MTH 9878

Interest Rate Models HW 3

Group 5

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Q1

① Apply Itô's formula to $\log(\sigma_1 F_t + \sigma_0)$ we have:

$$d(\log(\sigma_1 F_t+\sigma_0)) = \log\frac{\sigma_1 F_T+\sigma_0}{\sigma_1 F_0+\sigma_0} = \frac{\sigma_1}{\sigma_1 F_t+\sigma_0}\, dF_t - \frac{1}{2}\frac{\sigma_1^2}{(\sigma_1 F_t+\sigma_0)^2}\, dF_t\, dF_t$$

$$= \sigma_1 dW_t - \frac{1}{2}\sigma_1^2 dt$$

$$\Rightarrow \sigma_1 F_T + \sigma_0 = (\sigma_1 F_0 + \sigma_0)\cdot \exp\left(-\frac{1}{2}\sigma_1^2 dt + \sigma_1 dW_t\right)$$

$$\Rightarrow F_T = \frac{1}{\sigma_1}\left[(\sigma_1 F_0+\sigma_0)\exp\left(-\frac{1}{2}\sigma_1^2 \overset{T}{\cancel{dt}} + \overset{\sigma_1 W_T}{\cancel{\sigma_1 dW_t}}\right) - \sigma_0\right]. \quad \text{as } t=0,\ W_0=0.$$

② for call option, $P_{sln}^{call} = E(F_T - K, 0)^{+} \cdot N(0))$

$$F_T > K \Longleftrightarrow (\sigma_1 F_0+\sigma_0)\cdot e^{-\frac{1}{2}\sigma_1^2 dt + \sigma_1 dW_t} > K\sigma_1 + \sigma_0$$

$$\Longleftrightarrow -\frac{1}{2}\sigma_1^2 dt + \sigma_1 dW_t > \ln\frac{K\sigma_1+\sigma_0}{\sigma_1 F_0+\sigma_0}$$

$$\overset{t_0=0}{\underset{W_0=0}{\Longleftrightarrow}} -\frac{1}{2}\sigma_1^2 T + \sigma_1 W_T > \ln\frac{K\sigma_1+\sigma_0}{\sigma_1 F_0+\sigma_0}$$

$$\Longleftrightarrow W_T > \frac{\ln\frac{K\sigma_1+\sigma_0}{\sigma_1 F_0+\sigma_0} + \frac{1}{2}\sigma_1^2 T}{\sigma_1}$$

$$\overset{W_T=\sqrt{T}Z}{\underset{Z\sim N(0,1)}{\Longleftrightarrow}} Z > \frac{\ln\frac{\sigma_1 K+\sigma_0}{\sigma_1 F_0+\sigma_0} + \frac{1}{2}\sigma_1^2 T}{\sigma_1\sqrt{T}} = -d_-$$

where $d_- = \dfrac{\ln\frac{\sigma_1 F_0+\sigma_0}{\sigma_1 K+\sigma_0} - \frac{1}{2}\sigma_1^2 T}{\sigma_1\sqrt{T}}$

③ pcall:

$$p^{call} = N(0)\cdot\int_{-\infty}^{\infty} (F_T-k,0)^+ \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= N(0)\cdot\int_{-d_-}^{\infty} (F_T-k) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx.$$

$$= N(0)\cdot\int_{-d_-}^{\infty} \left[\frac{(\sigma_1 F_0+\sigma_0)\cdot e^{-\frac{1}{2}\sigma_1^2 T+\sigma_1 W_T} - \sigma_0}{\sigma_1} - k\right]\cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= N(0)\cdot\int_{-d_-}^{\infty} \frac{(\sigma_1 F_0+\sigma_0)e^{-\frac{1}{2}\sigma_1^2 T+\sigma_1 W_T} - (k\sigma_1+\sigma_0)}{\sigma_1}\cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$\overset{W_T=\sqrt{T}x}{\underset{x\sim N(0,1)}{=}} N(0)\cdot\int_{-d_-}^{\infty} \frac{\sigma_1 F_0+\sigma_0}{\sigma_1}\cdot e^{-\frac{1}{2}\sigma_1^2 T+\sigma_1\sqrt{T}x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$- N(0)\int_{-d_-}^{\infty} \frac{k\sigma_1+\sigma_0}{\sigma_1}\cdot\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= N(0)\left[\int_{-d_-}^{\infty} \frac{\sigma_1 F_0+\sigma_0}{\sigma_1}\cdot e^{-\frac{(x-\sigma_1\sqrt{T})^2}{2}}\cdot\frac{1}{\sqrt{2\pi}} dx - \left(k+\frac{\sigma_0}{\sigma_1}\right) N(d_-)\right]$$

$$= N(0)\left[\int_{-d_-+\sigma_1\sqrt{T}}^{\infty} \left(F_0+\frac{\sigma_0}{\sigma_1}\right) e^{-\frac{u^2}{2}}\cdot\frac{1}{\sqrt{2\pi}} du - \left(k+\frac{\sigma_0}{\sigma_1}\right) N(d_-)\right]$$

$$= N(0)\left[\left(F_0+\frac{\sigma_0}{\sigma_1}\right) N(d_+) - \left(k+\frac{\sigma_0}{\sigma_1}\right) N(d_-)\right]$$

where $\int_{-d_-+\sigma_1\sqrt{T}}^{\infty} \left(F_0+\frac{\sigma_0}{\sigma_1}\right) e^{-\frac{u^2}{2}}\frac{1}{\sqrt{2\pi}} du = \int_{-d_+}^{\infty} \left(F_0+\frac{\sigma_0}{\sigma_1}\right) e^{-\frac{u^2}{2}}\cdot\frac{1}{\sqrt{2\pi}} du = \left(F_0+\frac{\sigma_0}{\sigma_1}\right) N(d_+).$

and $d_+ = \dfrac{\ln \frac{F_0\sigma_1+\sigma_0}{k\sigma_1+\sigma_0} + \frac{1}{2}\sigma_1^2 T}{\sigma_1\sqrt{T}}$, $d_- = \dfrac{\ln \frac{F_0\sigma_1+\sigma_0}{k\sigma_1+\sigma_0} - \frac{1}{2}\sigma_1^2 T}{\sigma_1\sqrt{T}}$. (29)

Therefore

$$P_{call} = N(0)\left[(F_0+\frac{\sigma_0}{\sigma_1})N(d_+) - (k+\frac{\sigma_0}{\sigma_1})N(d_-)\right]$$

and $B_{sln}^{call} = (F_0+\frac{\sigma_0}{\sigma_1})N(d_+) - (k+\frac{\sigma_0}{\sigma_1})N(d_-)$ (28)

$$p^{call} = N(0)\cdot B_{sln}^{call} \quad (27).$$

From put-call parity, we know

$$B_{sln}^{call} - B_{sln}^{put} = F_0 - k.$$

$$\Rightarrow B_{sln}^{put} = B_{sln}^{call} - F_0 + k.$$

$$= (F_0+\frac{\sigma_0}{\sigma_1})N(d_+) - F_0 - \left[(k+\frac{\sigma_0}{\sigma_1})N(d_-) - k\right]$$

$$= F_0 - F_0 - \frac{\sigma_0}{\sigma_1} - (F_0+\frac{\sigma_0}{\sigma_1})N(-d_+) - \left[k - k - \frac{\sigma_0}{\sigma_1} - (k+\frac{\sigma_0}{\sigma_1})N(-d_-)\right]$$

$$= -(F_0+\frac{\sigma_0}{\sigma_1})N(-d_+) + (k+\frac{\sigma_0}{\sigma_1})N(-d_-) \quad (30)$$

$$p^{put} = N(0)\cdot B_{sln}^{put} \quad (27)$$

Therefore, all formulas have been approved.

Q2

(a) Let's consider Call option:

for Normal model:

$$P_n^{Call} = N(0)\,\sigma_n\sqrt{T}\,(d_+ N(d_+) + N'(d_+)) \quad , \; d_\pm = \pm\frac{F_0 - k}{\sigma_n\sqrt{T}}$$

for lognormal model:

$$P_{ln}^{Call} = N(0)\,(F_0 N(d_1) - k N(d_2)) \quad , \; d_{1,2} = \frac{\log\frac{F_0}{k} \pm \frac{1}{2}\sigma_{ln}^2 T}{\sigma_{ln}\sqrt{T}}$$

for ATM Call option: $F_0 = k$, $\Rightarrow d_\pm = 0$, $d_{1,2} = \pm\frac{1}{2}\sigma_{ln}\sqrt{T}$

Then $P_n^{Call} = N_0\,\sigma_n\sqrt{T}\cdot\left(0 + \frac{1}{\sqrt{2\pi}} e^{-\frac{(F_0-k)^2}{2\sigma_n^2 T}}\right) = N_0\cdot\frac{\sigma_n\sqrt{T}}{\sqrt{2\pi}}$

$$P_{ln}^{Call} = N(0)\,F_0\,(N(d_1) - N(d_2))$$

Let $P_n^{Call} = P_{ln}^{Call}$,

$$\sigma_n = F_0\cdot\sqrt{\frac{2\pi}{T}}\cdot(N(d_1) - N(d_2))$$

$$N(d_1) - N(d_2) = \int_{-\frac{1}{2}\sigma_{ln}\sqrt{T}}^{\frac{1}{2}\sigma_{ln}\sqrt{T}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx$$

$$\xrightarrow[u=\frac{\sqrt{2}}{2}x]{x=\sqrt{2}u} \int_{-\frac{1}{2\sqrt{2}}\sigma_{ln}\sqrt{T}}^{\frac{1}{2\sqrt{2}}\sigma_{ln}\sqrt{T}} \frac{1}{\sqrt{2\pi}} e^{-u^2}\cdot\sqrt{2}\,du = \frac{2}{\sqrt{\pi}}\int_0^{\frac{1}{2\sqrt{2}}\sigma_{ln}\sqrt{T}} e^{-u^2}\,du$$

Therefore, $\sigma_n = F_0\frac{\sqrt{2\pi}}{\sqrt{T}}\,\mathrm{erf}\left(\frac{\sqrt{T}}{2\sqrt{2}}\sigma_{ln}\right)$, where $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}\,du$.

(b) As we know, Taylor expansion of e^{-u^2} is following

$$e^{-u^2} \approx 1-u^2+\frac{(-u^2)^2}{2!}+\frac{(-u^2)^3}{3!}+\frac{(-u^2)^4}{4!}+\cdots \qquad u\to 0.$$

$$=1-u^2+\frac{u^4}{2!}-\frac{u^6}{3!}+\frac{u^8}{4!}+\cdots$$

Then: when $\sigma_{ln}\sqrt{T}$ is small.

$$\int_0^{\frac{1}{2\sqrt{2}}\sigma_{ln}\sqrt{T}} e^{-u^2}\,du \approx \int_0^{\frac{1}{2\sqrt{2}}\sigma_{ln}\sqrt{T}} \left(1-u^2+\frac{u^4}{2!}-\frac{u^6}{3!}+\frac{u^8}{4!}+\cdots\right)du$$

$$=\left(u-\frac{u^3}{3}+\frac{u^5}{2!\times 5}-\frac{u^7}{3!\times 7}+\frac{u^9}{4!\times 5}+\cdots\right)\Bigg|_0^{\frac{1}{2\sqrt{2}}\sigma_{ln}\sqrt{T}}$$

$$=\frac{1}{2\sqrt{2}}\sigma_{ln}\sqrt{T}\left[1-\frac{u^2}{3}+\frac{u^4}{10}-\frac{u^6}{42}+\frac{u^8}{120}+\cdots\right]\Bigg|_0^{\frac{1}{2\sqrt{2}}\sigma_{ln}\sqrt{T}}$$

Therefore

$$\sigma_n = F_0\cdot\sqrt{\frac{2\pi}{T}}\cdot\frac{2}{\sqrt{\pi}}\cdot\frac{1}{2\sqrt{2}}\sigma_{ln}\sqrt{T}\left[1-\frac{\left(\frac{1}{2\sqrt{2}}\sigma_{ln}\sqrt{T}\right)^2}{3}+\frac{\left(\frac{1}{2\sqrt{2}}\sigma_{ln}\sqrt{T}\right)^4}{10}+\cdots\right]$$

$$=F_0\cdot\sigma_{ln}\left(1-\frac{1}{24}\sigma_{ln}^2T+\frac{1}{640}\left(\sigma_{ln}^2T\right)^2+\cdots\right)$$

Q3:

(1) Normal SABR model

The asymptotic expression for implied normal Volatility is:

$$\sigma_n = \frac{\alpha(F_0-k)}{D(\xi)}(1+O(\varepsilon))$$

where $D(\xi) = \log\left(\frac{\sqrt{\xi^2-2\rho\xi+1}+\xi-\rho}{1-\rho}\right)$

$$\xi = \frac{\alpha(F_0-k)}{\sigma_0}$$

Therefore $$\sigma_n = \frac{\sigma_0\cdot\xi\cdot(1+O(\varepsilon))}{D(\xi)}$$

for $\xi \to 0$, we have

$$\lim_{\xi\to 0}\sigma_n = \lim_{\xi\to 0}\frac{\sigma_0\cdot\xi(1+O(\varepsilon))}{D(\xi)} = \lim_{\xi\to 0}\frac{\sigma_0(1+O(\varepsilon))}{D'(\xi)}$$

As $$D'(\xi) = \frac{1-\rho}{\sqrt{\xi^2-2\rho\xi+1}+\xi-\rho}\cdot\frac{1}{1-\rho}\cdot\left(1+\frac{2\xi-2\rho}{2\sqrt{\xi^2-2\rho\xi+1}}\right)$$

$$= \frac{1}{\sqrt{\xi^2-2\rho\xi+1}}$$

Then $$\sigma_n = \lim_{\xi\to 0}\frac{\sigma_0(1+O(\varepsilon))}{\frac{1}{\sqrt{\xi^2-2\rho\xi+1}}} = \sigma_0(1+O(\varepsilon))$$

(2) lognormal SABR model.

$$\sigma_{ln} = \frac{\alpha \log(F_0/k)}{D(\xi)} (1+O(\varepsilon))$$

where $D(\xi) = \log\left(\frac{\sqrt{\xi^2-2\rho\xi+1}+\xi-\rho}{1-\rho}\right)$

$$\xi = \frac{\alpha(F_0-k)}{\sigma_0}$$

As $\log(F_0/k) = \log\left(\frac{F_0-k}{k}+1\right) = \log\left(\frac{\xi\cdot\sigma_0}{\alpha k}+1\right)$

Then we have

$$\sigma_{ln} = \lim_{\xi\to 0} \frac{\alpha \log\left(\frac{\xi\cdot\sigma_0}{\alpha k}+1\right)(1+O(\varepsilon))}{D(\xi)}$$

$$= \lim_{\xi\to 0} \frac{\alpha(1+O(\varepsilon))\cdot\left(\frac{\alpha k}{\xi\cdot\sigma_0+\alpha k}\cdot\frac{1}{\alpha k}\cdot\sigma_0\right)}{\frac{1}{\sqrt{\xi^2-2\rho\xi+1}}}$$

$$= \frac{\sigma_0}{k}(1+O(\varepsilon))$$