Enhanced Logical Stochastic Resonance under Periodic Forcing

Vivek Kohar[1], K. Murali[2] and Sudeshna Sinha[1]
[1] *Indian Institute of Science Education and Research (IISER) Mohali, Knowledge City, SAS Nagar, Sector 81, Manauli PO 140 306, Punjab, India*
[2] *Department of Physics, Anna University, Chennai 600 025, India*

It was demonstrated recently that noise in an optimal window allows a bistable system to operate reliably as reconfigurable logic gates [Phys. Rev. Lett. 102, (2009) 104101], as well as a memory device [Phys. Lett. A, 376 (8) (2012) 957-962]. Namely, in a range of moderate noise, the system can operate flexibly, both as a NAND/AND/OR/NOR gate and a Set Reset latch. Here we demonstrate how the width of the optimal noise window can be increased by utilizing the constructive interplay of noise and periodic forcing, namely noise in conjunction with a periodic drive yields consistent logic outputs for all noise strengths below a certain threshold. Thus we establish that in scenarios where noise level is below the minimum threshold required for logical stochastic resonance (or stochastic resonance in general), we can add a periodic forcing to obtain the desired effects. Lastly, we also show how periodic forcing reduces the switching time, leading to faster operation of devices and lower latency effects.

I. INTRODUCTION

The phenomena of *logical stochastic resonance* (LSR) was demonstrated recently [1, 2, 5–10]: namely, when a bistable system is driven by two inputs it consistently yields a response mirroring a logic function of the two inputs in an optimal window of moderate noise. The same LSR elements can also be morphed into memory devices [11]. Further, the functionality of the LSR elements can be changed simply by varying an asymmetrizing bias. Thus it is clear that if noise is within some optimal window then it can play a constructive role in computational devices.

Unfortunately, the noise in a system doesn't always stay at the same level. Since LSR works in an optimal range, it is possible that the noise in the system is not sufficient to drive the system, i.e. the noise is below the minimum threshold of the optimal noise window. For instance, in case of thermal noise fluctuations in the environment such as ambient temperature, or certain internal processes, such as the work load of the device, may change the level of noise present in the system. So under very weak load or in very cold environments the system may not operate robustly.

Recently, Gupta *et al* [12] showed that dynamical behavior equivalent to LSR can be obtained in a *noise-free* bistable system, subjected only to periodic forcing, such as sinusoidal driving or rectangular pulse trains. This opens up the possibility of studying the behavior of bistable elements subject to a periodic signal, as well as noise. In this paper, we will demonstrate how periodic forcing and noise interact constructively, thereby allowing us to obtain consistent logic and memory operations over a *much larger noise window*. Thus, by adding a periodic signal to a noisy nonlinear system we can obtain LSR consistently even if noise level is lower than the minimum threshold required to obtain LSR. Further, we can use two coupled bistable systems in which the output of one bistable system controls the amplitude of the periodic forcing fed into the other system. This suggests a way in which to *adaptively adjust the strength of periodic forcing depending on the noise level present in the system.*

II. GENERAL PRINCIPLE

Consider the general nonlinear dynamical system,

$$\dot{x} = F(x) + b + I + D\eta(t) + A\ f(\omega t) \tag{1}$$

where $F(x)$ is a generic non linear function obtained via the negative gradient of a potential with two distinct stable energy wells. I is the input signal which is the sum of two square pulses encoding the two logic inputs, b is bias to asymmetrize the two potential wells, $\eta(t)$ is an additive zero-mean Gaussian noise with unit variance and D is the amplitude(intensity) of noise. The functional form of the periodic forcing is f, with ω being the frequency and A being the amplitude of the forcing.

A logical input- output can be obtained by driving the system with two trains of aperiodic square pulses: $I = I_1 + I_2$, where I_1 and I_2 encode the two logic inputs. Logic output can be obtained from the state x by defining a threshold value x^*. If $x > x^*$, then the logic output is interpreted to be 1, and 0 otherwise.

III. EXPLICIT EXAMPLE

We now explicitly demonstrate this phenomena in the system given by:

$$\dot{x} = a_1(x - a_2x^3) + b + I_1 + I_2 + A\sin(\omega t) + D\eta(t) \tag{2}$$

where D is the amplitude of noise, b the asymmetrizing bias and the functional form of periodic forcing is sinusoidal where A is amplitude of the sinusoidal forcing. The parameters a_1 and a_2 control the height of the potential barrier and the location of potential minima. In absence of other terms, the height of potential barrier is $a_1/4a_2$ and the wells are at $\pm\sqrt{1/a_2}$ as shown in figure 1. Here we have taken $a_1 = 4$, and $a_2 = 5$. This function $F(x)$ is reasonably insensitive to noise and its two stable states are close to the encoded values of inputs. This helps to *cascade the gates, and feed the output directly as input, without any scaling factors.*

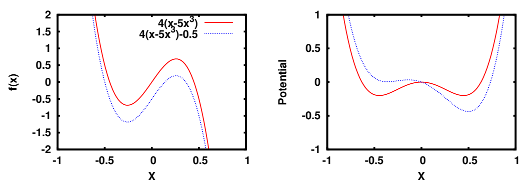

Figure 1. For the system (2): (left) the function $F(x)$ and (right) the effective potential obtained by integrating the function $F(x)$, for different bias: (a) $b = 0$ (red solid) and (b) $b = -0.5$ (blue dashed line).

The logic inputs are presented to the system with I_1 and I_2 switching levels in an uncorrelated aperiodic manner. The inputs being 0 or 1, produce 4 sets of binary inputs (I_1, I_2): $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$. These four distinct input conditions gives rise to three distinct values of I. Without loss of generality, consider the inputs to take value 0.5 when the logic input is 1, and value -0.5 when the logic input is 0. Hence, the input signal I, generated is a 3-level aperiodic wave form.

We choose 0 as our output determination threshold. If $x > 0$, i.e., when the system is in the positive potential well, then the logic output is interpreted to be 1, and 0 otherwise. Thus the logic output toggles as the system switches wells.

IV. RESULTS

We simulated the system in equation 2 for various possible frequencies and amplitude of the sinusoidal forcing and at various noise strengths. We used $b = -0.5$, so the system is biased to function as AND gate. We know that by changing the bias, we can easily switch to another logic operation. In this case, when bias is changed from -0.5 to 0.5, we obtain the OR gate. When bias is reduced to zero, we get a memory device. This effect arises from change in the symmetry and depths of the potential wells due to changing b. For brevity, we will show the results only for the representative AND gate.

We observe that for low noise strengths the system doesn't give the correct logical response in absence of periodic forcing as expected. However, as we apply some periodic forcing the system, gives the desired response. Notice that this response is obtained through interplay of noise and periodic forcing, as in absence of any one of these the system does not yield the desired response. *Only when both are present simultaneously, do we get the requisite output*, as shown in figure 2.

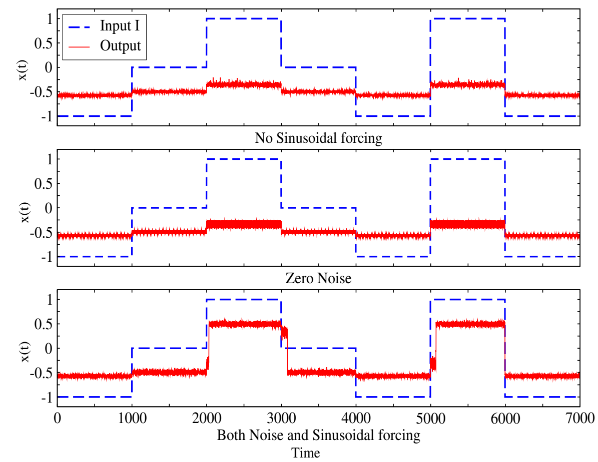

Figure 2. The panels show the waveforms of $x(t)$ obtained from simulating the system (2). In the top panel the amplitude of periodic forcing is zero and there is only noise in the system, with the noise intensity being below the minimum threshold required for LSR. In the middle panel the system is driven by periodic forcing only, and its amplitude and frequency are such that it alone can't drive the system to act as a robust logic gate. Only when there are both noise and periodic forcing, we get the desired AND gate (panel 3). The specific values of inputs used are $b = -0.5$, $D = 0.08$, $\omega = 4$ and $A = 0.5$. Dashed blue line shows the input I $(I_1 + I_2)$ and red line shows x(t).

So when the noise level is low, it is not sufficient to induce the desired switch from one well to the other. Similarly, for high frequency or low amplitude, the sinusoidal forcing cannot drive the required hopping. However, when both are present, they aid each other to give the appropriate switching. Thus at low noise, the periodic forcing helps the system to switch wells in the desired fashion, in response to the inputs.

We can quantify the consistency (or reliability) of obtaining a given logic output by calculating the probability of obtaining the desired logic output for different states of input. The probability, here, is the ratio of number of successful runs, i.e. when the desired logic output is obtained, to the total number of runs. In every run, we simulate the system for 7 different possible combinations of I_1 and I_2 such that we get all possible transitions of the inputs. Thus, any run is counted as successful only if the system is in the desired well for every time step for all seven possible combinations of the inputs, allowing for a small transience. Here we have chosen the transience time to be equal to 10 percent of the time for which an input is given. Thus the system must switch to the desired well within one-tenth of the time for which an input is applied. Thus we measure the output over the subsequent 90 percent of the input time. In this case, we consider the response correct if and only if the system remains in the correct well throughout this time, for all the seven possible combinations of inputs.

We vary the noise strength and amplitude of periodic forcing keeping the frequency of the sinusoidal forcing constant. We observe that when the amplitude of sinusoidal forcing is low, we get the correct logical response only when there is some noise in the system. For very low or very high values of the noise intensity, we get erroneous results. But for higher amplitudes of sinusoidal forcing, periodic forcing alone can drive the system to the desired well, even in absence of noise. Notice that as we keep on increasing the amplitude of sinusoidal forcing, the maximum noise intensity for which we obtain correct logical response decreases slightly. This is expected as the interplay of noise and periodic forcing is likely to worsen the response at higher values of noise strengths, as now the state of the output starts hopping randomly between the two wells as shown in figure 3.

Next we keep the amplitude of periodic forcing constant, and vary the frequency of the sine wave. We observe that for low frequencies we obtain the desired response for low noise levels and as the frequency is increased the optimal window reduces and also shifts upwards slowly as seen in figure 4. One can rationalize this as follows: when the frequency is higher, the system gets little time to respond to the sinusoidal forcing, thus limiting its effect. This is also evident from the bottom panel of figure 4 which clearly shows that if the noise level is constant, then as the frequency of periodic forcing is increased, we need higher and higher values of its amplitude to get the desired response.

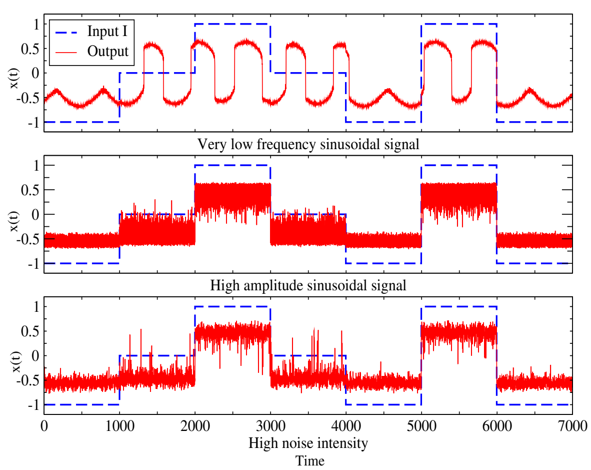

Figure 3. This figure shows how decreasing the frequency of periodic forcing to very low value $\omega = .01$ (top panel) or increasing the ampitude of periodic forcing to very high value $A = 2.0$ (middle panel) or very high noise intensity $D = 0.40$ can result in random hopping between the wells, leading to erratic response. Red line is the output of the system and dashed blue line is the sum of the two inputs that was fed into the system as defined in eq. 2.

Further, the addition of periodic forcing also results in lower switching times. We calculated the time the output takes to switch from low to high levels, or vice versa, in response to change in the input signal, averaged over 1000 such switches. We observed that for low noise intensities there is sharp reduction in the average switching time when a periodic signal is added (see figure 5). For faster operation of the logic gate, it is desirable that time of an input or bit time be as low as possible. This can be done by setting the bit time equal to the minimum switching time plus time required for reading the state of the output. This gives us the minimum time for which an input should be applied so that we get robust operation of the logic gate. We explored the minimum switching time for various combinations of modulation frequencies, modulation amplitude and noise strengths. We found that systems driven by a periodic forcing of appropriate frequency and amplitude can function robustly for bit times as low as one time unit (e.g by setting $A = 1$ and $\omega = 2.7$).

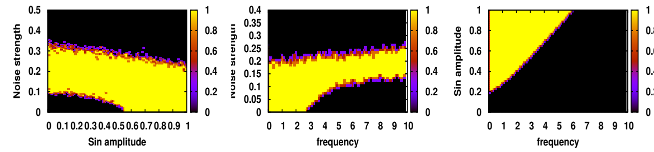

Figure 4. Probability of obtaining the AND logic operation with $b = -0.5$ Here $\omega = 4$ for left panel, $A = 0.5$ for middle panel and $D = 0.08$ for right panel. The system was simulated for 100 random combinations of the two inputs. The transience was set at 10 percent of the input timescale.

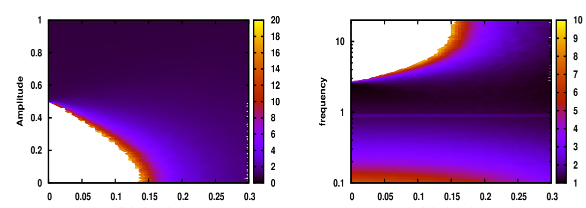

Figure 5. Switching time averaged over 1000 switches of output for the AND gate operation, over a range of noise strengths, amplitudes and frequencies. Here $b = -0.5$, $\omega = 2.7$ (left) and $A = 0.5$ (right). Note that for low amplitudes we don't get robust AND operation for low noise intensities. In the density plots, the colors represent the switching time, the white area corresponds to the range where the operation of the logic gate is not robust, and the frequency is plotted on a log scale in the density plot on the right.

V. ADAPTIVE LOGICAL STOCHASTIC RESONANCE

In the above results we have seen that noise and a periodic forcing interfere constructively and aid the switching of the system between the two wells. This on one hand, helps to obtain desired responses at lower noise strengths but at the same time the response deteriorates slightly at higher values of noise intensity. This deterioration is not so significant and is at the higher noise boundary where one typically does not operate. Still, if we want to ensure robust operation even in the higher noise window, one way to accomplish this would be to keep monitoring the noise levels in the system and then adjusting either the frequency or amplitude of periodic forcing, if the noise level crosses some threshold level. That is to say, we switch off the periodic forcing if the noise level is within the optimal window.

This approach will work fine if the changes in the noise strengths occur only occasionally. On the other hand, if the fluctuations in noise levels are quite frequent, then it will be better if the system could *automatically adapt itself to the changing noise levels for robust operation.*

Now we will show that by coupling two bistable systems we can enable robust operation of LSR elements over large spectrum of noise intensities. Here one of the bistable systems will be used to control the amplitude or frequency of the periodic forcing that is fed to the other system.

Consider the coupled systems:

$$\dot{x} = a_1(x - a_2x^3) + b + I_1 + I_2 + Ay\sin(\omega t) + D\eta(t) \quad (3)$$

$$\dot{y} = a_3(y - a_4y^3) + D\eta(t) \quad (4)$$

Here the output of second system modulates the sinusoidal forcing applied to the first system. The working principle in this form of coupling is that the second system has a lower potential barrier as compared to the first. When the noise is very low, the output of the second system will stay in the same well. Thus it will act as a constant signal and the sinusoidal forcing will simply be scaled by a factor, and the frequency of periodic forcing fed into first system will be same as that of the sinusoidal signal. This sinusoidal forcing then enables the operation of LSR elements even in low noise conditions, as was shown in [12].

When the noise level increases, y jumps between the two wells according to the Kramers rate, essentially behaving like a signal of high frequency as shown in figure 6 . As the modulating signal has a high frequency, the signal fed into the first system has a frequency much greater than the frequency of the sinusoidal signal. Thus the system doesn't get sufficient time to respond to this signal and hence the sinusoidal forcing doesn't lead to random hops. The minimum frequency that we can choose for the sinusoidal forcing is governed by the transience time in which the system must reach the desired state. This is so because within this time the system should receive at least one full cycle of the periodic forcing. As mentioned earlier, we set the potential barrier for the second system much lower compared to first by selecting appropriate values of a_3 and a_4. Further, the parameters chosen are such that the frequency of sinusoidal forcing is lower than the frequency obtained from Kramers rate for noise strengths where noise alone is sufficient for robust operation of the LSR elements.

We simulated the system as given by equations 3 and 4 for 100 different initial conditions. In particular, we choose $a_1 = 4$, $a_2 = 5$, $a_3 = 2$, $a_4 = 8$, $\omega = 0.05$, $A = 0.6$ and calculate the range of D over which we get the robust logic operation. As before, in each run we took 7 different possible combinations of I_1 and I_2. A run was counted as successful if and only if the time series matched the desired output 100 percent of the time, leaving out a small transient period. Then the ratio of successful runs to the total number of runs was defined as the probability of obtaining the correct logic. As shown in figure 7, we find that the probability of correct logic is 1 for much larger noise windows. Specifically, it is one for *all* noise intensities less than the maximum noise intensity for which correct logic was obtained in the absence of periodic forcing. Additionally the lower limit on noise intensity now no longer exists, and we can obtain robust logic operations even in noise-free case.

Moreover, if we use a slightly relaxed criteria for correct logic, that is, if we assume that logic is correct even if time series matches for a little less than 100 percent, then the critical noise intensity upto which correct logic is obtained increases further, as seen in the second and third panel of figure 7.

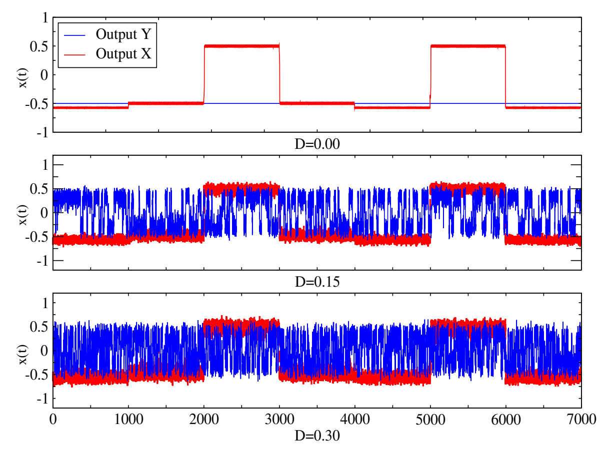

Figure 6. Time series for y (blue) and output (red) for the system given by eq. 3 and eq. 4 for noise intensities $D = 0$ (top), $D = 0.15$ (middle) and $D = 0.3$ (bottom).

It is evident from figure 7 that the noise intensity upto which we get robust operations is slightly lower than that obtained in absence of periodic forcing. This arises from the enhancement of random hopping due to the added

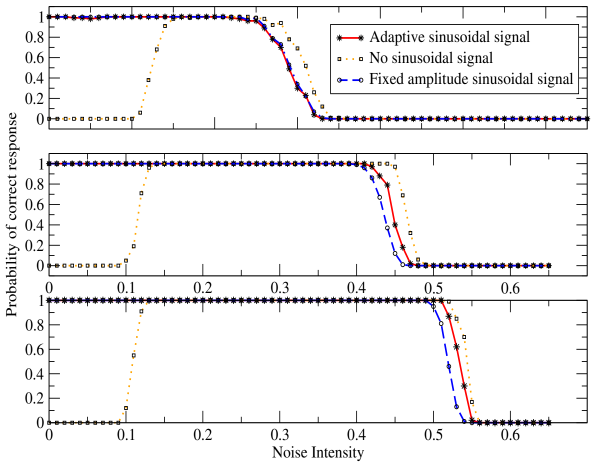

Figure 7. Probability of obtaining the AND logic operation with $b = -0.5$ Here $\omega = 0.05$, $A = 0.6$. The red line corresponds to simulations as given by eq. 3 and 4. Dashed orange and blue lines are obtained taking $A = 0$ and $A = 0.35$ in eq. 2. The system was simulated for 100 runs where each run was combination of seven different possible input sets of the two inputs. A run was taken to be successful if the time series matched the expected output 100 percent (top), 98 percent (middle) and 95 percent (bottom) of the time, after leaving out transience.

effect of periodic drive and noise.

Lastly, we consider adaptively changing the amplitude of sinusoidal forcing instead of the frequency. For this, consider the system given by:

$$\dot{x} = a_1(x - a_2x^3) + b + I_1 + I_2 + A\,\langle y\rangle\,\sin(\omega t) + D\eta(t) \quad (5)$$

$$\dot{y} = a_3(y - a_4y^3) + D\eta(t) \quad (6)$$

where $\langle y\rangle$ is the average of y over a short interval of time. This kind of coupling is particularly relevant in chemical and biological systems, where the instantaneous state of the system is not easily detectable, but a short-time average is more accesible. Here instead of instantaneous value of y, its average over previous few values is fed back to the coupled system. When the noise in the system is low, the average value of its state will be equal to that of the potential well in which it is lying. When the noise level increases, the system jumps between the two wells according to the Kramers rate. As the two wells are symmetric about zero, the average value approaches zero as the hopping rate increases, as shown in figure 8. Thus the amplitude of sinusoidal signal is reduced to zero.

We simulated the coupled systems given by equations 5 and 6 by averaging the value of y over 100 time units (the time for which an input is applied). Further, when the average is taken over a larger time interval, the response of these coupled systems will be better, provided that this time is smaller than the timescale over which the noise strength itself changes. The results are shown in figures 9. We see that in this case, not only does the lower

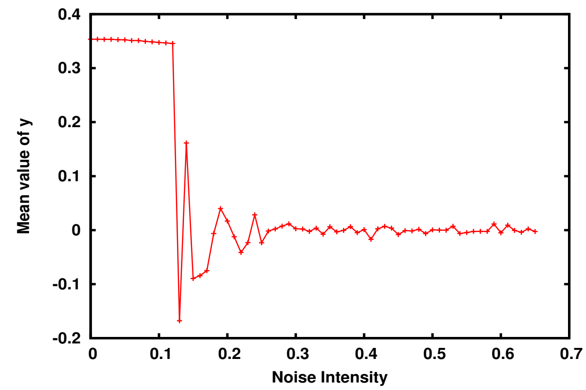

Figure 8. Every point represents mean value of y over previous 100 time units for that particular noise intensity, for the system given by eq. 6.

limit on optimal noise window vanish, but for higher noise intensities too the results match with those obtained in absence of a periodic forcing.

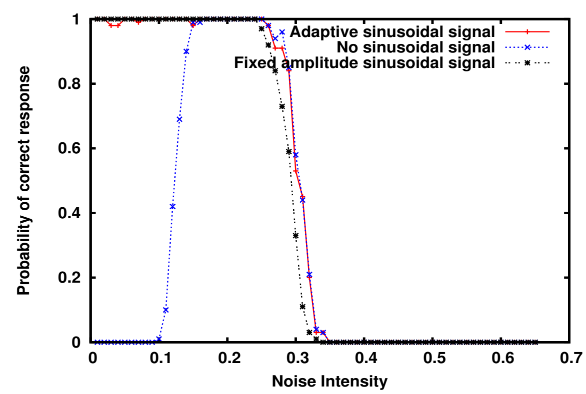

Figure 9. Probability of obtaining the AND logic operation, for the system given by eqs. $5-6$, (red line). Further we show the case with no sinusoidal forcing (green line) and the case where the amplitude of the periodic forcing is constant (blue line). The system was simulated for 100 runs where each run was combination of seven different possible input sets of the two inputs. A run was taken to be successful if the time series matched the expected output 100 percent of the time, after transience.

The results are qualitatively the same even if there is some delay in propagation of $\langle y\rangle$, provided it not so large that noise intensity itself changes in this time. Additionally, similar results were obtained if we use a moving or running average of y. This is particularly helpful in electronic systems where a running average is far more easy to implement vis-a-vis a normal average which may involve a lot more computations and consequently result in delays.

VI. CONCLUSION

In conclusion, we have explicitly shown that by utilizing the constructive interplay of noise and periodic forcing it is possible to obtain a logic response similar to LSR even when the strength of noise is lower then the minimum threshold. This enables us to use the LSR elements in subthreshold noise conditions. Further, by coupling the LSR element to another LSR element with a lower potential barrier we can make the systems adapt to varying noise intensity, so that its operation is robust even in high noise conditions. The results presented here are quite general, and can potentially be extended to other systems which show enhanced performance in the presence of noise, such as typically observed in generalized stochastic resonance phenomena.

[1] K. Murali, S. Sinha, W.L. Ditto and A.R. Bulsara, Reliable Logic Circuit Elements that Exploit Nonlinearity in the Presence of a Noise Floor, *Phys. Rev. Lett.* **102** (2009) 104101

[2] K. Murali, I. Raja Mohamed, S. Sinha, W.L. Ditto and A.R. Bulsara, Realization of reliable and flexible logic gates using noisy nonlinear circuits, *App. Phys. Lett.* **95** (2009) 194102;

[3] Worschech, L., F. Hartmann, T. Y. Kim, S. Hofling, M. Kamp, A. Forchel, J. Ahopelto, I. Neri, A. Dari, and L. Gammaitoni. Universal and reconfigurable logic gates in a compact three-terminal resonant tunneling diode. *Applied Physics Letters*, **96** (2010) 042112-042112.

[4] D. N. Guerra, A. R. Bulsara, W. L. Ditto, S. Sinha, K. Murali and P. Mohanty, A noise-assisted reprogrammable nanomechanical logic gate *Nano Letters* **10** (2010) 1168.

[5] S. Sinha, J.M. Cruz, T. Buhse and P. Parmananda, Exploiting the effect of noise on a chemical system to obtain logic gates, *Europhys. Lett.*, **86** (2009) 60003

[6] J. Zamora-Munt and C. Masoller, Numerical implementation of a VCSEL-based stochastic logic gate via polarization bistability *Optics Express* **18** (2010) 16418; S. Perrone, R. Vilaseca and C. Masoller, Stochastic logic gate that exploits noise and polarization bistability in an optically injected VCSEL, *Opt. Express* **20**, (2012) 22692-22699 M. F. Salvide, C. Masoller and M. S. Torre, All-optical stochastic logic gate based on a VCSEL with tunable optical injection, *IEEE J. Quantum Electron.*, **49** (2013) 886-893

[7] H. Ando, S. Sinha, R. Storni and K. Aihara, Synthetic gene networks as potential flexible parallel logic gates *Europhys. Lett.*, **93** (2011) 50001; E.H. Hellen, S.K. Dana, J. Kurths, E. Kehler and S. Sinha, Noise-aided Logic in an Electronic Analog of Synthetic Genetic Networks, PLoS One (2013) doi:10.1371/journal.pone.0076032; A. Sharma, V. Kohar, M. D. Shrimali, S. Sinha Realizing logic gates with time-delayed synthetic genetic networks *Nonlinear Dynamics* (2013) doi: 10.1007/s11071-013-1136-9

[8] K.P.Singh and S. Sinha, Enhancement of logical responses by noise in a bistable optical system, *Phys. Rev. E* **83** (2011) 046219

[9] Storni, Remo, et al. Manipulating potential wells in Logical Stochastic Resonance to obtain XOR logic. *Phys. Lett. A* **376** 8 (2012) 930-937

[10] Xu, Yong, et al. The Availability of Logical Operation Induced by Dichotomous Noise for a Nonlinear Bistable System *Journal of Statistical Physics* (2013) 1-16.

[11] V. Kohar and S. Sinha, Noise-assisted morphing of memory and logic function, *Phys. Lett. A*, 376 (8) (2012), 957-962

[12] A.Gupta, A. Sohane, V. Kohar, K. Murali and S.Sinha, Noise-free logical stochastic resonance, *Phys. Rev E* **84** (2011) 055201

[13] M. M. Mano *Computer System Architecture*, Prentice Hall (1993); T. C. Bartee *Computer Architecture and Logic Design*, Mc-Graw Hill (1991).

[14] Numerical simulations of the above stochastic differential equation were done by the standard Euler method.