LECTURE 16

SINGLE VIEW GEOMETRY

16-720B: COMPUTER VISION (Fall 2017)

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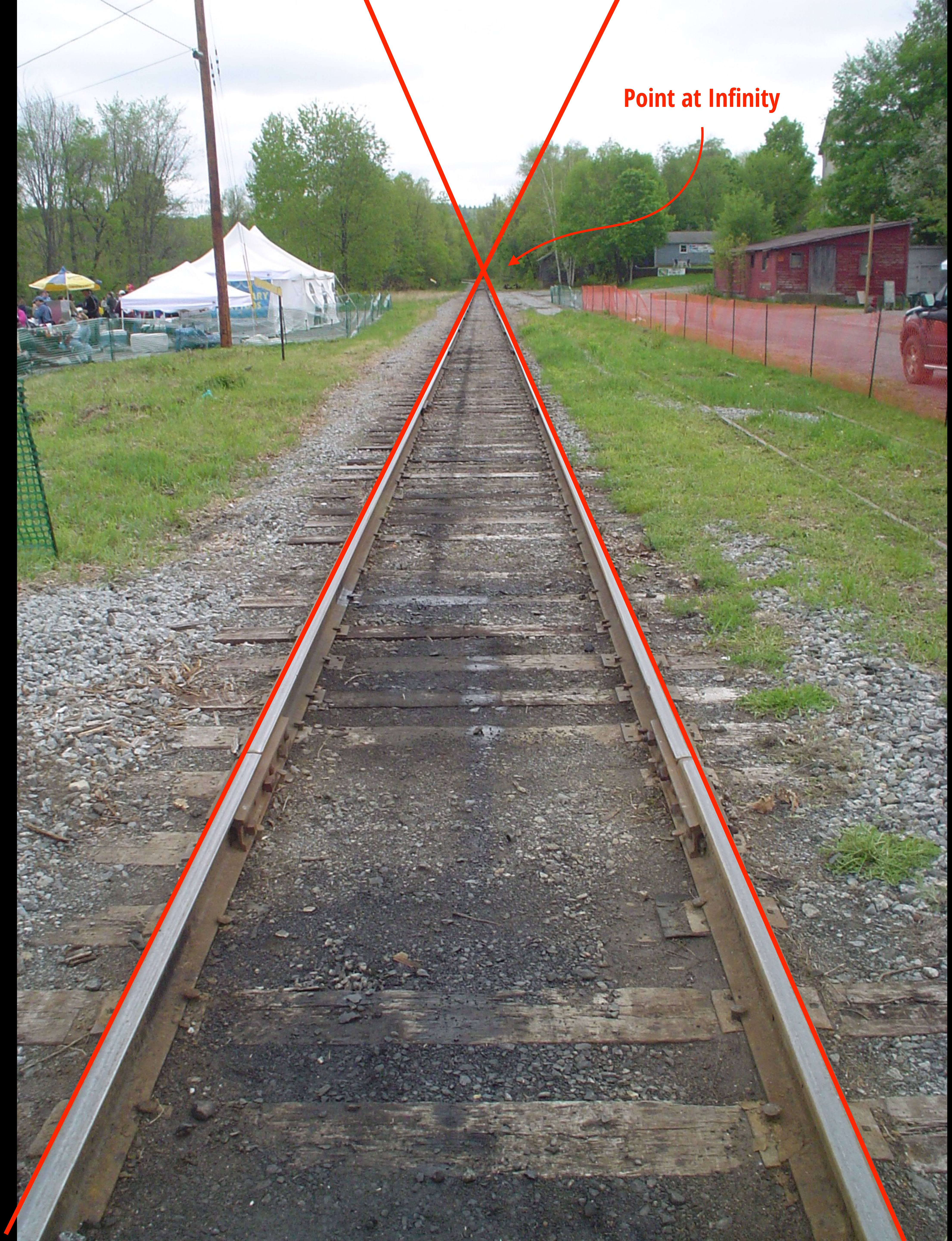
Based on slides by Kayvon Fatahlian, Alexei Efros, Robert Collins, and Srinivas Narasimhan


Image Size inversely proportional to distance
Why?

Photo Credit: Thomas Wolf


Point at Infinity

Old City of Dubrovnik: 57,845 photos

CAMERA GEOMETRY

Pinhole Cameras produce Perspective Projection

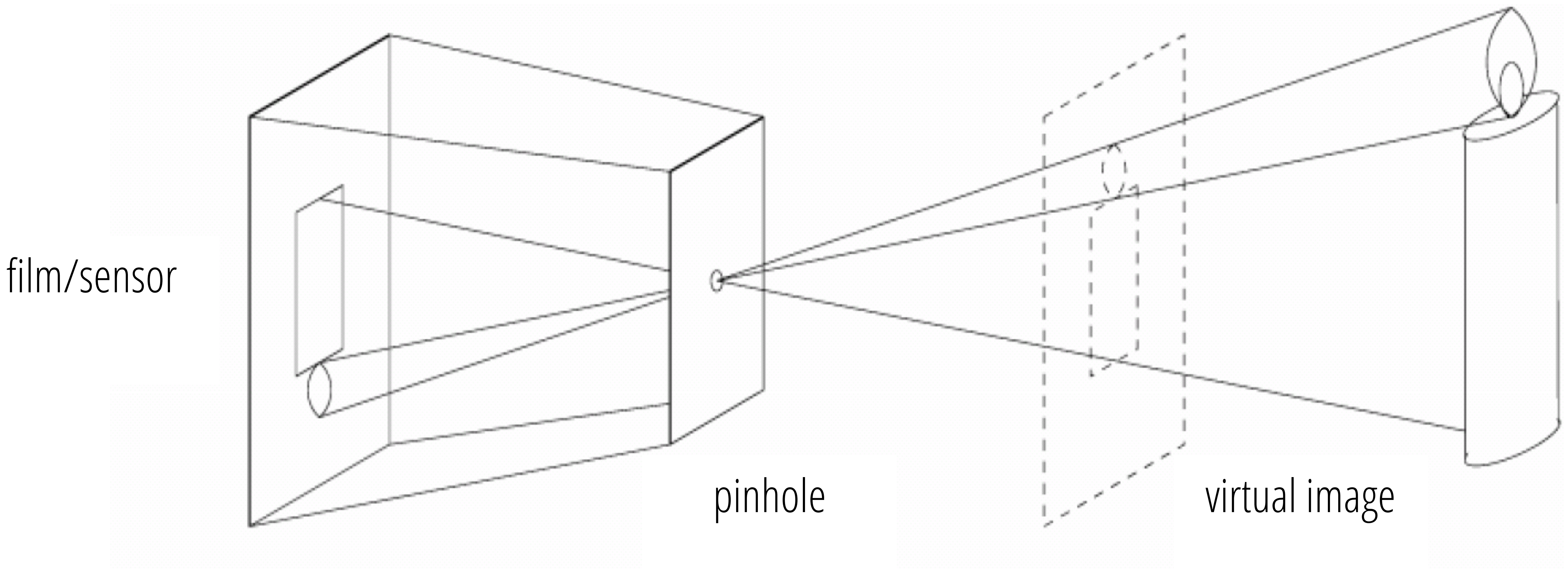
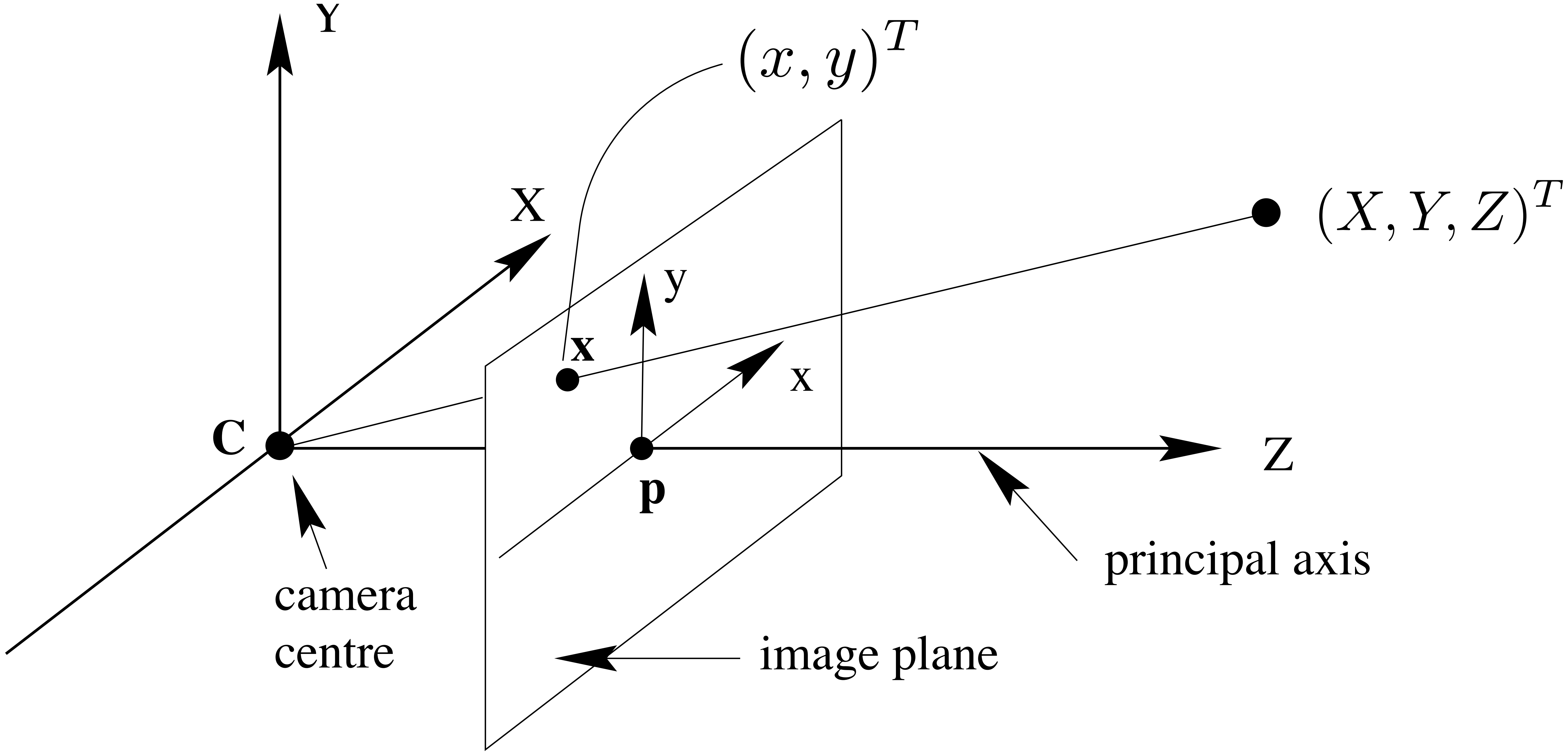


Illustration from Forsyth and Ponce

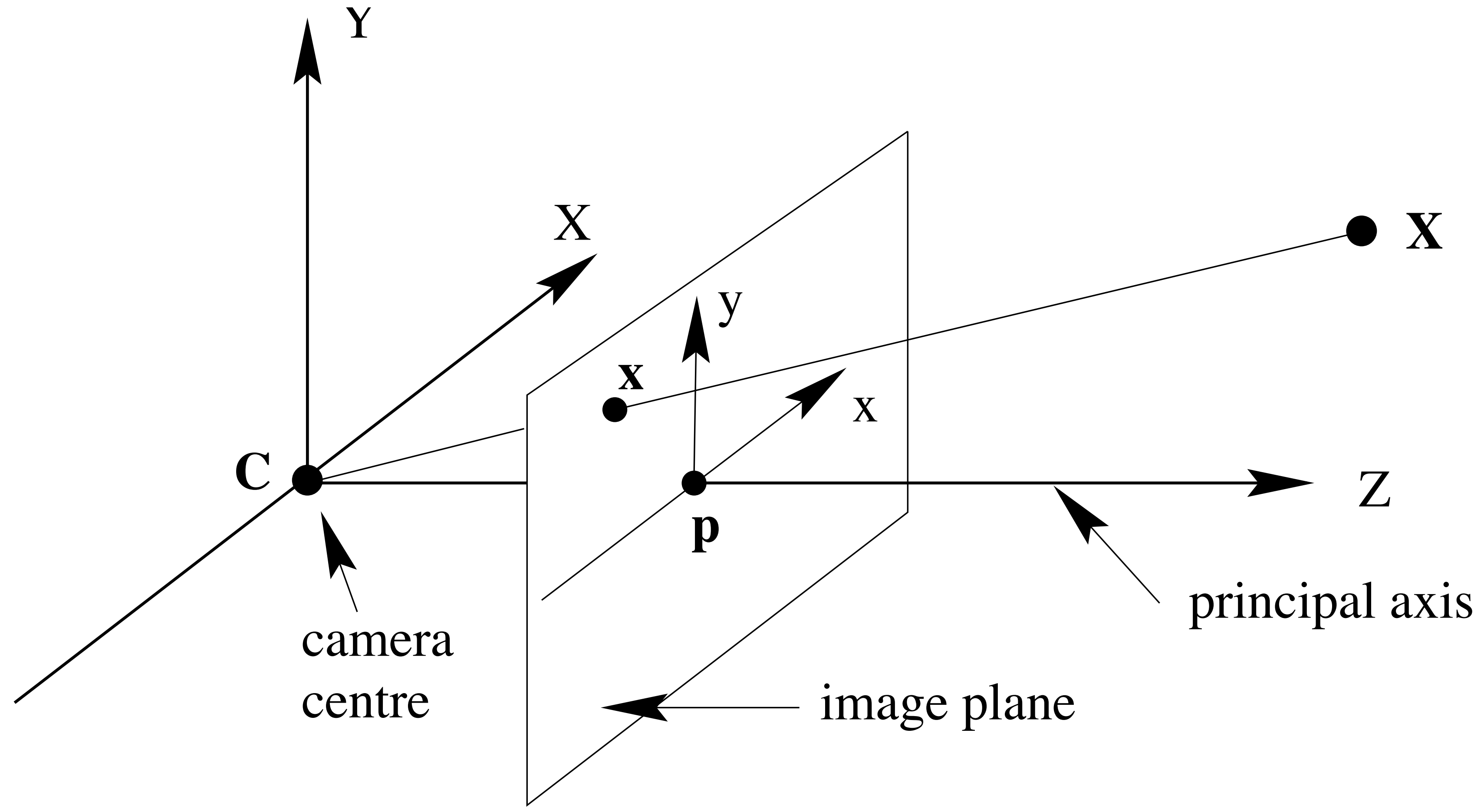
CAMERA GEOMETRY

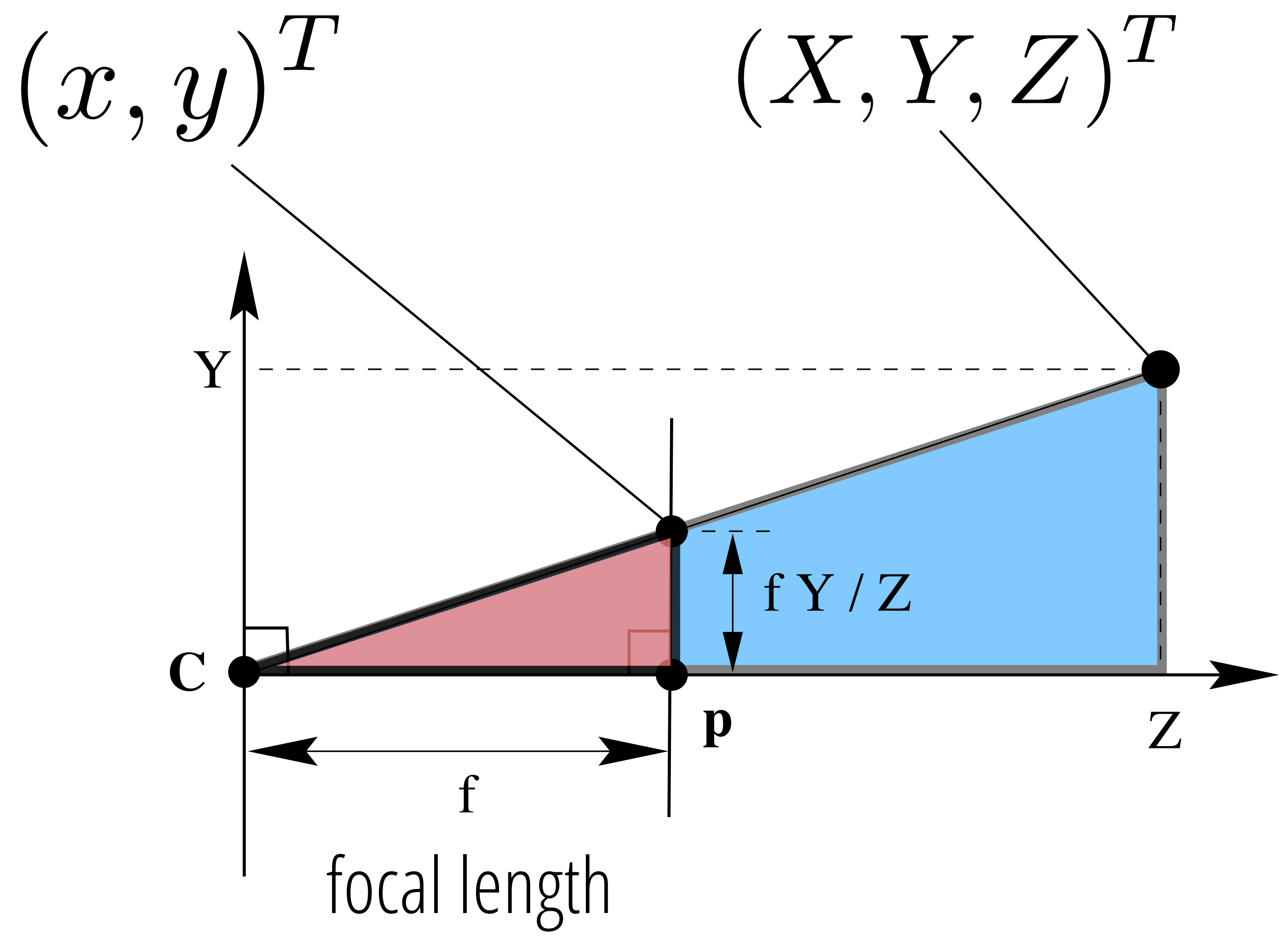
A Camera defines a mapping between the 3D world and a 2D image



CAMERA GEOMETRY

Perspective Projection can be defined using Similar Triangles





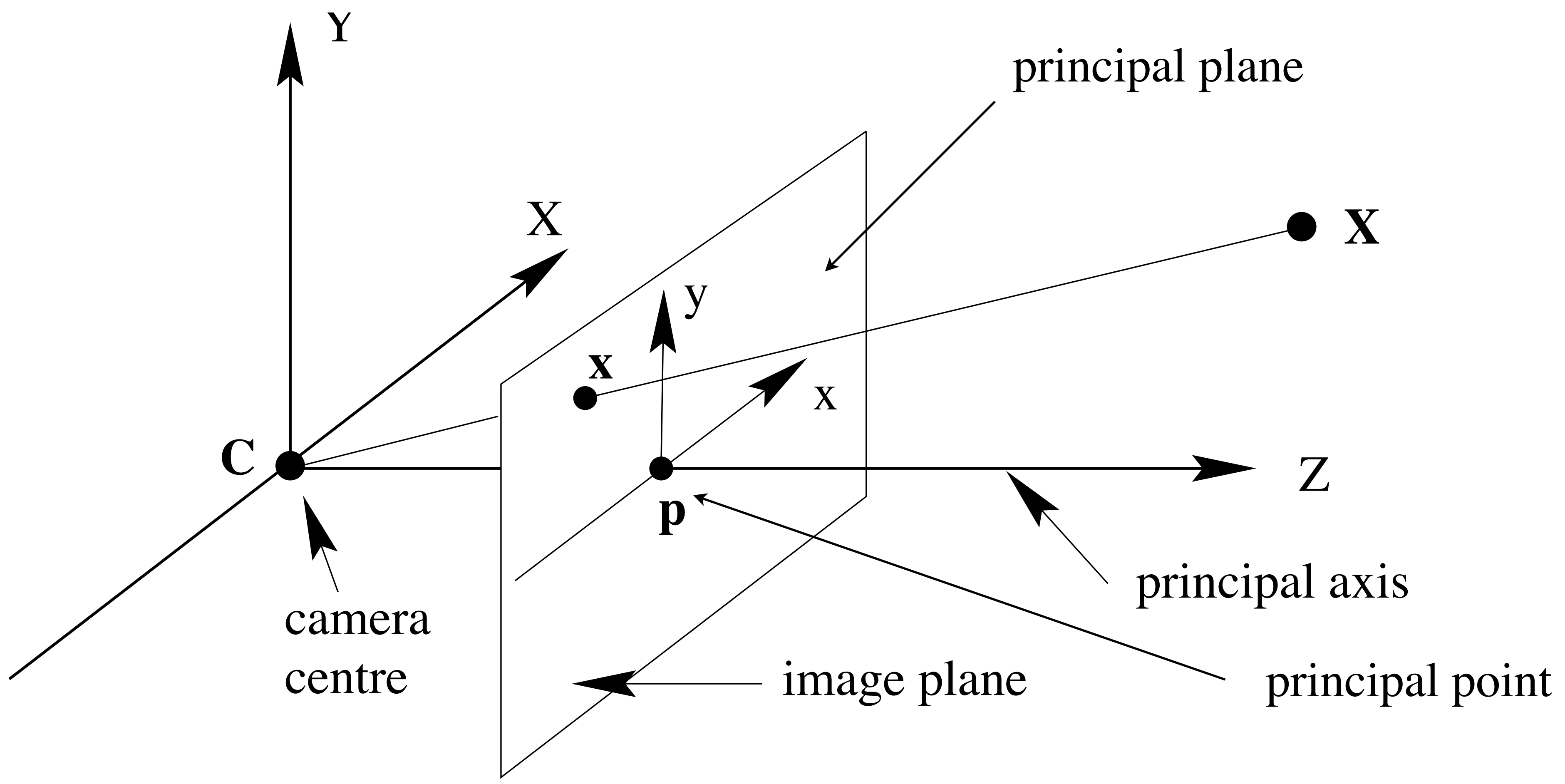
By similar triangles:

$$\frac{y}{Y} = \frac{f}{Z}$$

$$y = \frac{fY}{Z} \qquad x = \frac{fX}{Z}$$

CAMERA GEOMETRY

Central Projection Mapping



$$\mathbb{R}^3 \qquad\qquad \mathbb{R}^2$$

$$\mathbf{X} = (X, Y, Z)^T \longrightarrow \mathbf{x} = (x, y)^T$$

$$(X, Y, Z)^T \rightarrow (fX/Z, fY/Z)^T$$

Central Projection Mapping

CAMERA GEOMETRY

Revisiting Homogeneous Coordinates

$$(X, Y, Z)^T \rightarrow (fX/Z, fY/Z)^T$$

Central Projection Mapping

$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} fX \\ fY \\ Z \end{pmatrix} = \begin{bmatrix} f & & & 0 \\ & f & & 0 \\ & & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

$$\mathbf{x} = \mathtt{P}\mathbf{X}$$

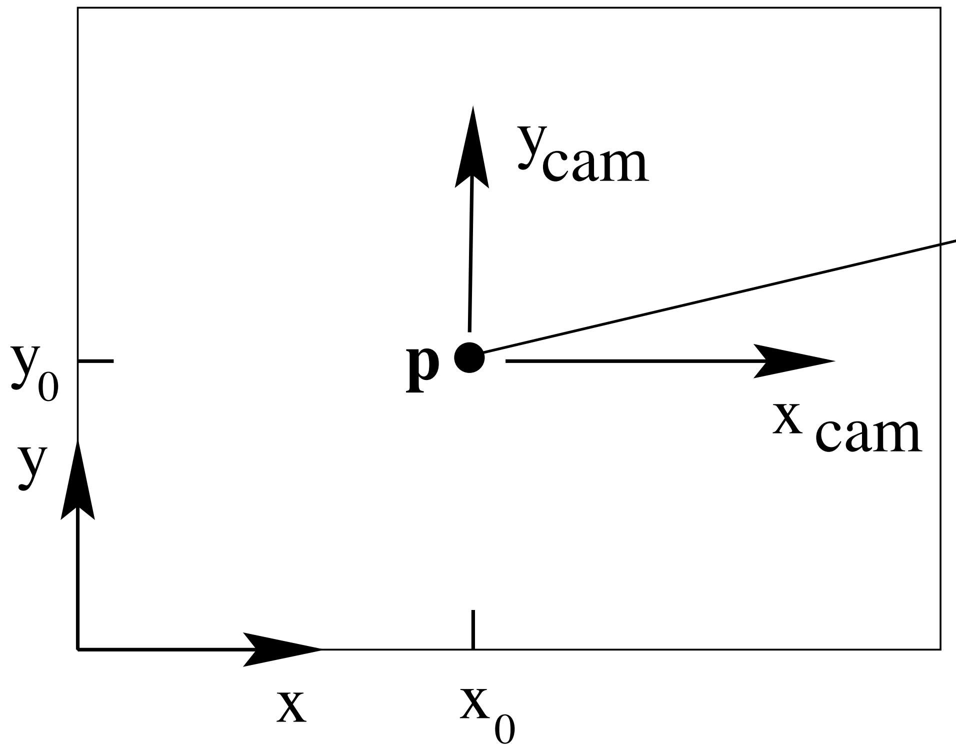
(<u>notation overload</u>: **x** now a homogeneous vector)

$$\mathtt{P} = \mathrm{diag}(f, f, 1)[\mathtt{I} \mid \mathbf{0}]$$

Camera Projection Matrix

CAMERA GEOMETRY

Principal Point Offset: Camera Coordinate vs Image Coordinate



$(p_x, p_y)^T$

principal point

$$(X, Y, Z)^T \rightarrow (fX/Z + p_x, fY/Z + p_y)^T$$

$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} fX + Zp_x \\ fY + Zp_y \\ Z \end{pmatrix} = \begin{bmatrix} f & & p_x & 0 \\ & f & p_y & 0 \\ & & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

$$\mathbf{x} = \mathtt{K}[\mathtt{I}|\mathbf{0}]\mathbf{X}_{\mathrm{cam}}$$

$$\mathtt{K} = \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix}$$

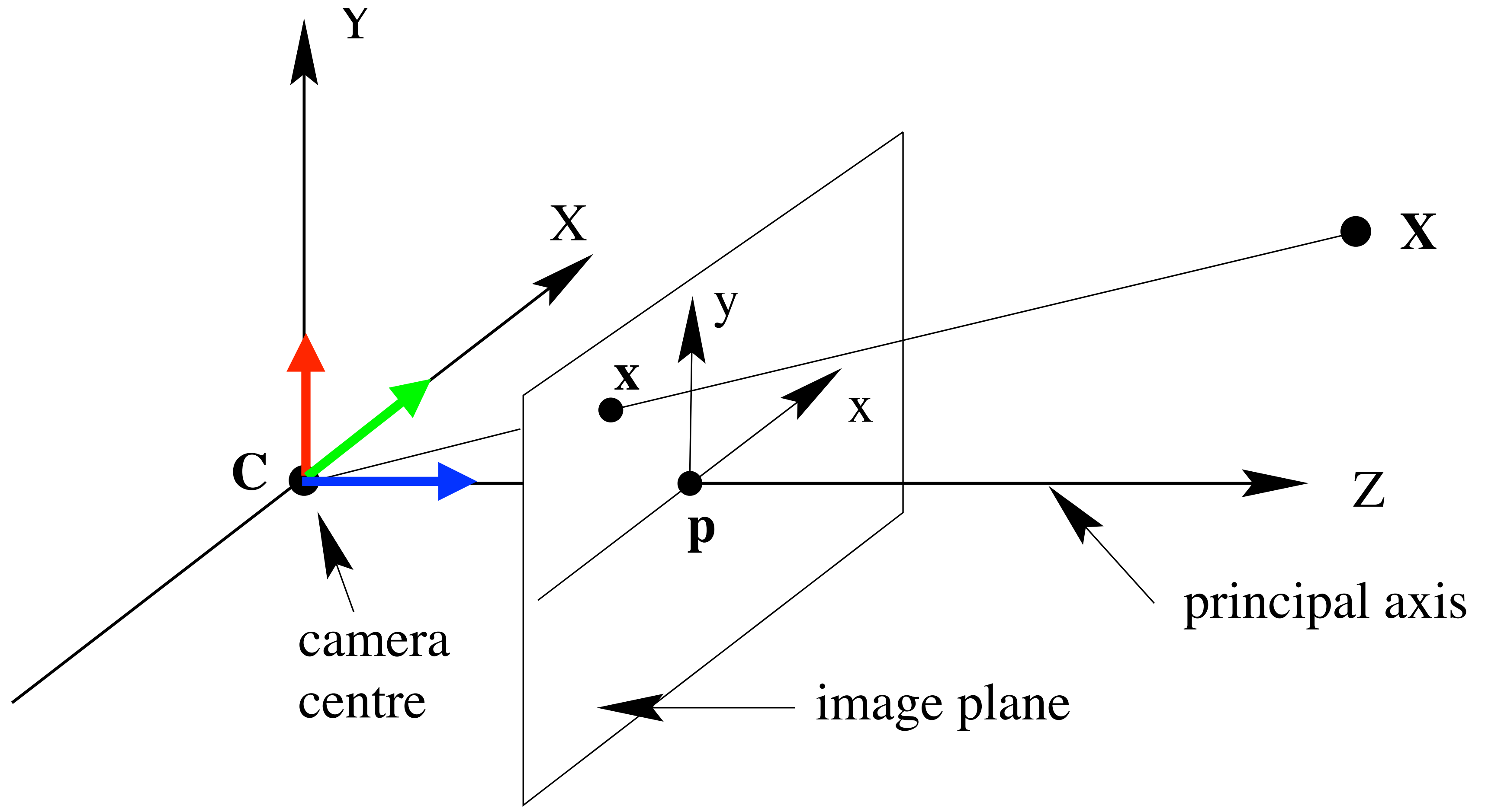
camera calibration matrix

CAMERA GEOMETRY

Camera Rotation and Translation

$$\mathbf{x} = \mathtt{K}[\mathtt{I}|\mathbf{0}]\mathbf{X}_{\text{cam}}$$

World Coordinate

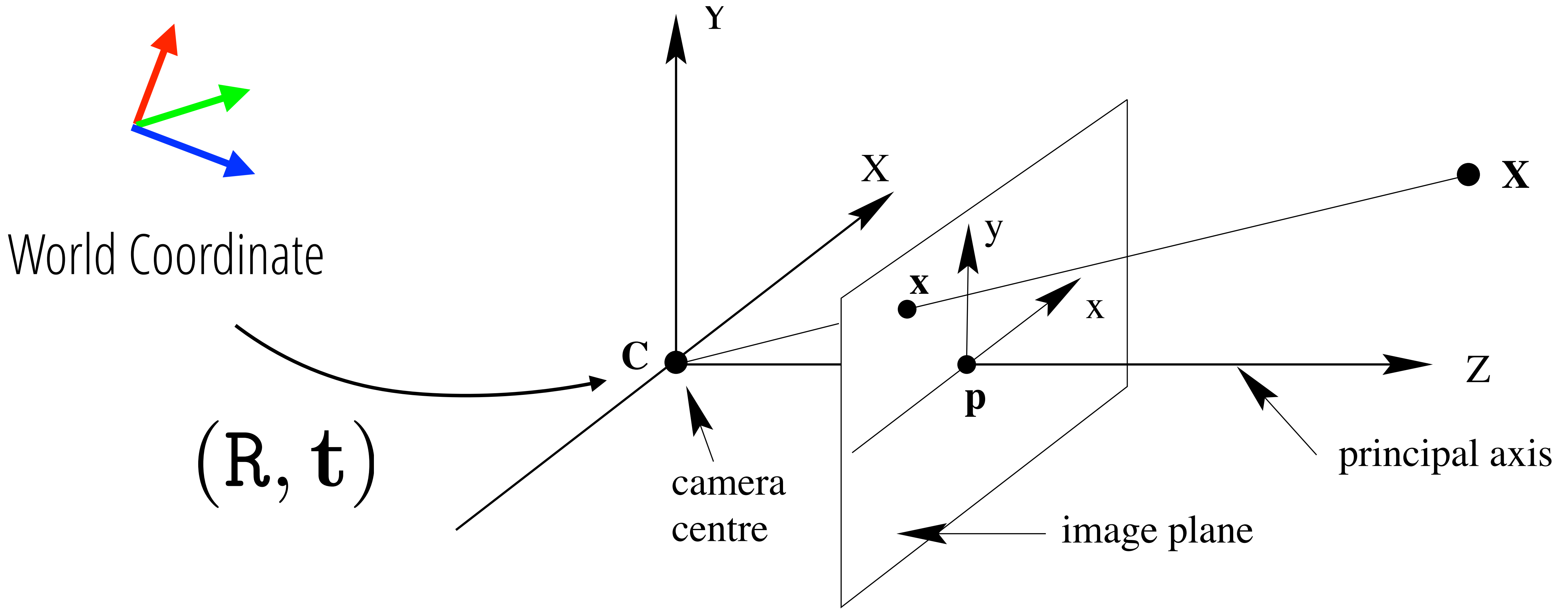


World Coordinate and Camera Coordinate System are Aligned

CAMERA GEOMETRY

Camera Rotation and Translation

$$\mathbf{x} = \mathtt{K}[\mathtt{I}|\mathbf{0}]\mathbf{X}_{\text{cam}}$$



$$\tilde{\mathbf{X}} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

inhomogeneous coordinate

$$\tilde{\mathbf{X}}_{\text{cam}} = \mathtt{R}(\tilde{\mathbf{X}} - \tilde{\mathbf{C}})$$

$$\mathbf{X}_{\text{cam}} = \begin{bmatrix} \mathtt{R} & -\mathtt{R}\tilde{\mathbf{C}} \\ 0 & 1 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} = \begin{bmatrix} \mathtt{R} & -\mathtt{R}\tilde{\mathbf{C}} \\ 0 & 1 \end{bmatrix} \mathbf{X}$$

$$\mathbf{x} = \mathtt{K}\mathtt{R}[\mathtt{I}| - \tilde{\mathbf{C}}]\mathbf{X}$$

$$\boxed{\mathbf{x} = \mathtt{K}[\mathtt{R}|\mathbf{t}]\mathbf{X}} \qquad \mathbf{t} = -\mathtt{R}\tilde{\mathbf{C}}$$

CAMERA GEOMETRY

General Pinhole Camera

$$\mathbf{x} = \mathtt{K}[\mathtt{R}|\mathbf{t}]\mathbf{X}$$

$$\mathtt{K} = \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix}$$

camera calibration matrix

3x3 rotation matrix

3x1 translation vector

3 parameters

3 parameters

3 parameters

9 degrees of freedom

CAMERA GEOMETRY

Non-square Pixels

$$\mathbf{x} = \mathtt{K}[\mathtt{R}|\mathbf{t}]\mathbf{X}$$

$$\mathtt{K} = \begin{bmatrix} \alpha_x & s & x_0 \\ & \alpha_y & y_0 \\ & & 1 \end{bmatrix}$$

3x3 rotation matrix

3x1 translation vector

5 parameters

3 parameters

3 parameters

11 degrees of freedom

$$\mathbf{x} = \mathbf{P}\mathbf{X}$$

image point

scene point

3x4 Matrix

CAMERA ANATOMY

The right null space of the camera matrix is the camera center

$$\mathbf{x} = \mathbf{P}\mathbf{X}$$

P is a rank 3

$$\boxed{\mathbf{P}\mathbf{C} = 0}$$

Example:

$$\begin{aligned}\mathbf{P}\mathbf{C} &= \mathtt{K}\mathtt{R}[\mathtt{I}|-\tilde{\mathbf{C}}]\mathbf{C} \\ &= \mathtt{K}\mathtt{R}[\mathtt{I}|-\tilde{\mathbf{C}}]\begin{pmatrix}\tilde{\mathbf{C}} \\ 1\end{pmatrix} \\ &= \mathtt{K}\mathtt{R}\left(\tilde{\mathbf{C}} - \tilde{\mathbf{C}}\right) \\ &= \mathbf{0}\end{aligned}$$

$$\tilde{\mathbf{X}} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

inhomogeneous coordinate

CAMERA ANATOMY

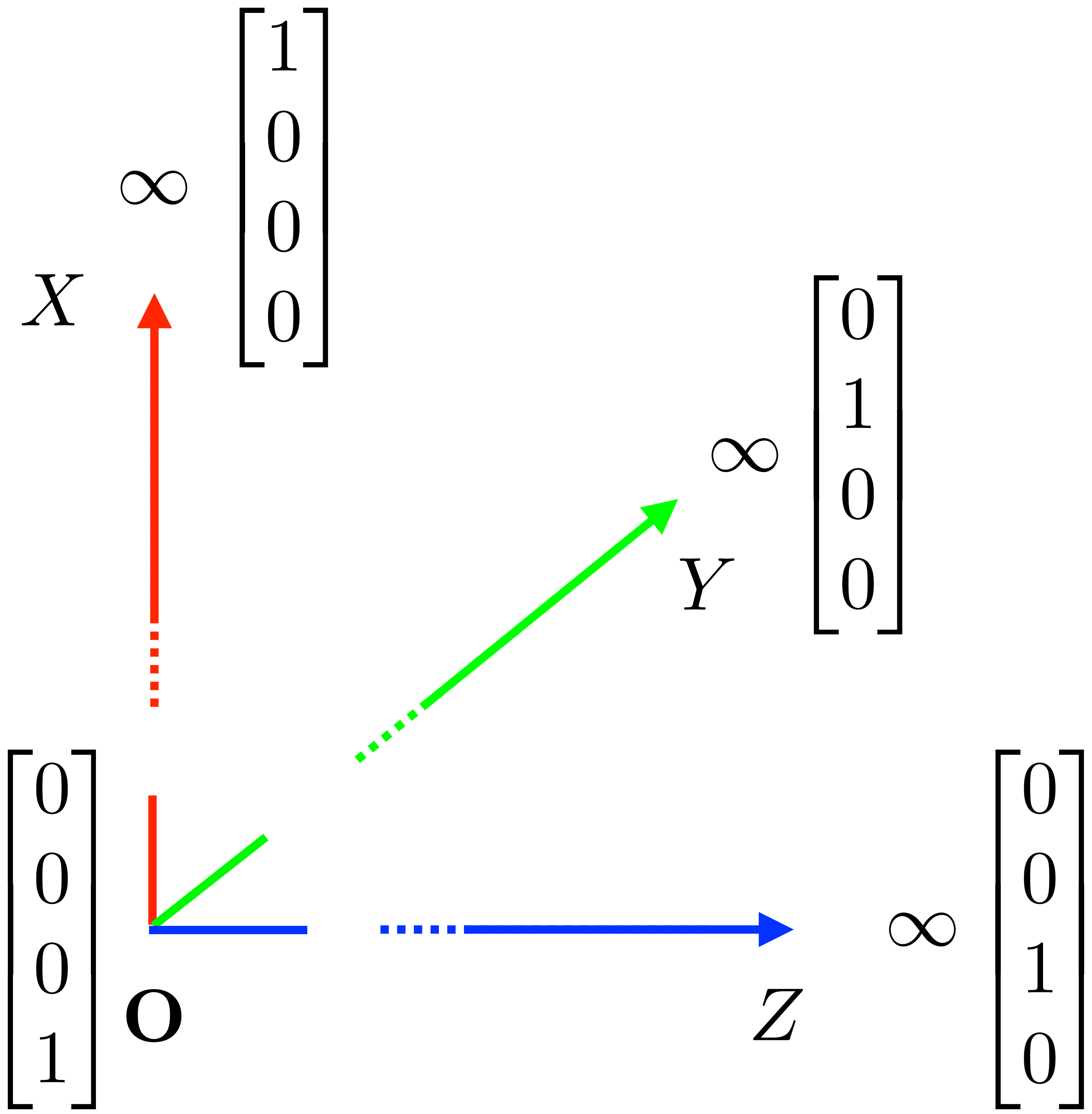
Columns vectors of P

Vanishing Point in the x-direction
$$\mathbf{p}_1 = \begin{bmatrix} | & | & | & | \\ \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 & \mathbf{p}_4 \\ | & | & | & | \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Vanishing Point in the y-direction
$$\mathbf{p}_2 = \begin{bmatrix} | & | & | & | \\ \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 & \mathbf{p}_4 \\ | & | & | & | \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Vanishing Point in the z-direction
$$\mathbf{p}_3 = \begin{bmatrix} | & | & | & | \\ \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 & \mathbf{p}_4 \\ | & | & | & | \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Image of the World Origin
$$\mathbf{p}_4 = \begin{bmatrix} | & | & | & | \\ \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 & \mathbf{p}_4 \\ | & | & | & | \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

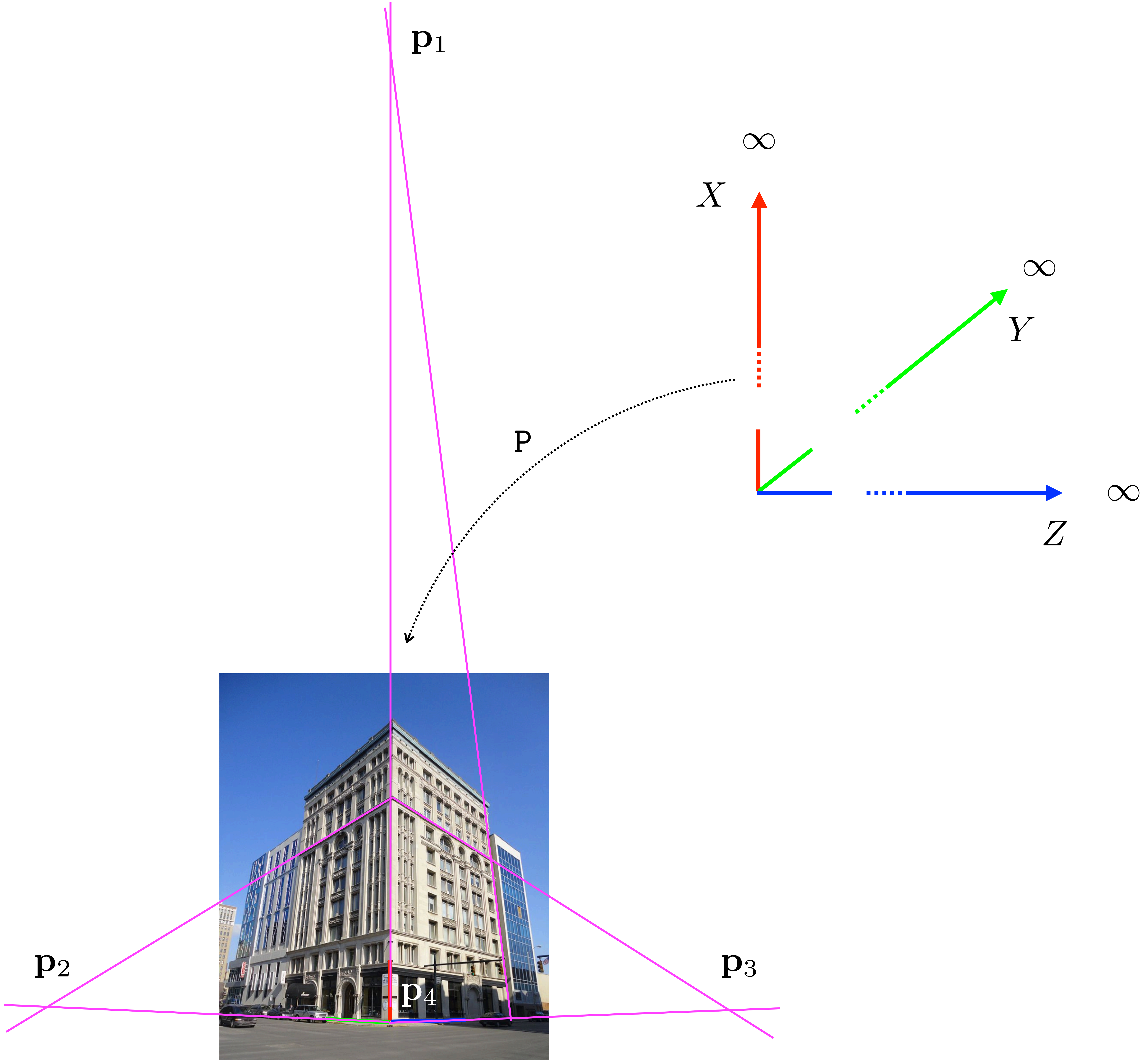


$$\mathbf{p}_1 = \begin{bmatrix} | & | & | & | \\ \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 & \mathbf{p}_4 \\ | & | & | & | \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{p}_2 = \begin{bmatrix} | & | & | & | \\ \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 & \mathbf{p}_4 \\ | & | & | & | \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

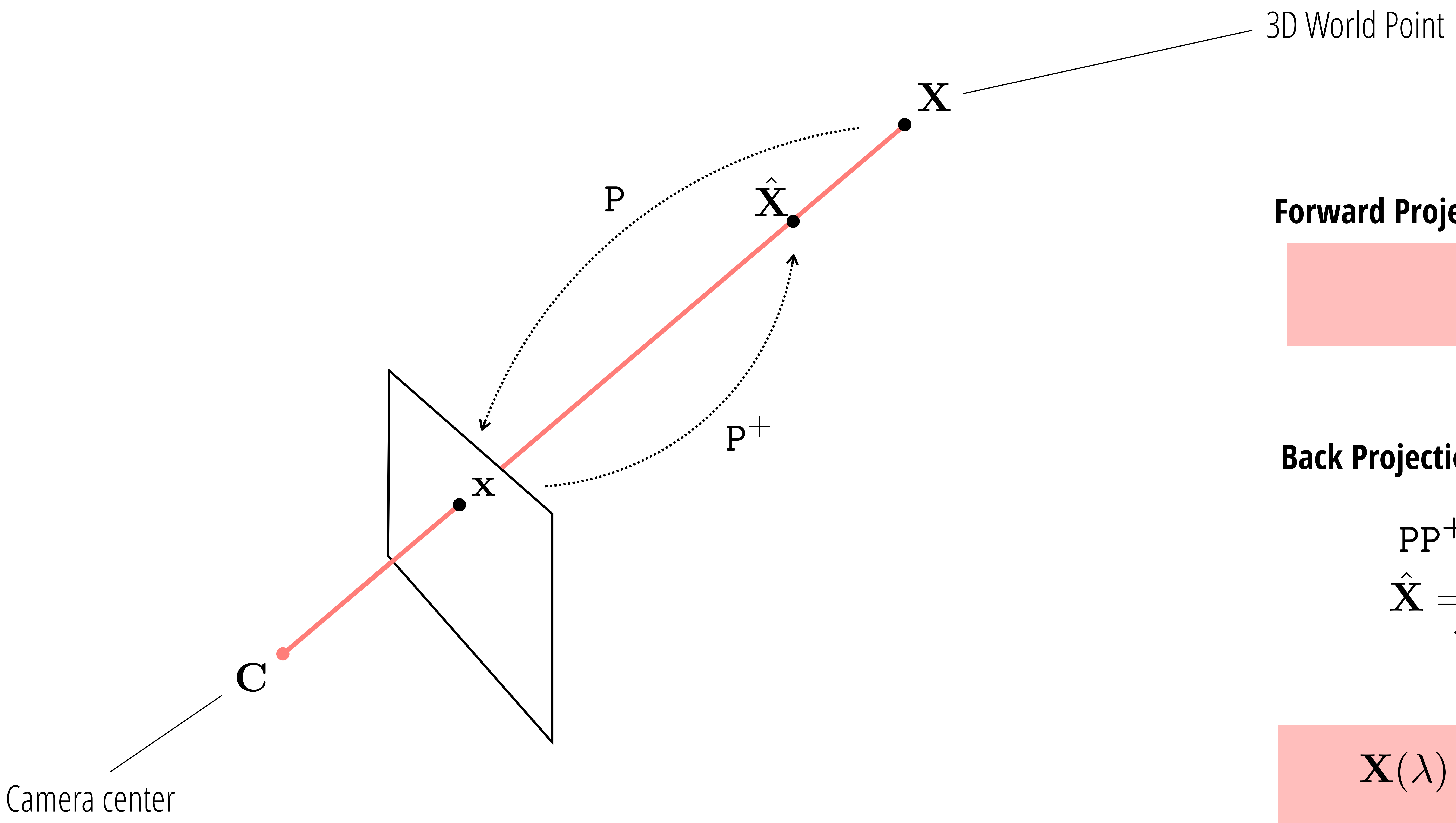
$$\mathbf{p}_3 = \begin{bmatrix} | & | & | & | \\ \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 & \mathbf{p}_4 \\ | & | & | & | \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathbf{p}_4 = \begin{bmatrix} | & | & | & | \\ \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 & \mathbf{p}_4 \\ | & | & | & | \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$



CAMERA ACTION

Forward-projection and Back-projection



Forward Projection

$$\mathbf{x} = \mathtt{P}\mathbf{X}$$

Back Projection

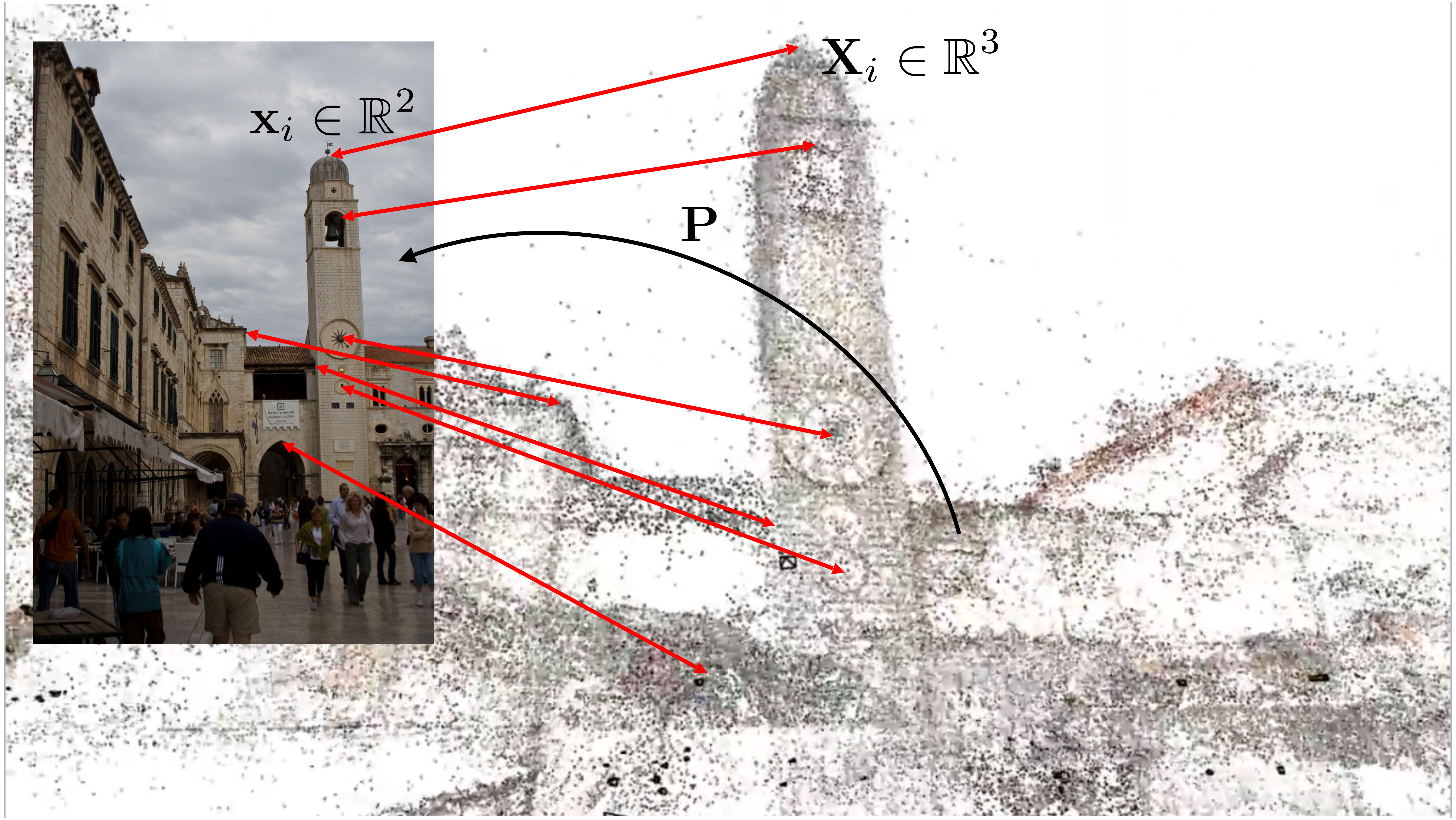
$$\mathtt{P}\mathtt{P}^+ = \mathtt{I} \qquad \mathtt{P}\mathbf{C} = 0$$

$$\hat{\mathbf{X}} = \mathtt{P}^+\mathbf{x} \qquad \mathtt{null}(\mathtt{P})$$

$$\mathbf{X}(\lambda) = \mathtt{P}^+\mathbf{x} + \lambda\mathbf{C}$$

WHERE WAS THE PHOTOGRAPHER?

Given 3D to 2D correspondences, estimate Camera matrix



CAMERA RE-SECTIONING

Calibrating the Camera Given 2D to 3D correspondences

Given: n correspondences $\mathbf{x}_i \leftrightarrow \mathbf{X}_i$

Compute: $\mathbf{P} = \mathtt{K}[\mathtt{R}|\mathbf{t}]$ such that $\mathbf{x}_i = \mathbf{P}\mathbf{X}_i$

Two step algorithm:

1. Compute **P** from the set of correspondences
2. Factorize **P** into **K, R**, and **t**.

CAMERA RE-SECTIONING

Step 1: Compute Camera Matrix

unknown

$$\mathbf{x}_i = \mathbf{P}\mathbf{X}_i$$

Each correspondence generates two nonlinear equations:

$$x_i = \frac{p_{11}X_i + p_{12}Y_i + p_{13}Z_i + p_{14}}{p_{31}X_i + p_{32}Y_i + p_{33}Z_i + p_{34}} \qquad y_i = \frac{p_{21}X_i + p_{22}Y_i + p_{23}Z_i + p_{24}}{p_{31}X_i + p_{32}Y_i + p_{33}Z_i + p_{34}}$$

Multiplying out by the denominator yields equations that are linear in the entries of **P**:

$$x_i(p_{31}X_i + p_{32}Y_i + p_{33}Z_i + p_{34}) = p_{11}X_i + p_{12}Y_i + p_{13}Z_i + p_{14}$$
$$y_i(p_{31}X_i + p_{32}Y_i + p_{33}Z_i + p_{34}) = p_{21}X_i + p_{22}Y_i + p_{23}Z_i + p_{24}$$

These two equations (produced by one correspondence) can be written in matrix form as:

$$\begin{pmatrix} X_i & Y_i & Z_i & 1 & 0 & 0 & 0 & 0 & -x_iX_i & -x_iY_i & -x_iZ_i & -x_i \\ 0 & 0 & 0 & 0 & X_i & Y_i & Z_i & 1 & -y_iX_i & -y_iY_i & -y_iZ_i & -y \end{pmatrix} \mathbf{p} = 0$$

where $\mathbf{p} = [p_{11}, \quad p_{12}, \quad \cdots \quad p_{34}]^T$

CAMERA RE-SECTIONING

Step 1 (continued): Compute Camera Matrix

These two equations (produced by one correspondence) can be written in matrix form as:

$$\begin{pmatrix} X_i & Y_i & Z_i & 1 & 0 & 0 & 0 & 0 & -x_iX_i & -x_iY_i & -x_iZ_i & -x_i \\ 0 & 0 & 0 & 0 & X_i & Y_i & Z_i & 1 & -y_iX_i & -y_iY_i & -y_iZ_i & -y \end{pmatrix} \mathbf{p} = 0$$

where $\mathbf{p} = [p_{11}, \quad p_{12}, \quad \cdots \quad p_{34}]^T$

Concatenate two equation from each correspondence (n greater than 6) to generate a 2n simultaneous equations:

$$\mathbf{A}_{(2n\times 12)}\mathbf{p} = 0$$

Will not have an exact solution, but a linear solution that minimizes |**Ap**| subject to |**p**| = 1 can be obtained.

Solving a homogeneous linear system

Solution can be obtained using the vector corresponding to the smallest singular value of the SVD of **A**

CAMERA RE-SECTIONING

Step 2: Factorize P into K,R, and t

$$\mathbf{P} = \mathtt{K}[\mathtt{R}|\mathbf{t}]$$

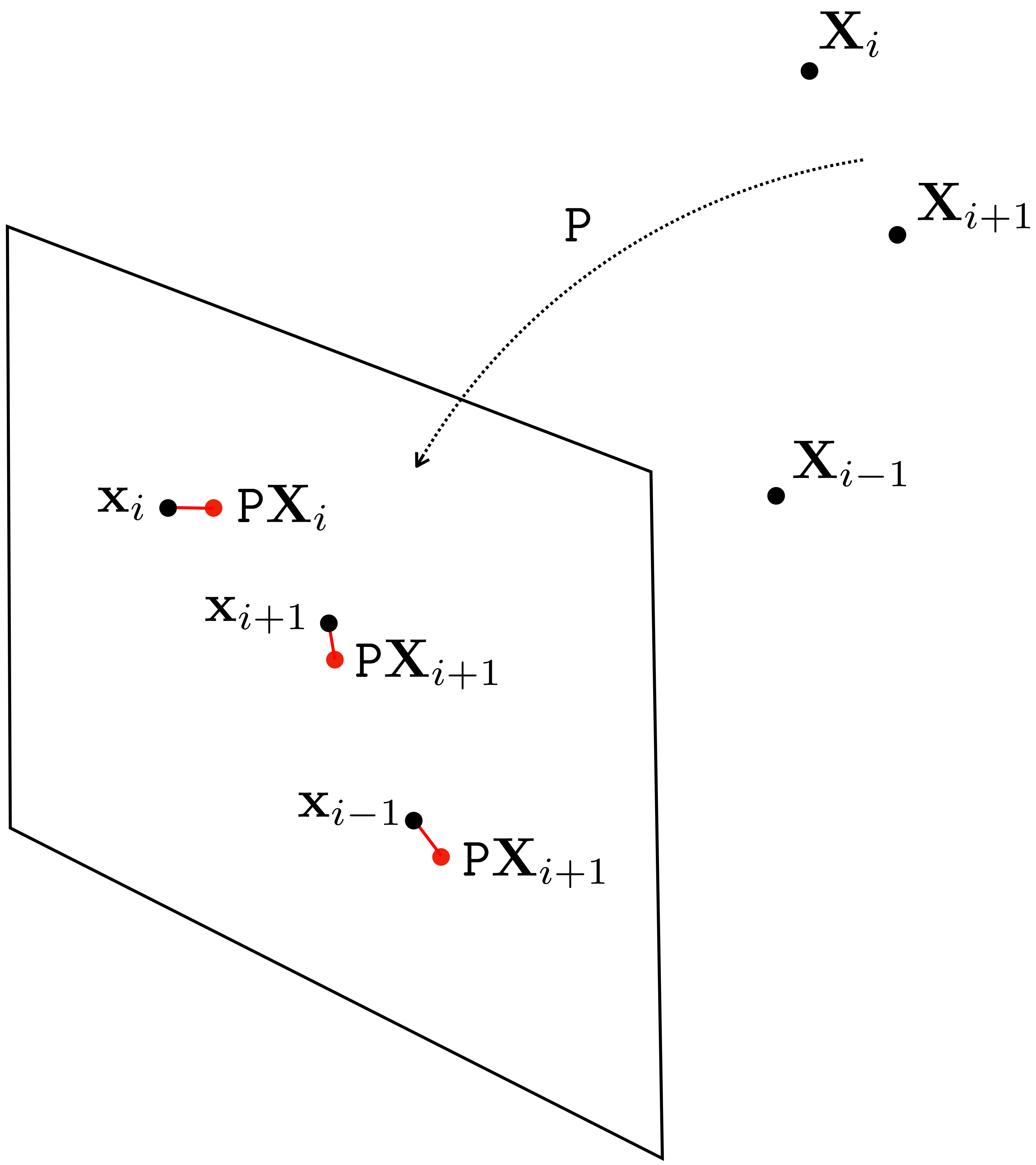
The first 3 x 3 submatrix **M** of **P** is a product (**M**=**KR**) of an upper triangular and rotation matrix

(i) Factor **M** into **K** and **R** using QR decomposition. This determines **K** and **R** (ambiguity can be removed by requiring diagonal entires of **K** to be positive).

(ii) Estimate **t** as $\mathbf{t} = \mathtt{K}^{-1}(p_{14}, p_{24}, p_{34})^T$

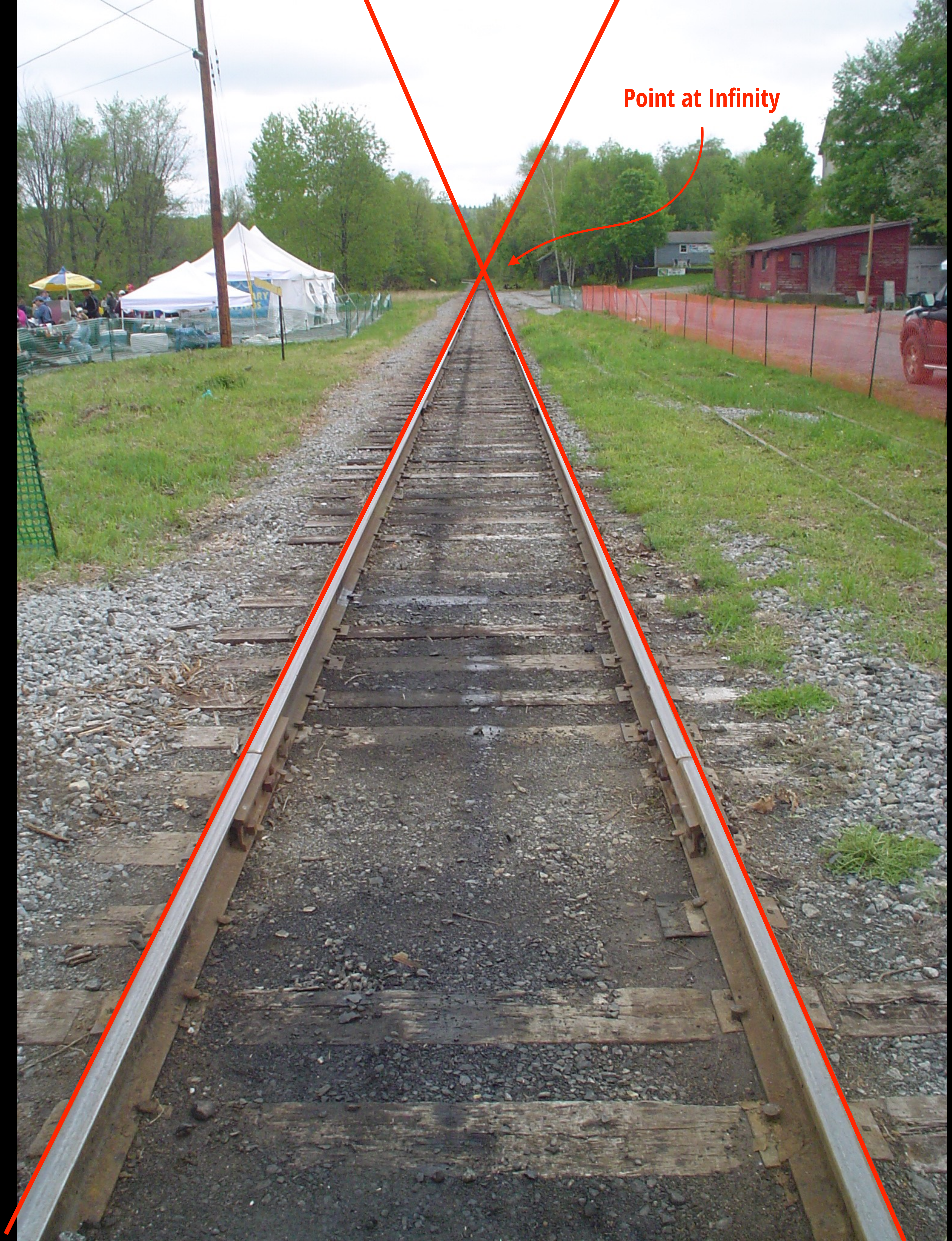
CAMERA RE-SECTIONING

Geometric distance measures error in projected point distances



$$\mathtt{P}^* = \arg\min_{\mathtt{P}} \sum_i d(\mathbf{x}_i, \mathtt{P}\mathbf{X}_i)^2$$

Note: Not the Maximum Likelihood Estimate!

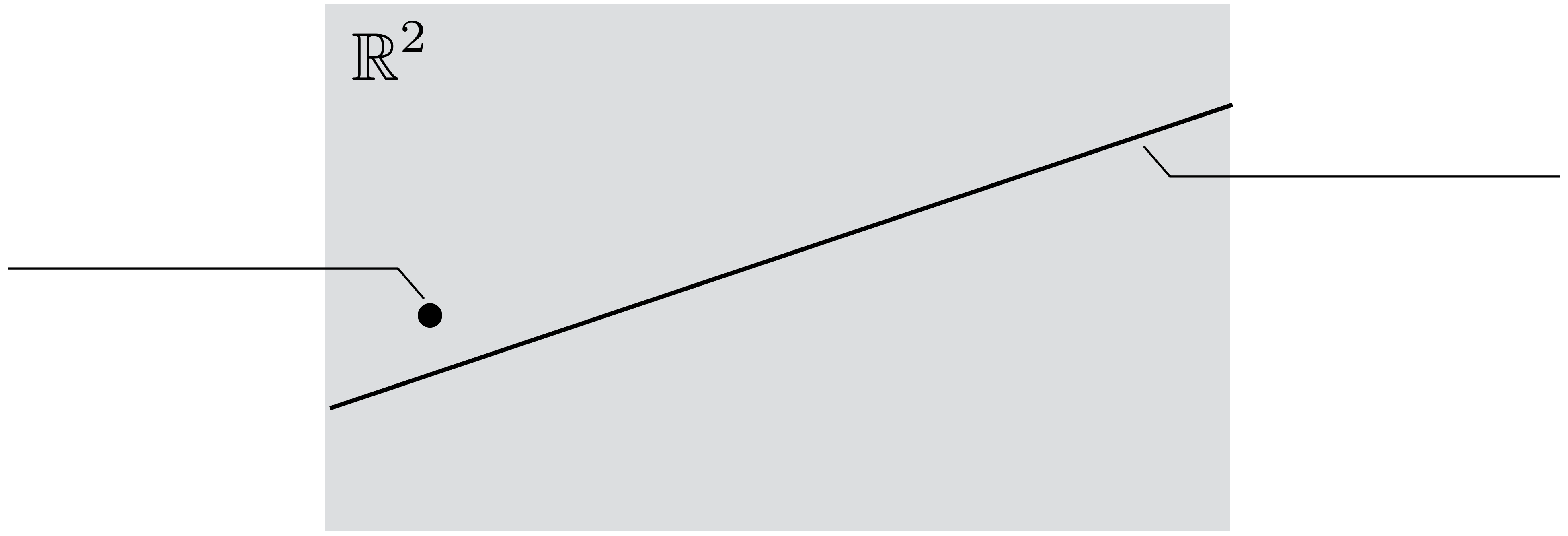

Point at Infinity

THE 2D PROJECTIVE PLANE

$\mathbb{P}^2$: Augmenting the concept of a plane with “points at infinity”

THE 2D PROJECTIVE PLANE

Homogeneous Representation of Lines denotes an Equivalence Class



$$ax + by + c = 0$$

Equation of a line

$$(ka)x + (kb)y + (kc) = 0$$

Representation not unique

$$\mathbf{l} = (a, b, c)^T \equiv (ka, kb, kc)^T$$

for a non-zero *k*

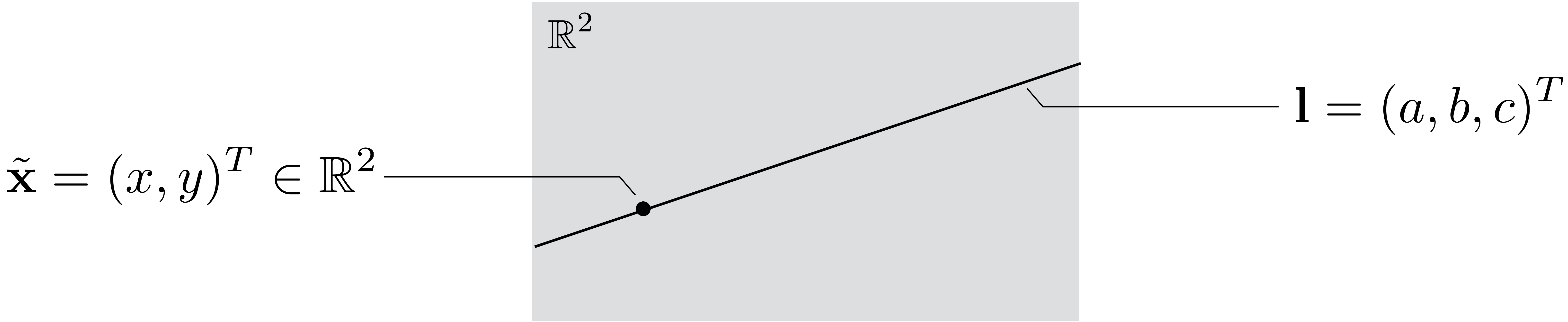
Equivalence Class

Homogeneous Vector

The set of equivalence classes of vectors in $\mathbb{R}^3 - (0, 0, 0)^T$ forms the projective space $\mathbb{P}^2$.

THE 2D PROJECTIVE PLANE

Homogeneous Representation of Points also denotes an Equivalence Class



Point $(x, y)^T$ lies on the line $(a, b, c)^T$ if and only if $ax + by + c = 0$

$$\begin{pmatrix} x & y & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \qquad \begin{pmatrix} kx & ky & k \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

Equivalence Class
Homogeneous Vector

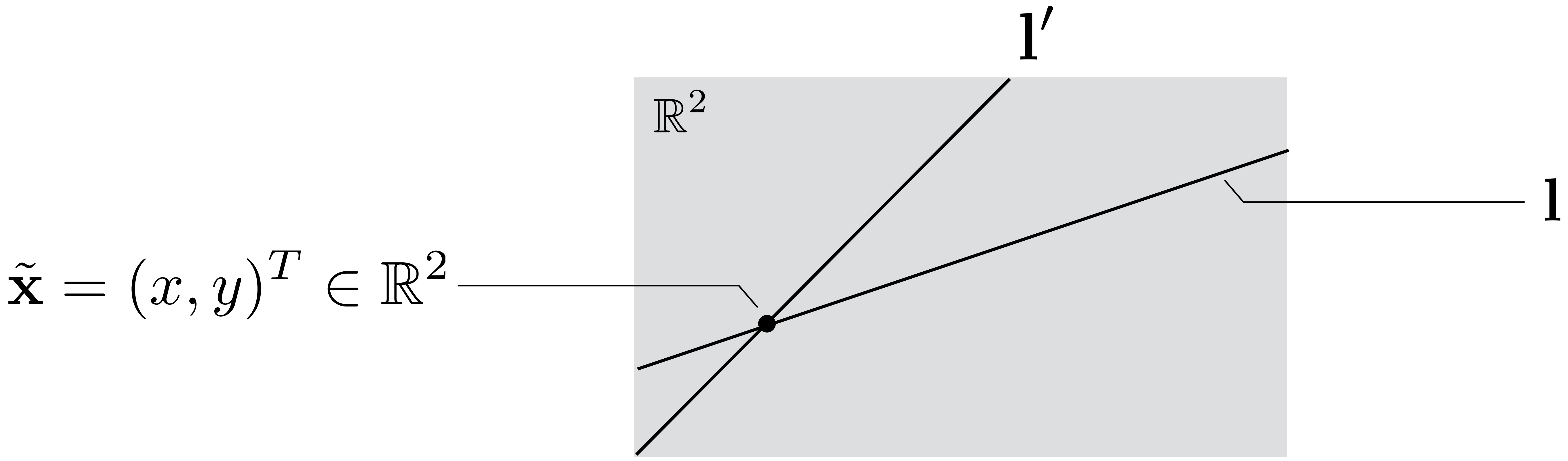
$$(x_1, x_2, x_3)^T \in \mathbb{P}^2 \leftrightarrow (x_1/x_3, x_2/x_3)^T \in \mathbb{R}^2$$

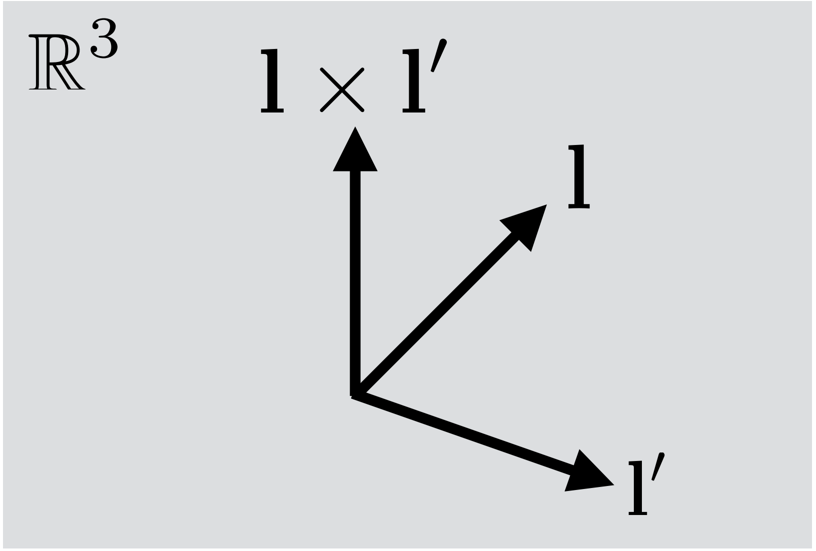
Homogeneous Point
2 degrees of freedom

Point $\mathbf{x}$ lies on the line $\mathbf{l}$ if and only if $\mathbf{x}^T\mathbf{l} = 0$

THE 2D PROJECTIVE PLANE

Intersection of Two Lines via a Cross Product





$$\mathbf{l} \cdot (\mathbf{l} \times \mathbf{l}') = \mathbf{l}' \cdot (\mathbf{l} \times \mathbf{l}') = 0$$

Triple Scalar Product Identity

$$\mathbf{x} = \mathbf{l} \times \mathbf{l}'$$

Definition

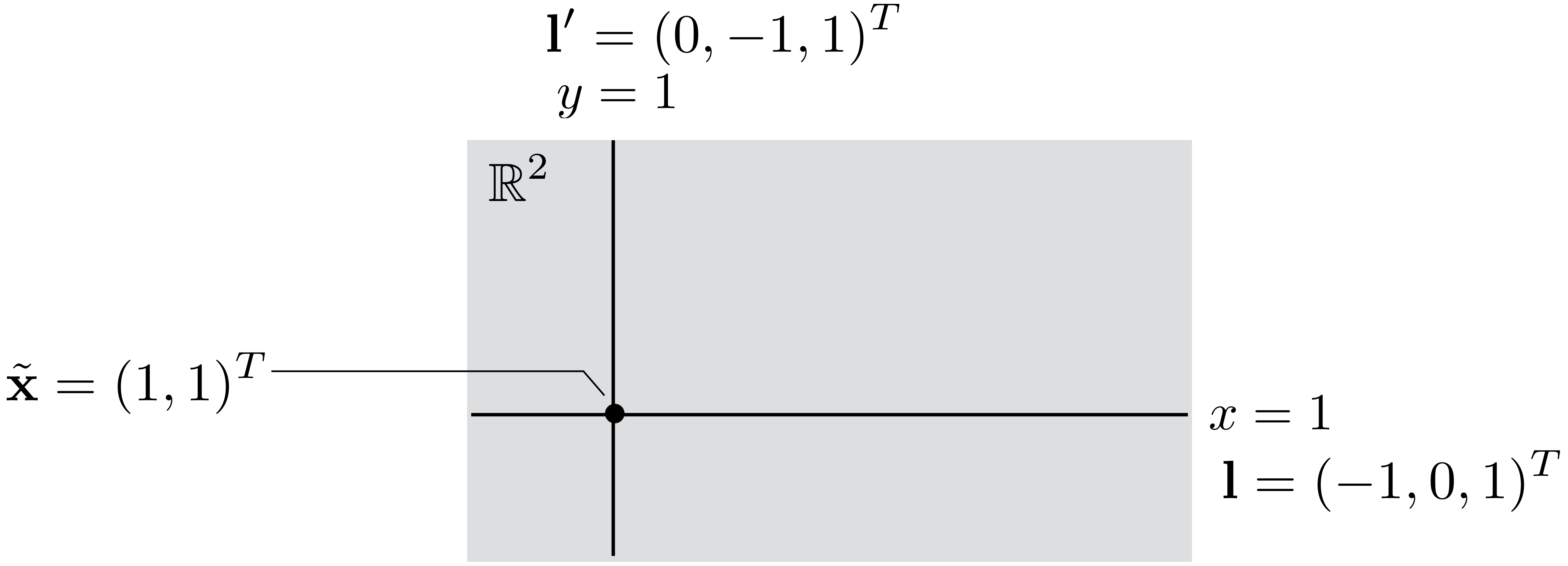
$$\mathbf{l}^T \mathbf{x} = \mathbf{l}'^T \mathbf{x} = 0$$

Intersection of Lines

The intersection of two lines $\mathbf{l}$ and $\mathbf{l}'$ is the point $\mathbf{x} = \mathbf{l} \times \mathbf{l}'$

THE 2D PROJECTIVE PLANE

Example: Intersection of lines *x*=1 and *y*=1



$$\mathbf{l} \cdot (\mathbf{l} \times \mathbf{l}') = \mathbf{l}' \cdot (\mathbf{l} \times \mathbf{l}') = 0$$

Triple Scalar Product Identity

$$\mathbf{x} = \mathbf{l} \times \mathbf{l}' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{vmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

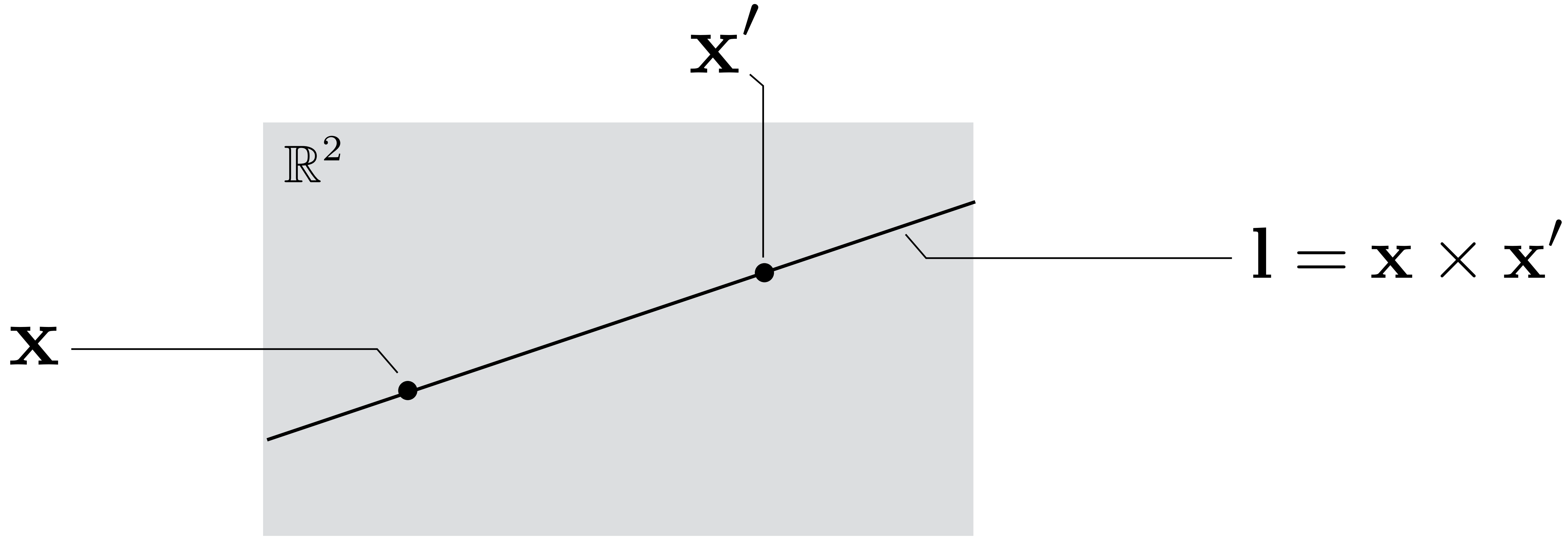
Cofactor Expansion

$$\tilde{\mathbf{x}} = (1, 1)^T$$

Fin.

THE 2D PROJECTIVE PLANE

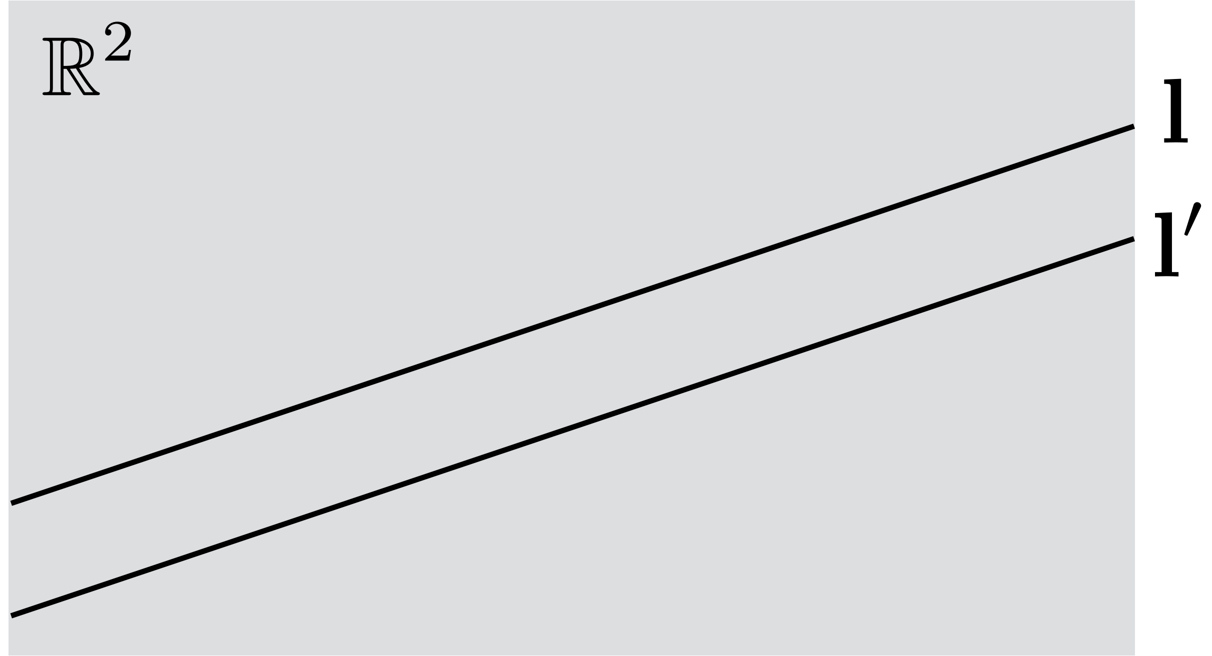
“Intersection” of Two Points



The intersection of two points $\mathbf{x}$ and $\mathbf{x}'$ is the line $\mathbf{l} = \mathbf{x} \times \mathbf{x}'$

THE 2D PROJECTIVE PLANE

Ideal Points and the Line at Infinity: Intersection of Parallel Lines



$$\mathbf{l} = (a, b, c) : ax + by + c = 0$$

$$\mathbf{l}' = (a, b, c') : ax + by + c' = 0$$

$$\mathbf{l} \times \mathbf{l}' = (c - c')(b, -a, 0)^T$$

$$\mathbf{x} = (b, -a, 0)^T$$

$$\tilde{\mathbf{x}} = (b/0, -a/0)^T$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ a & b & c' \end{vmatrix}$$

What point in $\mathbb{R}^2$ does this point correspond to?

A point at infinity

THE 2D PROJECTIVE PLANE

Example: Intersection of parallel lines *x*=1 and *x*=2

$x = 2$

$\mathbf{l}' = (-1, 0, 2)^T$

$x = 1$

$\mathbf{l} = (-1, 0, 1)^T$

$$\mathbf{x} = \mathbf{l} \times \mathbf{l}' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & 1 \\ -1 & 0 & 2 \end{vmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Point at Infinity in the direction of the y-axis

THE 2D PROJECTIVE PLANE

Ideal Points and the Line at Infinity

How many degrees of freedom?

Finite Points in $\mathbb{R}^2$: $(x_1, x_2, x_3)^T; x_3 \neq 0$

Ideal Points (or Points at Infinity): $(x_1, x_2, 0)^T$

Line at Infinity: $\mathbf{l}_\infty = (0, 0, 1)^T$

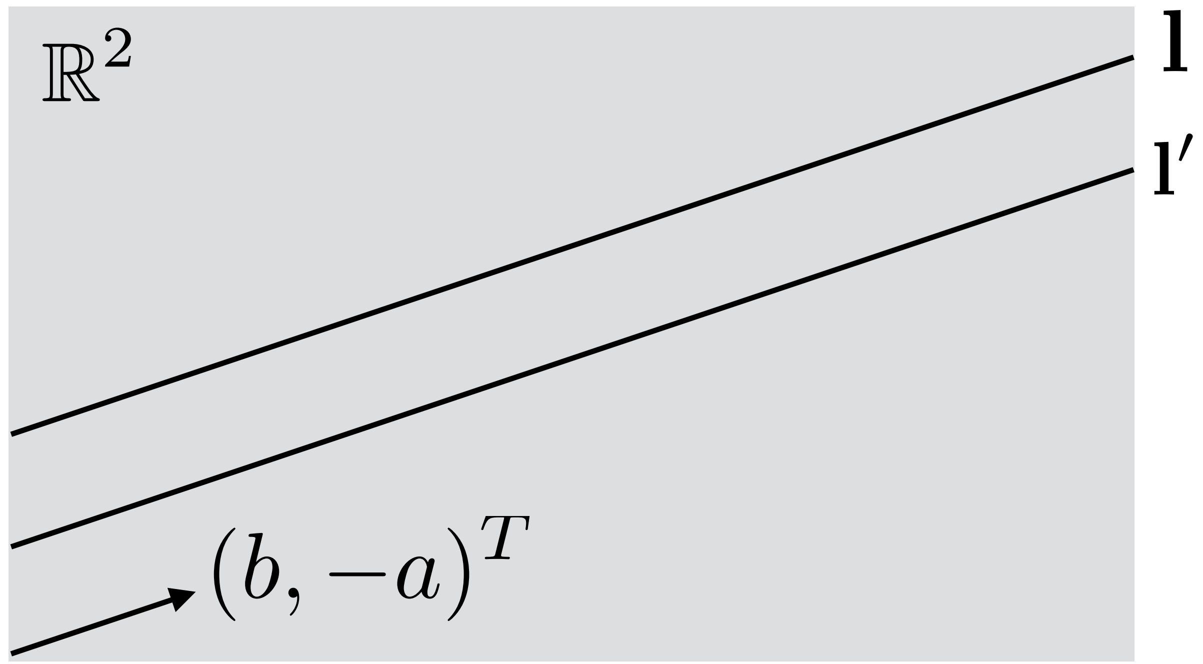
$$(0, 0, 1)(x_1, x_2, 0)^T = 0$$

All points at infinity intersect the line at infinity

THE 2D PROJECTIVE PLANE

Ideal Points and the Line at Infinity

$$\mathbf{l}_\infty = (0, 0, 1)^T$$



$$\mathbf{l} = (a, b, c)^T \qquad \mathbf{l}' = (a, b, c')^T$$

$$\mathbf{l} \times \mathbf{l}_\infty = (b, -a, 0)^T \qquad \mathbf{l}' \times \mathbf{l}_\infty = (b, -a, 0)^T$$

The point at infinity can be thought of as the direction of the parallel lines

The line at infinity can be thought of as the set of directions of lines in the plane

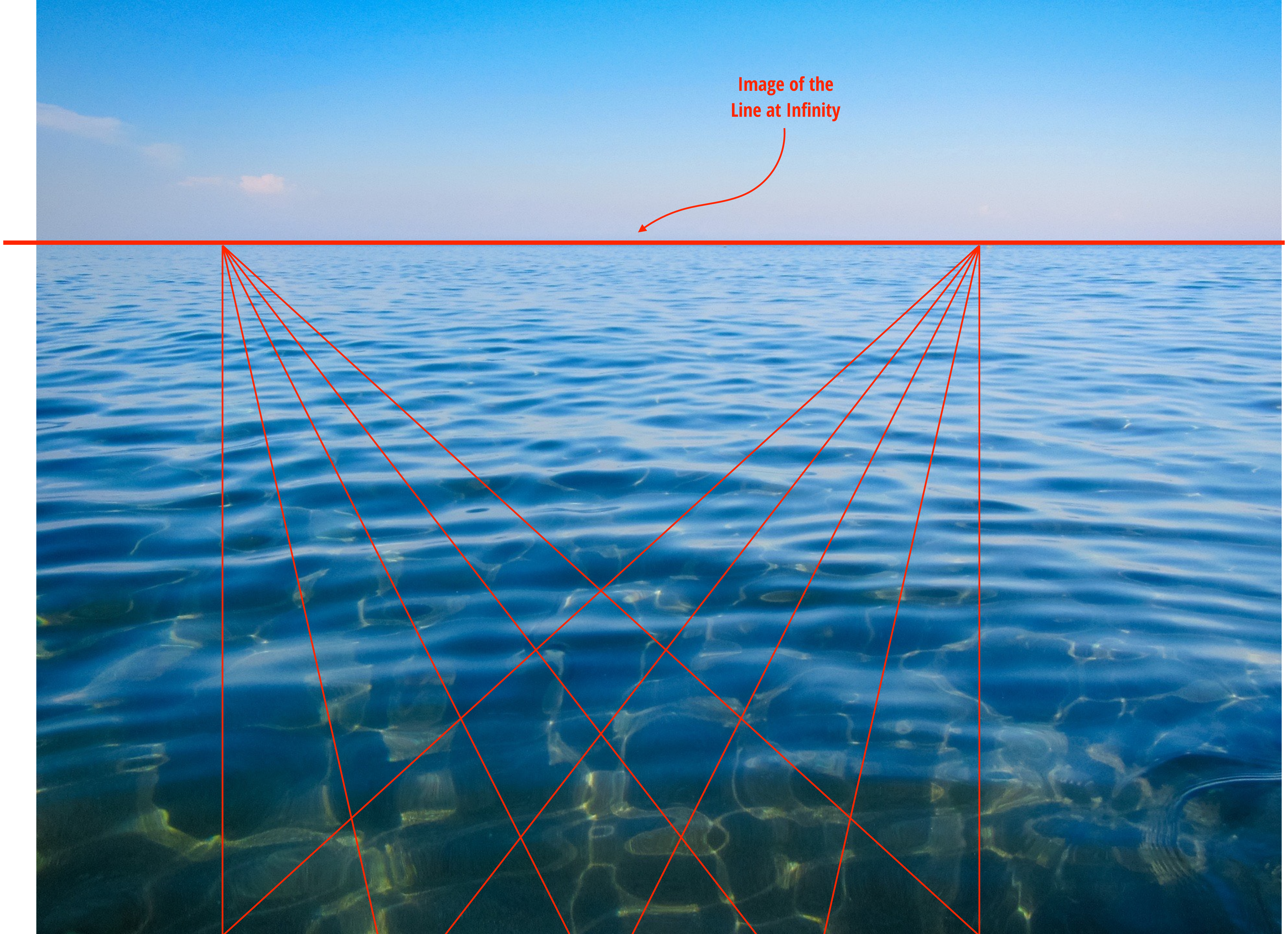


Image of the
Line at Infinity

THE 2D PROJECTIVE PLANE

Duality Principle

To any theorem of 2-dimensional projective geometry there corresponds a dual theorem, which may be derived by interchanging the roles of point and lines in the original theorem.

EXAMPLES

Incidence Equation: $\mathbf{l}^T\mathbf{x} = 0$ $\quad\quad$ $\mathbf{x}^T\mathbf{l} = 0$

Intersection: $\mathbf{x} \times \mathbf{x}' = \mathbf{l}$ $\quad\quad$ $\mathbf{l} \times \mathbf{l}' = \mathbf{x}$

THE 2D PROJECTIVE PLANE

The study of the geometry of $\mathbb{P}^N$ is known as projective geometry

Point $\mathbf{x}$ lies on the line $\mathbf{l}$ if and only if $\mathbf{x}^T\mathbf{l} = 0$

The intersection of two lines $\mathbf{l}$ and $\mathbf{l}'$ is the point $\mathbf{x} = \mathbf{l} \times \mathbf{l}'$

The intersection of two points $\mathbf{x}$ and $\mathbf{x}'$ is the line $\mathbf{l} = \mathbf{x} \times \mathbf{x}'$

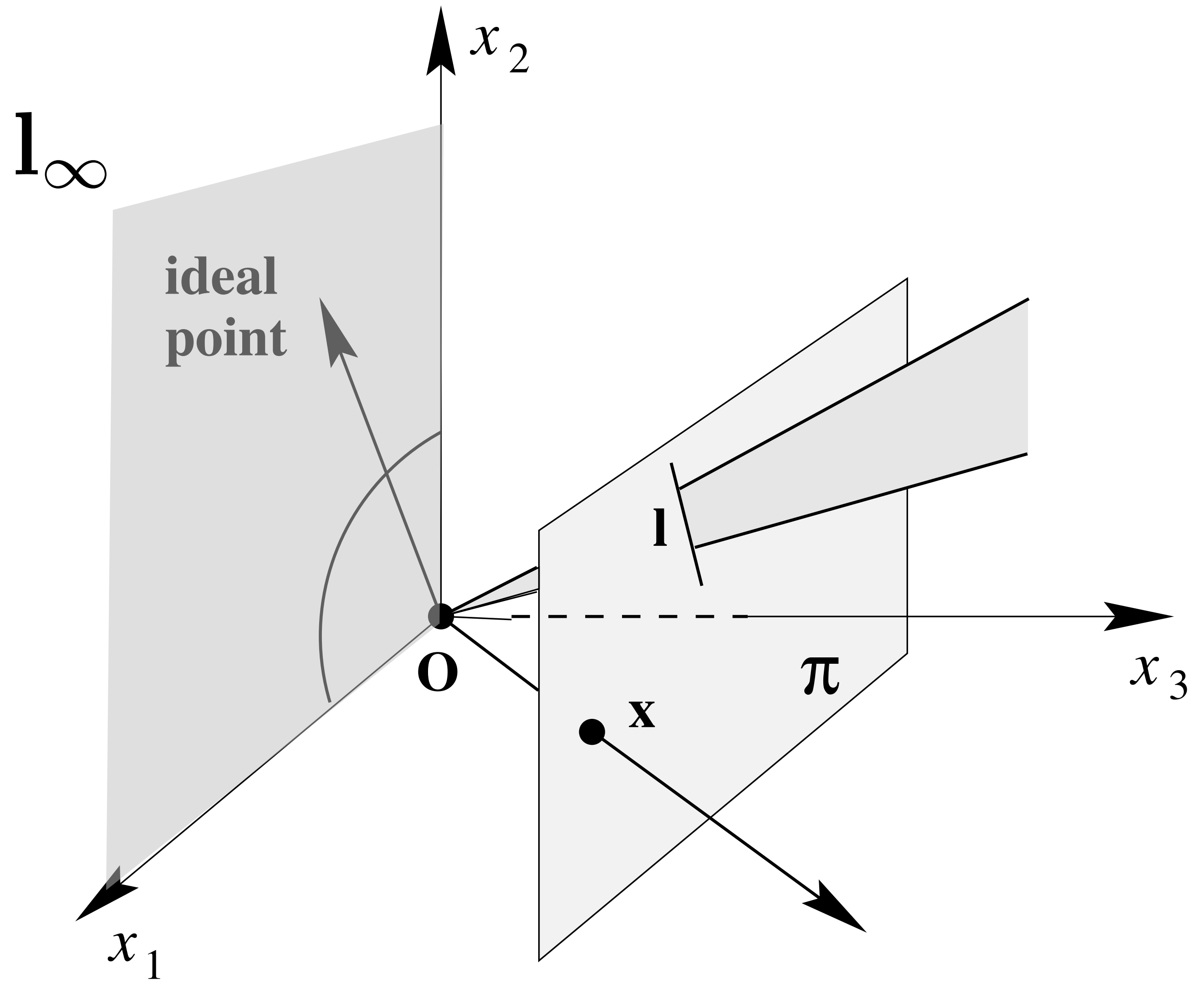
Equivalence class: $(x_1, x_2, x_3)^T \in \mathbb{P}^2 \leftrightarrow (x_1/x_3, x_2/x_3)^T \in \mathbb{R}^2$

Ideal Points (or Points at Infinity): $(x_1, x_2, 0)^T$

Line at Infinity: $\mathbf{l}_\infty = (0, 0, 1)^T$

THE 2D PROJECTIVE PLANE

The study of the geometry of $\mathbb{P}^N$ is known as projective geometry



$\mathbf{x}$: points corresponds to rays

$\mathbf{l}$: lines correspond to planes

Two points define a line?

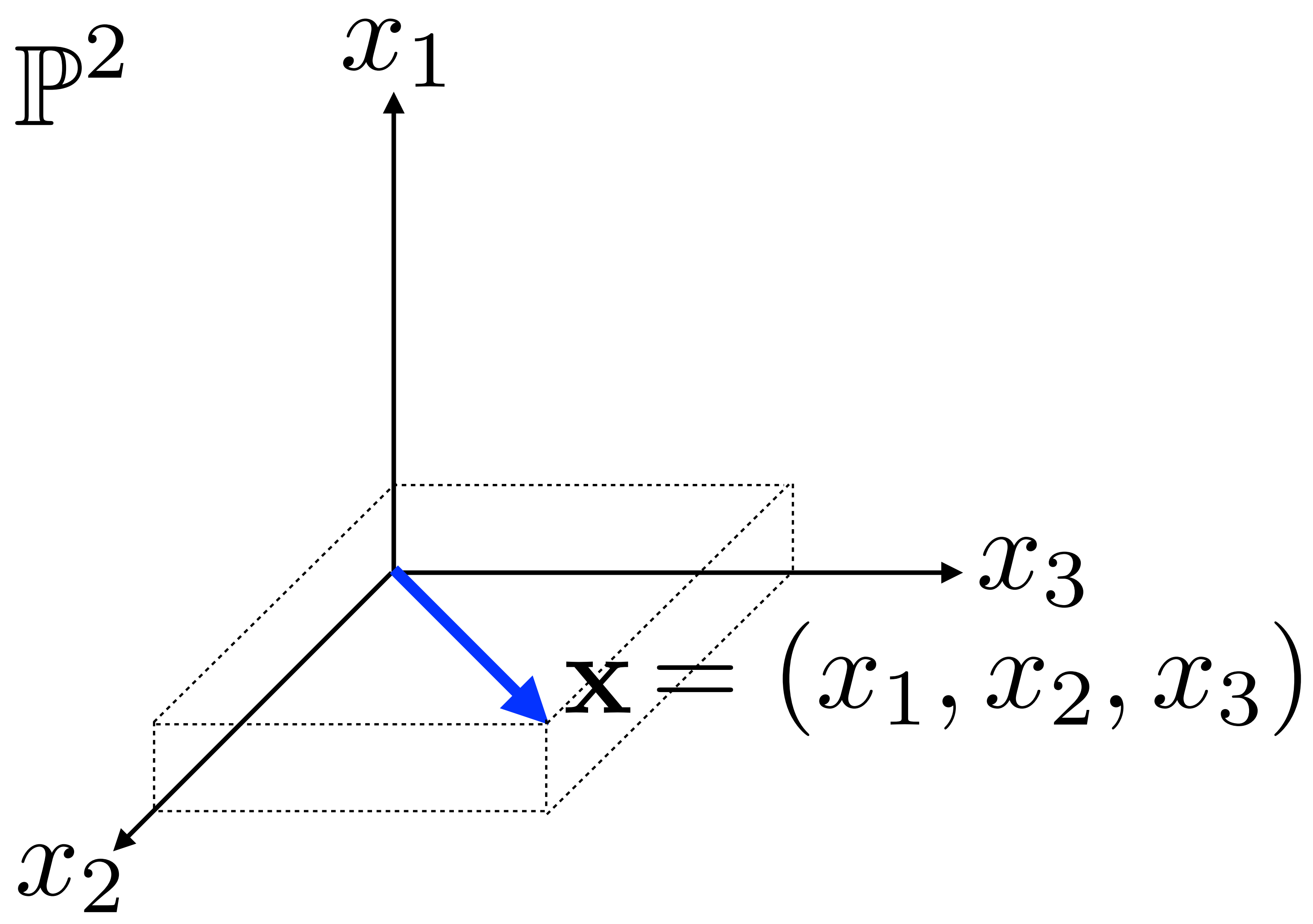
Two lines define a point?

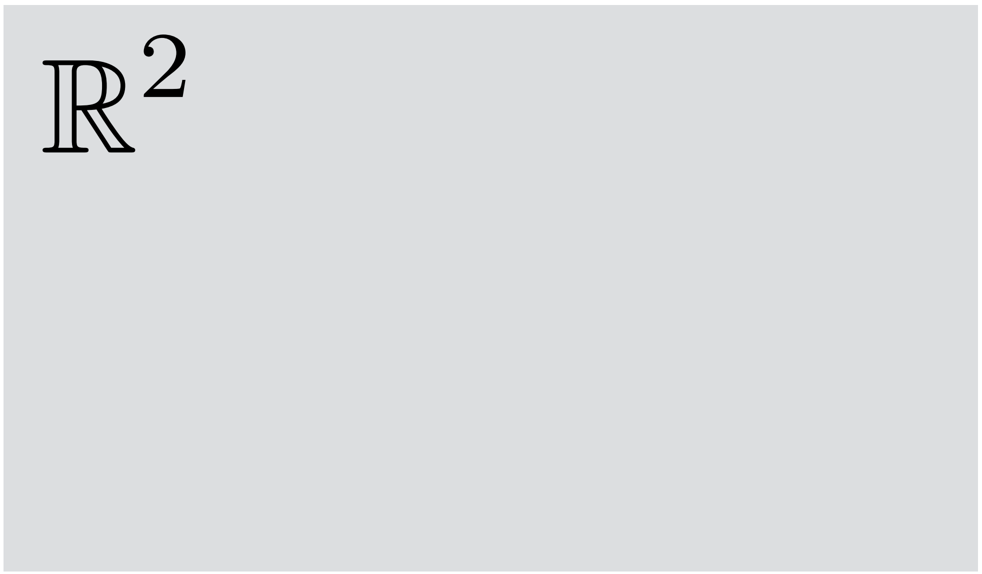
Intersection of two parallel lines?

Intersection of two ideal points?

THE 2D PROJECTIVE PLANE

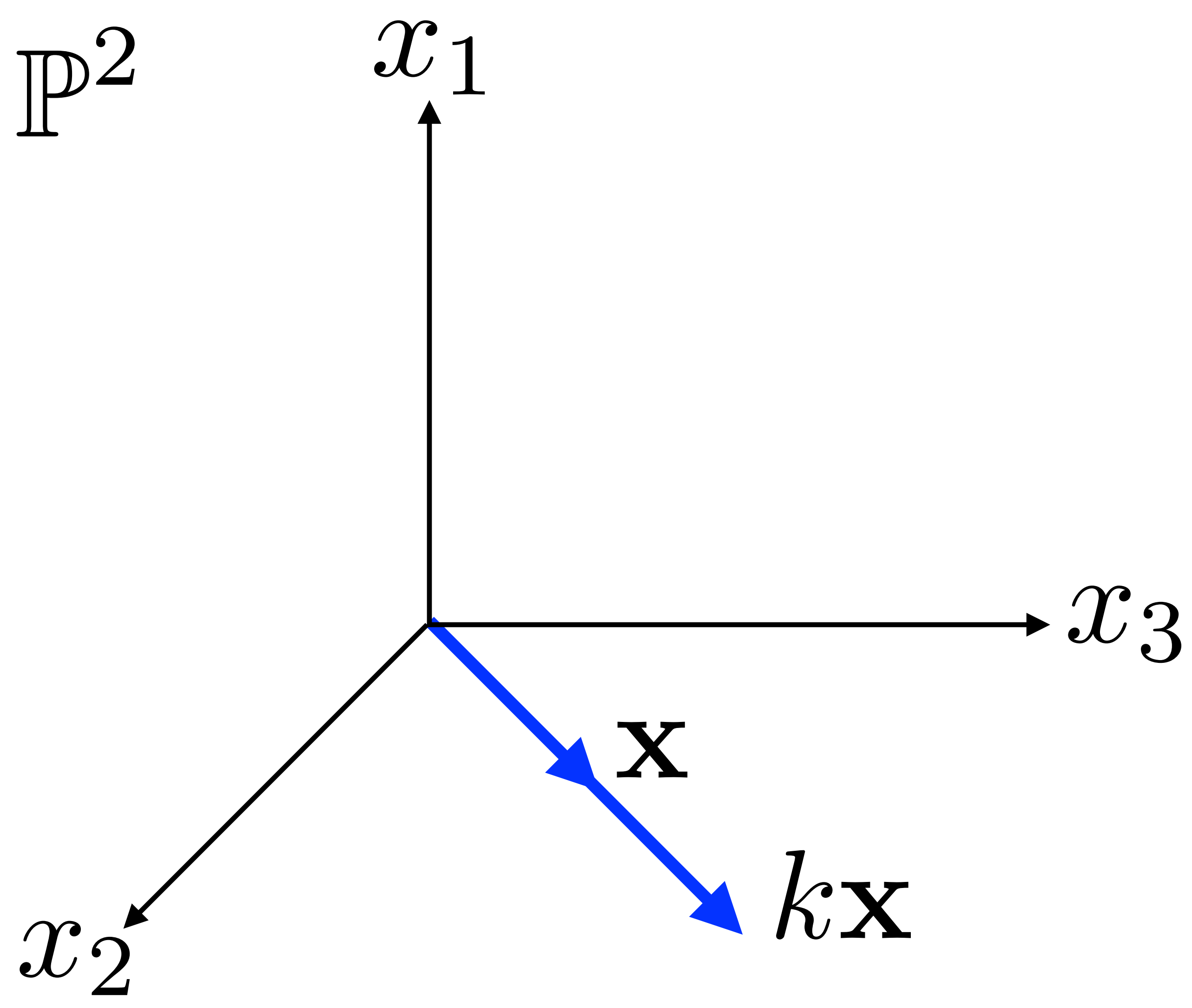
The study of the geometry of $\mathbb{P}^N$ is known as projective geometry

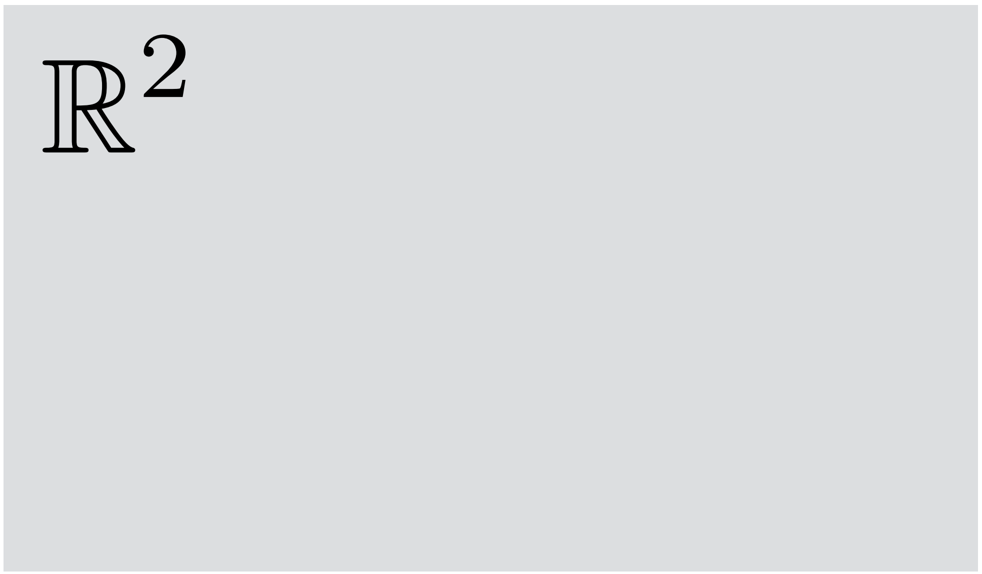




THE 2D PROJECTIVE PLANE

Homogeneous Vector: An Equivalence Class



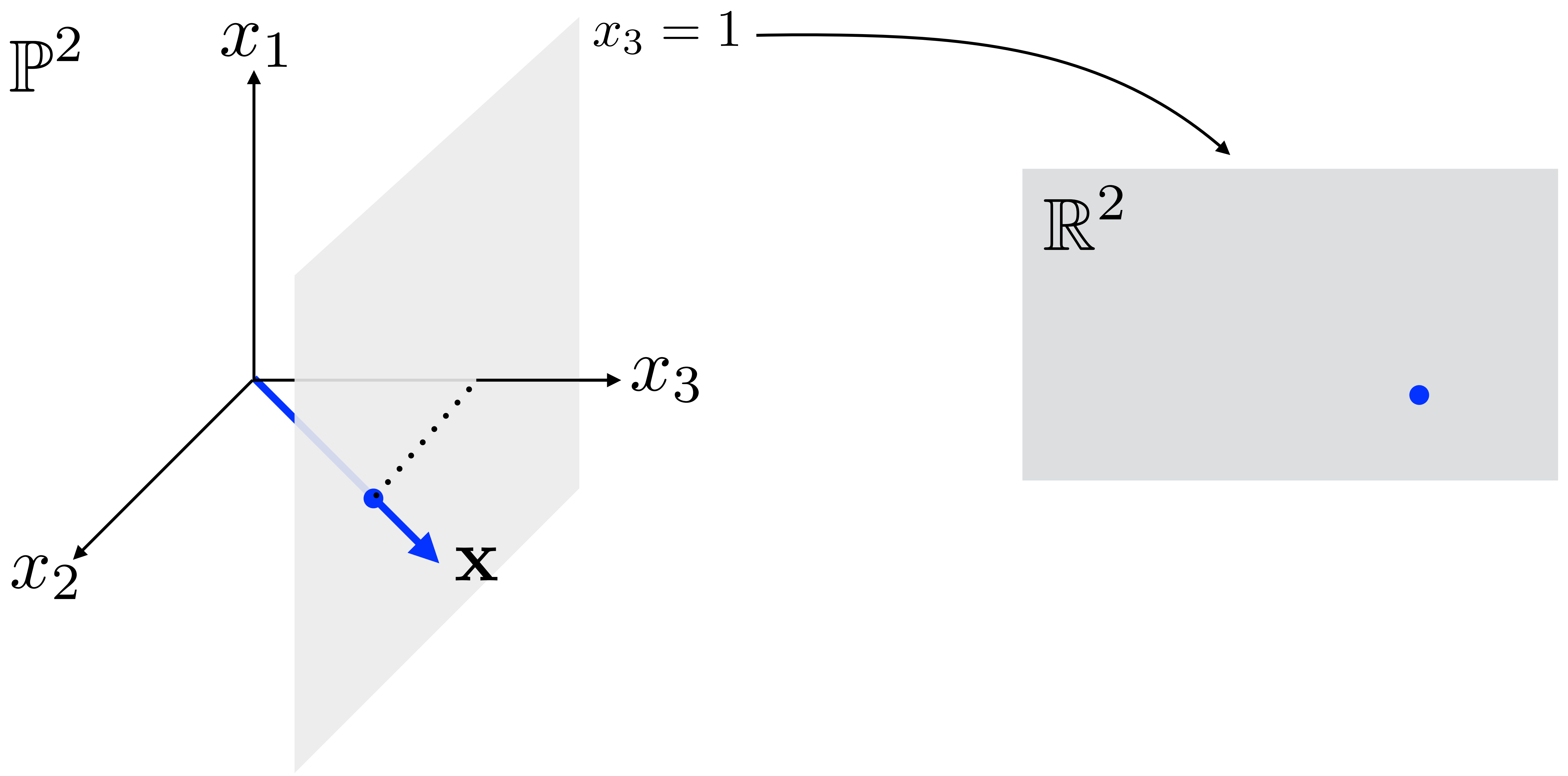


$$\mathbf{x} \equiv k\mathbf{x}$$

Equivalence class

THE 2D PROJECTIVE PLANE

A Ray in Projective Space Represents a Point in the 2D Image

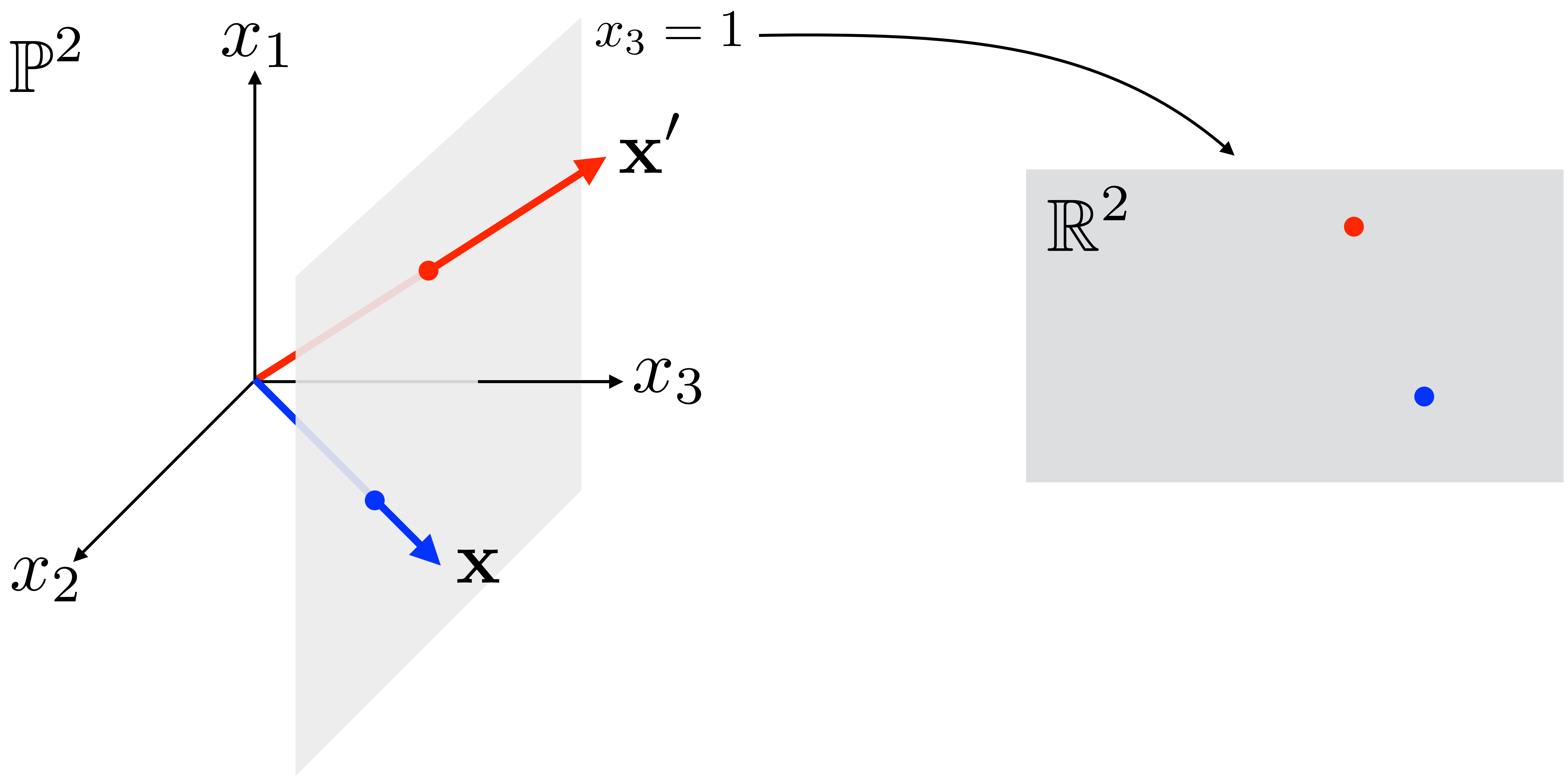


$$\mathbf{x} \equiv k\mathbf{x}$$

Equivalence class

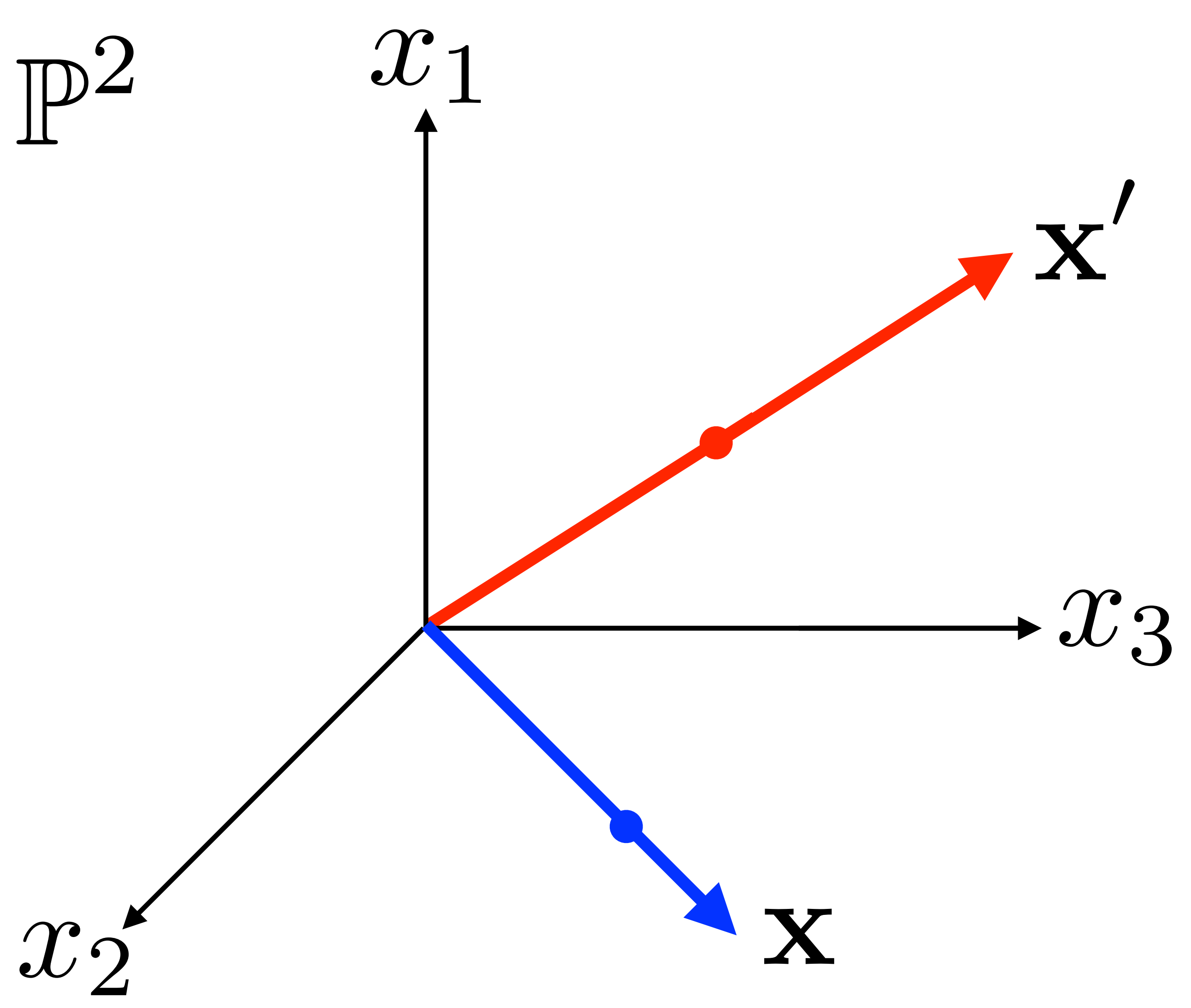
THE 2D PROJECTIVE PLANE

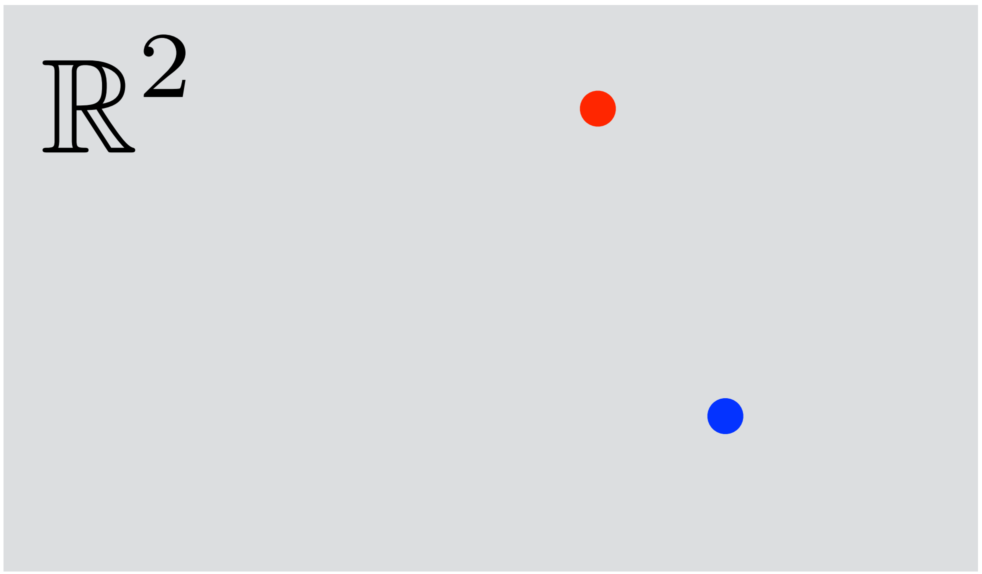
The study of the geometry of $\mathbb{P}^N$ is known as projective geometry



THE 2D PROJECTIVE PLANE

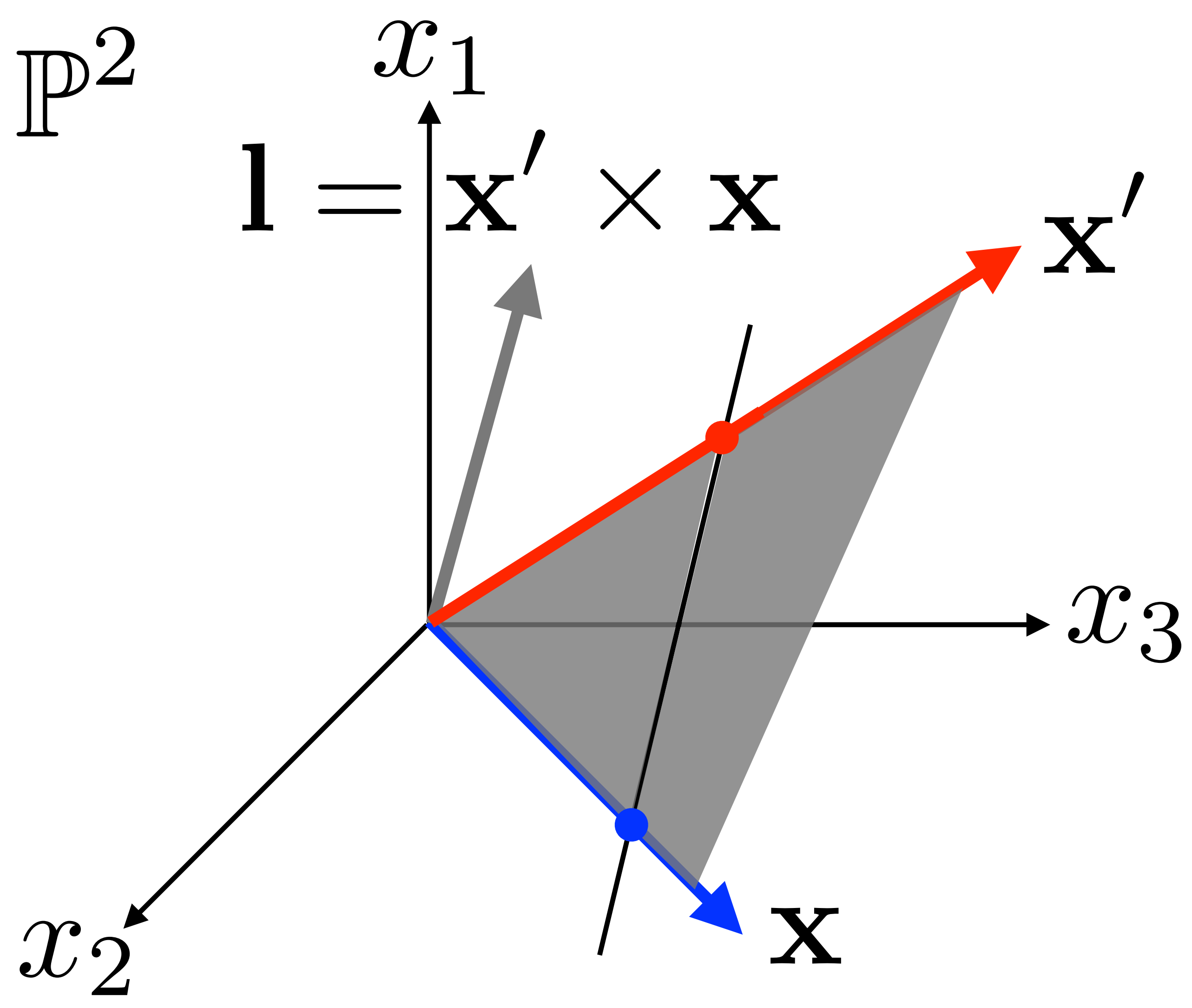
Join of two points?

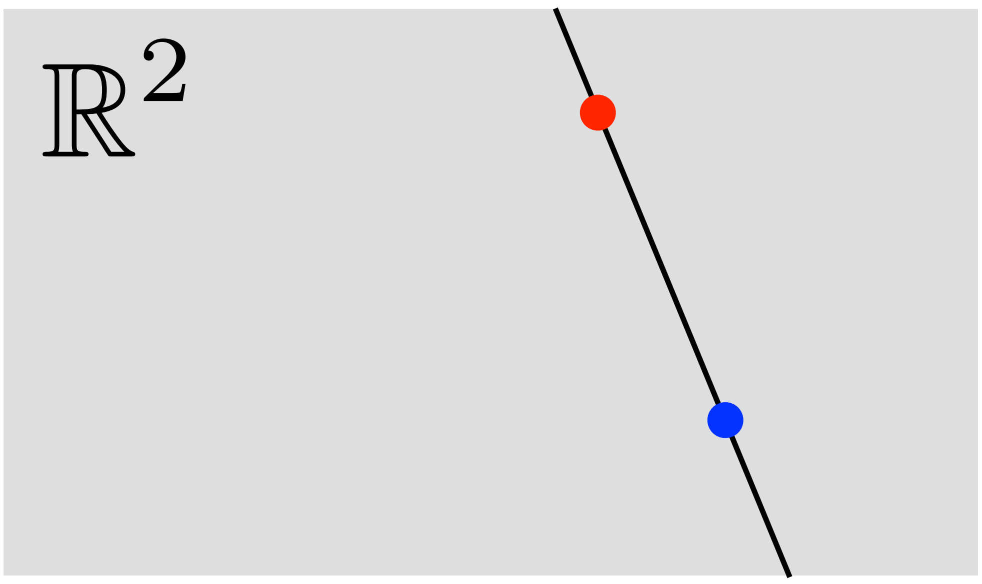




THE 2D PROJECTIVE PLANE

A plane in projective space represents a line in the 2D image

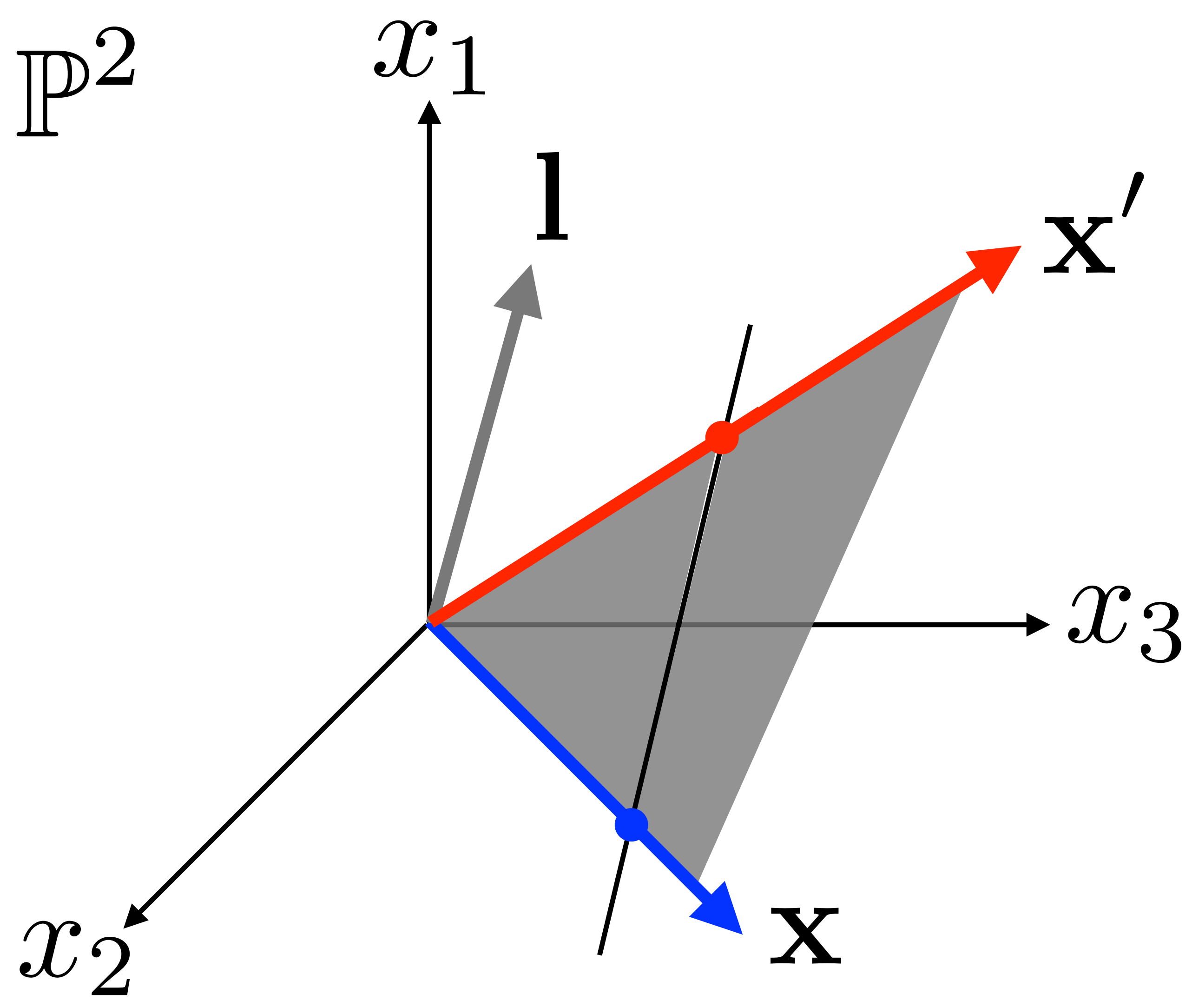


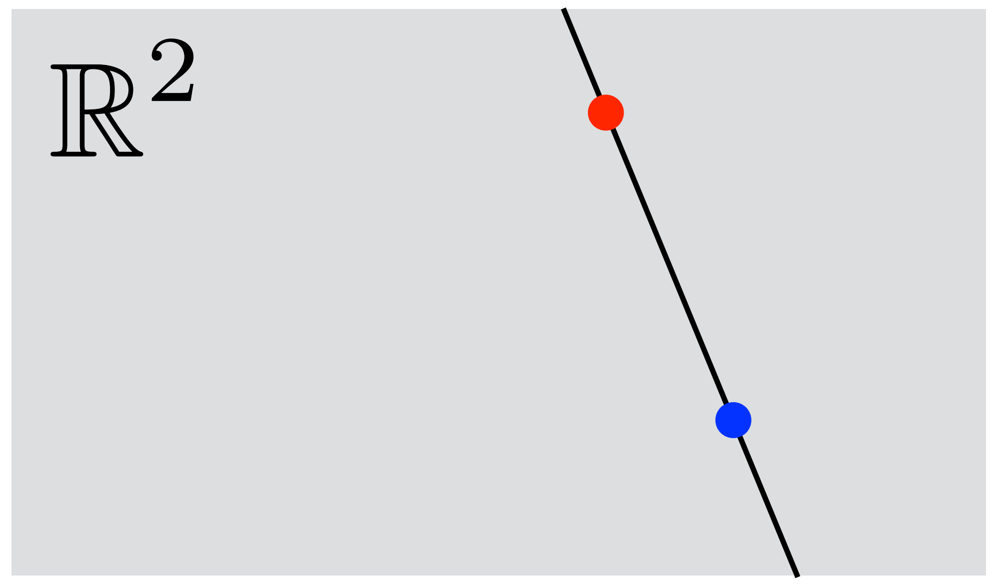


A plane represents a line in the image (plane normal)

THE 2D PROJECTIVE PLANE

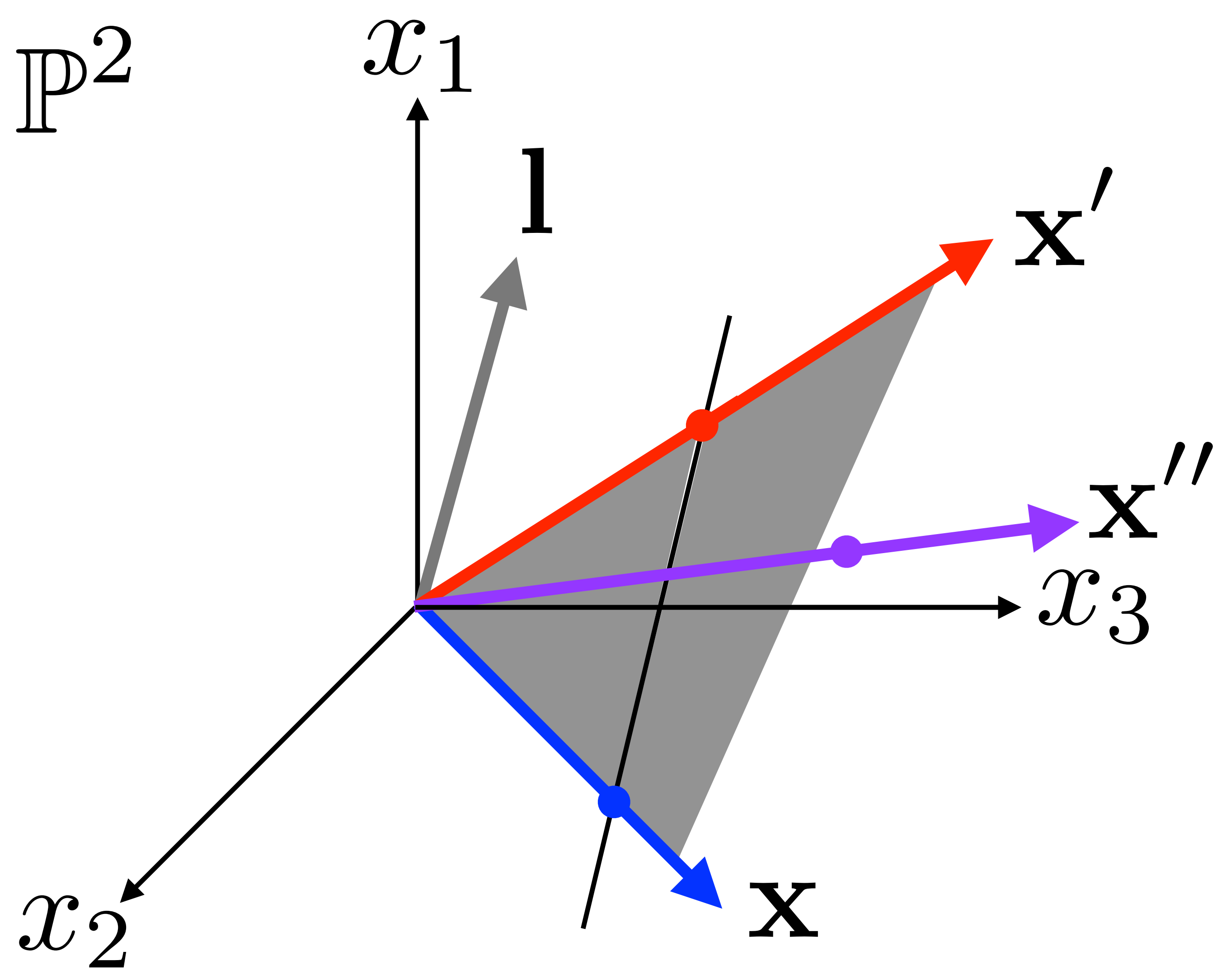
The study of the geometry of $\mathbb{P}^N$ is known as projective geometry

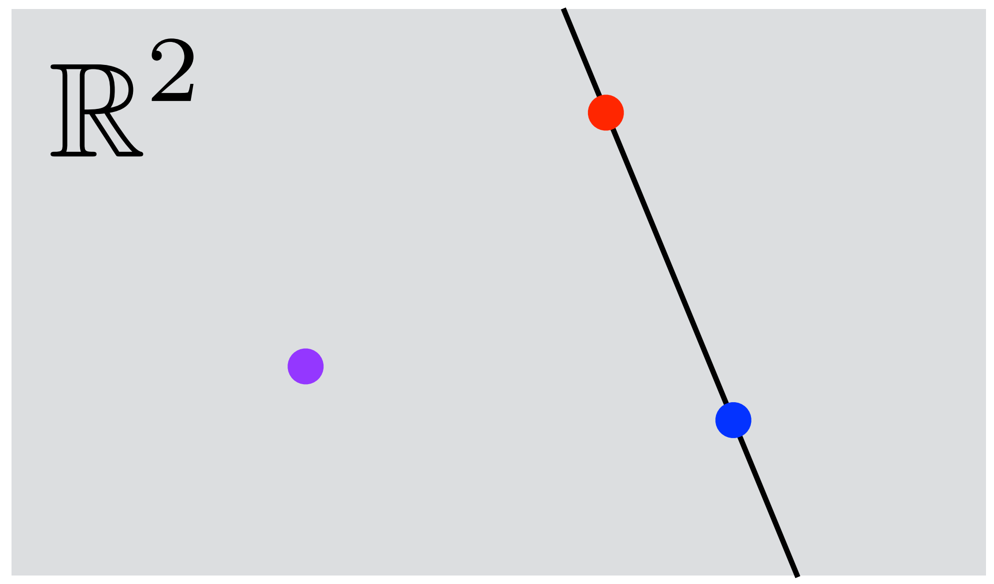




THE 2D PROJECTIVE PLANE

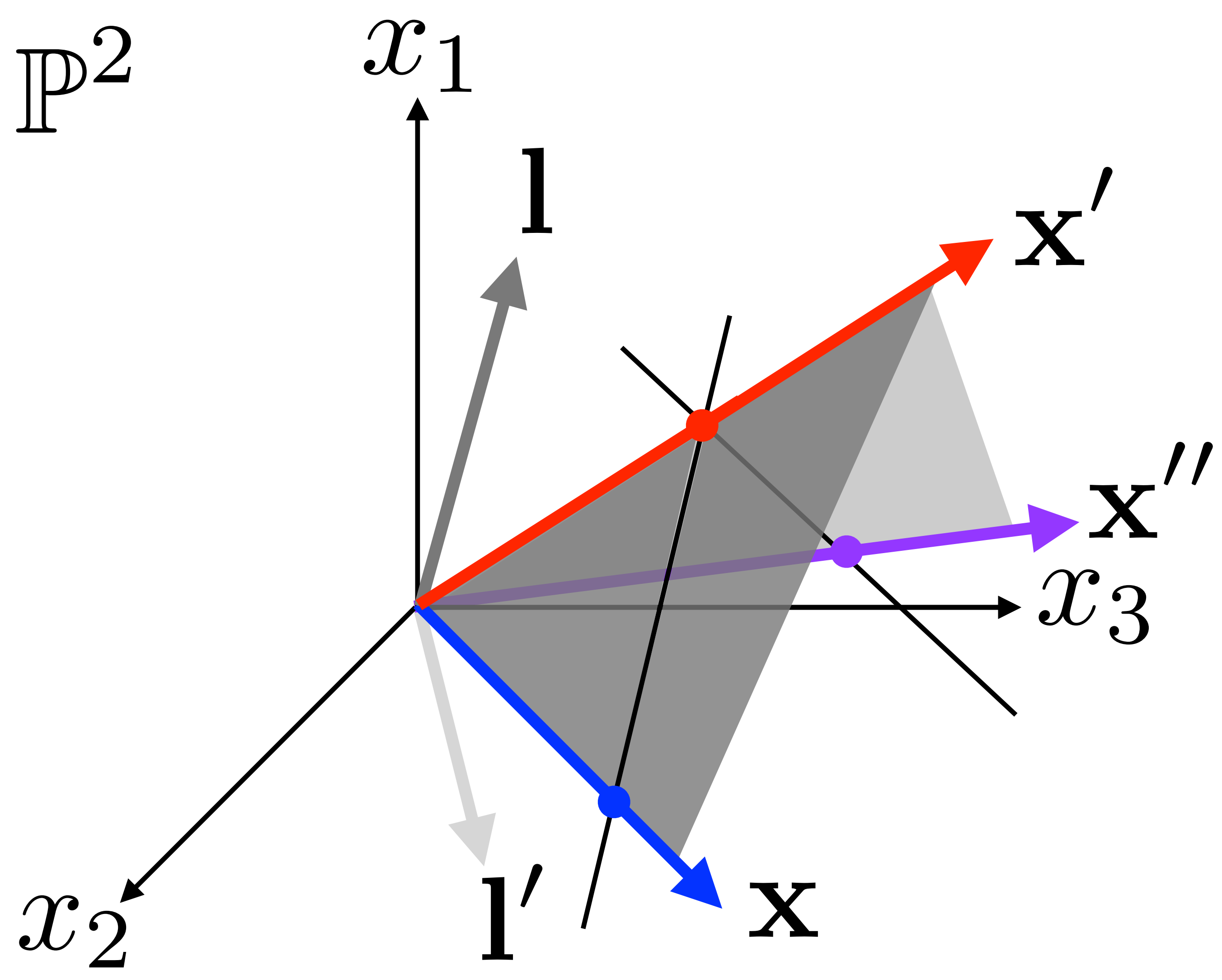
The study of the geometry of $\mathbb{P}^N$ is known as projective geometry

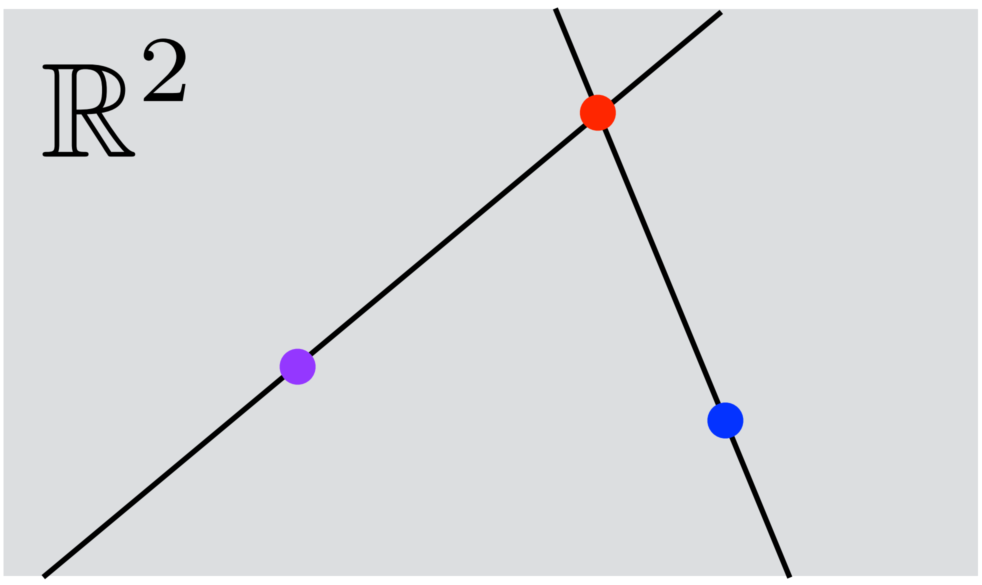




THE 2D PROJECTIVE PLANE

The intersection of two planes in projective space is a point in 2D image space

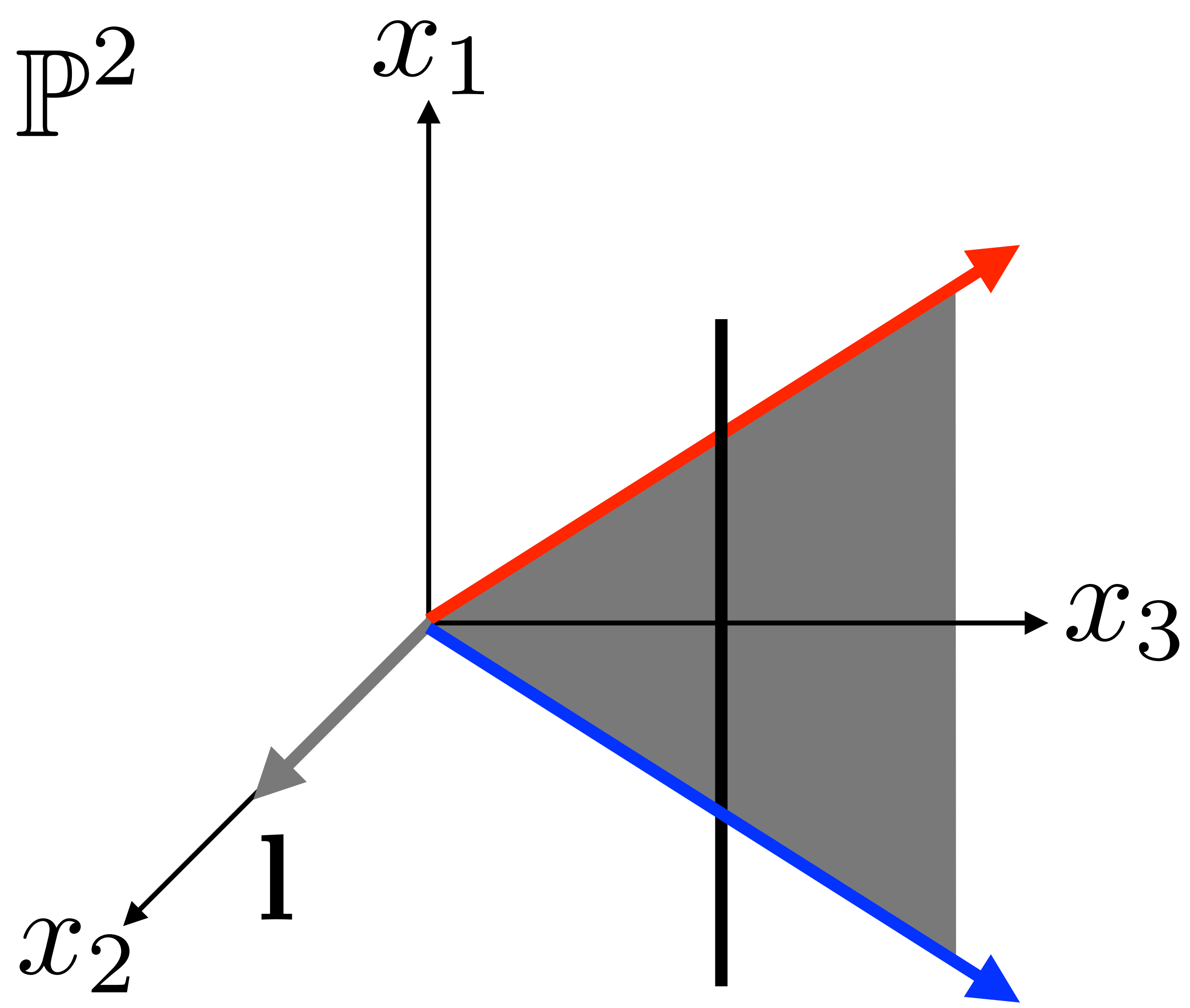


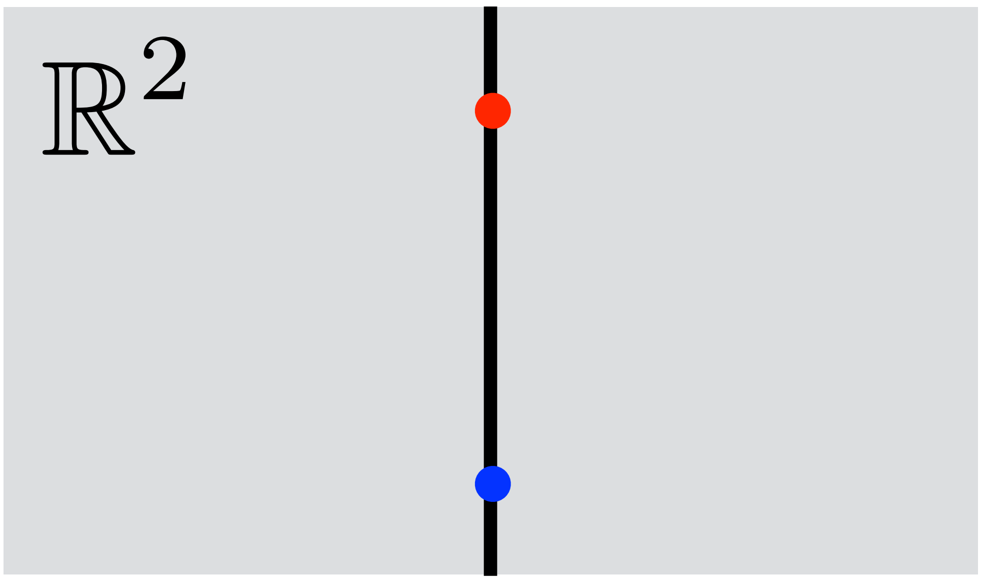


Intersection of two planes is a line

THE 2D PROJECTIVE PLANE

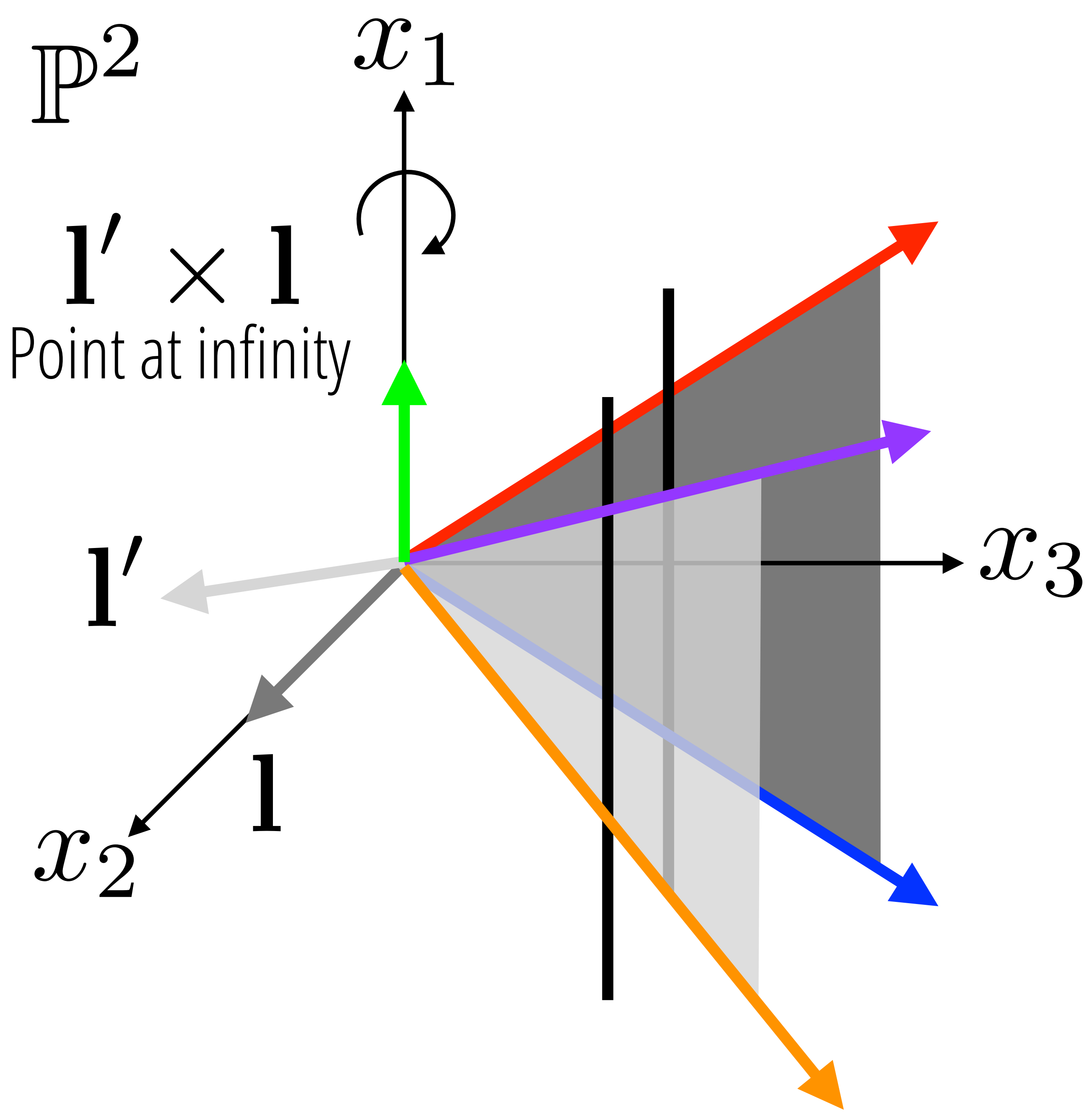
Consider a Vertical Line in the 2D image plane

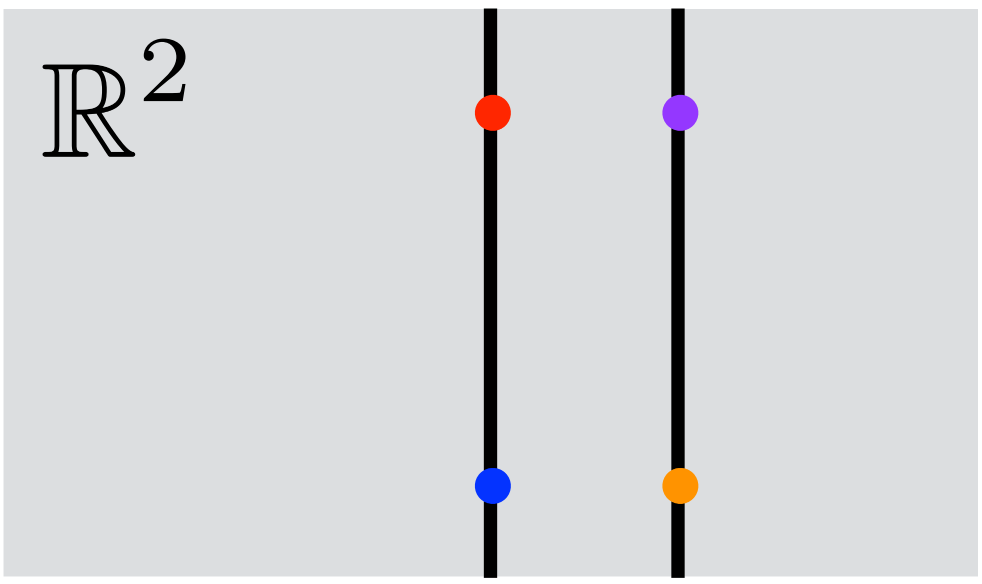




THE 2D PROJECTIVE PLANE

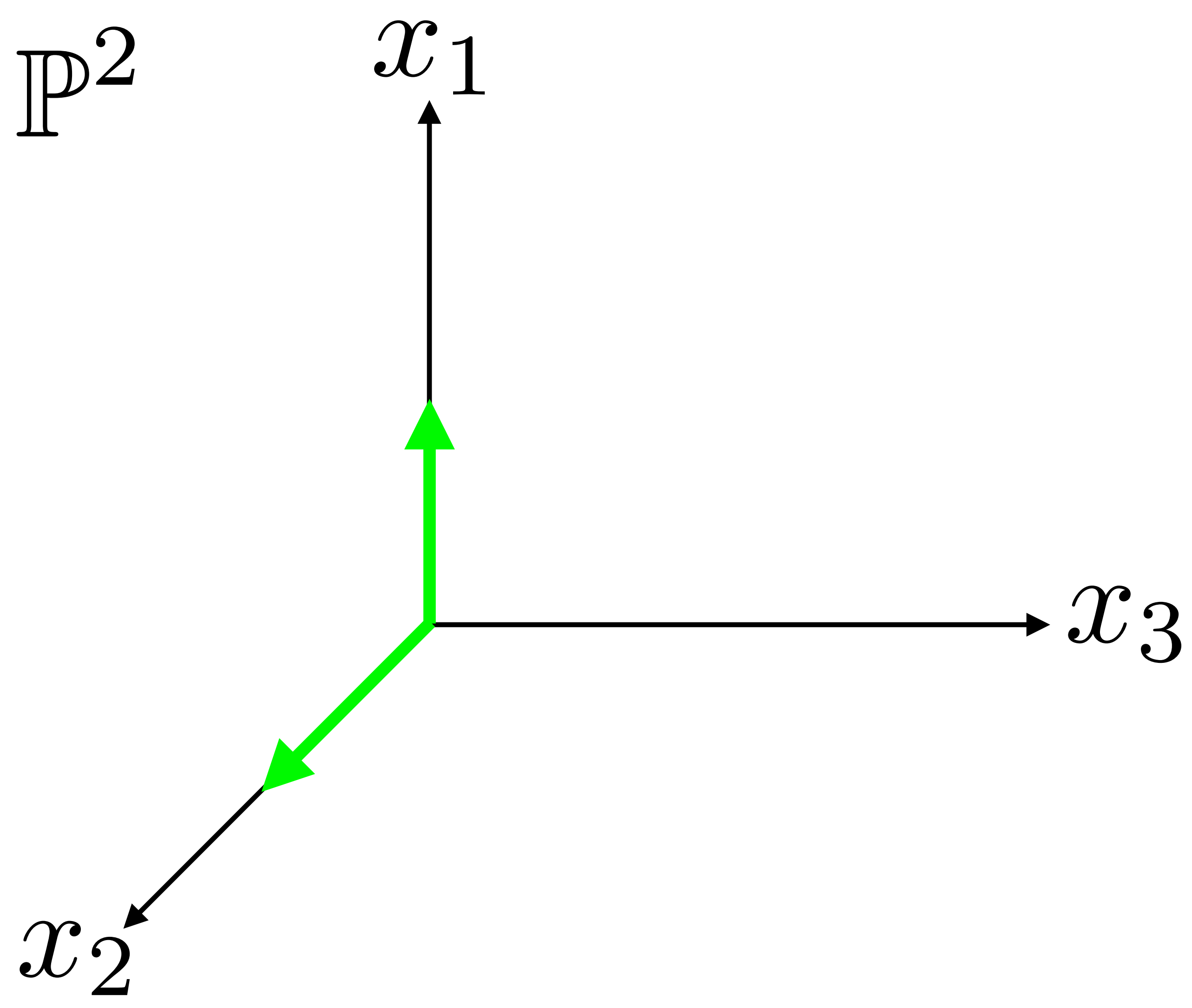
Point at infinity is in the direction of the lines

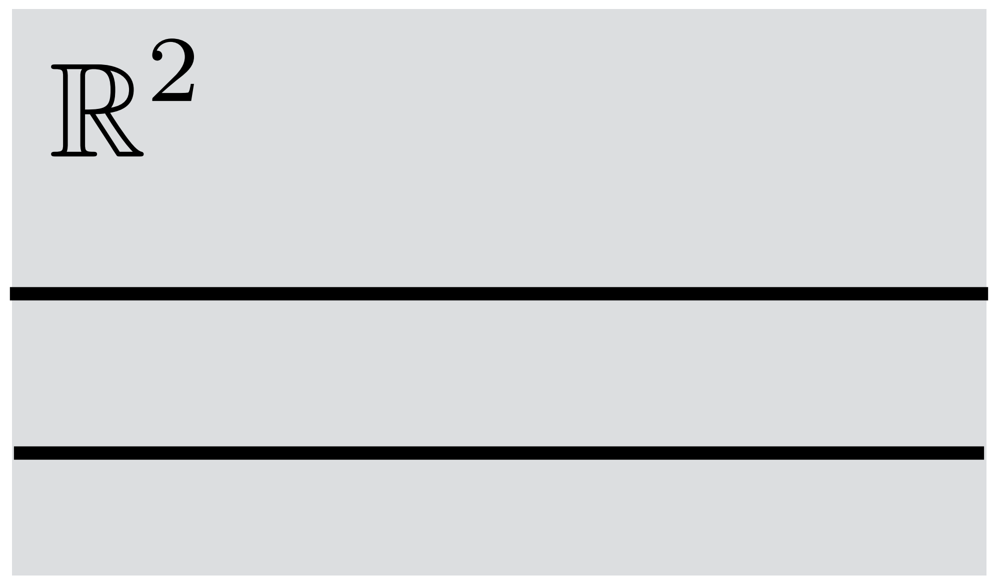




THE 2D PROJECTIVE PLANE

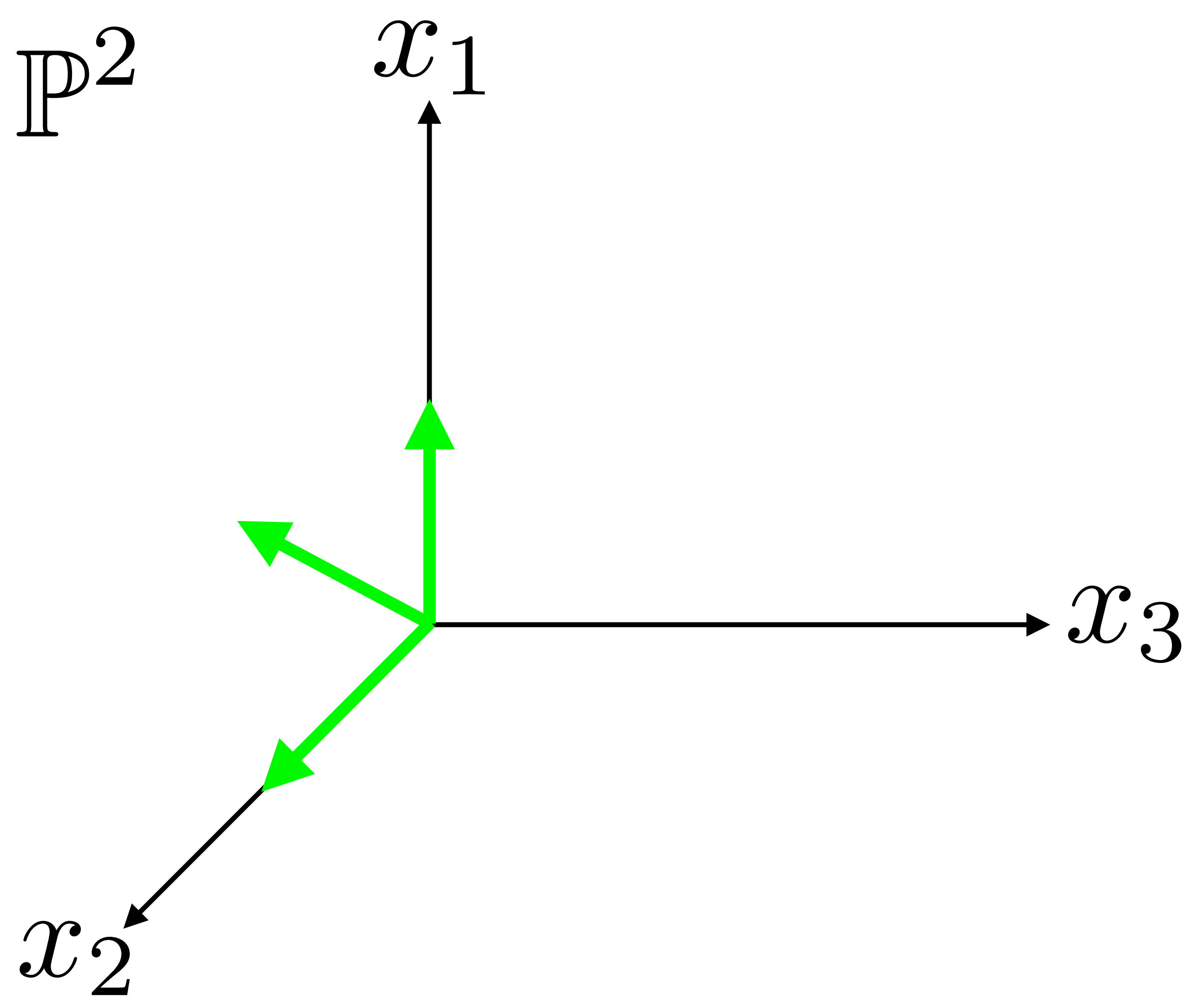
Point at infinity is in the direction of the lines

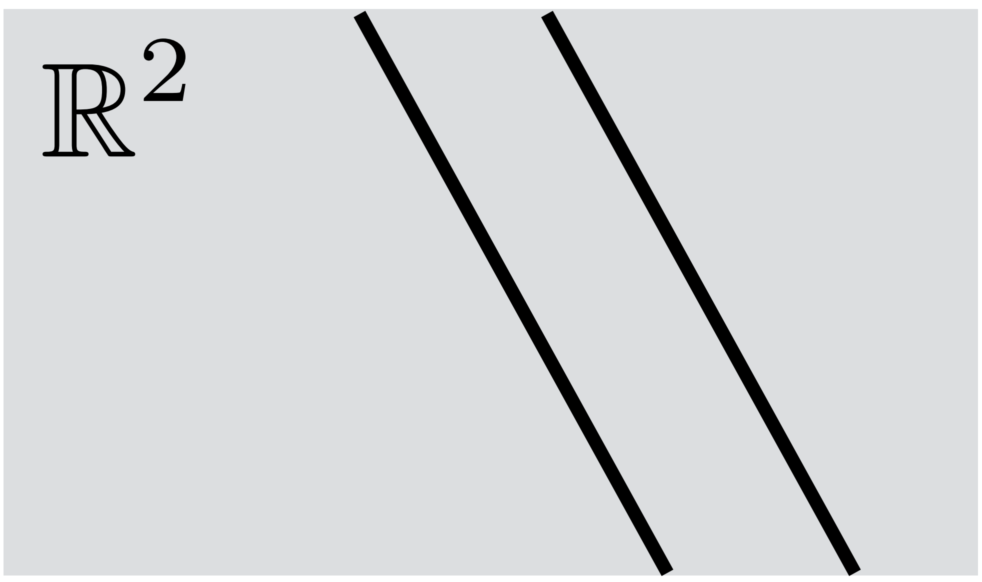




THE 2D PROJECTIVE PLANE

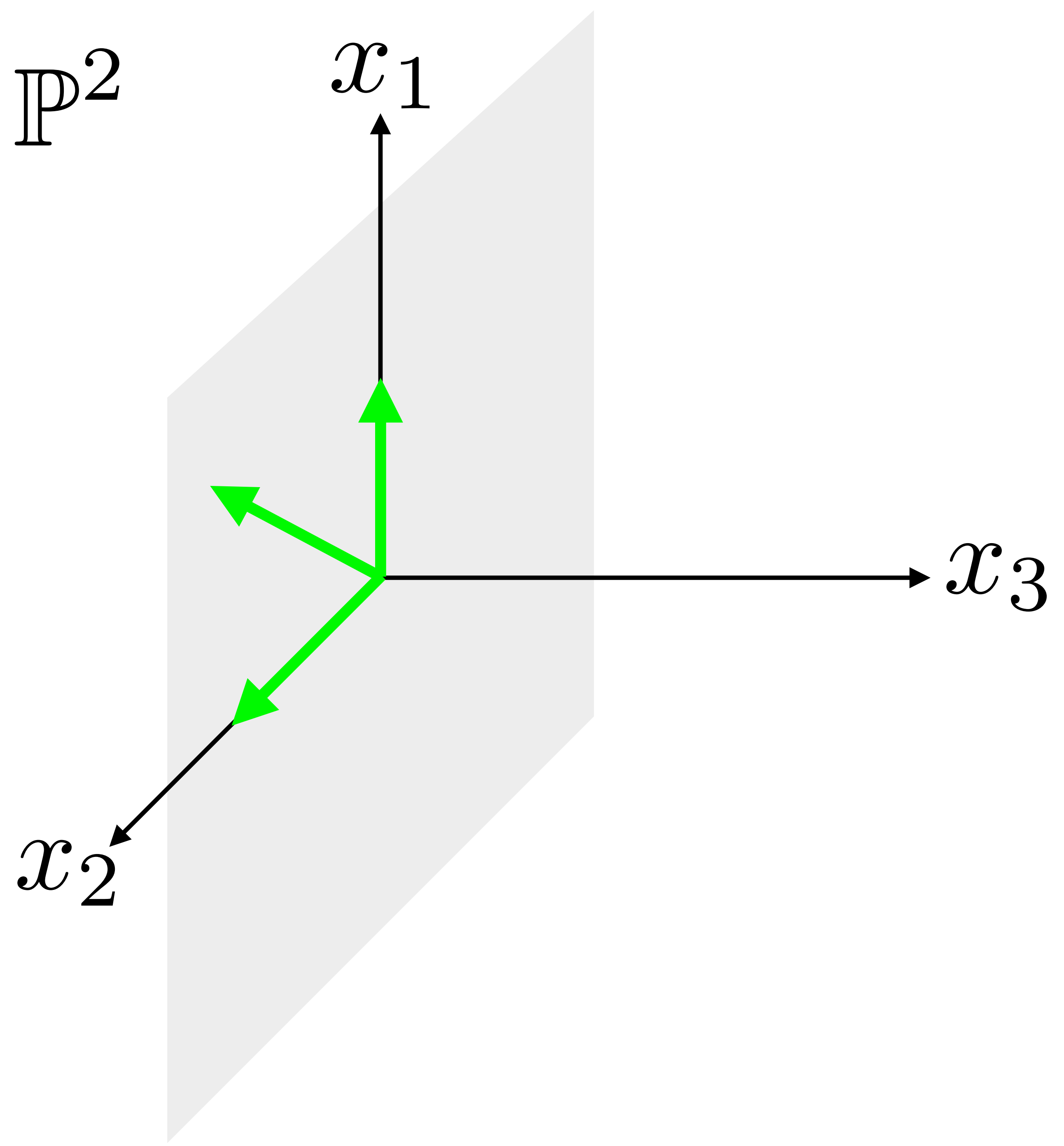
Point at infinity is in the direction of the lines

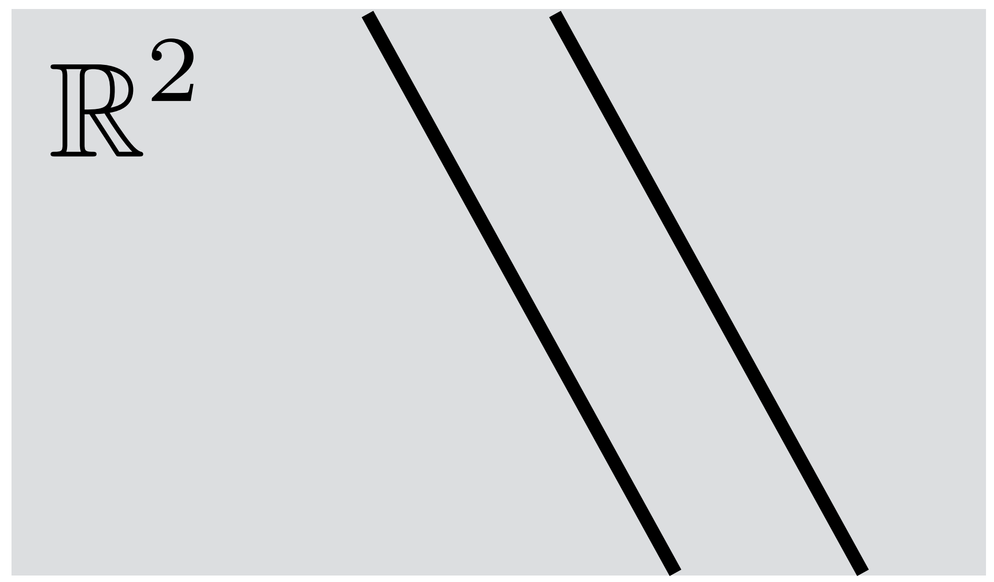




THE 2D PROJECTIVE PLANE

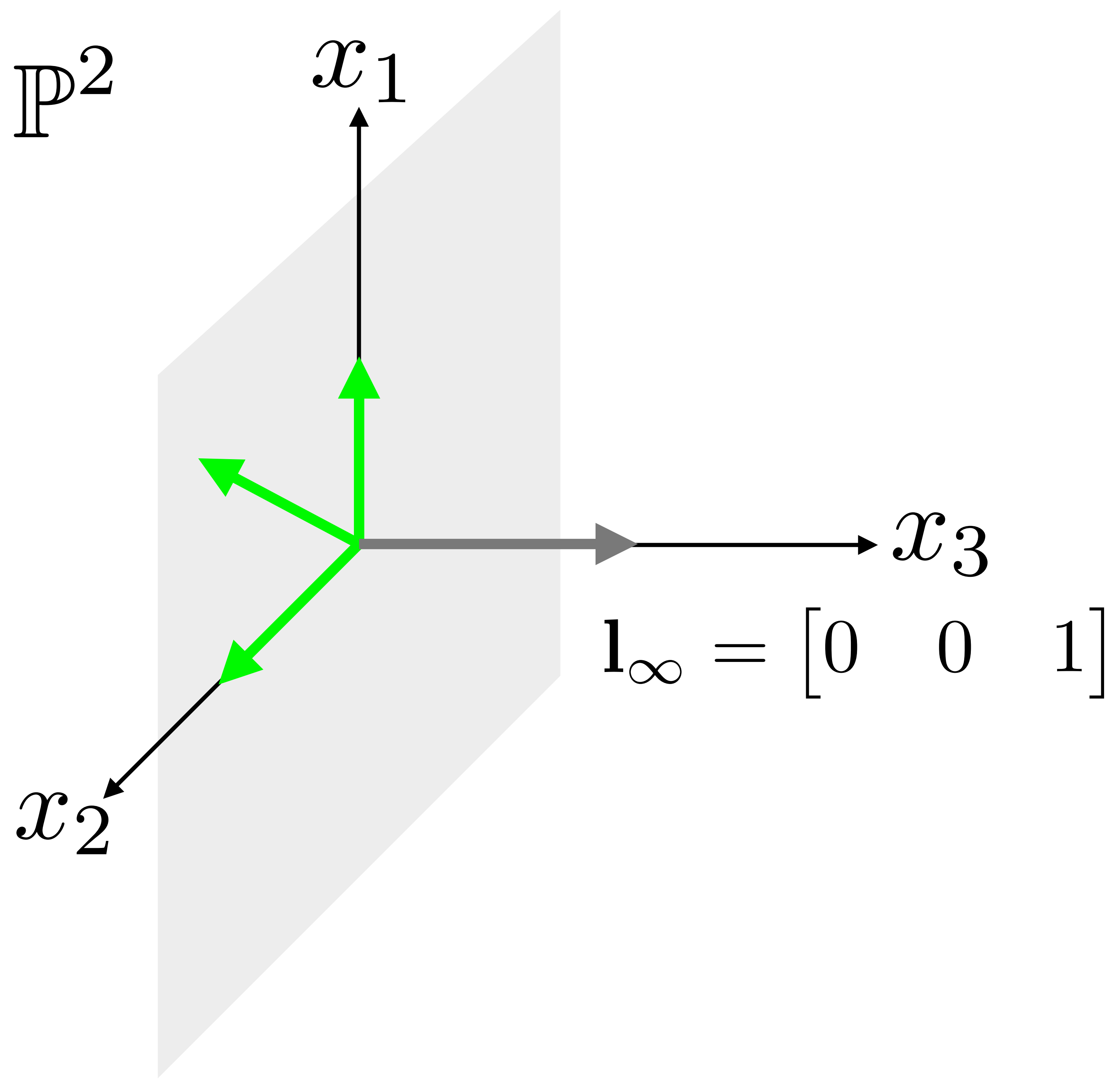
All points at infinity intersect the line at infinity

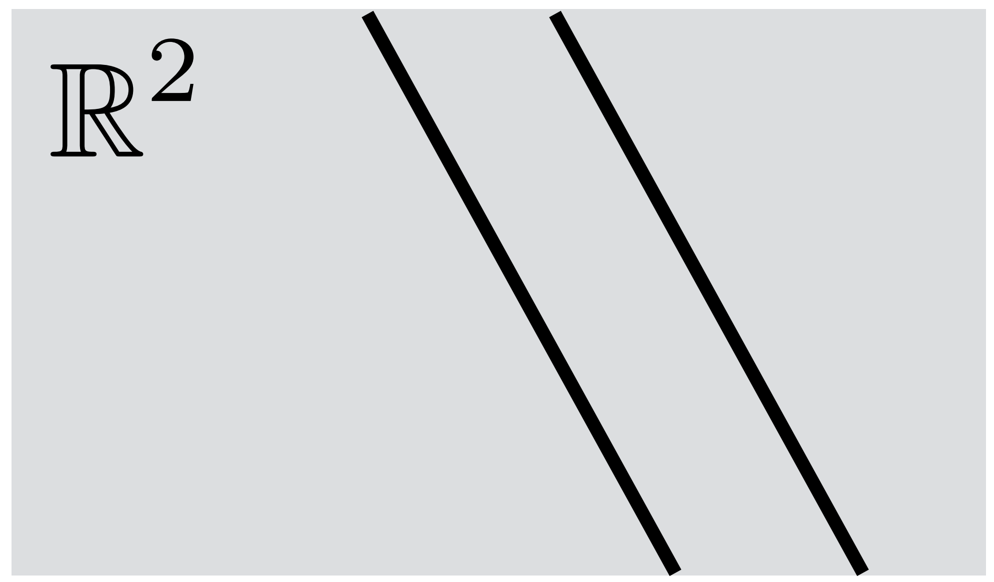




THE 2D PROJECTIVE PLANE

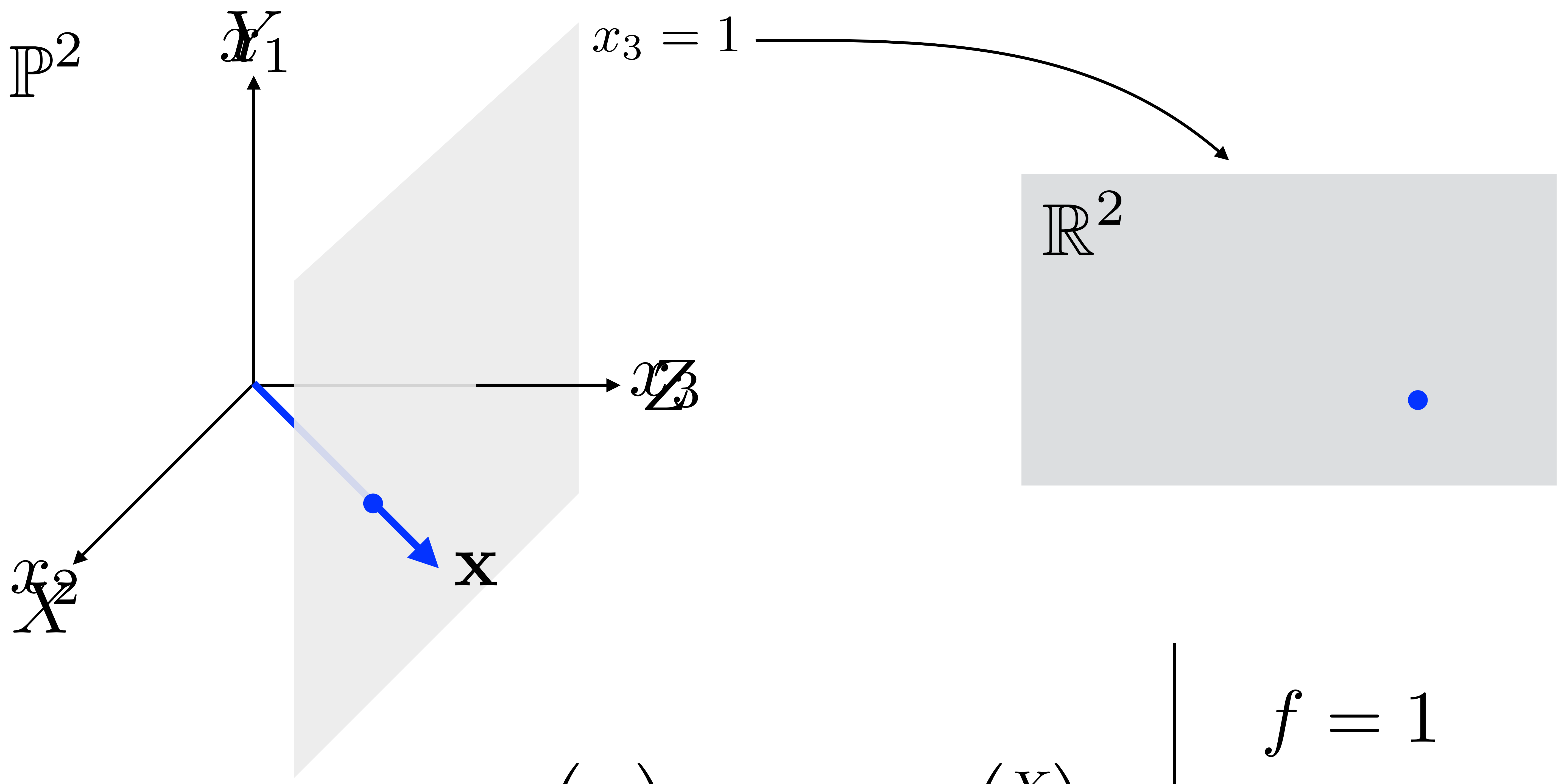
The study of the geometry of $\mathbb{P}^2$ is known as projective geometry





THE 2D PROJECTIVE PLANE

The study of the geometry of $\mathbb{P}^2$ is known as projective geometry



$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = [\mathtt{I} \mid \mathbf{0}] \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

$$f = 1$$
$$\mathtt{R} = \mathtt{I}$$
$$\mathbf{t} = \mathbf{0}$$

PROJECTIVE TRANSFORMATIONS

Projective Transformations = Homography = Collineations

PROJECTIVE TRANSFORMATIONS

Projective Transformations = Homography = Collineations

GEOMETRIC DEFINITION

A projectivity **is an invertible mapping** h **from** $\mathbb{P}^2$ **to itself, such that three points** $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ **lie on the same line if and only if** $h(\mathbf{x}_1), h(\mathbf{x}_2), h(\mathbf{x}_3)$ **do.**

ALGEBRAIC DEFINITION

A mapping $h : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ **is a** projectivity **if and only if there exists a non-singular 3 x 3 matrix** H **such that for any point in** $\mathbb{P}^2$ **represented by** $\mathbf{x}$ **it is true that** $h(\mathbf{x}) = \mathrm{H}\mathbf{x}$**.**

Homography/Projectivity/Collineation/Planar Projective Transformation

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\mathbf{x}' = \mathrm{H}\mathbf{x}$$

Degrees of freedom: 8

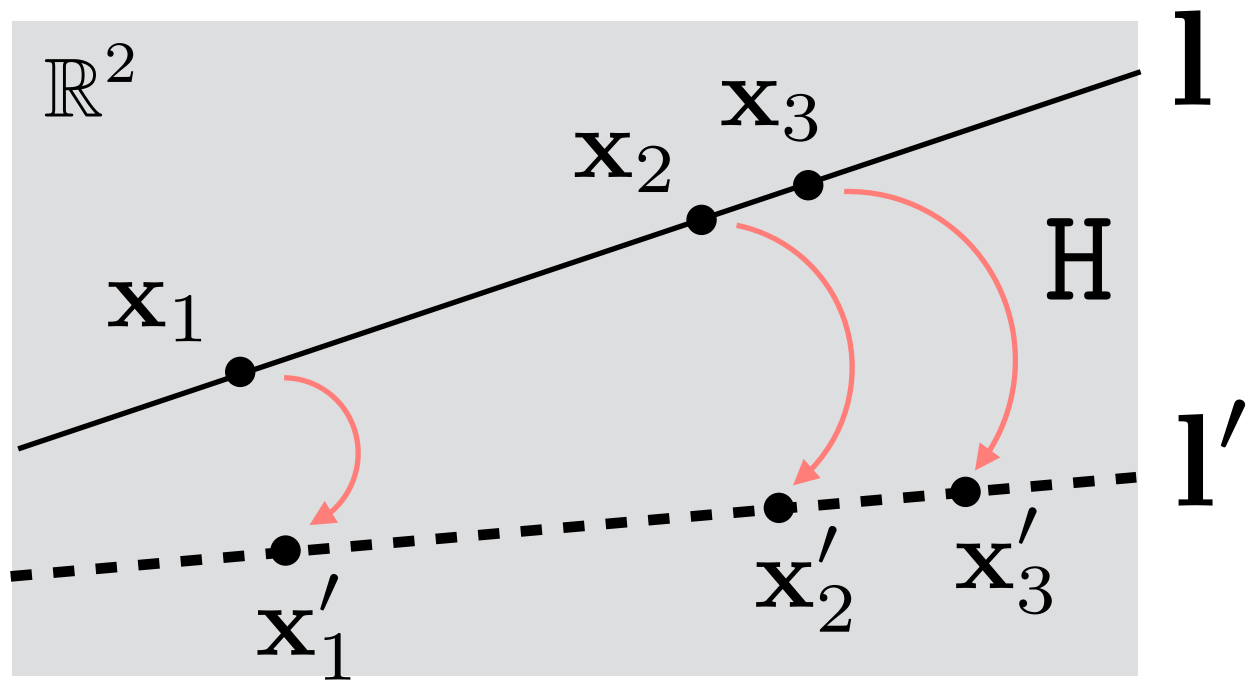
PROJECTIVE TRANSFORMATIONS

Collineation

GEOMETRIC DEFINITION

$$\mathbf{x}' = \mathrm{H}\mathbf{x}$$

A projectivity is an invertible mapping h from $\mathbb{P}^2$ to itself, such that three points $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ lie on the same line if and only if $h(\mathbf{x}_1), h(\mathbf{x}_2), h(\mathbf{x}_3)$ do.



$$\mathbf{l}^T\mathbf{x} = 0$$

$$\mathbf{l}'^T\mathrm{H}\mathbf{x} = 0$$

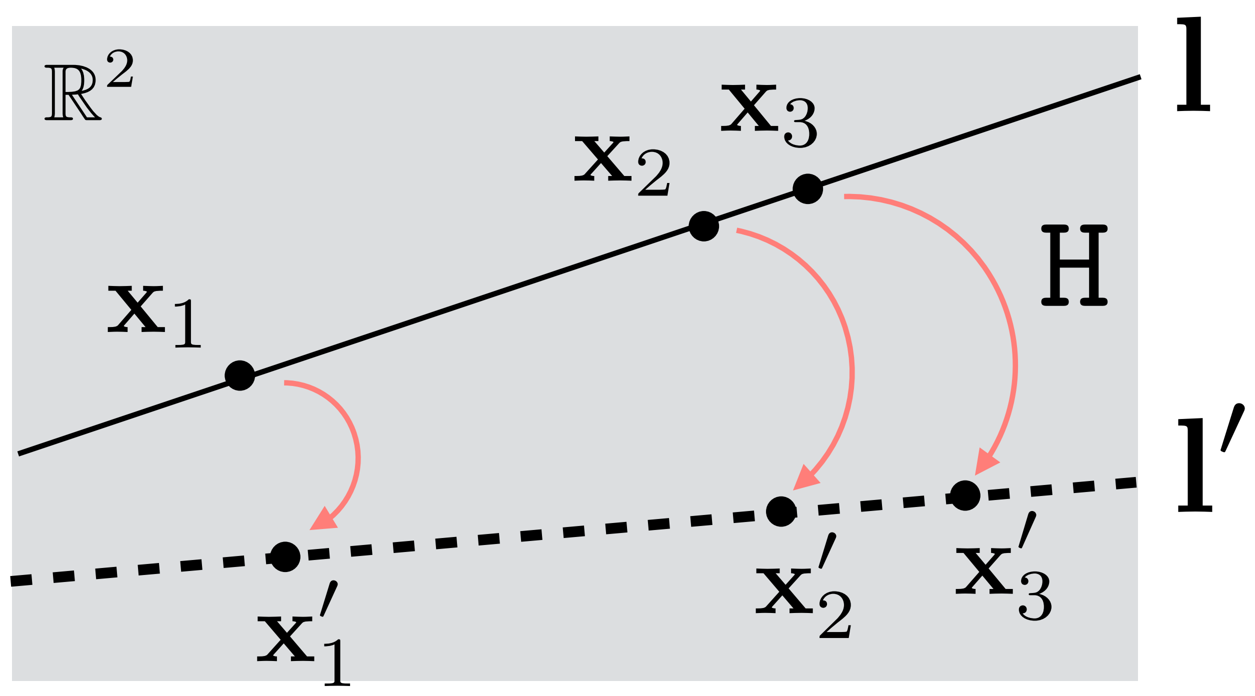
$$\mathbf{l}'^T = \mathbf{l}^T\mathrm{H}^{-1}$$

Does such an l' exist?

H must be non-singular

PROJECTIVE TRANSFORMATIONS

Transformations of Lines



$$\mathbf{l}^T\mathbf{x} = 0$$

$$\mathbf{l}'^T \mathtt{H}\mathbf{x} = 0$$

$$\mathbf{l}'^T = \mathbf{l}^T \mathtt{H}^{-1}$$

H must be non-singular

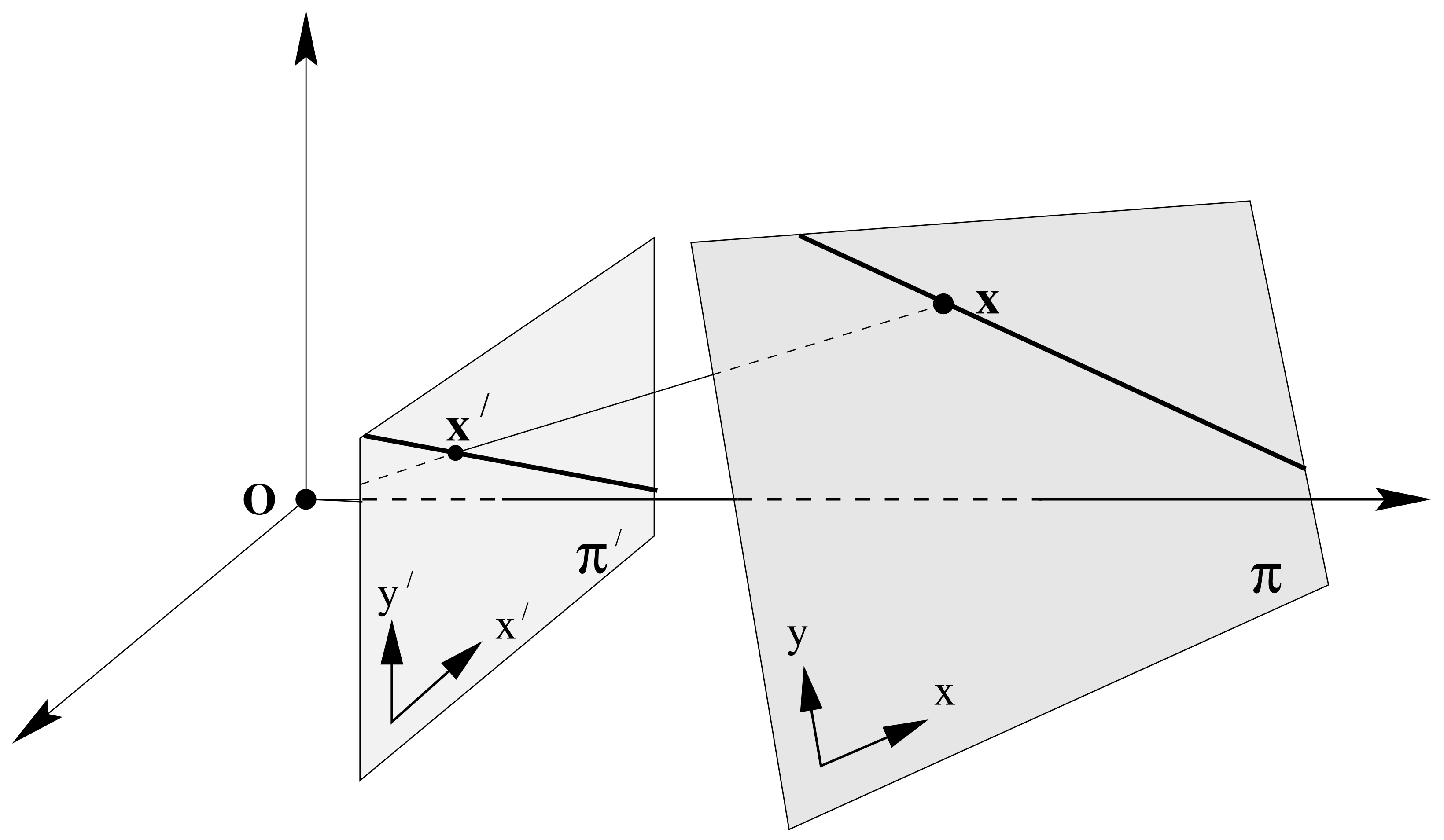
$$\boxed{\begin{aligned} \mathbf{l}' &= \mathtt{H}^{-T}\mathbf{l} \\ \mathbf{x}' &= \mathtt{H}\mathbf{x} \end{aligned}}$$

Planar Projective Transformation of Lines

Planar Projective Transformation of Points

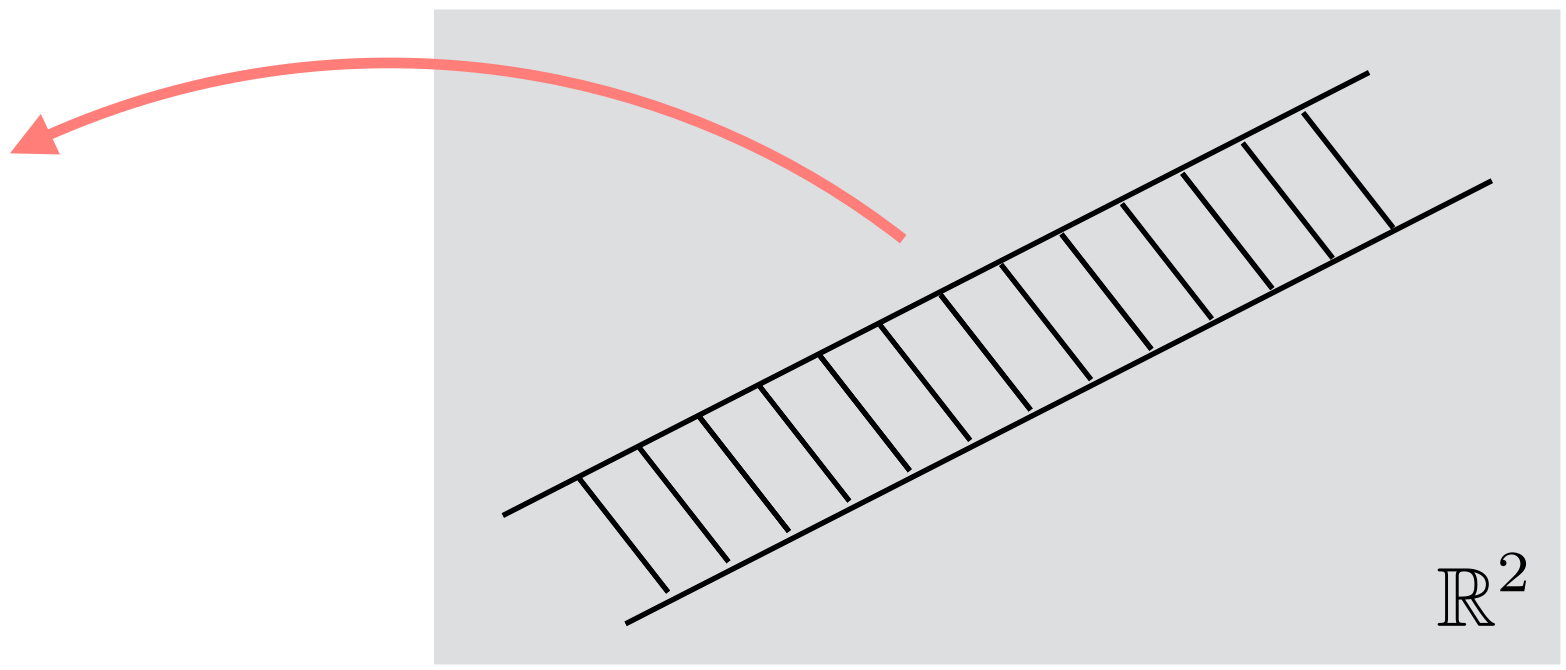
PROJECTIVE TRANSFORMATIONS

Mapping between Planes



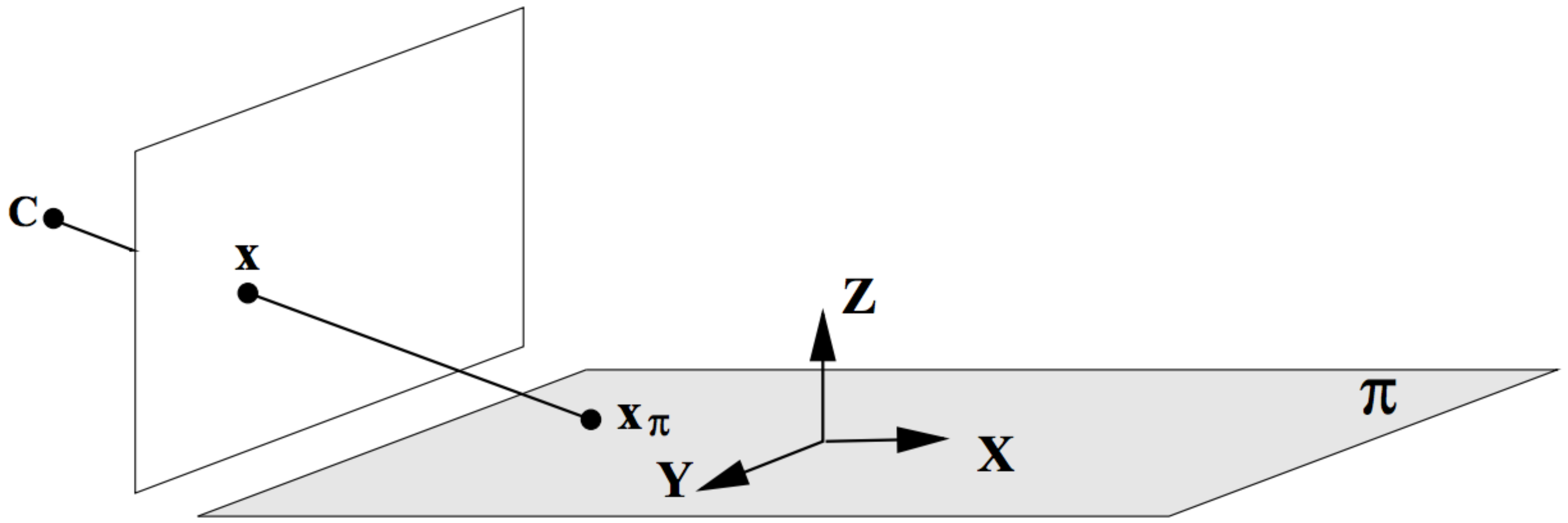
Central Projection between two planes can be expressed by a homography

$$\mathbf{x}' = \mathrm{H}\mathbf{x}$$



PROJECTIVE TRANSFORMATIONS

Case 1: Plane Projective Transformations



Choose the world coordinate system so that the world plane has Z=0 for all points.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} \begin{pmatrix} X \\ Y \\ 0 \\ 1 \end{pmatrix}$$

Homography/Projectivity/Collineation

PROJECTIVE TRANSFORMATIONS

Example: Removing Projective Distortion from a Perspective Image of a Plane

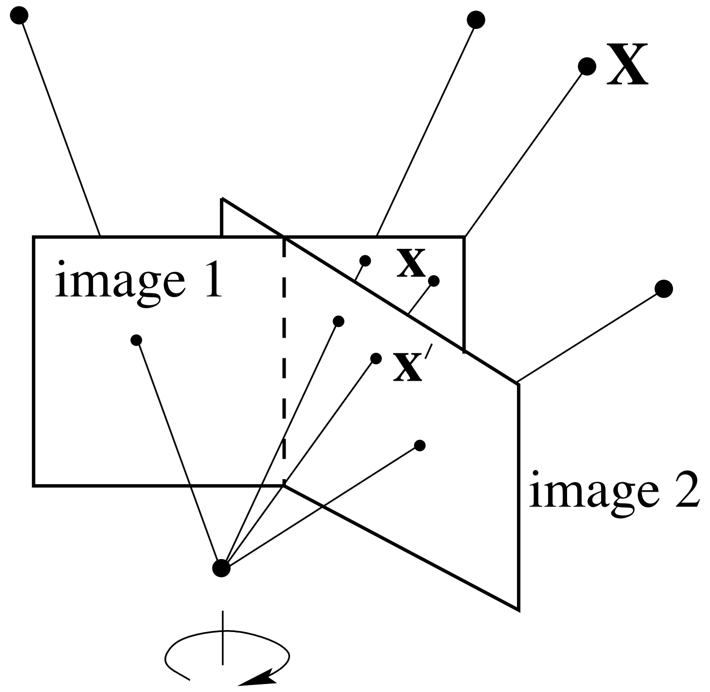
Original Image

Rectified Image

PROJECTIVE TRANSFORMATIONS

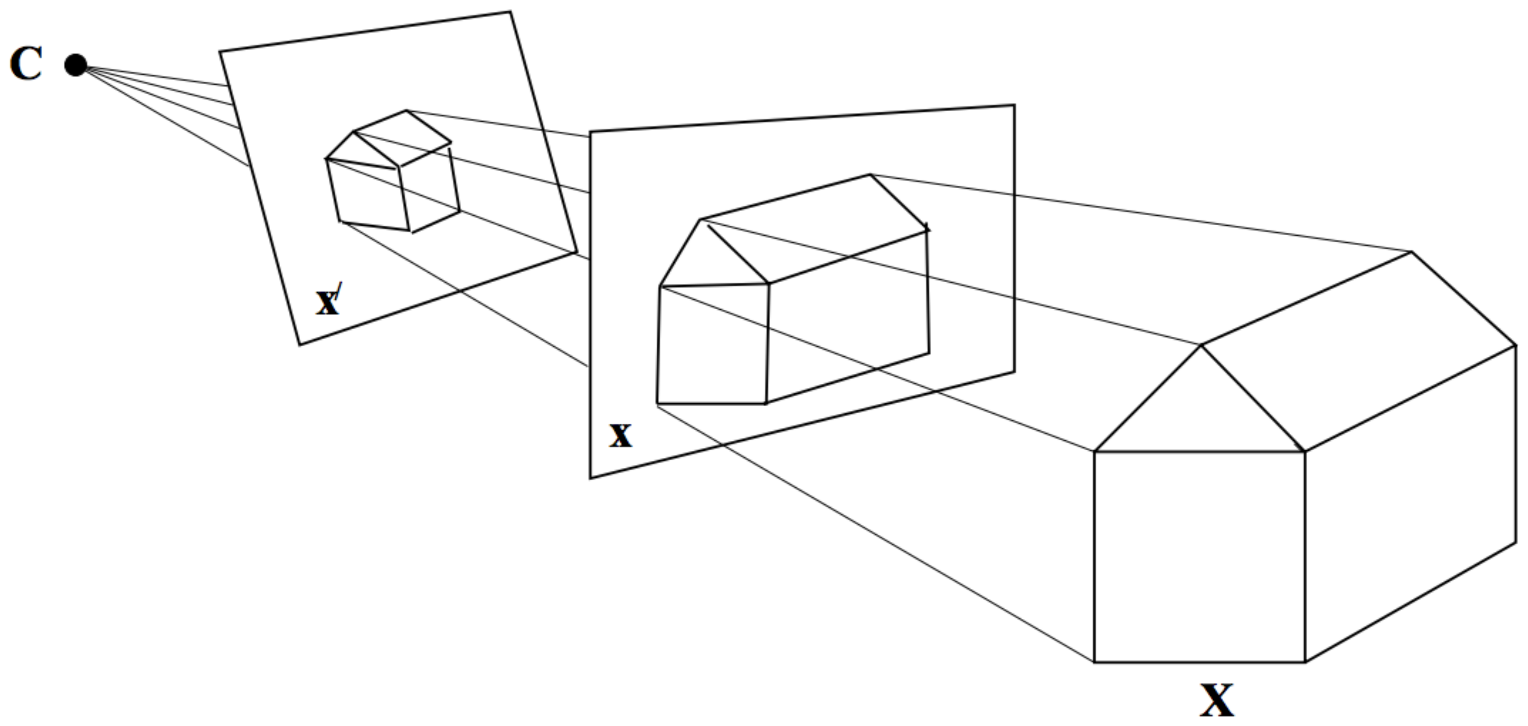
Case 2: Homography between two images with the same camera center

...even if the scene is three dimensional!



rotation about the center

$$\mathbf{x} = \mathtt{K}\left[\mathtt{I}|\mathbf{0}\right]\mathbf{X}$$
$$\mathbf{x}' = \mathtt{K}\left[\mathtt{R}|\mathbf{0}\right]\mathbf{X}$$
$$= \mathtt{K}\mathtt{R}\mathtt{K}^{-1}\mathtt{K}\left[\mathtt{I}|\mathbf{0}\right]\mathbf{X}$$
$$= \mathtt{K}\mathtt{R}\mathtt{K}^{-1}\mathbf{x}$$
$$= \mathtt{H}\mathbf{x}$$



zoom about the center

$$\mathbf{x} = \mathtt{K}\left[\mathtt{I}|\mathbf{0}\right]\mathbf{X}$$
$$\mathbf{x}' = \mathtt{K}'\left[\mathtt{I}|\mathbf{0}\right]\mathbf{X}$$
$$= \mathtt{K}'\mathtt{K}^{-1}\mathtt{K}\left[\mathtt{I}|\mathbf{0}\right]\mathbf{X}$$
$$= \mathtt{K}'\mathtt{K}^{-1}\mathbf{x}$$
$$= \mathtt{H}\mathbf{x}$$

[Refer to Lecture 7 for review of camera projection]

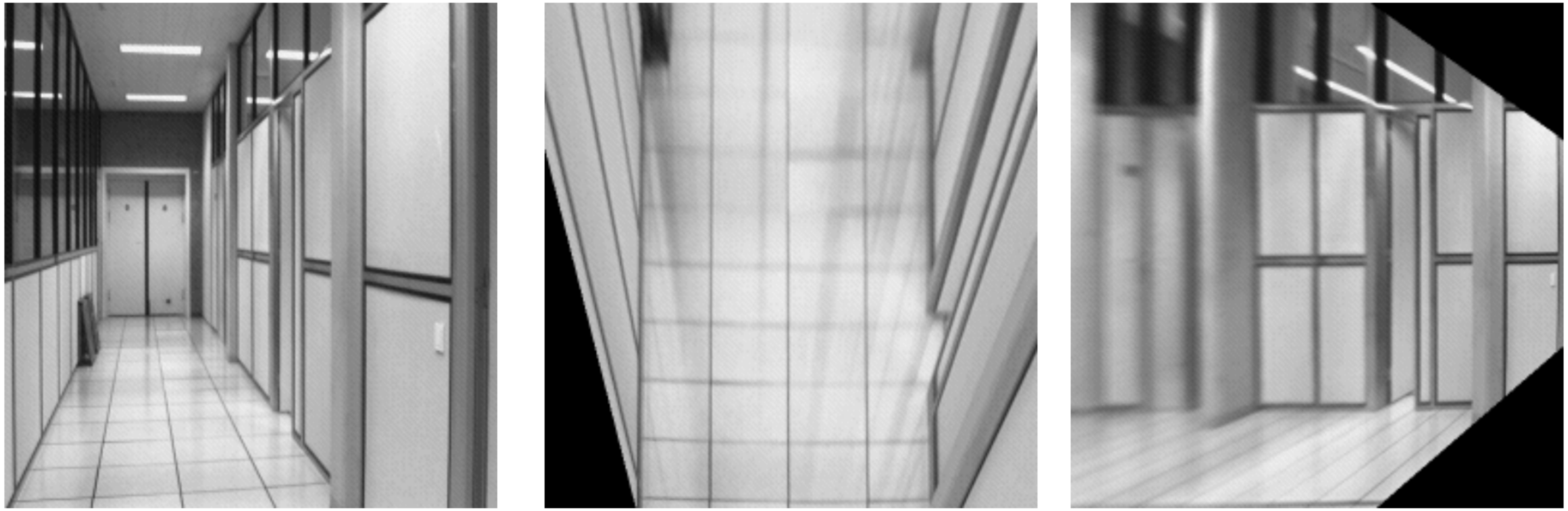
PROJECTIVE TRANSFORMATIONS

Case 2: Homography between two images with the same camera center

...even if the scene is three dimensional!

PROJECTIVE TRANSFORMATIONS

Synthetic Rotations



PROJECTIVE TRANSFORMATIONS

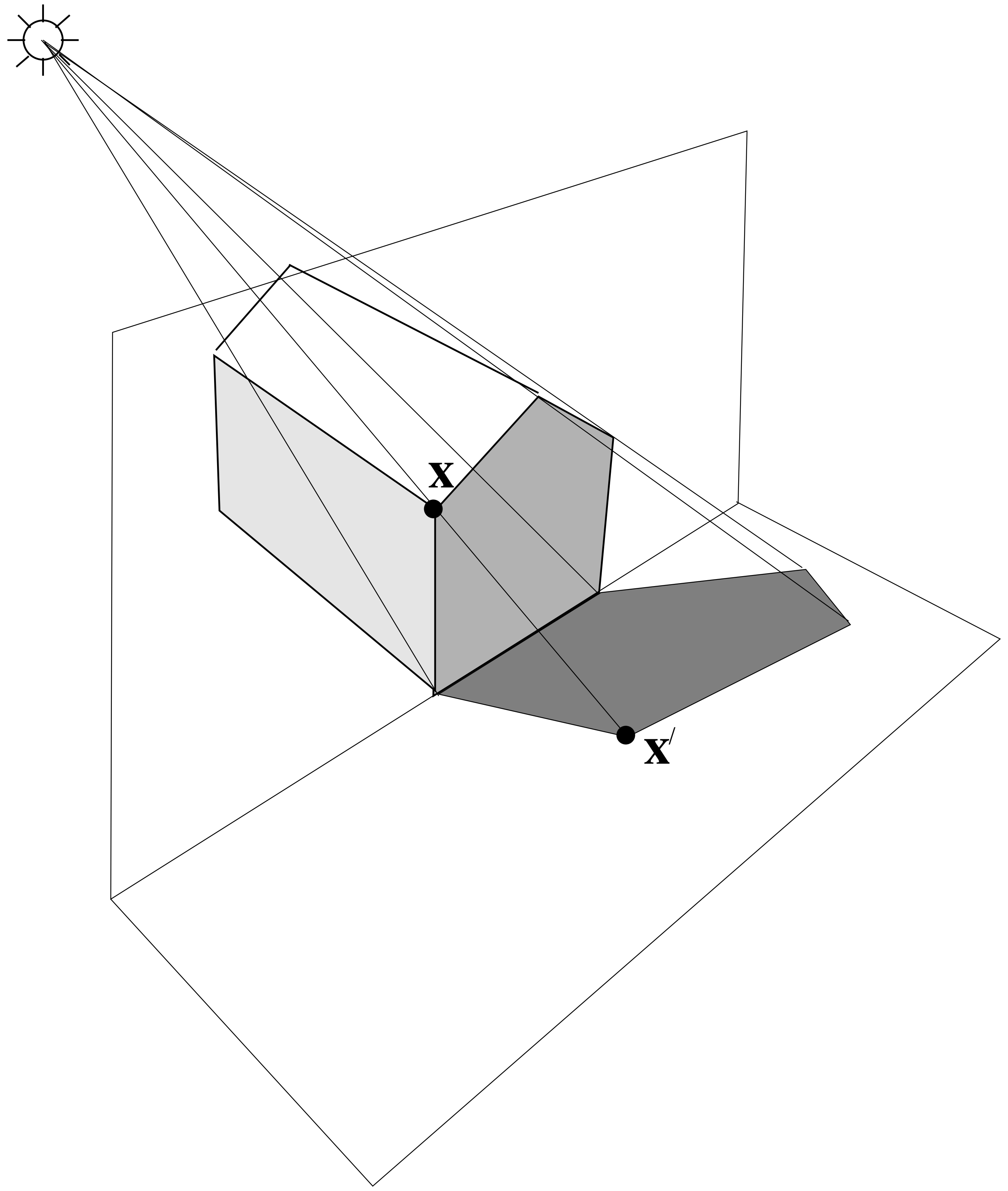
Gigapan

PROJECTIVE TRANSFORMATIONS

Gigapan

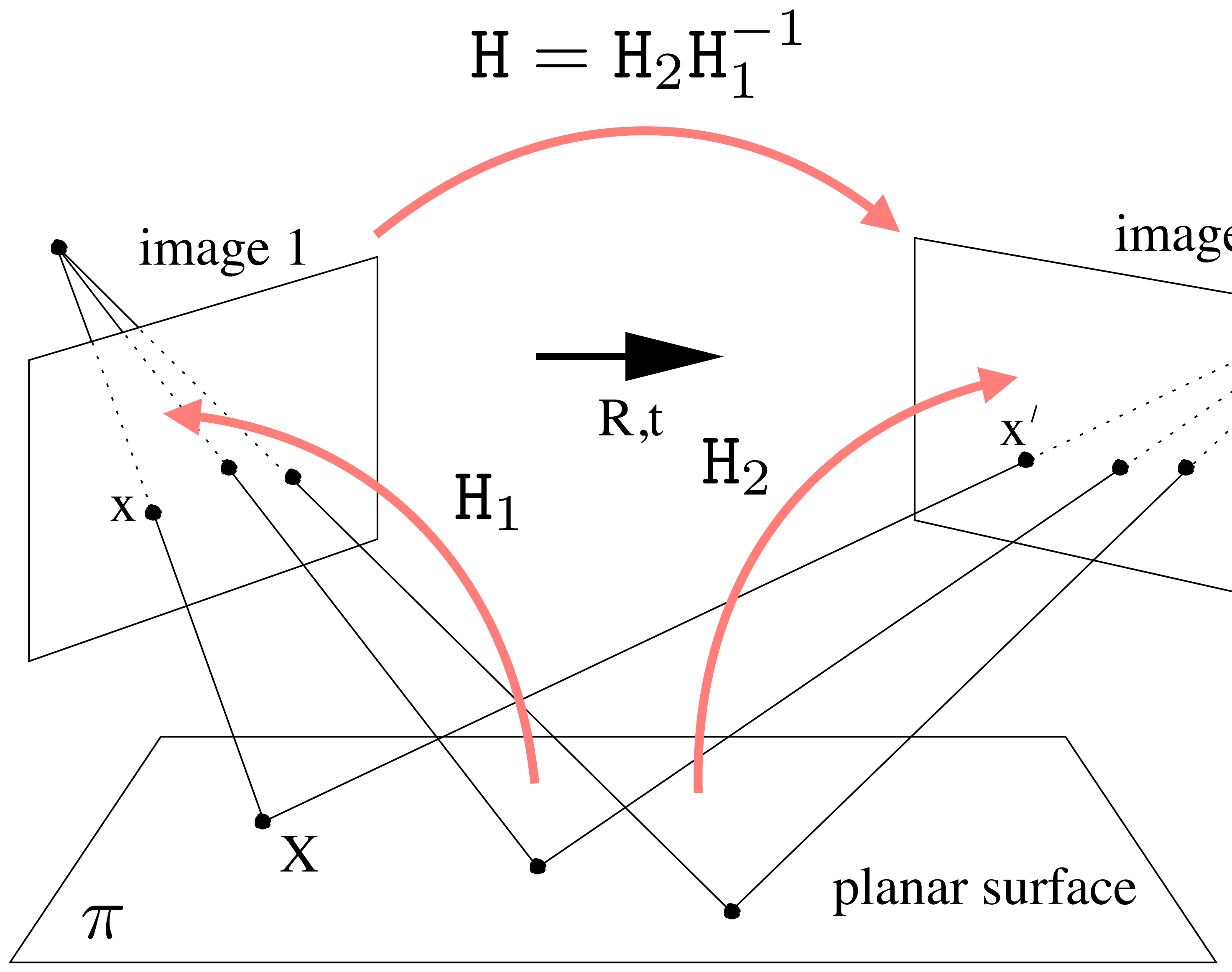
PROJECTIVE TRANSFORMATIONS

Case 3: Homography between the image of a plane and the image of its shadow onto another plane



PROJECTIVE TRANSFORMATIONS

Case 4: Homography between two images induced by a world plane



$$\mathbf{x} = \mathtt{H}_1\mathbf{x}_\pi \qquad \mathbf{x}' = \mathtt{H}_2\mathbf{x}_\pi$$

$$\mathbf{x}' = \mathtt{H}_2\mathtt{H}_1^{-1}\mathbf{x}$$

LECTURE SUMMARY

Single View Geometry and the 2D Projective Plane

1. **Single View Geometry**
    1. Similar Triangles
    2. Anatomy of the Camera Matrix
    3. Camera Resectioning
2. **The Projective Plane**
    1. Ideal points and the line at infinity
    2. Duality Principle
3. **Projective Transformations**
    1. Review of different cases where homographies arise
    2. Rectifying the images of planes