3D reconstruction

- Given a set of point correspondences $\mathbf{x}_i \leftrightarrow \mathbf{x}'_i$ we wish to reconstruct the world points $\mathbf{X}_i$ and the cameras $\mathtt{P}$, $\mathtt{P}'$ such that

$$\mathbf{x}_i = \mathtt{P}\mathbf{X}_i, \quad \mathbf{x}'_i = \mathtt{P}'\mathbf{X}_i, \quad \forall i$$

- Without any additional information it is possible to
  1. Estimate the fundamental matrix from point correspondences.
  2. Calculate camera pairs from the fundamental matrix.
  3. Estimate world coordinates $\mathbf{X}_i$ corresponding to each point pair $\mathbf{x}_i \leftrightarrow \mathbf{x}'_i$.
- The reconstructed cameras and points will be unique up to a projective homography of $\mathcal{P}^3$.
- Depending on which information we have about the world and/or the cameras (point coordinates, orthogonal line pairs, parallel line pairs, calibrated cameras, etc.) we may either
  - perform a stratified reconstruction — first projectively, then affinely, the metrically, or
  - directly perform a metric reconstruction.

Calculation of the fundamental matrix

- The defining equation for the fundamental matrix is

$$\mathbf{x}'^{\top}\mathtt{F}\mathbf{x} = 0$$

for each pair of corresponding points $\mathbf{x} \leftrightarrow \mathbf{x}'$. Given enough corresponding points we may calculated $\mathtt{F}$.

- Each point pair produces one equation linear in the elements in $\mathtt{F}$. If $\mathbf{x} = (x, y, 1)^{\top}$ and $\mathbf{x}' = (x', y', 1)^{\top}$ then

$$x'xf_{11} + x'yf_{12} + x'f_{13} + y'xf_{21} + y'yf_{22} + y'f_{23} + xf_{31} + yf_{32} + f_{33} = 0,$$

that may be written as

$$\begin{bmatrix} x'x & x'y & x' & y'x & y'y & y' & x & y & 1 \end{bmatrix} \mathbf{f} = 0,$$

where $\mathbf{f}$ is the 9-vector with elements $\mathtt{F}$ row-wise.

alculation of the fundamental matrix F

- Given n corresponding points we get a linear equation system on the form

$$\mathtt{A}\mathbf{f} = \begin{bmatrix} x'_1x_1 & x'_1y_1 & x'_1 & y'_1x_1 & y'_1y_1 & y'_1 & x_1 & y_1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x'_nx_n & x'_ny_n & x'_n & y'_nx_n & y'_ny_n & y'_n & x_n & y_n & 1 \end{bmatrix} \mathbf{f} = \mathbf{0}.$$

- The equation is homogenous, i.e. $\mathbf{f}$ can only be determined up to scale.
- In order for a homogenous solution to exist, the rank of $\mathtt{A}$ cannot be larger than 8. In that case a solution $\mathbf{f}$ exists in the null-space of $\mathtt{A}$.
- A solution may always be determined by solving

$$\min_{\mathbf{f}} ||\mathtt{A}\mathbf{f}||$$
$$\text{s.t. } ||\mathbf{f}|| = 1,$$

with solution $\mathbf{f} = \mathbf{v}_9$ where $\mathtt{A} = \mathtt{UDV}^{\top}$.

Minimal correspondence — the 7-point algorit

- If the matrix $\mathtt{A}$ is constructed from $n = 7$ correspondences, $\mathtt{A}$ will have a two-dimensional null-space.
- Let the vectors $\mathbf{f}_1$ and $\mathbf{f}_2$ be basis vectors for the null-space of $\mathtt{A}$. Then each vector

$$\mathbf{f} = \alpha\mathbf{f}_1 + (1-\alpha)\mathbf{f}_2$$

is a solution of $\mathtt{A}\mathbf{f} = \mathbf{0}$.

- the corresponding $\mathtt{F}$-matrices are

$$\mathtt{F} = \alpha\mathtt{F}_1 + (1-\alpha)\mathtt{F}_2.$$

- The condition $\det(\mathtt{F}) = 0$ leads to the equation

$$\det(\alpha\mathtt{F}_1 + (1-\alpha)\mathtt{F}_2) = 0,$$

that is a third order equation in α with one or three real roots, i.e. one or three solutions are possible.

The 7-point algorithm, 3 solutions

The 8-point algorithm

- The easiest way to calculate the fundamental matrix is to use $n \geq 8$ points and
  1. Calculate $\mathtt{F}$ that minimizes the algebraic error, i.e. solves

$$\min_{\mathbf{f}} \|\mathtt{A}\mathbf{f}\| \quad \text{s.t. } \|\mathbf{f}\| = 1.$$

  2. Find the "closest" matrix $\mathtt{F}'$ of rank 2, i.e. solve

$$\min_{\mathtt{F}'} \|\mathtt{F} - \mathtt{F}'\|_F \quad \text{s.t. } \operatorname{rank}(\mathtt{F}') = 2,$$

  where $\|\cdot\|_F$ is the Frobenius norm.

- If $\mathtt{A} = \mathtt{UDV}^\top$ is the singular value decomposition of $\mathtt{A}$, then the solution of problem 1 is

$$\mathbf{f} = \mathbf{v}_9.$$

- If $\mathtt{F} = \mathtt{UDV}^\top$ is the singular value decomposition of $\mathtt{F}$, where

$$\mathtt{D} = \operatorname{diag}(r, s, t), r \geq s \geq t,$$

  then

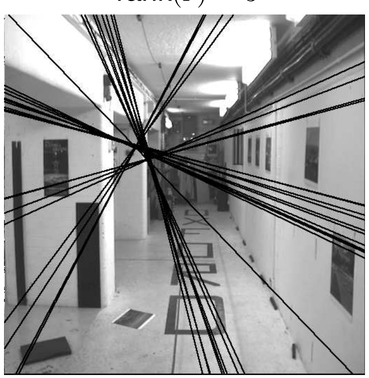
$$\mathtt{F}' = \mathtt{U} \operatorname{diag}(r, s, 0)\mathtt{V}^\top$$

  is a solution to problem 2.

Non rank-deficient $\mathtt{F}$ matrix

- If $\mathtt{F}$ has full rank it will have an empty null-space, i.e. not have any point that is on all lines, i.e. no epipole.

$\operatorname{rank}(\mathtt{F}) = 3$



$\operatorname{rank}(\mathtt{F}) = 2$

The normalized 8-point algorithm

- In order for the 8-point algorithm to work in practice, it has to be normalized. The algorithm then becomes:
  1. Determine a transformation $\mathtt{T}$ and $\mathtt{T}'$ such that $\hat{\mathbf{x}}_i = \mathtt{T}\mathbf{x}_i$ and $\hat{\mathbf{x}}'_i = \mathtt{T}\mathbf{x}'_i$ has center of gravity at the origin and a mean squared distance of 2 from the origin.
  2. Calculate a fundamental matrix $\hat{\mathtt{F}}'$ corresponding to the point pairs $\hat{\mathbf{x}}_i \leftrightarrow \hat{\mathbf{x}}'_i$:
     (a) Calculate $\hat{\mathtt{F}}$ from the solution $\hat{\mathbf{f}}$ that solves

$$\min_{\hat{\mathbf{f}}} \|\hat{\mathtt{A}}\hat{\mathbf{f}}\| \quad \text{s.t. } \|\hat{\mathbf{f}}\| = 1,$$

     where $\hat{\mathtt{A}}$ is calculated from the point pairs $\hat{\mathbf{x}}_h \leftrightarrow \hat{\mathbf{x}}'_i$.

     (b) Calculate $\hat{\mathtt{F}}'$ that solves

$$\min_{\hat{\mathtt{F}}'} \|\hat{\mathtt{F}} - \hat{\mathtt{F}}'\|_F \quad \text{s.t. } \operatorname{rank}(\hat{\mathtt{F}}') = 2.$$

  3. Calculate $\mathtt{F} = \mathtt{T}'^\top \hat{\mathtt{F}}'\mathtt{T}$ as the fundamental matrix corresponding to the original point pairs $\mathbf{x}_i \leftrightarrow \mathbf{x}'_i$.

Optimal F

- In order to determine an optimal $\mathtt{F}$, the reprojection error has to be minimized.
- This may be achieved through e.g. the following problem formulation:

$$\min_{\mathbf{u}} \sum_{i=1}^{m} d(\mathbf{x}_i, \hat{\mathbf{x}}_i)^2 + d(\mathbf{x}'_i, \hat{\mathbf{x}}'_i)^2$$
$$\text{s.t. } \hat{\mathbf{x}}_i'^{\top} \mathtt{F} \hat{\mathbf{x}}_i = 0, \; i = 1, \ldots, m$$
$$\|\mathtt{F}\|_F = 1, \det(\mathtt{F}) = 0,$$

where

$$\mathbf{u} = [f_1, \ldots, f_9, \hat{x}_{1x}, \hat{x}_{1y}, \ldots, \hat{x}_{mx}, \hat{x}_{my}, \hat{x}'_{1x}, \hat{x}'_{1y}, \ldots, \hat{x}'_{mx}, \hat{x}'_{my}]^{\top}.$$

Optimal F

- Given an algorithm to solve

$$\min_{\mathbf{u}} \frac{1}{2} f(\mathbf{u})^T W f(\mathbf{u})$$
$$\text{s.t. } c(\mathbf{u}) = 0$$

we get

$$f(\mathbf{u}) = \begin{bmatrix} \hat{\mathbf{x}}_1 - \mathbf{x}_1 \\ \vdots \\ \hat{\mathbf{x}}_m - \mathbf{x}_m \\ \hat{\mathbf{x}}'_1 - \mathbf{x}'_1 \\ \vdots \\ \hat{\mathbf{x}}'_m - \mathbf{x}'_m \end{bmatrix} \text{ and } c(\mathbf{u}) = \begin{bmatrix} \hat{\mathbf{x}}_1'^{\top} \mathtt{F} \hat{\mathbf{x}}_1 \\ \vdots \\ \hat{\mathbf{x}}_m'^{\top} \mathtt{F} \hat{\mathbf{x}}_m \\ \|\mathtt{F}\|_F - 1 \\ \det(F) \end{bmatrix}$$

- A starting approximation of $\mathtt{F}$ can be calculated by the normalized 8-point algorithm. The initial estimates of the line points $\hat{\mathbf{x}}_i$ and $\hat{\mathbf{x}}'_i$ are the corresponding measured points $\mathbf{x}_i$ and $\mathbf{x}'_i$.

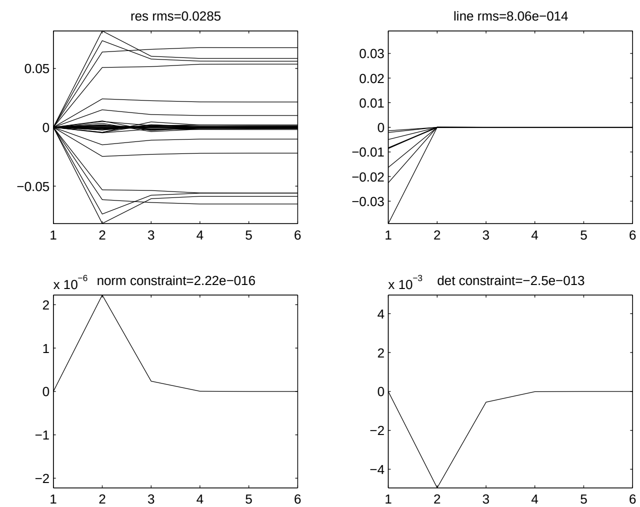
Optimal F, normal example

Starting approximation

One iteration

Solution

Optimal F, normal example



Iteration sequence.

Automatic calculation of $\mathtt{F}$

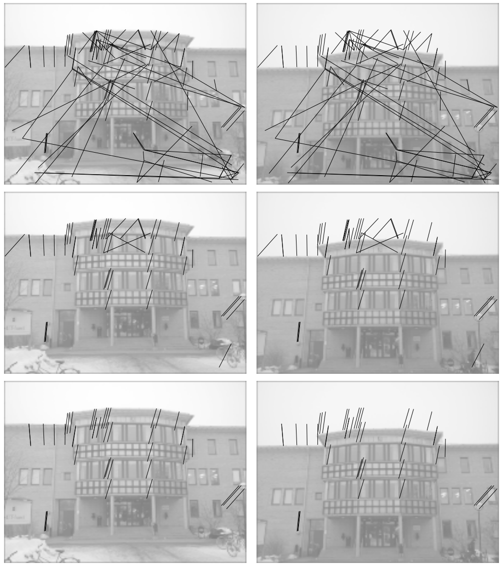
- By using RANSAC we may estimate $\mathtt{F}$ automatically. Given n preliminary point matches, a probability p and a distance threshold t:
  - Draw 7 point pairs randomly.
  - Calculate a fundamental matrix from the point pairs. We have 1 or 3 solutions. For each of the solutions
    - Calculate the distance d_i between each point pair and the corresponding epipolar lines.
    - Calculate the number of point pairs k_i that are inliers, i.e. with distance $d_i < t$.
    - If $k_i > k_{best}$ or $k_i = k_{best}$ and $\sigma(d_i) < \sigma_{best}$ then
      - $best = i$,
      - $\epsilon = k_{best}/(1-n),\ N_{max} = \frac{\log 1-p}{\log(1-(1-\epsilon)^7)}$
  - $i = i + 1$
  - Repeat until $i \geq N_{max}$.
- Then refine the best solution by minimizing the reprojection error.
- Add correspondences that now satisfies $d_i < t$. Optionally repeat the previous step.

Example

Suggested matches

After calculation of $\mathtt{F}$ with RANSAC

After manual cleaning



alculation of 3D coordinates, homogenous solution

- Given a corresponding point pair $\mathbf{x} \leftrightarrow \mathbf{x}'$ and two cameras $\mathtt{P}, \mathtt{P}'$, calculate the corresponding 3D point $\mathbf{X}$ such that $\mathbf{x} = \mathtt{P}\mathbf{X}$ and $\mathbf{x}' = \mathtt{P}'\mathbf{X}$.
- As previously with homography calculations we may dehomogenize the image points in e.g. image 1 and get $\mathbf{x} \times (\mathtt{P}\mathbf{X}) = 0$ or

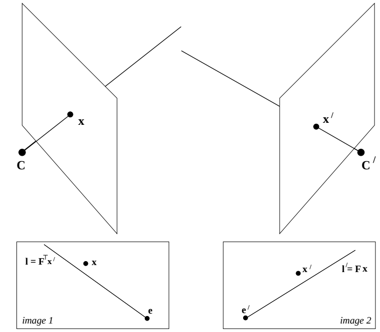
$$x(\mathbf{p}^{3\top}\mathbf{X}) - (\mathbf{p}^{1\top}\mathbf{X}) = 0$$
$$y(\mathbf{p}^{3\top}\mathbf{X}) - (\mathbf{p}^{2\top}\mathbf{X}) = 0$$
$$x(\mathbf{p}^{2\top}\mathbf{X}) - y(\mathbf{p}^{1\top}\mathbf{X}) = 0$$

- Of these, only two are linearly independent. If we remove the third and combine with the point in image 2 we get

$$\mathtt{A}\mathbf{X} = 0, \text{ where } \mathtt{A} = \begin{bmatrix} x\mathbf{p}^{3\top} - \mathbf{p}^{1\top} \\ y\mathbf{p}^{3\top} - \mathbf{p}^{2\top} \\ x'\mathbf{p}'^{3\top} - \mathbf{p}'^{1\top} \\ y'\mathbf{p}'^{3\top} - \mathbf{p}'^{2\top} \end{bmatrix}.$$

Calculation of 3D coordinates, homogenous solutio

- The equation system $\mathtt{A}\mathbf{X} = 0$ is overdetermined in the sense that $\mathbf{X}$ has 3 degrees of freedom, but we have 4 independent equations.
- If the points $\mathbf{x}$ and $\mathbf{x}'$ do not correspond *exactly*, $\mathbf{X} = 0$ will be the only solution.



- We thus have to solve $\mathtt{A}\mathbf{X} = 0$ approximately, either via SVD $\mathtt{A} = \mathtt{UDV}^\top = \mathtt{A}$, $\mathbf{X} = \mathbf{v}_4$ or by specifying $\mathbf{X} = (X, Y, Z, 1)^\top$.

Calculation of 3D coordinates, Euclidian solution

- We may also choose to minimize the Euclidian distance in $\mathcal{R}^3$. If camera 1 has center in $\tilde{\mathbf{C}}$ and we know another point $\tilde{\mathbf{U}}$ on the line $\mathbf{U} = \mathtt{P}^+\mathbf{x}$, then a point $\tilde{\mathbf{X}}$ on the line satisfies

$$\tilde{\mathbf{X}} = \tilde{\mathbf{C}} + \alpha(\tilde{\mathbf{U}} - \tilde{\mathbf{C}}).$$

- Similarly in image 2

$$\tilde{\mathbf{X}} = \tilde{\mathbf{C}}' + \alpha'(\tilde{\mathbf{U}}' - \tilde{\mathbf{C}}'),\ \text{where}\ \mathbf{U}' = \mathtt{P}^+\mathbf{x}'.$$

- If the lines do not intersect there is no point $\tilde{\mathbf{X}}$ that satisfies both equations. However, then we may minimize the distance to the lines

$$\min_{\alpha,\alpha',\tilde{\mathbf{x}}} \left\| \begin{bmatrix} \tilde{\mathbf{U}} - \tilde{\mathbf{C}} & \mathbf{0} & -\mathtt{I}_3 \\ \mathbf{0} & \tilde{\mathbf{U}}' - \tilde{\mathbf{C}}' & -\mathtt{I}_3 \end{bmatrix} \begin{bmatrix} \alpha \\ \alpha' \\ \tilde{\mathbf{X}} \end{bmatrix} - \begin{bmatrix} -\tilde{\mathbf{C}} \\ -\tilde{\mathbf{C}}' \end{bmatrix} \right\|^2.$$

- This method is called *forward intersection* in the photogrammetric community.

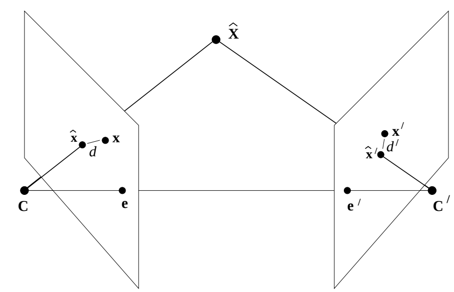
Minimization of the reprojection erro

- The optimal solution is obtained if we minimize the reprojection error, i.e. solves

$$\min_{\hat{\mathbf{x}},\hat{\mathbf{x}}',\mathbf{X}} d(\mathbf{x},\hat{\mathbf{x}})^2 + d(\mathbf{x}',\hat{\mathbf{x}}')^2$$
$$\text{s.t. } \hat{\mathbf{x}} = \mathtt{P}\mathbf{X}$$
$$\hat{\mathbf{x}}' = \mathtt{P}'\mathbf{X}$$

which exactly corresponds to

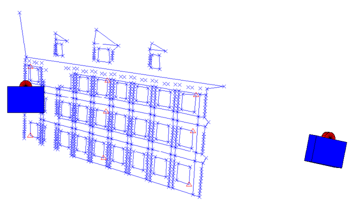
$$\min_{\hat{\mathbf{x}},\hat{\mathbf{x}}'} d(\mathbf{x},\hat{\mathbf{x}})^2 + d(\mathbf{x}',\hat{\mathbf{x}}')^2$$
$$\text{s.t. } \hat{\mathbf{x}}'^{\top}\mathtt{F}\hat{\mathbf{x}} = 0$$



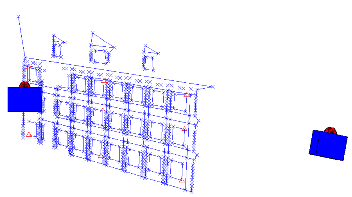
- If we have determined $\mathtt{F}$ by minimizing the reprojection error, the corresponding points have already been determined and the SVD solution of $\mathtt{A}\mathbf{X} = 0$ gives the null-space solution.

Example

Starting approximate from the normalized 8-point algorithm, via the essential matrix and forward intersection.



One iteration



Solution