Instance Space Analysis on Quantum Algorithms

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Introduction

Background

- Developing a framework to robustly evaluate quantum algorithms using Instance Space Analysis
- Explore the tools needed to develop Quantum Algorithms:
  - Adiabatic Quantum Algorithms (year 1)
  - Universal Gate Based Quantum Algorithms (year 2)
  - Explore various optimization problems, initialization techniques and instance classes (year 3)

Agenda

- Optimization Problems

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- Quantum Algorithms

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- Methodology

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- Research Plan

Optimisation Problems

- MAXCUT (will explore today)
- Other studied problems:
  - Traveling Salesperson Problem (TSP) and Vehicle Routing (VRP)
  - 3SAT - Exact Cover

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  - Potential for faster performance, as it can use advanced techniques such as error correction and quantum parallelism [3].

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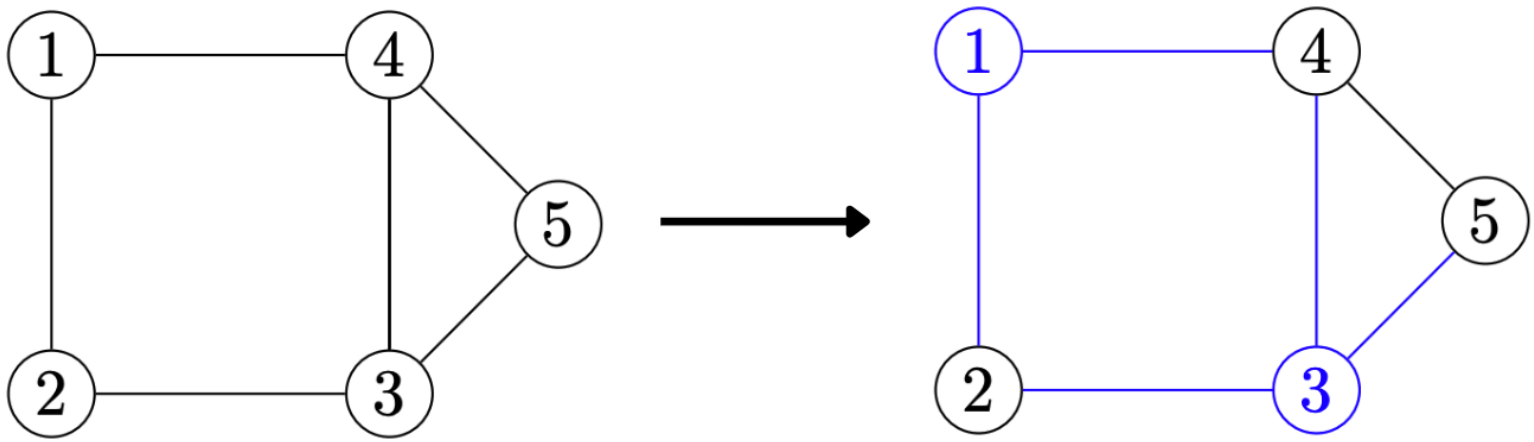
- This problem is NP-hard [5]

MAXCUT Formulation



Solution: Two partitions are $S = \{1, 2\}$ and $T = \{2, 4, 5\}$. The size of the cut is 5.

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- We can assign a qubit to each decision variable, make a the following substitution $x_i = \frac{(I - Z_i)}{2}$. This gives us the resulting encoding for a problem Hamiltonian H_c that can be encoded onto a quantum circuit.

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$$H_C = \sum_{i,j=1}^{n} \frac{1}{4} Q_{ij} Z_i Z_j - \sum_{i=1}^{n} \frac{1}{2} \left(c_i + \sum_{j=1}^{n} Q_{ij} \right) Z_i + \left(\sum_{i,j=1}^{n} \frac{Q_{ij}}{4} + \sum_{i=1}^{n} \frac{c_i}{2} \right)$$

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$$|\psi(\boldsymbol{\beta}, \boldsymbol{\gamma})\rangle = \underbrace{U(\boldsymbol{\beta})U(\boldsymbol{\gamma}) \cdots U(\boldsymbol{\beta})U(\boldsymbol{\gamma})}_{p \text{ times}} |\psi_0\rangle$$

- Where $U(\beta) = e^{-i\beta H_b}$ and $U(\gamma) = e^{-i\gamma H_c}$

QAOA Algorithm

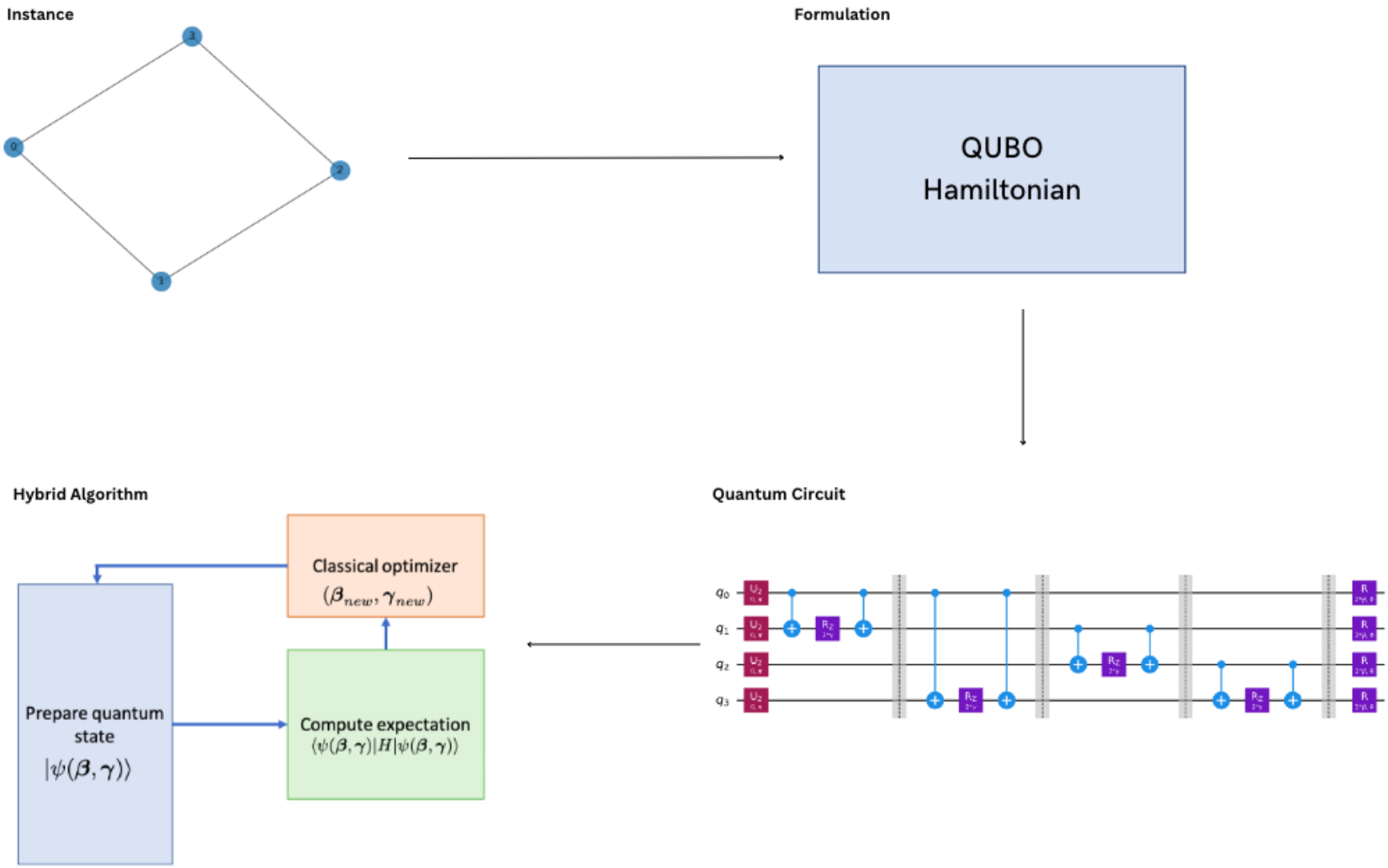
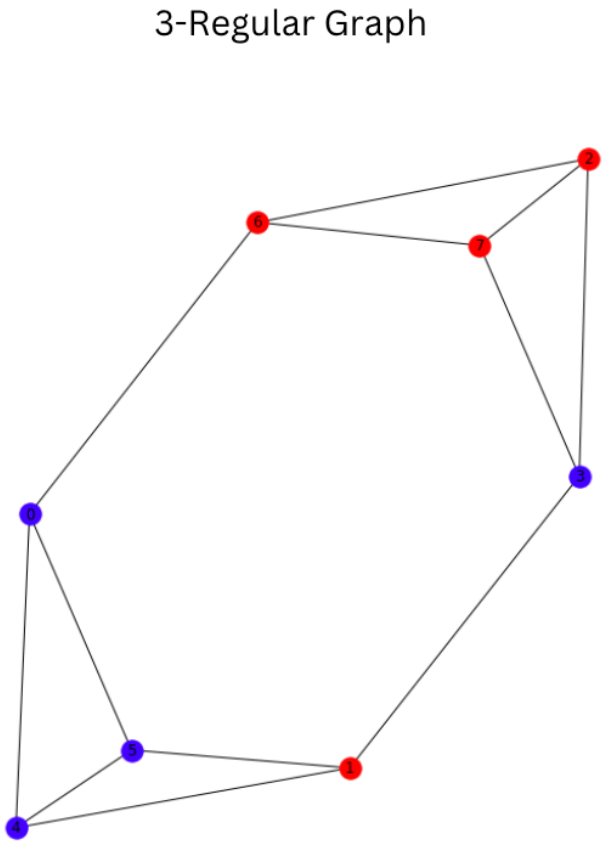
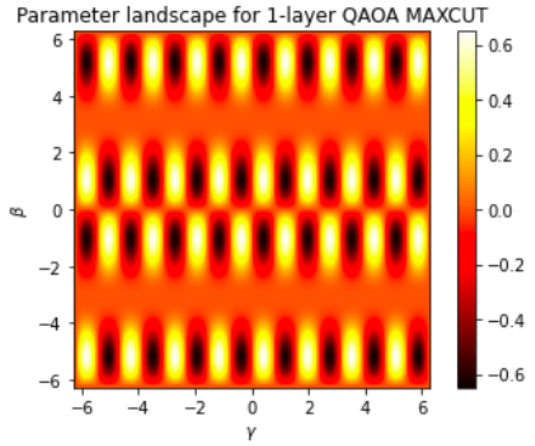


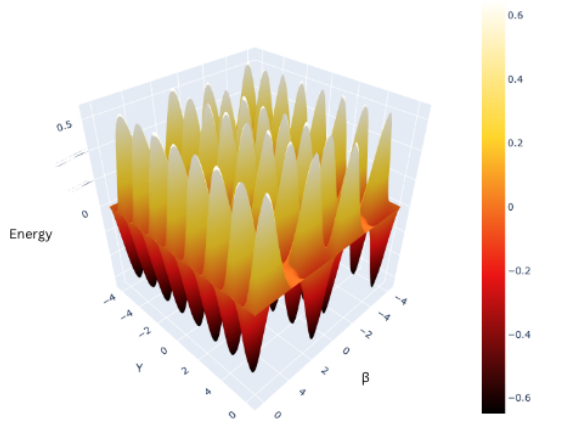
Figure 1: QAOA Strategy

Landscape for $p = 1$

- With QAOA we observe periodicity in γ and β







Applying ISA in the Quantum Context

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- **Can we measure characteristics of instances that make specific algorithms more suited to solve certain graphs?**

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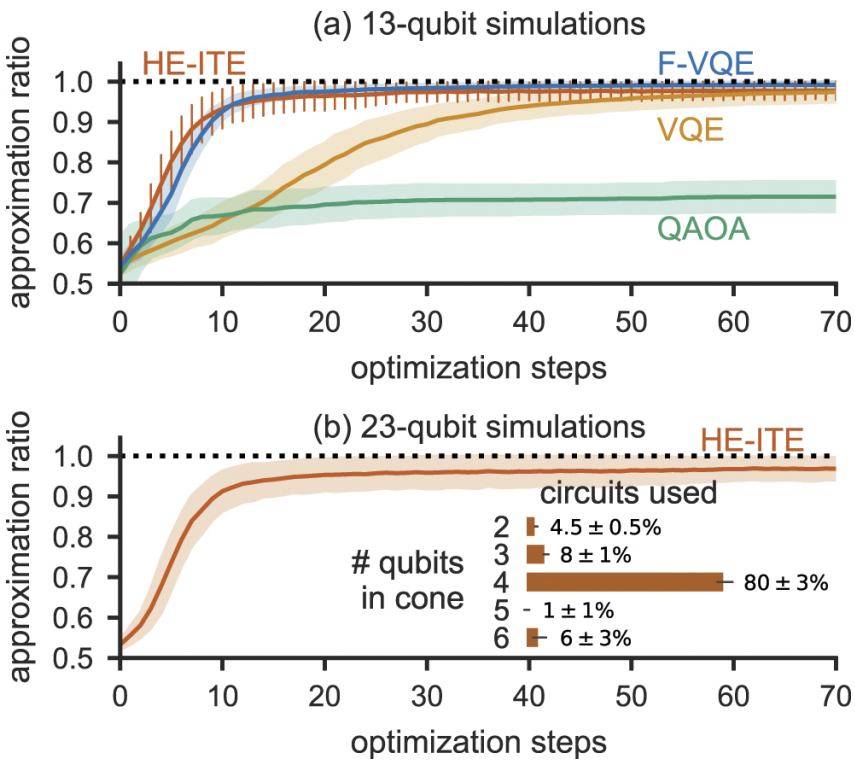

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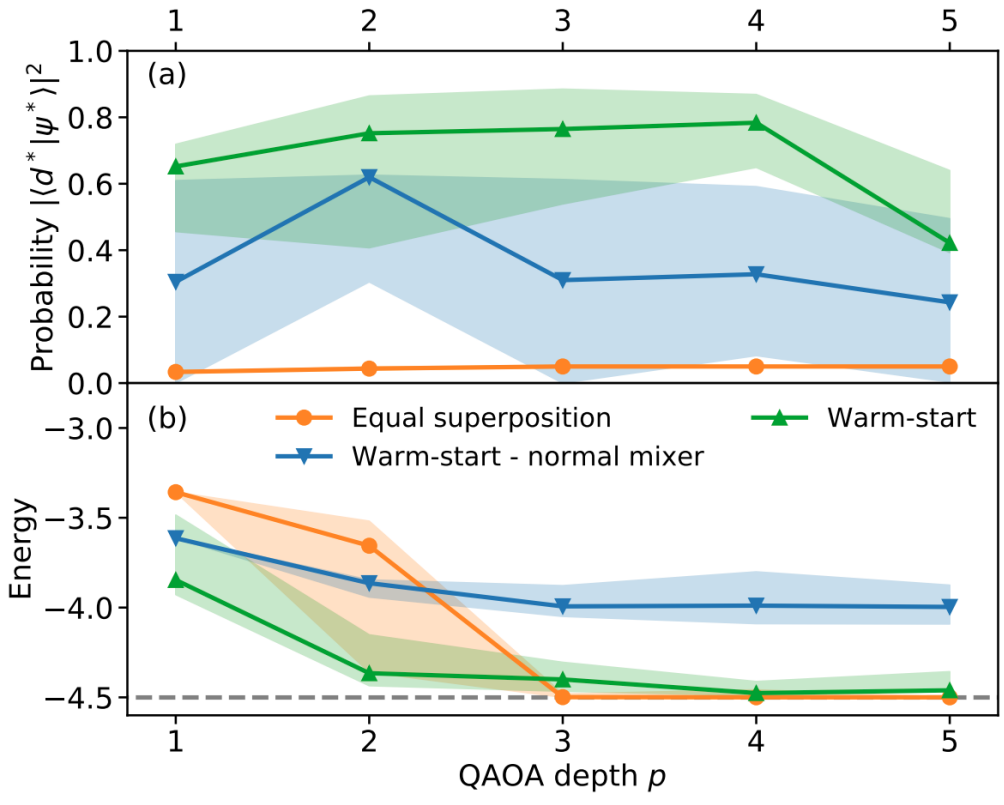

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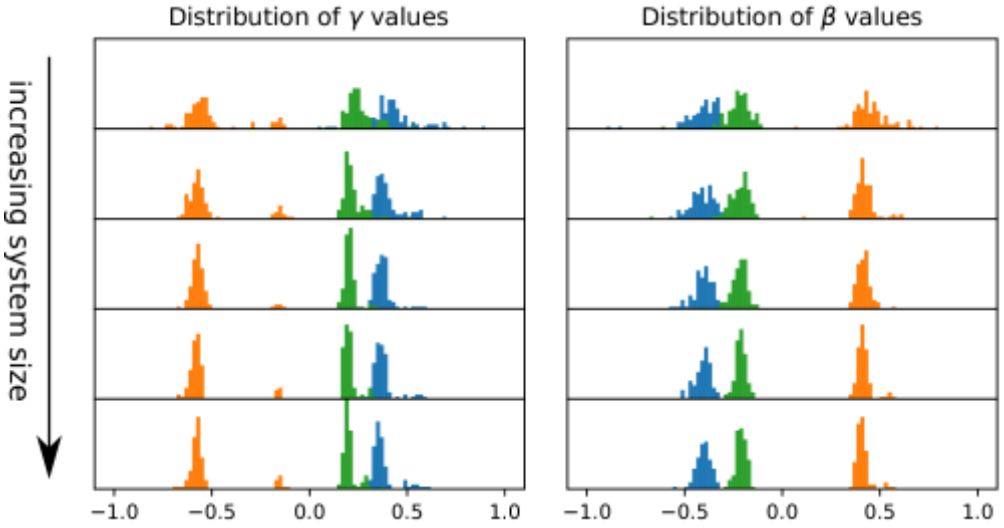

- **Do new developments in Quantum Algorithms in current literature extend for all classes of instances?**

Instance Space Analysis

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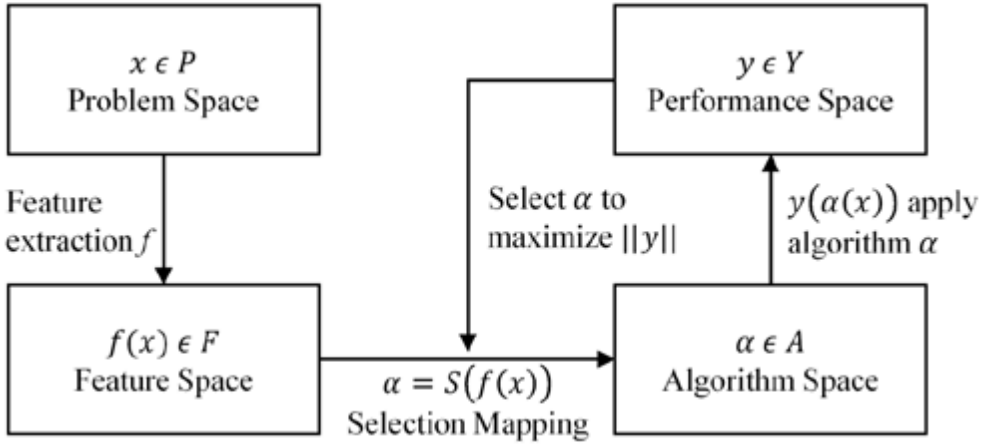

- However, what we are interested in is probing the strengths and weaknesses of Quantum Algorithms for different instances of MAXCUT, TSP and 3SAT.

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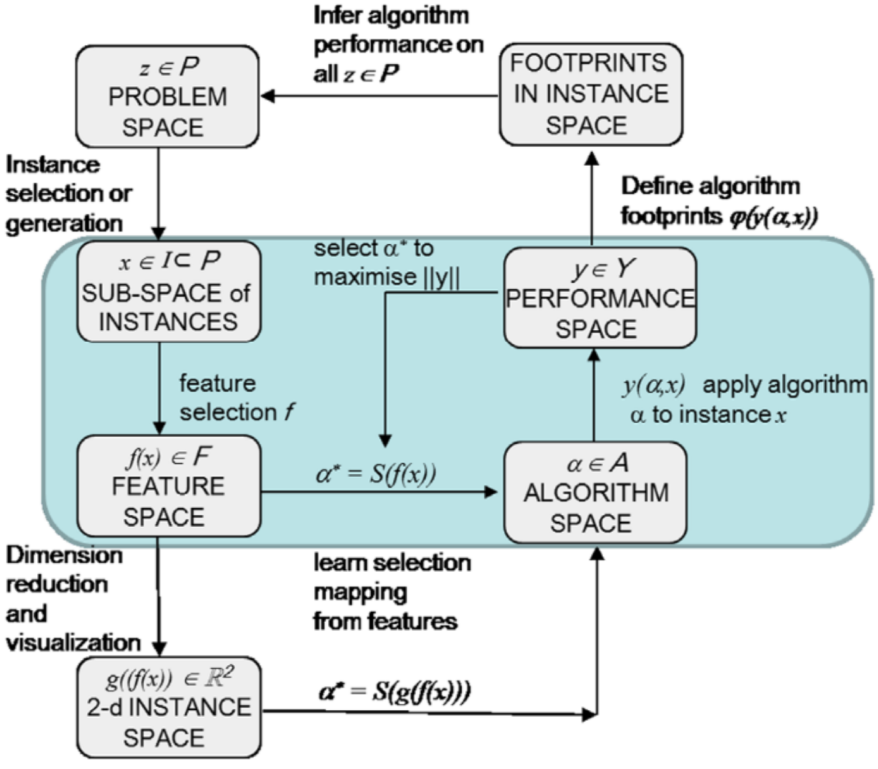

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Metadata: Features

- The feature space $\mathcal{F}$ consists of 21 different instance features generated 21 features were considered

Graph Based Features	**Boolean Features**
Average Distance	Bipartite
Clique Number	Connected
Algebraic Connectivity	Acyclic
Diameter	Eulerian
Edge Connectivity	Regularity
Vertex Connectivity	Planar
Maximum Node Degree	**Laplacian (Spectral Features)**
Minimum Node Degree	Largest Eigenvalue
Cardinality of minimal dominating set	Laplacian Second Largest Eigenvalue
Number of Components	Smallest Eigenvalue
Number of vertices	
Radius	

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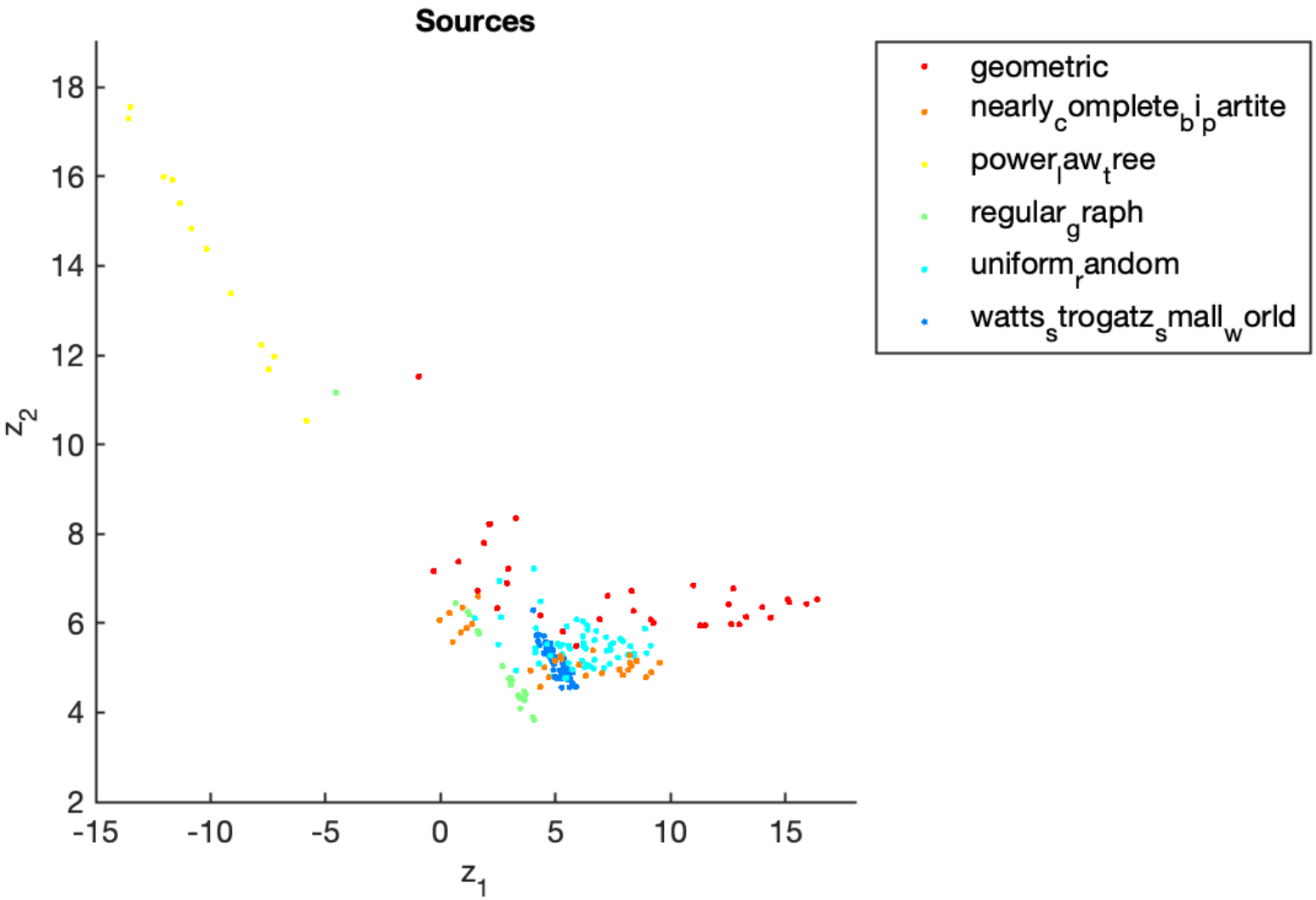
- The performance metric $y \in \mathcal{Y}$ is energy gap ($E_{\text{gap}} = \frac{\langle\psi|H|\psi\rangle}{\langle\psi_g|H|\psi_g\rangle}$)

Results

ISA


Sources
geometric
nearly_complete_bipartite
power_law_tree
regular_graph
uniform_random
watts_strogatz_small_world
z2
z1
18
16
14
12
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8
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2
-15
-10
-5
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15

Feature Distribution

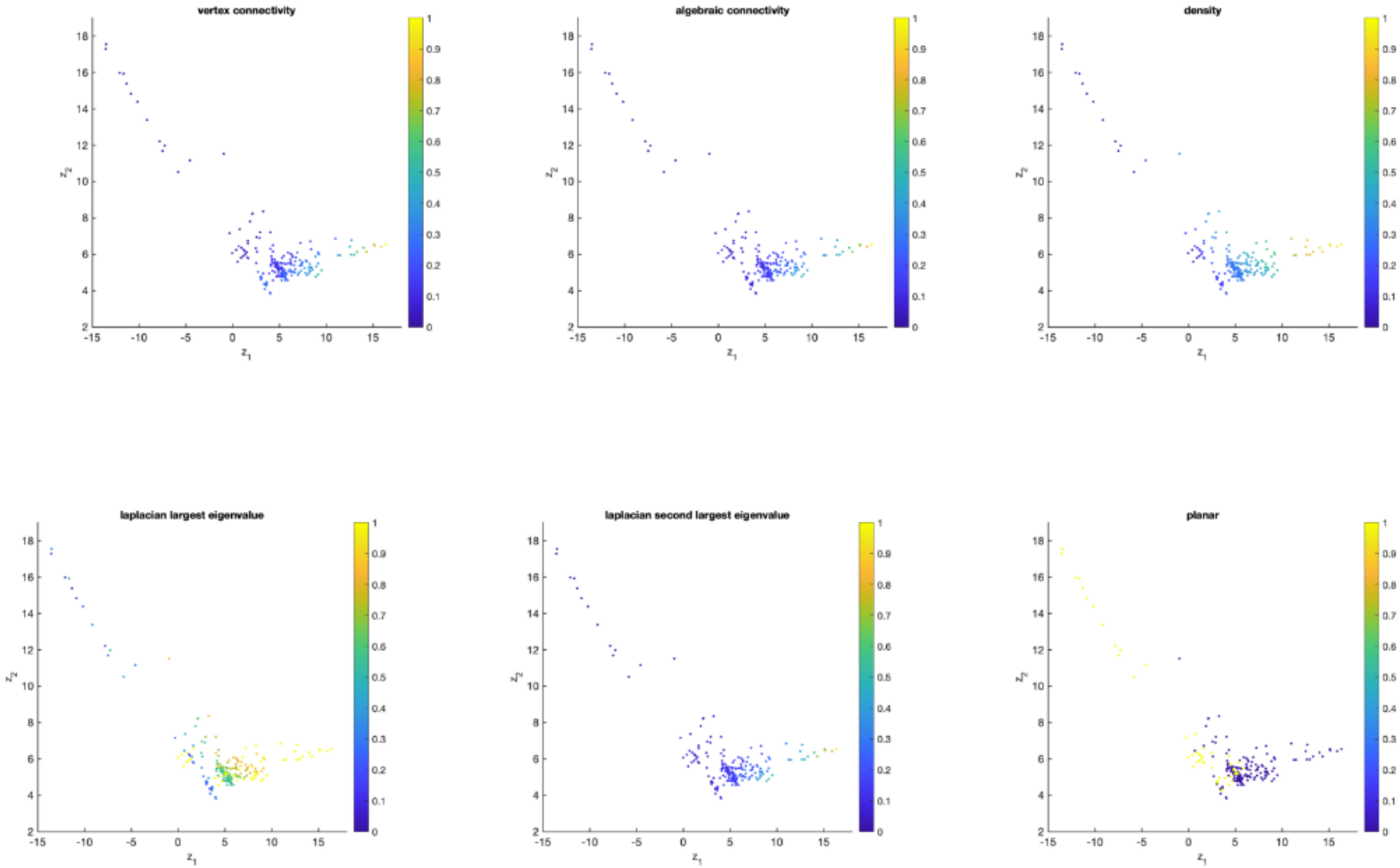
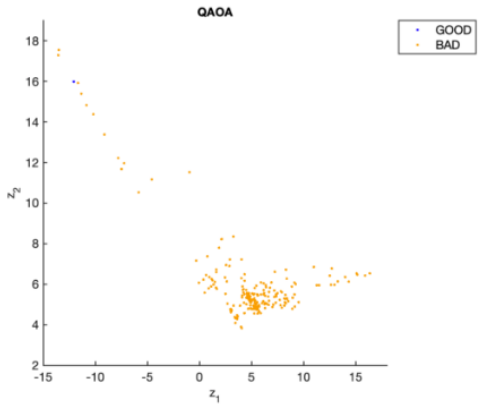
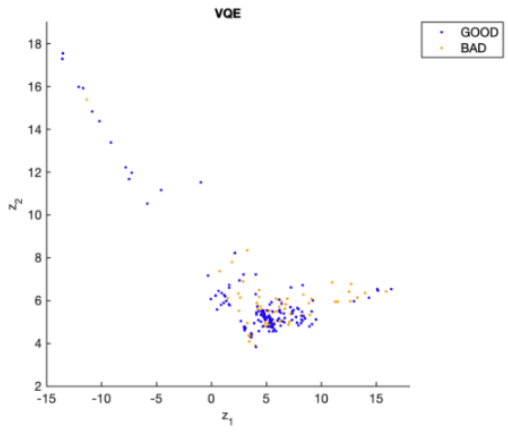
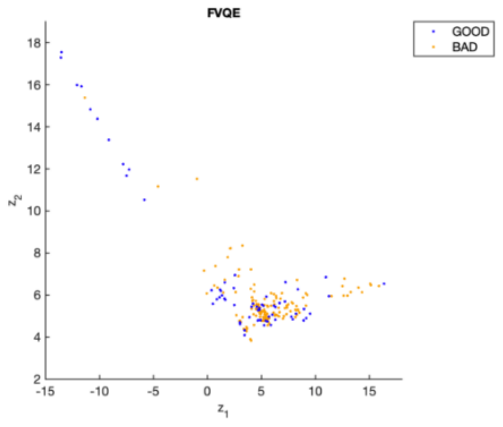


Figure 3: Features

Performance Distribution







Vanilla QAOA, increasing layers

- These experiments are still currently running on SPARTAN

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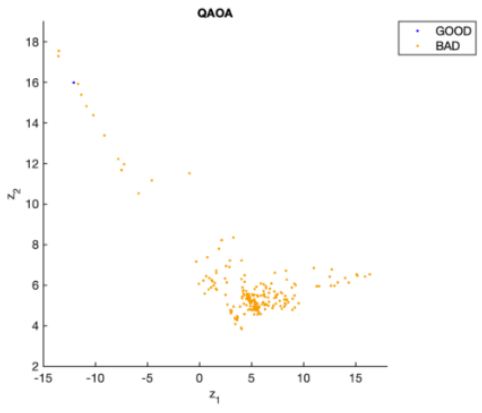

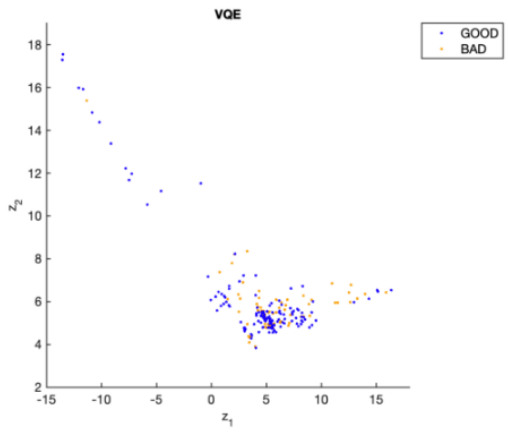

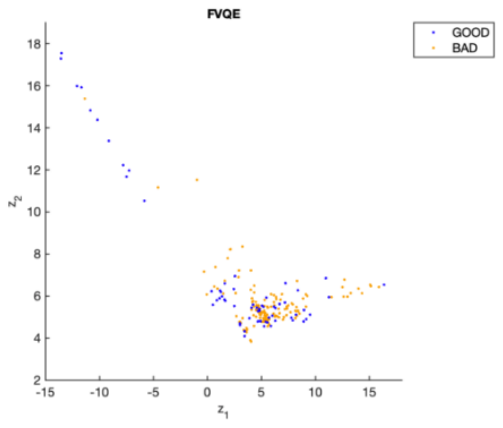

Vanilla QAOA, increasing layers

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- Run a vanilla QAOA instance for layers $p = 1, ..., 10$

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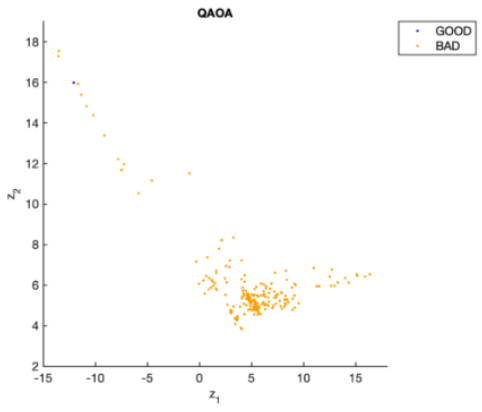

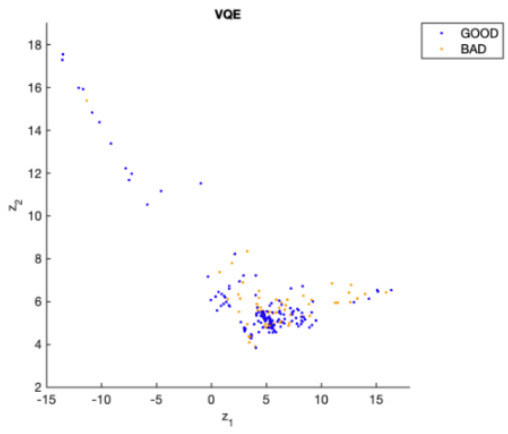

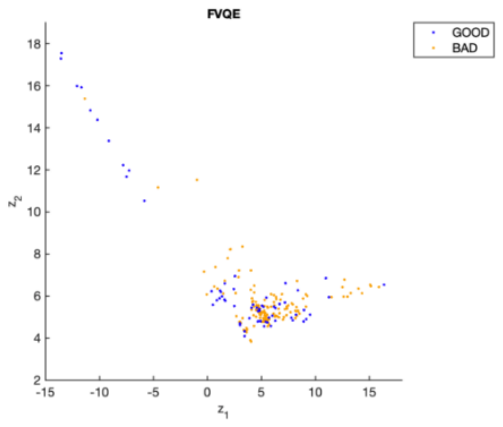

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  - 5 restarts

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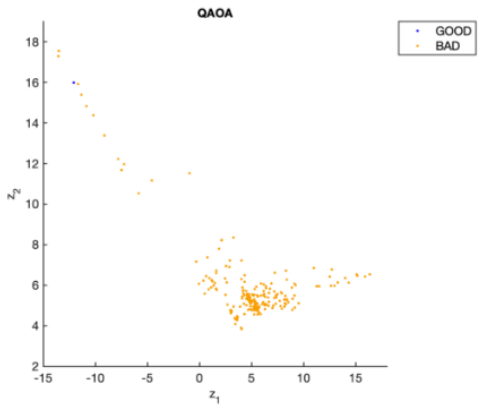

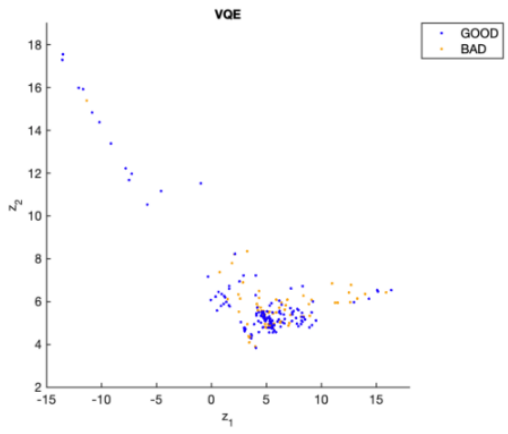

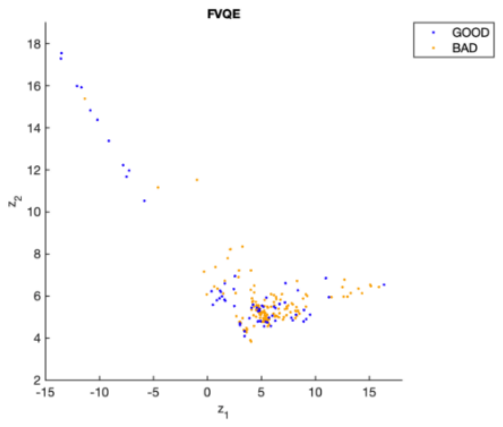

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Initialisation Techniques of the ground state (QAOA)

- Run a vanilla QAOA instance for layers $p = 7$ and a classical optimizer of `COBYLA`
- Trotterized Quantum Annealing (TQA) [17]
- Random Initialisation
- Perturb from previous layer
- Ramped up Initialisation,
- Optimise γ and β in the Fourier Space (FT)

Thesis Outline

-Thesis Structure - Introduction - Solving Optimisation Problems using a Quantum Computer - Quantum Algorithms Studied - Instance Space Analysis - ISA: MAXCUT - ISA: TSP - ISA: 3SAT - Conclusion

Discuss thesis outline

- Here is a link to the thesis outline

Next Steps

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Appendix

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$$\min_{\pi} \sum_{i=1}^{n} d(c_{\pi_i}, c_{\pi_{i+1}})$$

TSP Example

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- This QUBO can then solved by a quantum algorithm

TSP mapping to QUBO

Assign a qubit to each decision variable, make a the following substitution

$$x_{l,t} = \frac{(I - Z_{l,t})}{2}.$$

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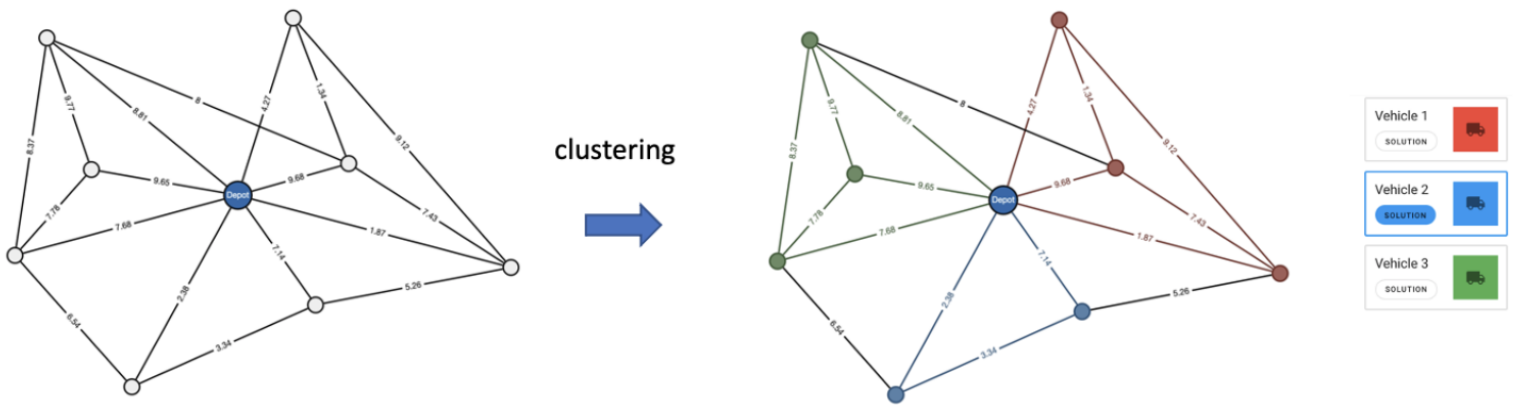

3SAT - Exact Cover

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- A clause is an expression which the variables must satisfy. For example $z_1 \wedge z_2 \implies z_1 = z_2 = 1$

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- Exact Cover implies that clauses are "exclusive" and in the form of $z_i + z_j + z_k = 1$

Mapping 3SAT to a Hamiltonian

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- Finally, complete measurement and our solution is our final state $|\psi(t = T)\rangle$

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