow$
PC1

New data $\Rightarrow$ $(-0.6778 \times \underbrace{0.69}_{(x-\bar{x})}) + (-0.7351 \times \underbrace{0.49}_{(y-\bar{y})})$

1st Row, $= -0.827$

2nd row, $(-0.6778 \times (-1.31)) + (-0.7351 \times \underbrace{}_{(-1.21)}$

~~RCZ~~ $\cancel{(-0.}$

$\simeq 1.77$

Take another PC and Calculate new feature (Y)

Singular value decomposition: (SVD)

The SVD of matrix A $(m\times n)$ is given by $A = U\Sigma V'$

→ U is a $m\times m$ matrix, the colums of U are orthogonal eigen vectors of AA^T.

→ V' is transpose of $n\times n$ matrix V, the columns of V are orthogonals eigen vectors of A^TA.

→ Σ is $n\times n$ diagonal matrix $\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$

$\sigma_1, \sigma_2, \sigma_n$ are called singular values.

Condition ⇒ $\sigma_1 \geq \sigma_2 \geq \sigma_3 \ldots$

$\sigma_1, \sigma_2 \ldots \geq 0$ (non negative).

Singular values are square roots of eigen values of A^TA or AA^T.

Example:

let $A = \begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ Calculate SVD of A.

$A = \begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 3 \\ -1 & 1 \\ 3 & 1 \end{bmatrix}$

(i) Determine V and then V^T

(ii) Determine $\sigma_1, \sigma_2 \ldots$ & Σ.

(iii) Determine U.

$A = U\Sigma V'$

$AV = U\Sigma V'V = U\Sigma$

$U = \frac{AV_i}{\sigma_1}$

To calculate V, $A = \begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 3 \\ -1 & 1 \\ 3 & 1 \end{bmatrix}$

$$A^TA = \begin{bmatrix} 1 & 3 \\ -1 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 2 & 6 \\ 2 & 2 & -2 \\ 6 & -2 & 10 \end{bmatrix}$$

$$\begin{vmatrix} 10-\lambda & 2 & 6 \\ 2 & 2-\lambda & -2 \\ 6 & -2 & 10-\lambda \end{vmatrix} = 0.$$

$$((10-\lambda)((10-\lambda)(2-\lambda)-4)) - 2(20-2\lambda+12) + 6(-4-12+6\lambda)) = 0$$

$\lambda = 0$ or,

$(\lambda^2 - 12\lambda + 20 - 4 - 40 + 4\lambda - 24 - 24 - 72 + 36\lambda) = 0$

$\lambda = 6, 16.$

$\cancel{\lambda^2 + 28\lambda}$

$AX = \lambda X$

$$\begin{bmatrix} 10 & 2 & 6 \\ 2 & 2 & -2 \\ 6 & -2 & 10 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 16\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} 10x_1 + 2x_2 + 6x_3 \\ 2x_1 + 2x_2 - 2x_3 \\ 6x_1 - 2x_2 + 10x_3 \end{bmatrix} = \begin{bmatrix} 16x_1 \\ 16x_2 \\ 16x_3 \end{bmatrix}$$

$2x_2 + 6x_3 = 6x_1$

$x_2 + 3x_3 = 3x_1$

$x_2 + 3x_3 = 3(x_3 + 7x_2)$

$x_2 + 3x_3 = 3x_3 + 21x_2$

$x_2 = 21x_2$

$x_2 = 0.$

$x_1 - x_3 = 7x_2$

$x_1 = x_3$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} t \\ 0 \\ t \end{bmatrix}$$

$$= t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

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Consider $t = 1/\sqrt{2}$ $\Rightarrow$ $\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$ $\Rightarrow$ becomes an unit vector.

$\downarrow$ 1st column

Consider Eigenvalue = 6.

$$\begin{pmatrix} 10 & 2 & 6 \\ 2 & 2 & -2 \\ 6 & -2 & 10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 6 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\begin{bmatrix} 10x_1 + 2x_2 + 6x_3 \\ 2x_1 + 2x_2 - 2x_3 \\ 6x_1 + 2x_2 + 10x_3 \end{bmatrix} = \begin{bmatrix} 6x_1 \\ 6x_2 \\ 6x_3 \end{bmatrix}$$

$x_2 + 3x_3 = 2x_1$

$x_1 - x_3 = 2x_2$

$3x_1 - x_2 + 2x_3 = 0$

$3x_1 - x_2 + 6x_2 = 0.$

$3x_1 = -5x_2.$

$x_2 + 3x_3 = 2(x_3 + 2x_2)$

$x_2 + 3x_3 = 2x_3 + 4x_2$

$x_3 = 3x_2.$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} t \\ -3t/5 \\ -9t/5 \end{bmatrix} = t \begin{bmatrix} 1 \\ -3/5 \\ -9/5 \end{bmatrix} = t_1 \begin{bmatrix} 5 \\ -3 \\ -9 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}.$$ $\rightarrow$ 2nd column.

$-8x_2$

$\lambda = 0.$

$$\begin{cases} 10x_1 + 2x_2 + 6x_3 = 0 \\ x_1 + x_2 = x_3 \\ 3x_1 + 5x_3 = x_2 \end{cases}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ 2t \\ t \end{bmatrix} = \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

↓
3rd Column.

$$V = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{3} & -1/\sqrt{6} \\ 0 & -1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$$

$$V' = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}$$

for calculating Σ ~~matrix~~

$$\Sigma = \begin{bmatrix} 4 & 0 & 0 \\ 0 & \sqrt{6} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad \sigma_1 = \sqrt{16}. \quad \sigma_2 = \sqrt{6}.$$

$$\begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

$$u_1 = \frac{AV_1}{\sigma_1} = \frac{\begin{bmatrix} 10 & 2 & 6 \\ 2 & 2 & -2 \\ 6 & -2 & 10 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}}{4}$$

$$= \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$u_2 = \frac{AV_2}{\sigma_2} = \frac{\begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} -1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}}{\sqrt{6}} = \frac{\begin{bmatrix} \sqrt{3} \\ -\sqrt{3} \end{bmatrix}}{\sqrt{6}} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}.$$

$$A = U \Sigma V'$$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & \sqrt{6} & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}.$$

$$= \begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{bmatrix}.$$