Applied Linear Algebra

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Vectors$_{e}xamples$

- A 3-vector can represent a color, with its entries giving the Red, Green, and Blue (RGB) intensity values (often between 0 and 1). The vector (0, 0, 0) represents black, the vector (0, 1, 0) represents a bright pure green color, and the vector (1, 0.5, 0.5) represents a shade of pink.

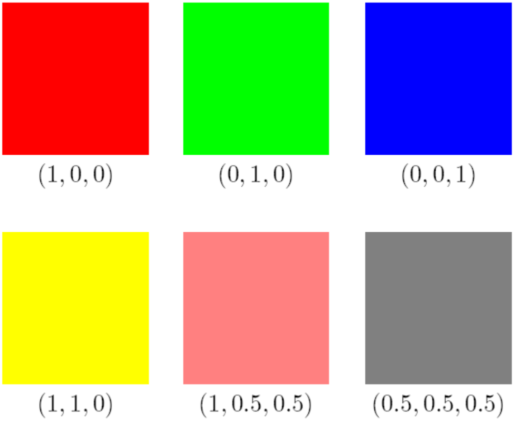

Figure: Six colors and their RGB vectors

Vectors Examples

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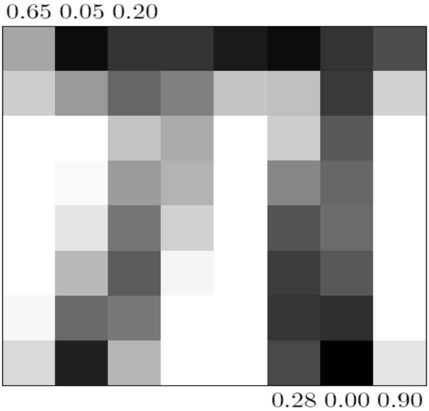


Figure: 8 x 8 image and the grayscale levels at six pixels

Vectors Examples

- An image can be represented by a vector of length MN, with the elements giving grayscale levels at the pixel locations, typically ordered column-wise or row-wise.
- With the vector entries arranged row-wise, the associated 64-vector is $x = (0.65, 0.05, 0.20, \ldots, 0.28, 0.00, 0{:}90)$.
- A color M x N pixel image is described by a vector of length 3MN, with the entries giving the R, G, and B values for each pixel, in some agreed-upon order.
- **Video**: A monochrome video, i.e., a sequence of length K of images with M x N pixels, can be represented by a vector of length KMN (again, in some particular order).

Vector Examples

- **Word count and histogram**. A vector of length n can represent the number of times each word in a dictionary of n words appears in a document.

- For example,(25, 2, 0) means that the first dictionary word appears 25 times, the second one twice, and the third one not at all. (Typical dictionaries used for document word counts have many more than 3 elements.).

- A small example is shown in figure.

$$
\begin{array}{l}
\text{word} \\
\text{in} \\
\text{number} \\
\text{horse} \\
\text{the} \\
\text{document}
\end{array}
\qquad
\begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \\ 4 \\ 2 \end{bmatrix}
$$

Figure: A snippet of text (top), the dictionary (bottom left), and word count vector (bottom right).

Vector addition

$$\begin{bmatrix} 0 \\ 7 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 3 \end{bmatrix}.$$

Vector subtraction is similar. As an example,

$$\begin{bmatrix} 1 \\ 9 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \end{bmatrix}.$$

The result of vector subtraction is called the *difference* of the two vectors.

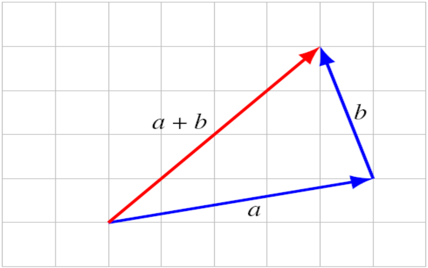
Vector addition

Properties. Several properties of vector addition are easily verified. For any vectors a, b, and c of the same size we have the following.

- Vector addition is *commutative*: $a + b = b + a$.
- Vector addition is *associative*: $(a + b) + c = a + (b + c)$. We can therefore write both as $a + b + c$.
- $a + 0 = 0 + a = a$. Adding the zero vector to a vector has no effect. (This is an example where the size of the zero vector follows from the context: It must be the same as the size of a.)
- $a - a = 0$. Subtracting a vector from itself yields the zero vector. (Here too the size of 0 is the size of a.)

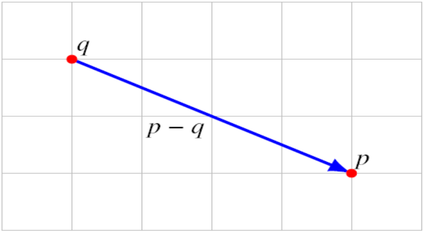
Adding displacements

if 3-vectors a and b are displacements, $a + b$ is the sum displacement



Displacement from one point to another

displacement from point q to point p is $p - q$



Scalar-vector multiplication

- scalar β and n-vector a can be multiplied

$$\beta a = (\beta a_1, \ldots, \beta a_n)$$

- also denoted $a\beta$
- example:

$$(-2)\begin{bmatrix} 1 \\ 9 \\ 6 \end{bmatrix} = \begin{bmatrix} -2 \\ -18 \\ -12 \end{bmatrix}$$

Properties of scalar-vector multiplication

- associative: $(\beta\gamma)a = \beta(\gamma a)$
- left distributive: $(\beta + \gamma)a = \beta a + \gamma a$
- right distributive: $\beta(a + b) = \beta a + \beta b$

these equations look innocent, but be sure you understand them perfectly

Scalar-vector multiplication

Word count vectors

- a short document:

  > **Word** count vectors are used **in** computer based **document** analysis. Each entry of the **word** count vector is the **number** of times the associated dictionary **word** appears **in** the **document**.

- a small dictionary (left) and word count vector (right)

$$\begin{array}{l} \text{word} \\ \text{in} \\ \text{number} \\ \text{horse} \\ \text{the} \\ \text{document} \end{array} \qquad \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \\ 4 \\ 2 \end{bmatrix}$$

- dictionaries used in practice are much larger

Scalar-vector multiplication

Scalar multiplication obeys several other laws that are easy to figure out from the definition. For example, it satisfies the associative property: If a is a vector and β and γ are scalars, we have

$$(\beta\gamma)a = \beta(\gamma a).$$

On the left-hand side we see scalar-scalar multiplication $(\beta\gamma)$ and scalar-vector multiplication; on the right-hand side we see two scalar-vector products. As a consequence, we can write the vector above as $\beta\gamma a$, since it does not matter whether we interpret this as $\beta(\gamma a)$ or $(\beta\gamma)a$.

The associative property holds also when we denote scalar-vector multiplication with the scalar on the right. For example, we have $\beta(\gamma a) = (\beta a)\gamma$, and consequently we can write both as $\beta a\gamma$. As a convention, however, this vector is normally written as $\beta\gamma a$ or as $(\beta\gamma)a$.

Scalar-vector multiplication

If a is a vector and β, γ are scalars, then

$$(\beta + \gamma)a = \beta a + \gamma a.$$

(This is the left-distributive property of scalar-vector multiplication.) Scalar multiplication, like ordinary multiplication, has higher precedence in equations than vector addition, so the right-hand side here, $\beta a + \gamma a$, means $(\beta a) + (\gamma a)$. It is useful to identify the symbols appearing in this formula above. The $+$ symbol on the left is addition of scalars, while the $+$ symbol on the right denotes vector addition. When scalar multiplication is written with the scalar on the right, we have the right-distributive property:

$$a(\beta + \gamma) = a\beta + a\gamma.$$

Scalar-vector multiplication also satisfies another version of the right-distributive property:

$$\beta(a + b) = \beta a + \beta b$$

for any scalar β and any n-vectors a and b. In this equation, both of the $+$ symbols refer to the addition of n-vectors.

Scalar-vector multiplication

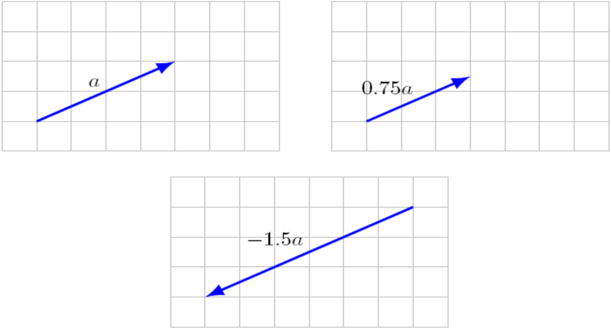


Figure The vector $0.75a$ represents the displacement in the direction of the displacement a, with magnitude scaled by 0.75; $(-1.5)a$ represents the displacement in the opposite direction, with magnitude scaled by 1.5.

Linear combinations

- for vectors $a_1, \ldots, a_m$ and scalars $\beta_1, \ldots, \beta_m$,

$$\beta_1 a_1 + \cdots + \beta_m a_m$$

  is a *linear combination* of the vectors
- $\beta_1, \ldots, \beta_m$ are the *coefficients*
- a *very* important concept
- a simple identity: for any n-vector b,

$$b = b_1 e_1 + \cdots + b_n e_n$$

Linear combination of unit vectors. We can write any n-vector b as a linear combination of the standard unit vectors, as

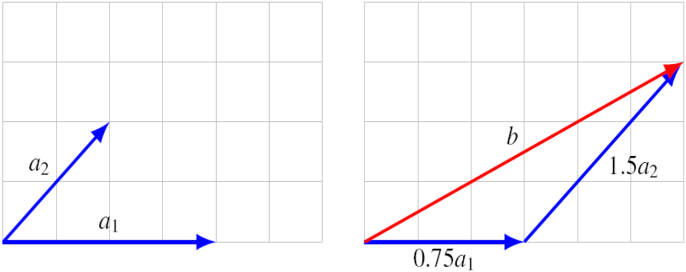
$$b = b_1 e_1 + \cdots + b_n e_n. \tag{1.1}$$

In this equation b_i is the ith entry of b (*i.e.*, a scalar), and e_i is the ith unit vector. In the linear combination of $e_1, \ldots, e_n$ given in (1.1), the coefficients are the entries of the vector b. A specific example is

$$\begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix} = (-1) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Scalar-vector multiplication

two vectors a_1 and a_2, and linear combination $b = 0.75a_1 + 1.5a_2$



Replicating a cash flow

- $c_1 = (1, -1.1, 0)$ is a \$1 loan from period 1 to 2 with 10% interest
- $c_2 = (0, 1, -1.1)$ is a \$1 loan from period 2 to 3 with 10% interest
- linear combination

$$d = c_1 + 1.1c_2 = (1, 0, -1.21)$$

  is a two period loan with 10% compounded interest rate
- we have *replicated* a two period loan from two one period loans

Special linear combinations. Some linear combinations of the vectors $a_1, \ldots, a_m$ have special names. For example, the linear combination with $\beta_1 = \cdots = \beta_m = 1$, given by $a_1 + \cdots + a_m$, is the *sum* of the vectors, and the linear combination with $\beta_1 = \cdots = \beta_m = 1/m$, given by $(1/m)(a_1 + \cdots + a_m)$, is the *average* of the vectors. When the coefficients sum to one, *i.e.*, $\beta_1 + \cdots + \beta_m = 1$, the linear combination is called an *affine combination*. When the coefficients in an affine combination are nonnegative, it is called a *convex combination*, a *mixture*, or a *weighted average*. The coefficients in an affine or convex combination are sometimes given as percentages, which add up to 100%.

Linear Algebra - study of systems of linear equations

A **linear equation** is of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \quad \text{where}$$

- $x_1, x_2, \ldots, x_n$ are variables
- $a_1, a_2, \ldots, a_n$ are called coefficients
- b is the right-hand-side value
- $a_1, a_2, \ldots, a_n$ and b are typically real numbers, i.e., numbers found on a number line

Linear Equation

Linear Algebra - study of systems of linear equations

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Terms like x_1^2 or x_1x_2 are nonlinear and therefore not allowed.

Some sample linear equations: $5x_1 + 3x_2 = 10$
$6x_1 + 8x_2 - 9x_3 = 14$

System of linear equations—multiple equations.