Group Work Project

Submission 3

American Options

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Abstract

The following paper consist of the brief study of the American options, this report is the solutions to the group research project for the course Discrete Time Stochastic Process, a constituent of master of science in Financial Engineering is further divided into the three parts. The first parts reflects study of the theoretical analysis of the American call options with comparison with the European counterparts. After the introduction of the American options, we tried to go some details about the American Put options with the some numerical examples and theorems which includes the example and brief explanation of the different methods and technique. Furthermore, the final part of the report consist of the short explanation about the perpetual American put described with the Binomial Tree Model.

Solution to Question No 1

An American option can be exercised at any time, whereas a European option can only be exercised at the expiration date. This added flexibility of American options increases their value over European options in certain situations. Thus, we can say

$$American\ Options\ =\ European\ Options\ +\ Premium$$

Where, the Premium is greater than or equal to zero.

For standard American call options without dividends, there are several reasons why the call should never be exercised before the expiration date. Because of the time value of money, it costs more to exercise the option today at a fixed strike price K than in the future at K. Finally, there is an intrinsic time value of the option that would be lost by exercising the option prior to the expiration date. Let's explain the above statements mathematically.

For an American call or put, the decision to exercise or hold at any time t depends just on the time value t and the underlying stock value $S(t)$. The exercise time τ which can be termed as the stopping timre is chosen to maximize the value of the option by the option holder.

For an American call (on a stock without dividends), early exercise is never optimal. The reason is that exercise requires payment of the strike price X. By holding onto X until the expiration time, the option holder saves the interest on$X$. To see this mathematically, consider two portfolios:

E: one American call c, $X * e^r(T\ - t)$ cash

F: one share S .

If the exercise time is $\tau < T$, the value of E is

$$E\ =\ (S\ -\ X)\ +\ X * e^r(T - \tau)\ \ <\ S\ =\ F$$

If the exercise is at $\tau\ =\ T,$ then

$$E\ =\ max(S\ -\ X, 0)\ +\ X =\ max(S, X)\ \ \geq S\ =\ F$$

It follows that $\mathbb{E}\ \geq\ F$ for any maturity time T however not for $\tau < T$, so that one should never take $\tau < T$.

Part B of Question 1

The early exercise of either an American call leads to the loss of insurance value associated with holding of the option. For an American call, the holder gains on the dividend yield from the asset but loses on the time value of the strike price. There is no advantage to exercise an American call prematurely when the asset received upon early exercise does not pay dividends. In this case, the American call has the same value as that of its European counterpart.

Solution to Question 2

For an American put option, the early exercise leads to some gain on time value of strike. Therefore, when the riskless interest rate is positive, there always exists an optimal exercise price below which it becomes optimal to exercise the American put prematurely. This can be mathematically represented as:

If the exercise time is τ<T, the value of E is

$$\mathbb{E}_1 = (X - S) + X * e^r(T - \tau) \quad > (X - S)$$

If the exercise is at τ = T, then

$$\mathbb{E}_2 = max(X - S, 0)$$

It follows that $\mathbb{E}_1 \geq \mathbb{E}_2$ for all $\tau < T$, so it is always better to exercise the American put option before maturity.

We consider a simple two – period market defined as follows $\Omega = \{a, b, c, d\}$, $T = 2$, and only one risky asset $X = \{X_0, X_1, X_2\}$ defined as follows:

ω	$\mathbb{P}(\{\omega\})$	$X_0(\omega)$	$X_1(\omega)$	$X_2(\omega)$
a	¼	10	20	30
b	¼	10	20	15
c	¼	10	5	12
d	¼	10	5	3

We also pick, $\mathbb{F} = \mathbb{F}^X$

Solving the simultaneous equations, we find the unique EEM to be:

$$\mathbb{P}^* = \frac{1}{9}\,\boldsymbol{\delta_a} + \frac{2}{9}\,\boldsymbol{\delta_b} + \frac{4}{27}\,\boldsymbol{\delta_c} + \frac{14}{27}\,\boldsymbol{\delta_d}$$

Consider the following American derivatives $H = \{H_0, H_1, H_2\}$

ω	$H_0(\omega)$	$H_1(\omega)$	$H_2(\omega)$
a	1/5	1	2
b	1/5	1	0
c	1/5	0	1

d	1/5	0	0

We first calculate the Snell envelope U of H. At time $t = 2$,
We have $U_2 = H_2$.
Let $t = 1$. Then,

$$U_1 = \max\{E^*(U_2|\mathcal{F}_1), H_1\}$$

Now,

$$E^*(U_2|\mathcal{F}_2) = \frac{2*\frac{1}{9}+0*\frac{2}{9}}{\frac{3}{9}} I_{\{a,b\}} + \frac{1*\frac{4}{27}+0*\frac{14}{27}}{\frac{13}{27}} I_{\{c,d\}} = \frac{2}{3} I_{\{a,b\}} + \frac{2}{9} I_{\{c,d\}}$$

Since $H_1 = I_{\{a,b\}}$, it follows that

$$U_1 = \max\{E^*(U_2|\mathcal{F}_1), H_1\} = I_{\{a,b\}} + \frac{2}{9} I_{\{c,d\}}$$

Finally, let t = 0. Then

$$E^*(U_1|\mathcal{F}_0) = E^*(U_1) = 1 * \frac{1}{3} + \frac{2}{9} * \frac{2}{3} = \frac{13}{27}$$

Since $H_0 = \frac{1}{5} < \frac{13}{27}$, it follows that

$$U_0 = \max\{E^*(U_2|\mathcal{F}_0), H_1\} = E^*(U_2|\mathcal{F}_0) = \frac{13}{27}$$

So, the price of H is $U_0 = \frac{13}{27}$. The price of the corresponding European derivatives is

$$2 * \frac{1}{9} + 1 * \frac{4}{27} = \frac{10}{27}$$

Which is less than its American counterpart. Let us now calculate τ_0, the smallest optimal strategy. Recall the τ_0 is defined as:

$$\tau_0(\omega) := \inf\{t \in \{0,1,2\} : U_t(\omega)\} = H_t(\omega)\}$$

In this example we have:

ω	$\tau_0(\omega)$
a	1
b	1
c	2
d	2

Finally, we calculate the hedging strategy, φ, we write

$$U = U_0 + M - A$$

Where M is a martingale and A is a positive increasing process. The formula for the Doob decomposition is,

$A_0 = 0,$

$A_1 = A_0 + E^*(U_1 - U_0|\mathcal{F}_0)$ And

$A_2 = A_1 + E^*(U_2 - U_1|\mathcal{F}_1)$

With

$$M_n = U_n - U_0 - A_n$$

It is easy to check that these components are as given below:

ω	$A_0(ω)$	$A_1(ω)$	$A_2(ω)$	$M_0(ω)$	$M_1(ω)$	$M_2(ω)$
a	0	0	1/3	0	14/27	2-(4/27)
b	0	0	1/3	0	14/27	-(4/27)
c	0	0	0	0	-7/27	14/27
d	0	0	0	0	-7/27	-13/27

We need to find a predictable process, φ, such that $M = (φ\,.\,X)$. Note that this is equivalent to finding a hedging strategy for a European derivative with payoff M_2. Hence, we get

$$φ = \frac{M_1 - M_0}{X_1 - X_0} = \frac{7}{135}$$

And

$$φ_1 = \frac{M_1 - M_0}{X_1 - X_0} = \frac{2}{5} I_{\{a,b\}} + \frac{2}{9} I_{\{c,d\}}$$

So, a hedging strategy for H begins with an initial capital of $\boldsymbol{\pi(H) = U_0 = \frac{13}{27}}$ **,** and we invest $φ_1$ at time 0 and $φ_2$ at time 1.

Solution to Question 3

The price of an American put option with fixed expiration time T>0 and strike price K can be expressed as the expected value of its discounted payoff:

$$f(t, S_t) = \underbrace{sup}_{\substack{t \ll \tau \ll T \\ \hline \tau\ stopping\ time}} \quad \mathbb{E}^*[e^{-(\tau - t)r}(K - S_t)^+|S_t]$$

Under the risk natural probability measure $\mathbb{P}^*$, where the supremum is taken over stopping times between t and fixed maturity T.

In this section we take T=+infinity, in which case we refer to these options as perpetual options, and the corresponding put and call options ate respectively priced as

$$f(t, S_t) = \underset{\tau \text{ stopping time}}{\underbrace{sup}_{t \ll \tau \ll T}} \quad \mathbb{E}^*[e^{-(\tau - t)r}(K - S_\tau)^+ | S_t]$$

The boundary conditions for the pricing model of the perpetual American call are

$$C_\infty\ (0) = 0\ and\ C_\infty\ (S_\infty^*) = S_\infty^* - X$$

This option price and its optimal exercise boundary can be reduced to a steady state-like free boundary problem, which is independent of time.

Here is a closed form solution for pricing the perpetual American options using the binomial tree method:

Stock price = 40 = S
Stock Price = K
Standard deviation (volatility) = 30 % = r
Risk Free rate = 5 % = div
b- Negative = 0.5 $- (\frac{r-div}{\sigma^2}) - (((\frac{r-div}{\sigma^2}) - 0.5)^2 + \frac{2*r}{\sigma^2})^2$
a = b-negative

$$S_{bar} = \frac{strike * a}{a - 1}$$

If S > S_{bar} then

$$Price = (k - S_{bar}) * (\frac{S}{S_{bar}})^a$$

In this case,

b-negative = -1.17704
S_{bar} = 21.62645
Price = 8.0909116

Thus the American option is explained with the some numerical and theoretical concepts with addressing the solutions to the all the question asked in the research topic.

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