Nonlinear Dyn manuscript No.
(will be inserted by the editor)

Role of network topology in noise reduction using coupled dynamics

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Received: date / Accepted: date

Abstract We study the usefulness of coupled redundancy as a mechanism for reduction of local noise in coupled map lattices and investigate the role of network topology, coupling strength and iteration number in this mechanism. Explicit numerical simulations to measure noise reduction in coupled units connected in different topologies such as ring, star, small world, random and grid networks have been carried out. We study both symmetric as well as asymmetric networks. Linear stability analysis is presented to identify an optimal symmetric topology. The effect of rewiring is also investigated and we find that dynamic links enhance the noise reduction capabilities.

Keywords Network, Topology, Noise, Coupled map lattices, Coupled Dynamics

Acknowledgements We gratefully acknowledge support from the Office of Naval Research under Grant No. N00014-12-1-0026 and STTR grant No. N00014-14-C-0033.

1 Introduction

Coupling can serve as noise control mechanism in coupled dynamical systems [5, 32, 33, 4, 15]. If the systems are nonlinear then using redundancy and linear averaging may not yield a meaningful signal [33, 25]. Bouvrie et al. have shown that synchronization due to coupling between redundant nonlinear dynamical systems enhances the noise robustness and plays an important part in neural learning and decision making [4]. Zhai [38] found that the disturbance attenuation performance in a network of nonlinear systems can be enhanced by maximizing the second smallest eigenvalue of the graph Laplacian when feedback gain is positive or by minimizing the largest eigenvalue of the graph Laplacian when feedback gain is negative. Coupling of dynamical units results in reduction of deviations caused by white as well as colored noise in globally coupled maps (GCM) [15, 19]. In other words, GCM evolving in presence of local noise suffer lower deviations from noise free evolution as compared to isolated maps evolving in same environment. Such redundant coupled systems act as averaging filter and enhance the noise robustness of dynamical systems without use of any additional averaging hardware. This coupling based noise reduction can be useful in applications based on chaos or nonlinear systems [31, 3] including but not limited to computing [14, 16, 28, 17, 34], communications [29, 35, 1, 7], harmonic oscillation generation [21], microfluidic mixers [30], biological systems [22, 13, 24], and social systems [10]. A practical application of this phenomenon was demonstrated in chaos computing where it was shown that for fixed noise variance, the number of errors in chaos computing reduces if some chaos computing units are coupled together. In other words, if the number of errors that can be tolerated is fixed, then the maximum noise variance in which our system can operate, increases by a factor equal to the number of coupled units [14].

GCM are appropriate for situations where the number of coupled units N is small. As N increases, the

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number of connections or links that are needed increases as $O(N^2)$. Such large number of links may not be at all feasible in practical applications or they will be prohibitively costly. In this paper, we explore different network topologies to identify which topology will result in better noise control when the total number of nodes or dynamical units and the number of connection or links to couple them is constrained. We investigate symmetric as well as asymmetric topologies. In symmetric networks, noise reduction is same for all nodes whereas noise reduction may be different for different nodes in an asymmetric network and by choosing the maximally robust node for output measurement, a better noise control can be achieved with lesser number of links and/or nodes.

In the next section we discuss an asymmetric topology, namely the star network and evaluate the noise reduction at different nodes. In section III, we discuss the symmetric topologies where noise reduction is the same for all the nodes and output can be measured from any of the nodes. We also present the linear stability analysis of such networks and present conditions for maximum noise reduction. We present our results for the dynamic or switching networks in section IV and provide a summary of results in the last section.

2 Non-Uniform Noise Reduction

In this section, we investigate the noise robustness in a star network. In such a network, there is one central node which is connected to all other nodes and the other nodes are peripheral nodes as they are not connected amongst each other directly but only via the central node. Such a topology is useful when output can be measured from a particular node and the goal is to maximize the noise reduction for that particular node. Noise reduction for other nodes can be equal to or less than the noise reduction at the selected node. A star topology can be represented as:

$$x_i(t+1) = (1-c)f(x_i(t)) + cf(x_0(t)) + \sigma\delta_i(t), \quad (1)$$

$$x_0(t+1) = (1-c_0)f(x_0(t)) + \frac{c_0}{N-1}\sum_{i=1}^{N-1} f(x_i(t)) + \sigma\delta_0(t), \quad (2)$$

where $x_i(t+1)$ denotes the state variable of node i at iteration $(t+1)$ for $i = 1, ..., N-1$ and $i = 0$ is the central node, f is a nonlinear function describing the evolution of the map, c_0 $(0 \leq c_0 \leq 1)$ is coupling term for the central node, c $(0 \leq c \leq 1)$ is coupling for peripheral nodes, σ^2 is the noise variance representing noise strength of the independent and identically distributed additive white Gaussian noise so that $\sigma\delta_i(t)$ represents the instantaneous noise term at i^{th} node at iteration t.

As an explicit example we consider a specific functional form of $f(x)$, namely the quadratic map, and evaluate noise reduction ability of coupled map lattice (CML) by explicit calculation of deviations due to noise. The quadratic map is given by:

$$x(t+1) = 1 - a\,(x(t))^2\,. \quad (3)$$

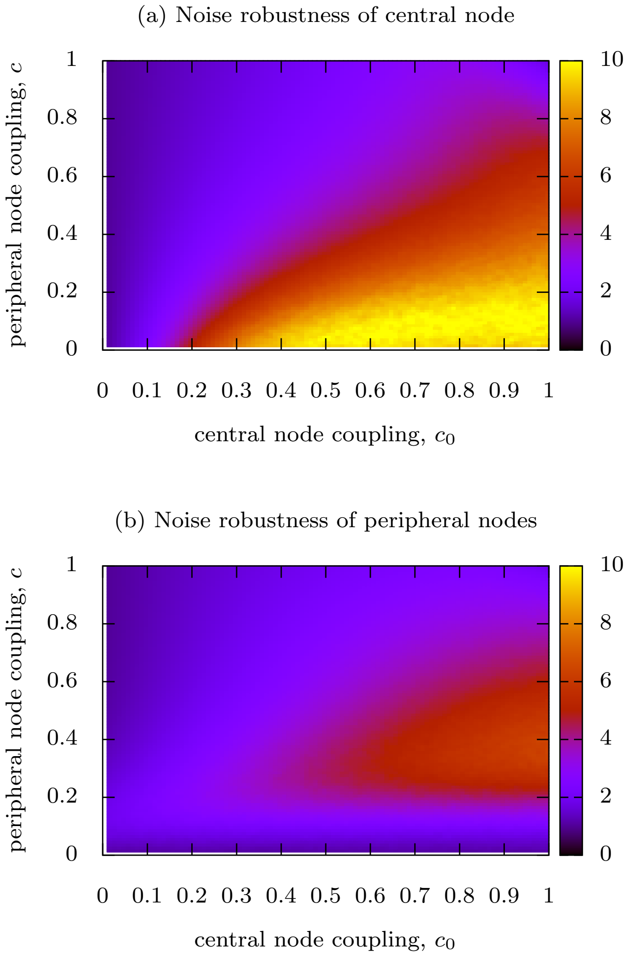

Fig. 1 Noise robustness (color scale) for quadratic maps on a star lattice for different coupling terms for (a) central node and (b) peripheral nodes. Here $N = 10$, $x = 0.4$, $t = 6$ and $\sigma^2 = 10^{-10}$. Variance of deviations from noise free evolution for CML ($\sigma_c^2(t)$) and for single map ($\sigma_s^2(t)$) were computed for 10^4 different noise realizations. Noise robustness was calculated by taking the average of ratio of these variances. Similar results were obtained for other initial conditions in the interval $(-1, 1)$.

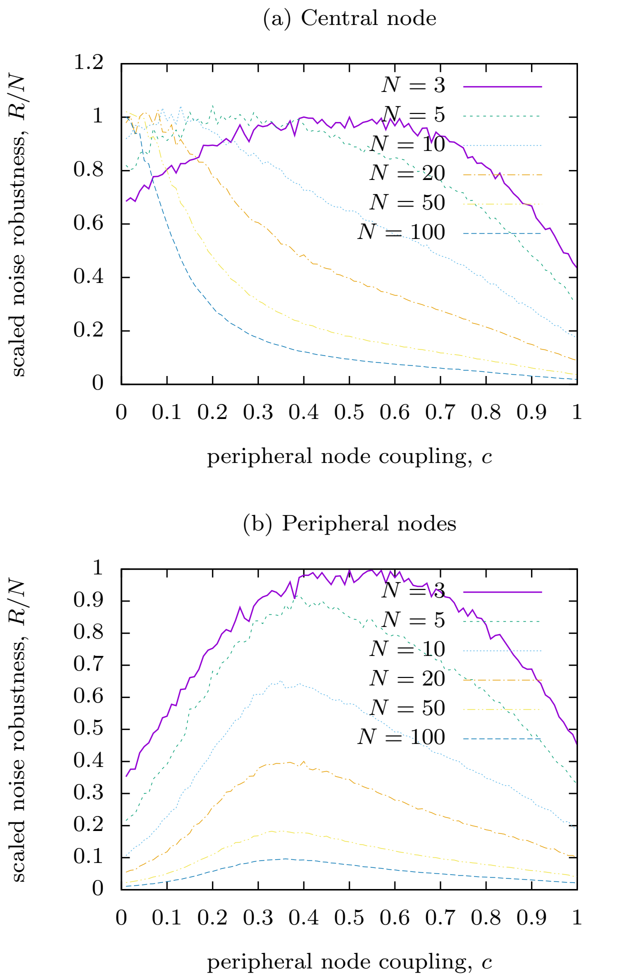

Fig. 2 Noise robustness R for (a) central node and (b) peripheral nodes scaled by the number of nodes N for quadratic maps on a star lattice for different coupling strength for peripheral nodes. Here $c_0 = 1$, $x = 0.4$, $t = 6$ and $\sigma^2 = 10^{-10}$.

This map is chaotic in the interval $x^i \in [-1, 1]$ for $a = 2$. To understand how noise affects the evolution of this system, we initialize all the nodes of the coupled system with the same initial condition and simulate the system given by above equations for t iterations. As the noise at a particular node is independent of noise terms at other nodes, the noise terms will vary at different nodes. We compare the final state of this system with a single map or an uncoupled system such that $c = c_0 = 0$ evolving in noiseless environment ($\sigma = 0$). This difference between the final state of a coupled system evolving in noisy environment from that of an uncoupled system in noiseless environment is called the deviation of the coupled system Δ_c. We also compare the final state of uncoupled system evolving in noisy environment to that of an uncoupled system in noiseless environment and call this as the deviation of a single map Δ_s given by

$$\begin{aligned}\Delta_s(t+1) = f(...f(f(x + \sigma\delta(0)) + \sigma\delta(1))... \\ +\sigma\delta(t)) - f(...f(f(x(0))...).\end{aligned} \tag{4}$$

We calculate Δ_c and Δ_s for different realizations of noise and calculate the variance of these deviations. Each time we initialize the map with a specific initial condition $\in (-1, 1)$ and calculate the deviations Δ_c and Δ_s for that initial condition after t iterations. Then we calculate the variances of Δ_c and Δ_s and call these variances as $\sigma_c^2(t)$ and $\sigma_s^2(t)$ respectively. Next, we calculate the ratio of these variances which is defined as noise robustness, R at iteration t.

$$R(t) = \frac{\sigma_s^2}{\sigma_c^2}. \tag{5}$$

This averaged R for the star network with $N = 10$ at $t = 6$ is shown in Fig. 1 for different c_0 and c. Similar $R(6)$ is obtained for other initial conditions in the interval $(-1, 1)$ which are not the pre images of the fixed point. If a pre image is selected as starting initial condition, then the linearization about that point will not be valid as first derivative will be zero at that point. In fact the noise robustness for such initial conditions increases nonlinearly with the number of nodes in the lattice. These results will be discussed elsewhere.

We also observe that the central and peripheral nodes have different noise robustness. R for central node approaches the theoretical maximum of N for sufficiently wide window of c_0 and c whereas R for peripheral nodes remains less than N. This can be expected because the central node is connected to all other nodes and thus all nodes contribute to its noise robustness. Highest noise robustness is obtained when $c_0 = 1$. R is observed to be maximum when $c_0 = 1$ for other values of N also. Thus it is clear that very high noise robustness can be achieved with just $N - 1$ links in a star network. Notice that $c_0 = 1$ is really the mean-field coupling of the peripheral nodes which has been extensively studied in literature [12].

Now we fix $c_0 = 1$ and investigate how changing N and c impacts R. In this case the central node averages the peripheral nodes. The results are shown in Fig. 2 and we observe that c for which R is maximum depends on the number of nodes in the array. As the number of nodes increases, the optimal c decreases such that optimal $c \sim 1/(N-1)$. At optimal c, R approaches N for the central node. The noise reduction in peripheral nodes is much smaller as compared to central node and increasing the number of nodes doesn't translate to corresponding increase in noise robustness.

3 Uniform Noise Reduction

If noise reduction is desired for all the nodes then one has to look for other topologies. Further, in a star network, the central node connects to $N-1$ nodes which may not be practically feasible as the maximum number of links per node may be limited. For such cases, we study the noise robustness in common topologies like the ring, small world, random and grid networks. Similar to the previous section, we consider the network of CML represented as:

$$x_i(t+1) = (1-c)f(x_i(t)) + \frac{c}{\sum_j A_{ij}} \sum_{j=1}^{N} A_{ij} f(x_j(t)), \quad (6)$$

where $x_j(t+1)$ denotes the state variable of node i at iteration $(t+1)$ for $i = 1, ..., N$, f is a nonlinear function describing the evolution of the map and c is coupling strength such that $0 \leq c \leq 1$. The topology of the CML is expressed in the form of adjacency matrix A whose elements $A_{ij} = 1$ if nodes i and j are connected and 0 otherwise.

First we investigate the noise reduction capability of Watts-Strogatz (WS) [36] networks. To construct these networks, initially all nodes are connected to their k nearest neighbors on either side such that each node is connected with $2k$ neighbors. Then each link is rewired to some distant random node with probability p. By varying p between 0 and 1 we can obtain ring topology (for $p = 0$), small-world (SW) topology (for low p) and random networks (for high p). In our explicit example, we fix $N = 100$ and study the effect of rewiring probability p, coupling parameter, c, number of neighbors k and number of iterations i.

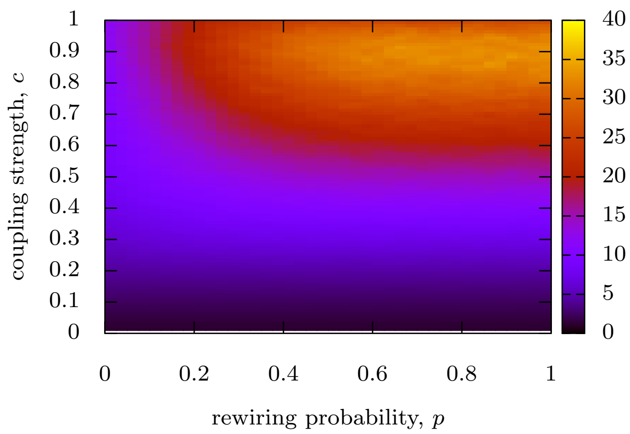

Fig. 3 Noise robustness R for WS networks for different coupling strength, c and rewiring probability p for $N = 100$ and $k = 2$.

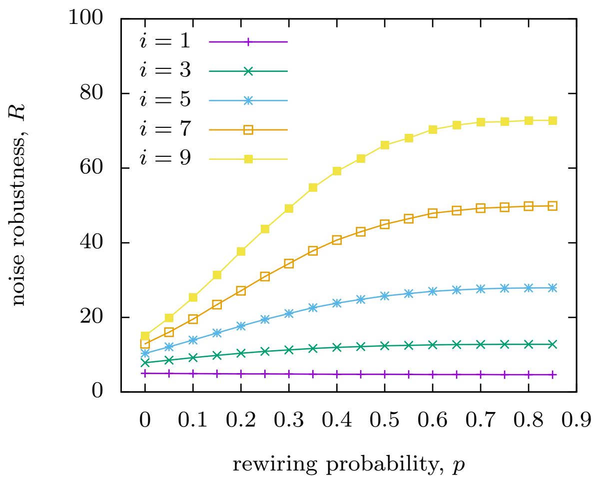

Fig. 4 Noise robustness R for WS networks for different iterations, i and rewiring probability p for $N = 100$, $c = 0.9$ and $k = 2$

As described in the previous section, we evolve the quadratic map in presence of noise with variance $\sigma^2 = 10^{-10}$ for 10^4 different noise realizations and find variances for these 10^4 noise realizations. We repeat the same procedure for coupled maps and find the variance of deviations observed in coupled maps. The ratio of these variances is the noise robustness R. The R for WS networks for different coupling strength and rewiring probability p for $N = 100$ is plotted in Fig. 3. We see that R is maximum when p is large. If c is large, then R increases rapidly with increase in p. For low c, increasing the randomness or p has no or little effect on R. Only when c is sufficient, the network topology comes into play and random networks result in higher noise robustness compared to ring or small world networks. We also observe that R is much lower than N.

We also studied the impact of iteration number, i on the noise robustness as shown in Fig. 4 and found that the maximum R increases with increase in iteration number i.

Then we examined the effect of different number of neighbors in the WS networks and the noise robustness for different iteration number and number of neighbors is shown in Fig. 5. In general, noise robustness increases with increase in number of neighbors. For first iteration we see that noise robustness $R \sim 2k$ for all k. For higher iteration numbers, noise robustness increases sharply as the number of neighbors is increased.

These observations can be understood in terms of path length of the network. It was shown in [36] that average path length decreases sharply with increase in p. In our case, one can argue that the number of nodes for which the path length is less than or equal to iteration number are the ones that contribute to reduction

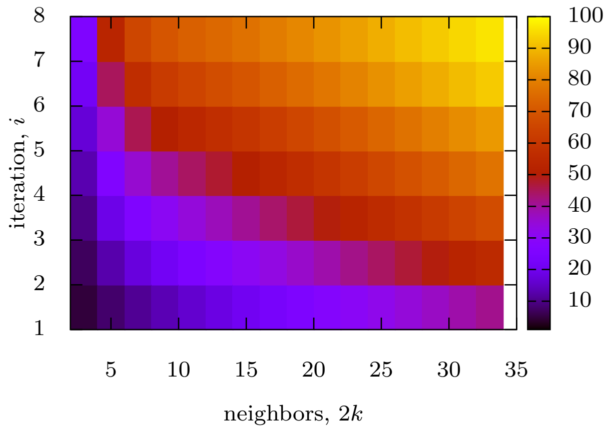

Fig. 5 Noise robustness (color scale, R for WS networks with different number of neigbors, $2k$ and iteration number i. Here $N = 100$, $c = 0.9$ and $p = 0.8$.

in noise levels. With each iteration, the noise diffuses over the nearest neighbors and so the total number of nodes over which the noise diffuses will be related to the path length of the network. As p increases, path length decreases and consequently higher noise robustness is observed. Similarly, at higher iteration numbers, nodes with higher path length also contribute in reduction of noise levels. Further, as the coupling strength increases, the contribution of the connected nodes becomes more significant resulting in higher noise robustness. Similarly, an increase in number of neighbors decreases the average path length.

3.1 Time varying topologies

Several studies have demonstrated that time varying connections result in faster diffusion of information and the network behaves like an averaged network of different individual network realizations if the links are rewiring very fast [11, 37, 6, 20, 18, 23, 8]. Such temporal variation in topologies can help us to obtain similar performance metric as that of a GCM network but with very few links. We studied the effect of time varying connections in the WS network. To implement the time varying network, we simply generated a new configuration of the WS network at each iteration with same p. This means that at every iteration, the number of random links and nearest neighbor links is same but whether a particular link will be random or not can change after every iteration. If a link is common in previous and new configuration, that implies that the link did not rewire. The links which were in previous configuration but are not in new configuration are the ones that have been rewired. The results of noise robustness for such rewiring networks are presented in Fig. 6. We observe that noise robustness increases considerably in case of time varying networks and high noise robustness can be achieved with very few links. Previous studies have demonstrated that for sufficiently fast rewiring, the rewiring network can be approximated by a network obtained by taking the weight of connections equal to time average of the connections. To verify if this holds true for noise robustness, we created a weighted GCM and measured R for such a network. We constructed a network obtained by equating the elements of the connectivity matrix by the ensemble average. For WS network the averaged network is represented by a network in which the entries corresponding to nearest neighbors are $(1 - p)$ and those to distant nodes are $2kp/(N - 2k - 1)$. The noise robustness for such networks with different p is also plotted in Fig. 6 indicating that R in such time averaging is indeed close to that of the rewired networks. Larger iteration numbers result in better convergence of two as more iterations result in better time averaging. We also studied slower rewiring in which links do not rewire after every iteration but only after a fixed number of iteration. We observed that faster rewiring results in higher noise robustness. Thus a network in which links can rewire after every iteration shows a higher noise robustness as compared to one in which links can rewire after two or more iterations.

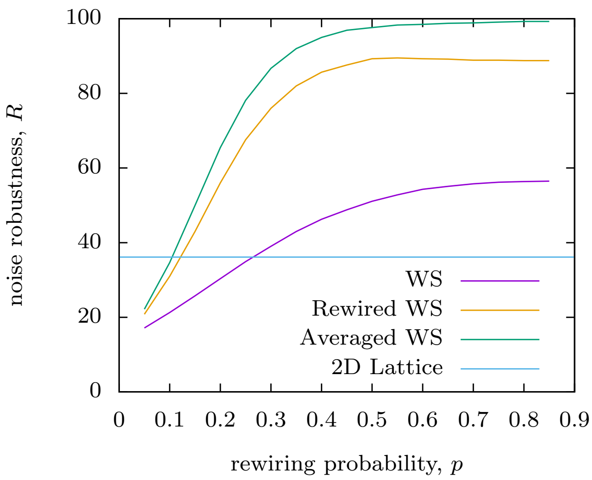

Fig. 6 Noise robustness for static, time varying and time averaged WS networks for different rewiring probability p. Noise robustness for 2D lattice is also plotted. Here $N = 100$, $c = 0.8$, $i = 8$ and $k = 2$.

Lastly we calculate the noise robustness for $2 - D$ lattice network in which each node is connected to its left, right, top and bottom neighbors. Noise robustness for such a network is also shown in Fig. 6.

3.2 Linear Stability Analysis

By linearizing the coupled system about the initial condition, Kia et. al [15] have shown that when the dynamical units are globally connected, noise robustness increases by a factor equal to the number of units N that are coupled. In this section, we present the linear stability analysis of the configurations studied in this paper and show how we can identify an optimal topology through linear stability analysis. Let $\mathbf{x} = [x_1, x_2, ..., x_N]^T$, $\mathbf{F}(\mathbf{x}) = [f(x_1), f(x_2), ..., f(x_N)]^T$, then, Eq. 6 can be written as,

$$\mathbf{x}(t+1) = \mathbf{G}\mathbf{F}(\mathbf{x}(t)) \tag{7}$$

where $\mathbf{G}$ is a N dimensional matrix such that

$$G_{ij} = (1-c)I_{ij} + \frac{c}{\sum_j A_{ij}} A_{ij}, \tag{8}$$

where I is an identity matrix of dimension N. The row sum of the elements of matrix $\mathbf{G}$ is 1. In presence of local noise, the dynamics of the coupled systems is given by:

$$\mathbf{x}(t+1) = \mathbf{G}\mathbf{F}(\mathbf{x}(t) + \delta(t)), \tag{9}$$

where $\delta = [\delta_1, \delta_2, ..., \delta_N]^T$ is the noise at different nodes at time t.

When the CML is used for noise reduction, all the nodes are initialized to same initial condition. Then each map evolves in presence of local noise and its state at next iteration depends on the instantaneous noise term at that iteration and the contribution from nodes connected to it. The instantaneous noise is different at different nodes and thus coupling of the maps results in averaging of noise terms. As all the nodes are initialized with same initial condition, the CML can be considered in a synchronized state and the local noise present at a node can be considered as perturbation of that node. One can consider the mechanism of noise reduction similar to analysis of stability of synchronized state of coupled dynamical units but there are some subtle differences. Noise reduction focuses on minimizing the drift of the dynamical units from noise free behavior. This means that even the perturbations that are along the synchronization manifold are also unwarranted. In synchronization stability analysis, such perturbations along the synchronized state do not affect the stability of synchronized state but here these will result in deviations from the noise free evolution. This is also evident from our results in Section II where the central and peripheral nodes are not synchronized and still the central node exhibits very high noise robustness.

The deviation of CML at first iteration, $\Delta(1)$, from noise free evolution of isolated maps can be written as

$$\mathbf{\Delta}(1) = \delta(0)\mathbf{G}f'(x_0). \tag{10}$$

Now these deviations $\mathbf{\Delta}(1)$ can be resolved as perturbations along the synchronized state and along transversal directions. The amplitude of these perturbations is minimum when the eigenvalues of the matrix $\mathbf{G}$ are minimum[26, 27, 2, 9]. In this case, the maximum eigenvalue corresponds to the perturbations along the synchronized state. To reduce the effects of noise, ideally we would like to minimize these also. But as the row sum of elements of matrix $\mathbf{G}$ is unity, the maximum eigenvalue is always 1, irrespective of the topology. Thus any perturbations along the maximum eigenvalue do not affect the stability of the synchronized state but they result in amplification of noise effects. Further, the nodes are affected by perturbations at each iteration. Thus irrespective of topology that we consider, the effect of noise will grow exponentially and the deviations remains within bound only for limited number of iterations. The role of topology of the CML comes into play when we consider the perturbations along transversal directions and the deviations due to noise will be minimum when the second largest eigenvalue, λ_2, is minimum.

We randomly generate a configuration of the WS network for a fixed p and calculate the second largest eigenvalue of G. We repeat this process 500 times and take the average of these 500 eigenvalues as the second largest eigenvalue of WS network for that value of p. We repeat this process for different rewiring probability $p \in [0, 1]$ and coupling strength $c \in [0, 1]$ and these second largest eigenvalues of G for WS networks are plotted in Fig. 7. It can be seen that the second largest eigenvalue decreases with increase in either rewiring probability p or coupling strength c or both.

For $2-D$ lattice with 100 nodes (10×10 lattice), the second largest eigenvalue was found to be 0.9236 which is close to a WS network for $p = 0.3$ with same number of nodes. We also calculated the second largest eigenvalues for averaged network. The eigenvalues of these networks are plotted in Fig. 8. Comparing Figs. 3 & 6 with Figs. 7 & 8 respectively, we observe that a lower λ_2 results in high R indicating an inverse relationship between the two. It can be inferred that for uniform noise reduction second largest eigenvalue is indeed a good predictor for noise reduction capability of a given configuration and networks with comparable λ_2 exhibit comparable noise robustness.

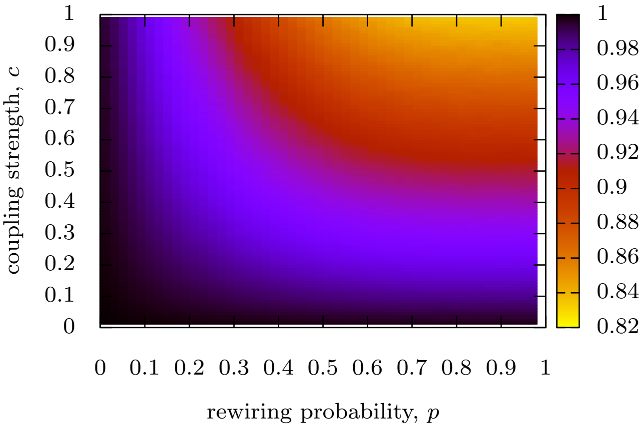

Fig. 7 Second largest eigenvalue, λ_2 represented by color scale for WS networks for different values of coupling strength, c and rewiring probability p with $N = 100$ and $k = 2$. We observe that R follows a similar trend as evident from Fig. 3

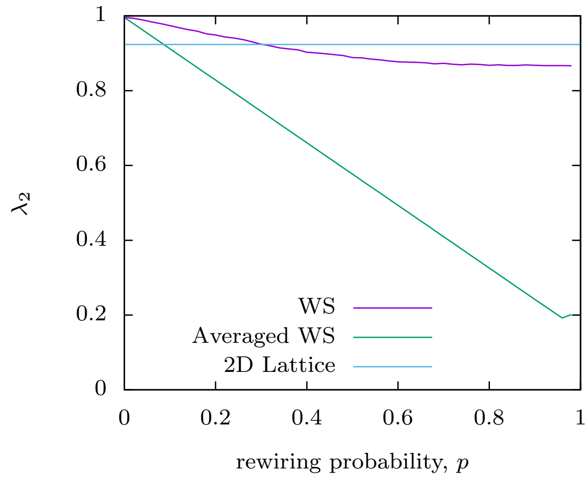

Fig. 8 Eigenvalues of static WS networks for different rewiring probability p and those of time averaged WS networks. Eigenvalue for 2D lattice is also plotted. Here $N = 100$, $c = 1$ and $k = 2$.

4 Conclusions

We studied the effect of coupling strength, iteration number and topology on the noise reduction capability of a coupled system. We found that noise reduction comparable to that of a GCM can be obtained for the central node in a star network when the coupling is asymmetric. The noise reduction is not same for all nodes and peripheral nodes have lower noise reduction. If similar noise reduction is desired for all the nodes then we found that noise robustness is higher for networks with higher randomness amongst networks having same number of links. Consequently, random networks were found to perform much better as compared to ring, small world or 2D lattices. In such cases, the second largest eigenvalue of the effective topology matrix is a good predictor for identifying optimal topology and lower values of second largest eigenvalue result in higher noise robustness.

We also studied dynamic networks and demonstrated that if the links vary with time, then much higher noise robustness can be obtained in same topologies. Similar noise reduction mechanism has been observed in continuous systems involving coupled ordinary differential equations results for which will be presented elsewhere.

Acknowledgements We gratefully acknowledge support from the Office of Naval Research under Grant No. N00014-12-1-0026 and STTR grant No. N00014-14-C-0033.

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