Group Work Project

Submission 2

***A TRINOMIAL TREE: A DISCRETE TIME MODEL FOR STOCK PRICES***

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## Abstract

In this document we are trying to express short review on a Trinomial Tree. First, we have tried to construct the Concrete Trinomial tree of given stock price of X with given conditions by defining the probability spaces and random variable Zn. After the tree we have studied the conditions of arbitrage free and EEM's. The third part of this paper includes the brief study of the attainable claim in the market.

Given,

Stock price X, defined as

$X_{n+1} = X_n Z_{n+1}$ , $X_0$ = Constant, And

$(Z_n)_{n=1}^T$ is a sequence of independent random variable

Inductive we can write for each n ≥ 1 i.e. For n = 1, 2, 3….. , T

$$
\begin{aligned}
X_{n+1} &= X_n Z_{n+1} \\
&= X_{n-1} Z_n Z_{n+1} \\
&\quad \ldots\ldots \\
&= X_0 Z_1 Z_2 Z_{n+1} \\
&= X_0 \prod_{t=1}^{n} Z_{t+1}
\end{aligned}
$$

**PART 1**

Let Ω := {w = (w1, w2) : $w_i$ ∈ { -1, 0,1 }}

And P ({w}) > 0 for each w ∈ Ω

In trinomial tree, $Z_n$ can take 3 distinct values as per below

$$
Z_n = \begin{cases} u & \text{if } w = 1 \\ 1 & \text{if } w = 0 \\ d & \text{if } w = -1 \end{cases}
$$

Where, $u$ and d are positive real numbers

**Defining probability space** (Ω, $\mathcal{F}$, F, P)

Collection of events $\mathcal{F} := 2^{\Omega}$

Filtration $F = (\mathcal{F})_{n \geq 0}$

For T= 2, values of $\mathcal{F}$ are as given below:

$\mathcal{F}_0 := \mathcal{F}_1 := \sigma(\{X_0\}) := \{\varnothing , \Omega\}$

Blocks of $\mathcal{F}_0$ : {11, 10,1-1,01,00,0-1,-11,-10,-1-1}

$\mathcal{F}_1 := \sigma(\{X_0, X_1\})$

Blocks of $\mathcal{F}_1$ : {{11, 10,1-1}, {01,00,0-1}, {-11,-10,-1-1}}

$\mathcal{F}_2 := \sigma(\{X_0, X_1, X_2\})$

Blocks of $\mathcal{F}_2$ : {{11}, {10}, {1-1}, {01}, {00}, {0-1}, {-11}, {-10}, {-1-1}}

Without loss of any generality, let's define probabilities

$$
P(w) = \begin{cases} p_u & \text{if } w = 1 \\ 1 - p_u - p_d & \text{if } w = 0 \\ P_d & \text{if } w = -1 \end{cases}
$$

Figure below is the diagram summarizing above for T = 2 for risky asset d =1

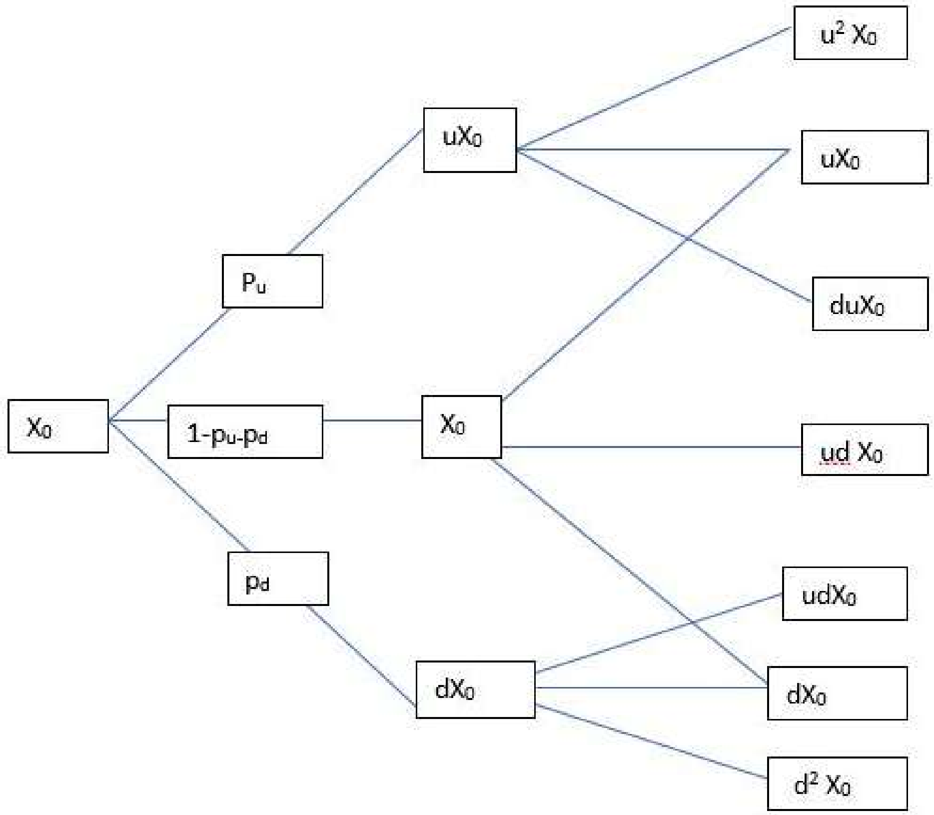



Figure: Trinomial Tree

| n= 0 | n =1 | n =2 | ω |
|---|---|---|---|
| $X_0$ | u $X_0$ | $u^2 X_0$ | 1 1 |
| $X_0$ | u $X_0$ | u $X_0$ | 1 0 |
| $X_0$ | u $X_0$ | du $X_0$ | 1 -1 |
| $X_0$ | $X_0$ | u $X_0$ | 0 1 |
| $X_0$ | $X_0$ | $X_0$ | 0 0 |
| $X_0$ | $X_0$ | d $X_0$ | 0 -1 |
| $X_0$ | d $X_0$ | u d $X_0$ | -1 1 |
| $X_0$ | d $X_0$ | $dX_0$ | -1 0 |
| $X_0$ | d $X_0$ | $d^2X_0$ | -1 -1 |

**PART 2:**

**ARBITRAGE FREE CONDITION**

Assuming self-financing trading strategy in assets 1 risk free and another risky asset (Stock). Also assume risk free rate is r.

r = constant, known from market

To rule out arbitrate we must assume below condition

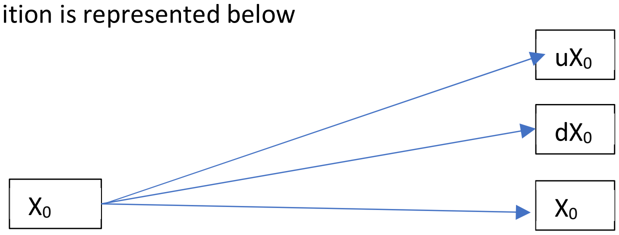
$$0 < d < 1 + r < u$$

The inequality d>0 follows from the positivity of stock price. Now we will show if above condition is not true then there would be arbitrage

**If d >= 1+r**

One could begin with 0 wealth at time n = 0, borrow from bank at risk free rate to buy stock. Even in the worst case he will have stock worth not less than 1+r, and has a positive probability of being worth strictly more since, $0 < 1+r <= d < u$

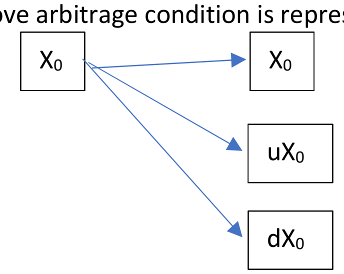
Above arbitrage condition is represented below




On the other hand, **if u <= 1+ r**

One could sell the stock short and deposit the proceeds in bank and earn risk free rate. Even in the best case of stock, the cost of buying the stock at time n = 1, will be less than or equal to value of deposits in the bank. And since $d < u < = 1+r$, there is positive probability that the cost of buying back the stock will be strictly less than then value of deposits in the bank.

Above arbitrage condition is represented below




**EQUIVALENT MARTINGALE MEASURE**

Now, since we have got no arbitrage condition, Let's find EMM

$$P(w) = \begin{cases} p_u & \text{if } w=1 \\ 1-p_u-p_d & \text{if } w=0 \\ P_d & \text{if } w=-1 \end{cases}$$

From no arbitrage condition, the path has positive probability so

$p_u , p_m , p_d > 0$ and

And since trinomial tree has three possible paths

$$p_u + p_m + p_d =1 \qquad ...1$$

now we consider martingale measure on $X_n$, P is martingale for every n>= 0

$$X_n = E(X_{n+1} | F_n)$$

For n=0, assuming r = 0

$$X_0 = u\, p_u X_0 + p_m X_0 + d\, p_u X_0$$

$$u\, p_u + p_m + d\, p_d =1 \qquad ...2$$

where, u, d >0 and constant.

Since there are 2 equations for 3 unknowns. There is no unique solution. This is in contrast to binomial model where we could uniquely get risk neutral probabilities.

In trinomial tree $\{p_u, p_m, p_d\}$ are risk neutral probabilities which we should satisfy

$0 <= p_u, p_m, p_d <= 1$

Also $E(|X|) < \infty$

## COMPLETENESS OF MARKETS

As we have seen above, market cannot satisfy completeness condition until we introduce additional condition such that, we get unique $p_u, p_m, p_d$

Let consider variance of X, as we know

$Var(x) = E(X^2) - (E(X))^2$

Let's assume, stock has volatility of Sigma

So that $Var(X) = \sigma^2 * \Delta t$

And time step $\Delta t = t_{n+1} - t_n$

$E(X^2) = u^2 p_u + p_m + d^2 p_d$

So, putting in equations of Var (X)

$u^2 p_u + p_m + d^2 p_d - 1 = \sigma^2 \Delta t$ ....3

now we got 3 equations and 3 unknowns' risk neutral probabilities to solve.

replacing $p_m$ from equation 1 into equation 2 and equation 3

$u\, p_u - p_u + d\, p_d - p_d = 0$

$p_u (u-1) = (1-d)\, p_d$

$p_u = ((1-d) / (u-1))\, p_d$ .....4

from equation 3

$u^2 p_u - p_u - p_d + d^2 p_d = \sigma^2 \Delta t$

$(u^2 - 1) p_u - p_d (1-d^2) = \sigma^2 \Delta t$

Substituting $p_u$ from equation 4, we get,

$[(1-d) / (u-1)] (u^2-1) - p_d (1-d^2) = \sigma^2 \Delta t$

$p_d (1-d^2) = [(u+1) / (1+d)] - \sigma^2 \Delta t$

$= (1-d)(u+1) - \sigma^2 \Delta t$

Hence,

$p_d = (u+1) / (1+d) - (\sigma^2 \Delta t) / (1-d^2)$

$p_u = (1-d)(u+1) / (1+d)(u-1) - (\sigma^2 \Delta t) / (1+d)(u-1)$

$p_m = 1 - p_d - p_u$

we should make sure certain conditions so, above probabilities remain reasonable.

$0 <= p_u, p_m, p_d <= 1$

Based on above condition we can find boundary conditions for u & d

## PART 3:

Lets assume H is a contingent claim on underlying X. H is measurable with respect to $\mathcal{F}$

**Classification of attainable claims**

There may be 2 types of contingent claims

1. Path independent
2. Path Dependent

*Path Independent*

Call Option: $H = (X_T - K)^+$, where K>0 is called the strike price

Put Option: $H = (K - X_T)^+$, where K>0 is called the strike price

*Path Dependent*

Asian Call option

Digital option
American option
Barrier Options - Knock in/ Knock outs

Trinomial tree model can price some of the exotic options in a better way than binomial tree model. For example, barrier options.

There are 2 important aspect of an **attainable contingent claim**

1. Price at time = 0 ➔ how much should one pay at t=0 in return to random payment at H at time T
2. Replication – having sold the contingent claim at time 0, how can seller protect himself by investing in primary assets one risky asset - X and another - risk free.

Actually above 2 are related. An attainable contingent claim H has a unique no-arbitrage price and that price is

*E(H) for any EMM Risk Neutral Probabilities.*

**Numerical Example of attainable claim**

**$X_0$ = 100**

| | |
|---|---|
| u | 1.5 |
| d | 0.7 |
| Std Dev | 10% |
| Time step - delta t | 50 |
| p_u | 0.294118 |
| p_m | 0.215686 |
| p_d | 0.490196 |

| X_0 | X_1 | X_2 |
|---|---|---|
| | | 225 |
| | 150 | 150 |
| | | 105 |
| | | 150 |
| 100 | 100 | 100 |
| | | 70 |
| | | 105 |
| | 70 | 70 |
| | | 49 |

Define Contingent Claim H = max (X_2 - 110, 0)

| H_0 | H_1 | H_2 |
|---|---|---|
| | | 115 |
| | 42.45098 | 40 |
| | | 0 |
| | | 40 |
| 12.48558 | 0 | 0 |
| | | 0 |
| | | 0 |
| | 0 | 0 |
| | | 0 |

## Conclusion

In our study, we developed a concrete 2 step trinomial model. Trinomial tree gives an extra degree of freedom compared to binomial trees. We found trinomial tree models are better in pricing since they also incorporate 2$^{nd}$ moment that is variance in calculation. While this increases accuracy, we should carefully select up, down movement and calculate associated risk neutral probabilities.

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