# Collaborative Review Task M2 Solution


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Solution to Question 1 Natural filtration of $\mathbb{F}^X = \left\{\mathcal{F}_n^X : n \in \mathbb{I}\right\} = \left\{\mathcal{F}_0^X, \mathcal{F}_1^X, \mathcal{F}_2^X\right\}$ can be

constructed using the below process where $\mathcal{F}_0^X$, $\mathcal{F}_1^X$,$\mathrm{F}_2^X$ have been constructed as follows.

$\mathcal{F}_0^X$=$\{\emptyset, \Omega\}$ where $\Omega$ is the sample space. This is the smallest filtration with respect to which X is adapted.

Since $\mathcal{F}_0^X$ does not differentiate between the values of a,b,c,d and $\mathcal{F}_1^X$ does not differentiate between a,b & c,d pairwise. $\mathcal{F}_1$ will be the $\sigma$- algebra generated by below:

$\mathcal{F}_1 = \sigma\ (\{\{a, b\}, \{c, d\}\})$

and

$\mathcal{F}_2^X = \sigma\left(\{\{a\}, \{b\}, \{c\}, \{d\}\}\right) = 2^\Omega$

Solution to Question 2

In this example we have four different possible realizations of , that is $\Omega = \{a, b, c, d\}$
We can interpret each realization $\Omega = \{a, b, c, d\}$ as a potential path that the discrete stochastic process can follow. In this case,
$= \{X0, X1, X2\}$ only is allowed to follow four different paths provided in the problem statement.

For instance, in the path a, the realizations of $X(a) = \{X0, X1, X2\}$ is $X(a) = \{12, 18, 36\}$
As the problem defined the stopping time as: $\tau(\omega) = \inf\{n \in \mathbb{I} : X_n(\omega) < 10\}$

We just have to find the value of t (take into account that we have only three times in this) that is $t = \{0, 1, 2\}$ such that $X_n(\omega) < 10$ , and repeat the process for each path, $\{a, b, c, d\}$, in order to compute $\{\tau(a), \tau(b), \tau(c), \tau(d)\}$

So, for path a: we can see that the process never goes below 10 so the stopping time for this will be $\infty$.

For path b, at t=2, the process achieves a value of 9 which is less than 10. In this case stopping time will be 2.

For path b, at t=2, the process achieves a value of 9 which is less than 10. In this case stopping time will be 2.

For path c, at t=1, the process achieves a value of 9 which is less than 10. In this case stopping time will be 1.

For path d, at t=1 the process achieves the value of 9 and at t=2, it goes to 8 however for the purposes of stopping processes, we will consider t=1 as applying the theory of filtration we have the required information already available at t=1.

Hence $\{\tau(a), \tau(b), \tau(c), \tau(d)\} = \{\infty\,, 2, 1, 1\}$.

Solution to Question 3

To show that $\tau$ is stopping time with respect to $\mathbb{F}^X$, we calculate the event $\{\tau \leq t\}$ and this should be $\in \mathcal{F}_t$ for every $t \in \mathbb{I}$. To start , when t=0, the event that $\{\tau \leq t\}$ is empty because $\tau$ is never less than or equal to 0. This belongs to $\mathcal{F}_0^X$. When t=1, the event that $\{\tau \; 1\}$ is equal to $\{a, b\}$ and $\{c, d\}$, which belongs to $\mathcal{F}_1^X$ because $\mathcal{F}_1^X$ is generated by these two blocks. Finally when t=2, the event that $\{\tau \; 2\}$ is exactly the same as before as $\{a, b\}$ which belongs to $\mathcal{F}_2^X$ because $\mathcal{F}_2^X$ i a power set of $\Omega$.

Solution to Question 4

We can calculate the expected value of X at each point of interval using all the four sample paths at each time, if all these expected values as per condition that expectation of X at t will be same as expectation of X at s where s < , then X is a martingale.

X(0)= 12*[(1/9)+(2/9)+(1/6)+(1/2)] = 12

X(1)= 18*[(1/9)+(2/9)]+9*[(1/6)+(1/2)]=12

X(2)= 36*(1/9)+9*(2/9)+ 12*(1/6) +8 *(1/2)= 12 Since expected value of X is same

across all times given, we say that X is a martingale with respect to $\mathrm{F}^X$.

Solution to Question 5

Let $((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space, and
X=$\{X_t : t \in \mathbb{I}\} beastochasticprocess.WecallXamartingaleif(oran(\mathbb{F}, \mathbb{P}) - martingale)if$
1. X is adapted to F,
2. E(X$_t$) $< \infty forallt \in \mathbb{I}$
3. E(X$_t|\mathcal{F}_s) = X_s \mathbb{P} - ass \leq tWecallXasub(resp.super) - martingaleifitsatisfies1, 2and$

3'. E(X$_t|\mathcal{F}_s) \geq X_s(resp\mathbb{E}(X_t|\mathcal{F}_s) \leq X_s\mathbb{P} - ass \leq t$
Since all Z from Z=$\{Z_0, Z_1, Z_2\}$
are independent and identically distributed random variables,
their expected value at all t will be same.
Hence we can safely say that Z is an F$^X martingale$.