Collaborative Review Task M1 Solution

Vivek Srivastava
Student, MSQF, World Quant University

We start with mentioning that a sample space Ω on a real number line can be generated between -∞ and ∞ and can be used for constructing a probability space (Ω,F, $\mathbb{P}$) and generating a random variable X: $\Omega \rightarrow \mathbb{R} on(\omega$,F, $\mathbb{P}$) such that X has a standard normal distribution. As indicated by the differential equation provided in the first portion of the problem statement, we need to generate a probability distribution such that its derivative is $\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$, $x \in \mathbb{R}$.

Selection of a target function

We start by assuming a function P(x)=$\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}dx,\ x \in \mathbb{R}$. Differentiating P(X) with respect to X will yield us the desired function

$$\frac{d\mathbb{P}_X}{d\lambda_1}(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2},\ x \in \mathbb{R}$$

Generating the probability Space and a σ- Algebra

In this case our sample space is Ω = (-∞, ∞) and the smallest σ - algebra that can be generated is the borel algebra on open interval (-∞, ∞) is $\mathcal{F} = (\Omega, \phi)$.

This σ - algebra satisfies all the properties necessary to become a σ - algebra as it contains ϕ, is closed under complement (complement of ϕ is Ω = (-∞, ∞) and vice-versa) and closed under union of the subsets (Union of ϕ and Ω = (-∞, ∞) is Ω) .

From this sample space, we generate a random variable X which is measurable on the real line using the function $\frac{d\mathbb{P}_X}{d\lambda_1}(x)$.

Prove that X follows standard Normal Distribution

We can see that the right hand side of the equation $\frac{d\mathbb{P}_X}{d\lambda_1}(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2},\ x \in \mathbb{R}$ i.e. $\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$, is ≥ 0, x$\in \mathbb{R}$ and $\frac{d\mathbb{P}_X}{d\lambda_1}(x)$ is always continuous for $x \in \mathbb{R}$.

We can show that the area under the curve of function $\frac{d\mathbb{P}_X}{d\lambda_1}(x)$ from (-∞, ∞) is 1.

Let P(x)=I

$$I=\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}dx$$

$$I*I=(\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}dx) * (\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}dy)$$

$$I*I=(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}dxdy)$$

Evaluating the above integral in polar coordinates we get, I*I=1, hence, I=1 which basically means that the P(-∞, $\frac{d\mathbb{P}_X}{d\lambda_1}(x)$,∞)=1. This proves that X indeed has the standard normal distribution.