Applied Linear Algebra

by
Dr.Dinesh Naik
(B.E, M.Tech, Ph.D)

Assistant Professor, Dept. of Information Technology
National Institute of Technology Karnataka, Surathkal

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Linear Equation

Linear Algebra - study of systems of linear equations

A **linear equation** is of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \quad \text{where}$$

- $x_1, x_2, \ldots, x_n$ are variables
- $a_1, a_2, \ldots, a_n$ are called coefficients
- b is the right-hand-side value
- $a_1, a_2, \ldots, a_n$ and b are typically real numbers, i.e., numbers found on a number line

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Terms like x_1^2 or x_1x_2 are nonlinear and therefore not allowed.

Some sample linear equations:

$$5x_1 + 3x_2 = 10$$
$$6x_1 + 8x_2 - 9x_3 = 14$$

System of linear equations—multiple equations.

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A system is said to be consistent if it has at least one solution; otherwise it is inconsistent.

Need a systematic method to solve a system of equations.

Elementary row operations:

1. Interchange (swap) two rows.
2. Multiply all entries in a row by a nonzero constant.
3. Replace one row by the sum of itself and another row.

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If the augmented matrices of two systems are row equivalent, then the two systems have the same solution set.

Two Fundamental Questions:

1. Is the system consistent, i.e., does at least one solution exist?
2. If the system is consistent, then is there a unique solution or an infinite number of solutions?

Inner product

- *inner product* (or *dot product*) of n-vectors a and b is

$$a^T b = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

- other notation used: $\langle a, b\rangle$, $\langle a|b\rangle$, (a, b), $a \cdot b$
- example:

$$\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}^T \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} = (-1)(1) + (2)(0) + (2)(-3) = -7$$

Inner product. When a and b are n-vectors, $a^T b$ is exactly the inner product of a and b, obtained from the rules for transposing matrices and forming a matrix-vector product. We start with the (column) n-vector a, consider it as an $n \times 1$ matrix, and transpose it to obtain the n-row-vector a^T. Now we multiply this $1 \times n$ matrix by the n-vector b, to obtain the 1-vector $a^T b$, which we also consider a scalar. So the notation $a^T b$ for the inner product is just a special case of matrix-vector multiplication.

Properties of inner product

- $a^T b = b^T a$
- $(\gamma a)^T b = \gamma(a^T b)$
- $(a + b)^T c = a^T c + b^T c$

can combine these to get, for example,

$$(a + b)^T (c + d) = a^T c + a^T d + b^T c + b^T d$$

General examples

- $e_i^T a = a_i$ (picks out ith entry)
- $\mathbf{1}^T a = a_1 + \cdots + a_n$ (sum of entries)
- $a^T a = a_1^2 + \cdots + a_n^2$ (sum of squares of entries)

Examples

- w is weight vector, f is feature vector; $w^T f$ is score
- p is vector of prices, q is vector of quantities; $p^T q$ is total cost
- c is cash flow, d is discount vector (with interest rate r):

  $$d = (1, 1/(1+r), \ldots, 1/(1+r)^{n-1})$$

  $d^T c$ is net present value (NPV) of cash flow
- s gives portfolio holdings (in shares), p gives asset prices; $p^T s$ is total portfolio value

Scalar-vector multiplication

Block vectors. If the vectors a and b are block vectors, and the corresponding blocks have the same sizes (in which case we say they *conform*), then we have

$$a^T b = \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix}^T \begin{bmatrix} b_1 \\ \vdots \\ b_k \end{bmatrix} = a_1^T b_1 + \cdots + a_k^T b_k.$$

The inner product of block vectors is the sum of the inner products of the blocks.

Applications. The inner product is useful in many applications, a few of which we list here.

- *Co-occurrence.* If a and b are n-vectors that describe occurrence, *i.e.*, each of their elements is either 0 or 1, then $a^T b$ gives the total number of indices for which a_i and b_i are both one, that is, the total number of co-occurrences. If we interpret the vectors a and b as describing subsets of n objects, then $a^T b$ gives the number of objects in the intersection of the two subsets. This is illustrated in figure 1.13, for two subsets A and B of 7 objects, labeled $1, \ldots, 7$, with corresponding occurrence vectors

$$a = (0, 1, 1, 1, 1, 1, 1), \qquad b = (1, 0, 1, 0, 1, 0, 0).$$

  Here we have $a^T b = 2$, which is the number of objects in both A and B (*i.e.*, objects 3 and 5).

Scalar-vector multiplication

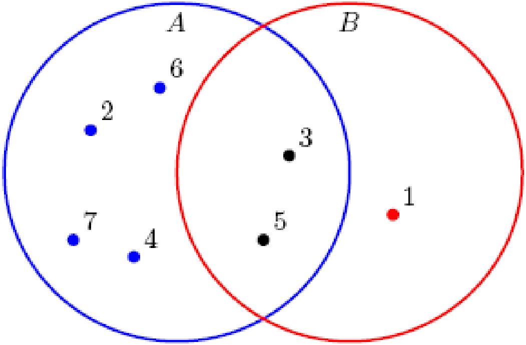

Figure : Two sets A and B, containing seven objects.