# Applied Linear Algebra

by
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- A vector is an ordered finite list of numbers. Vectors are usually written as vertical arrays, surrounded by square or curved brackets, as in.

$$\begin{bmatrix} -1.1 \\ 0.0 \\ 3.6 \\ -7.2 \end{bmatrix} \quad \text{or} \quad \begin{pmatrix} -1.1 \\ 0.0 \\ 3.6 \\ -7.2 \end{pmatrix}.$$

Figure: Vector Representation

- They can also be written as numbers separated by commas and surrounded by parentheses. In this notation style, the vector above is written as.

$$(-1.1, 0.0, 3.6, -7.2).$$

Figure: Vector Representation

- The elements (or entries, coefficients, components) of a vector are the values in the array. The size (also called dimension or length) of the vector is the number of elements it contains.

- The vector above, for example, has size four; its third entry is 3.6. A vector of size n is called an n-vector. A 1-vector is considered to be the same as a number, i.e., we do not distinguish between the 1-vector [ 1.3 ] and the number 1.3.
- We often use symbols to denote vectors. If we denote an n-vector using the symbol a, the $i^{th}$ element of the vector a is denoted $a_i$, where the subscript i is an integer index that runs from 1 to n, the size of the vector.
- Two vectors a and b are equal, which we denote a = b, if they have the same size, and each of the corresponding entries is the same. If a and b are n-vectors, then a = b means $a_1 = b_1, \ldots, a_n = b_n$.
- The numbers or values of the elements in a vector are called scalars.

- Sometimes the case that arises in most applications, where the scalars are real numbers. In this case we refer to vectors as real vectors. (Occasionally other types of scalars arise, for example, complex numbers, in which case we refer to the vector as a complex vector.)

- The set of all real numbers is written as R, and the set of all real n-vectors is denoted $\mathbf{R}^n$, so $a\varepsilon\mathbf{R}^n$ is another way to say that a is an n-vector with real entries. Here we use set notation: $a\varepsilon\mathbf{R}^n$ means that a is an element of the set $\mathbf{R}^n$;

**Block or stacked vectors.** It is sometimes useful to define vectors by *concatenating* or *stacking* two or more vectors, as in

$$a = \begin{bmatrix} b \\ c \\ d \end{bmatrix},$$

where $a$, $b$, $c$, and $d$ are vectors. If $b$ is an $m$-vector, $c$ is an $n$-vector, and $d$ is a $p$-vector, this defines the $(m+n+p)$-vector

$$a = (b_1, b_2, \dots, b_m, c_1, c_2, \dots, c_n, d_1, d_2, \dots, d_p).$$

The stacked vector $a$ is also written as $a = (b, c, d)$.

Stacked vectors can include scalars (numbers). For example if $a$ is a 3-vector, $(1, a)$ is the 4-vector $(1, a_1, a_2, a_3)$.

# Block Vectors

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# Sub Vectors

**Subvectors.** In the equation above, we say that $b$, $c$, and $d$ are *subvectors* or *slices* of $a$, with sizes $m$, $n$, and $p$, respectively. *Colon notation* is used to denote subvectors. If $a$ is a vector, then $a_{r:s}$ is the vector of size $s - r + 1$, with entries $a_r, \ldots, a_s$:

$$a_{r:s} = (a_r, \ldots, a_s).$$

The subscript $r{:}s$ is called the *index range*. Thus, in our example above, we have

$$b = a_{1:m}, \qquad c = a_{(m+1):(m+n)}, \qquad d = a_{(m+n+1):(m+n+p)}.$$

As a more concrete example, if $z$ is the 4-vector $(1, -1, 2, 0)$, the slice $z_{2:3}$ is $z_{2:3} = (-1, 2)$. Colon notation is not completely standard, but it is growing in popularity.

**Notational conventions.** Some authors try to use notation that helps the reader distinguish between vectors and scalars (numbers). For example, Greek letters $(\alpha, \beta, \ldots)$ might be used for numbers, and lower-case letters $(a, x, f, \ldots)$ for vectors. Other notational conventions include vectors given in bold font ($\mathbf{g}$), or vectors written with arrows above them ($\vec{a}$). These notational conventions are not standardized, so you should be prepared to figure out what things are (*i.e.*, scalars or vectors) despite the author's notational scheme (if any exists).

**Indexing.** We should give a couple of warnings concerning the subscripted index notation $a_i$. The first warning concerns the range of the index. In many computer languages, arrays of length $n$ are indexed from $i = 0$ to $i = n - 1$. But in standard mathematical notation, $n$-vectors are indexed from $i = 1$ to $i = n$, so in this book, vectors will be indexed from $i = 1$ to $i = n$.

The next warning concerns an ambiguity in the notation $a_i$, used for the $i$th element of a vector $a$. The same notation will occasionally refer to the $i$th vector in a collection or list of $k$ vectors $a_1, \ldots, a_k$. Whether $a_3$ means the third element of a vector $a$ (in which case $a_3$ is a number), or the third vector in some list of vectors (in which case $a_3$ is a vector) should be clear from the context. When we need to refer to an element of a vector that is in an indexed collection of vectors, we can write $(a_i)_j$ to refer to the $j$th entry of $a_i$, the $i$th vector in our list.

# Zero Vectors

- A zero vector is a vector with all elements equal to zero. Sometimes the zero vector of size n is written as $0_n$, where the subscript denotes the size.
- But usually a zero vector is denoted just 0, the same symbol used to denote the number 0. In this case you have to figure out the size of the zero vector from the context. As a simple example, if a is a 9-vector, and we are told that $a = 0$, the 0 vector on the right-hand side must be the one of size 9.
- Even though zero vectors of different sizes are different vectors, we use the same symbol 0 to denote them. In computer programming this is called overloading: The symbol 0 is overloaded because it can mean different things depending on the context.

**Unit vectors.** A (standard) *unit vector* is a vector with all elements equal to zero, except one element which is equal to one. The $i$th unit vector (of size $n$) is the unit vector with $i$th element one, and denoted $e_i$. For example, the vectors

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

are the three unit vectors of size 3. The notation for unit vectors is an example of the ambiguity in notation noted above. Here, $e_i$ denotes the $i$th unit vector, and not the $i$th element of a vector $e$. Thus we can describe the $i$th unit $n$-vector $e_i$ as

# Unit vectors.

$$(e_i)_j = \begin{cases} 1 & j = i \\ 0 & j \neq i, \end{cases}$$

for $i, j = 1, \ldots, n$. On the left-hand side $e_i$ is an $n$-vector; $(e_i)_j$ is a number, its $j$th entry. As with zero vectors, the size of $e_i$ is usually determined from the context.

## Ones vectors.

- We use the notation $1_n$ for the n-vector with all its elements equal to one.
- We also write 1 if the size of the vector can be determined from the context.
- Some authors use e to denote a vector of all ones, but we will not use this notation.
- The vector 1 is sometimes called the ones vector.

# Sparse Vectors

- A vector is said to be sparse if many of its entries are zero;
- Its sparsity pattern is the set of indices of nonzero entries. The number of the nonzero entries of an n-vector $x$ is denoted $\mathrm{nnz}(x)$.
- Unit vectors are sparse, since they have only one nonzero entry.
- The zero vector is the sparsest possible vector, since it has no nonzero entries.
- Sparse vectors arise in many applications.