NYU TANDON SCHOOL OF ENGINEERING

Autonomous Mobile Robots

Lecture 6: Nonlinear Control Methods

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Today's Topics

- Feedback Linearization
- Robust Control
- Sliding Mode Control
- Adaptive Control

Goal: Control nonlinear systems in the presence of uncertainties and unknown parameters.

Feedback Linearization

Controllability of Nonlinear Systems via Lie Brackets

Linear System

$$\dot{x} = Ax + bu$$

Controllability Matrix

$$\mathcal{C} = \begin{bmatrix} b & Ab & A^2 b & \cdots & A^{n-1} b \end{bmatrix}$$

Controllability Condition

The system is **controllable** if and only if:

$$\text{rank}(\mathcal{C}) = n \quad \Leftrightarrow \quad |\mathcal{C}| \neq 0$$

Nonlinear Control-Affine System

$$\dot{x} = f(x) + g(x)u$$

Controllability via Lie Brackets

$$\mathcal{C} = \{ad_f^0 g,\ ad_f^1 g,\ ad_f^2 g,\ \ldots,\ ad_f^{n-1} g\}$$

The **Lie bracket** is the nonlinear analog of the $A^k b$ terms in linear controllability.

Definition

For vector fields $f(x)$ and $g(x)$, the **Lie bracket** is:

$$[f, g] = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g$$

Iterated Lie Brackets (ad notation)

$$ad_f^0 g = g$$

$$ad_f^1 g = [f, g]$$

$$ad_f^2 g = [f, [f, g]]$$

Distribution

$$\Delta = \text{span}\{g,\ [f,g],\ [f,[f,g]],\ \ldots\}$$

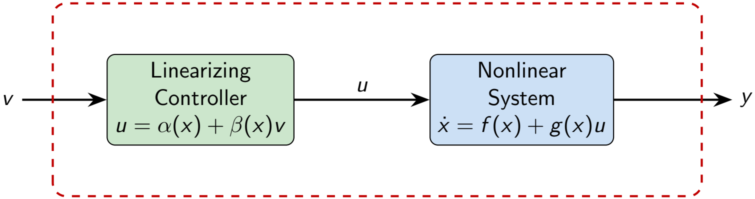
Controllability Rank Condition

The system $\dot{x} = f(x) + g(x)u$ is **locally controllable** at x_0 if:

$$\dim(\Delta(x_0)) = n$$

This is analogous to the rank condition for linear systems.

Feedback Linearization Concept



Equivalent Linear System: $\dot{z} = Az + Bv$

Key Idea

Find $u = \alpha(x) + \beta(x)v$ such that the closed-loop system is linear in new coordinates.

Consider the nonlinear system:

$$\dot{x} = f(x) + g(x)u$$

Where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and:

$$f(x) = \begin{bmatrix} x_2 \\ x_1^2 + 2x_1x_2 \end{bmatrix}, \qquad g(x) = \begin{bmatrix} x_1^2 \\ x_2^2 \end{bmatrix}$$

Computing $\frac{\partial g}{\partial x}$

$$\frac{\partial g}{\partial x} = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x_1 & 0 \\ 0 & 2x_2 \end{bmatrix}$$

Computing $\frac{\partial f}{\partial x}$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2x_1 + 2x_2 & 2x_1 \end{bmatrix}$$

$$[f, g] = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g$$

$$[f, g] = \begin{bmatrix} 2x_1 & 0 \\ 0 & 2x_2 \end{bmatrix} \begin{bmatrix} x_2 \\ x_1^2 + 2x_1 x_2 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 2x_1 + 2x_2 & 2x_1 \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_2^2 \end{bmatrix}$$

First term:

$$\frac{\partial g}{\partial x} f = \begin{bmatrix} 2x_1x_2 \\ 2x_2(x_1^2 + 2x_1x_2) \end{bmatrix}$$

Second term:

$$\frac{\partial f}{\partial x} g = \begin{bmatrix} x_2^2 \\ (2x_1 + 2x_2)x_1^2 + 2x_1x_2^2 \end{bmatrix}$$

Therefore:

$$[f, g] = \begin{bmatrix} 2x_1x_2 - x_2^2 \\ 2x_1^2x_2 + 4x_1x_2^2 - 2x_1^3 - 2x_1^2x_2 - 2x_1x_2^2 \end{bmatrix}$$

$$\mathcal{C} = \begin{bmatrix} g & [f, g] \end{bmatrix} = \begin{bmatrix} x_1^2 & 2x_1x_2 - x_2^2 \\ x_2^2 & 2x_1^2x_2 - 2x_1^3 \end{bmatrix}$$

Feedback Linearizability Conditions

1. $|\mathcal{C}| \neq 0$ (full rank)
2. The distribution $\{g, [f, g], \ldots, ad_f^{n-2}g\}$ must be **involutive**

Robust Control

Handling Bounded Uncertainties

Real System with Uncertainties

$$\dot{x} = f(x, u) + \Delta$$

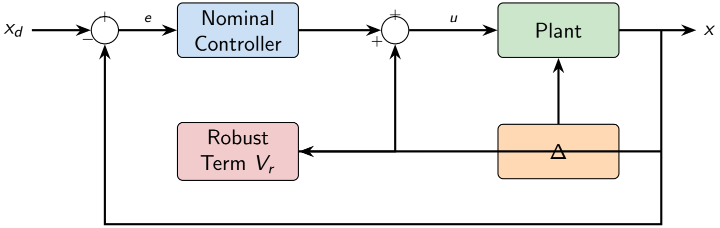
where Δ represents unknown uncertainties.

Example: Nonlinear Spring-Damper

$$\ddot{x} + a\dot{x} + bx^3 + d(x) = u$$

- $a\dot{x}$: damping term (uncertain coefficient)
- bx^3: nonlinear spring (uncertain coefficient)
- $d(x)$: unknown disturbance

Robust Control Architecture



Goal

Find control u such that $x \to x_d$ despite uncertainties.

Nominal Model

$$\ddot{x} + \hat{a}\dot{x} + \hat{b}x^3 = u$$

where $\hat{a}$, $\hat{b}$ are estimated parameters.

Proposed Controller

$$u = \ddot{x}_d + \hat{a}\dot{x} + \hat{b}x^3 + k_d(\dot{x}_d - \dot{x}) + k_p(x_d - x) + V_r$$

Define tracking error: $e = x_d - x$

Substituting controller into the actual system dynamics:

$$\ddot{x}_d - \ddot{x} + k_d\dot{e} + k_p e = (a - \hat{a})\dot{x} + (b - \hat{b})x^3 + d(x) - V_r$$

Error Dynamics

$$\ddot{e} + k_d\dot{e} + k_p e = \Delta - V_r$$

where $\Delta = (a - \hat{a})\dot{x} + (b - \hat{b})x^3 + d(x)$

Define state vector:

$$z = \begin{bmatrix} e \\ \dot{e} \end{bmatrix}$$

State Space Form

$$\dot{z} = Az + bw$$

Where:

$$A = \begin{bmatrix} 0 & 1 \\ -k_p & -k_d \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$w = (a - \hat{a})\dot{x} + (b - \hat{b})x^3 + d(x) - V_r$$

Lyapunov Function Candidate

$$V = \frac{1}{2} z^T P z$$

where $P > 0$ satisfies $A^T P + PA = -Q$ for some $Q > 0$.

Time Derivative

$$\begin{aligned}
\dot{V} &= \frac{1}{2}[\dot{z}^T P z + z^T P \dot{z}] \\
&= \frac{1}{2} z^T [A^T P + PA] z + z^T P b w \\
&= -\frac{1}{2} z^T Q z + z^T P b w
\end{aligned}$$

Assumptions on Uncertainty

- $|a - \hat{a}| < \alpha$
- $|b - \hat{b}| < \beta$
- $|d(x)| < \rho$

Total Uncertainty Bound

$$|(a - \hat{a})\dot{x} + (b - \hat{b})x^3 + d(x)| < \alpha|\dot{x}| + \beta|x|^3 + \rho = F(x, \dot{x})$$

Robust Term Design

$$V_r = \frac{z^T Pb}{|z^T Pb|} F(x, \dot{x})$$

This choice ensures:

$$z^T Pb \cdot w < 0$$

Stability Guarantee

With this V_r:

$$\dot{V} = -\frac{1}{2} z^T Q z + z^T Pbw < 0$$

guaranteeing asymptotic stability.

Sliding Mode Control

Robustness Through Discontinuous Control

Sliding Surface

$$S = \dot{e} + \lambda e$$

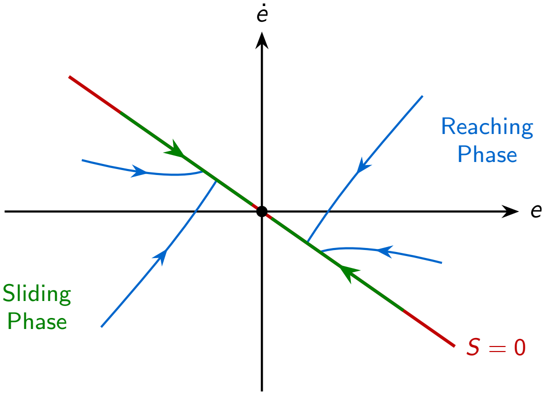
where $\lambda > 0$ is a design parameter.

Control Objective

Design control such that $S \to 0$.

Convergence Property

If $S = 0$: $\quad \dot{e} + \lambda e = 0 \implies e(t) = e(0)e^{-\lambda t} \to 0$


$\dot{e}$
e
Reaching
Phase
Sliding
Phase
$S = 0$

Lyapunov Function

$$V = \frac{1}{2}S^2$$

Reaching Condition

We want:

$$\dot{V} = S\dot{S} = -\eta|S|, \quad \eta > 0$$

This implies:

$$\dot{S} = -\eta\,\mathrm{sgn}(S)$$

Consider the second-order system:

$$\ddot{x} = f(x) + u$$

Sliding surface derivative:

$$\dot{S} = \ddot{e} + \lambda\dot{e} = \ddot{x}_d - \ddot{x} + \lambda\dot{e}$$

Substituting dynamics:

$$\dot{S} = \ddot{x}_d - f(x) - u + \lambda\dot{e}$$

From the reaching condition $\dot{S} = -\eta\,\mathrm{sgn}(S)$:

$$\ddot{x}_d - f(x) - u + \lambda\dot{e} = -\eta\,\mathrm{sgn}(S)$$

Control Law

$$u = \ddot{x}_d - f(x) + \lambda\dot{e} + \eta\,\mathrm{sgn}(S)$$

The $\mathrm{sgn}(S)$ term creates the discontinuous switching action.

For system with uncertainty:

$$\ddot{x} = \hat{f}(x) + \Delta + u, \quad |\Delta| < F$$

Robust Control Law

$$u = \ddot{x}_d - \hat{f}(x) + \lambda\dot{e} + (\eta + F)\,\mathrm{sgn}(S)$$

This guarantees:

$$S\dot{S} < -\eta|S|$$

ensuring finite-time reaching of the sliding surface.

Adaptive Control

Online Parameter Estimation

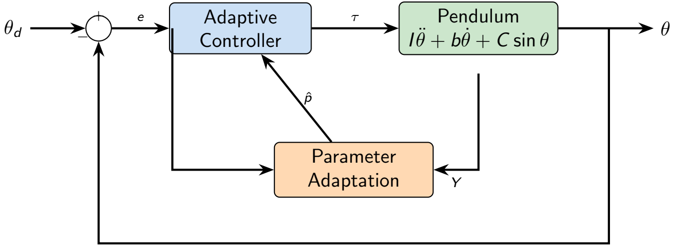
Example: Pendulum System

$$I\ddot{\theta} + b\dot{\theta} + C \sin \theta = \tau$$

- I: unknown inertia
- b: unknown damping
- C: unknown gravitational term

Control Objective

Design τ and parameter update laws to track desired trajectory θ_d.


θ_d
$-$
e
Adaptive Controller
τ
Pendulum
$I\ddot{\theta} + b\dot{\theta} + C \sin\theta$
θ
$\hat{p}$
Parameter Adaptation
Y

Proposed Controller

$$\tau = \hat{I}[\ddot{\theta}_d + k_d(\dot{\theta}_d - \dot{\theta}) + k_p(\theta_d - \theta)] + \hat{b}\dot{\theta} + \hat{C}\sin\theta$$

Where:

- $\hat{I}, \hat{b}, \hat{C}$: parameter estimates (updated online)
- k_d, k_p: feedback gains (fixed)

Define parameter vector and regressor:

$$p = \begin{bmatrix} I \\ b \\ C \end{bmatrix}, \qquad Y = \begin{bmatrix} \ddot{\theta} \\ \dot{\theta} \\ \sin\theta \end{bmatrix}$$

Then the dynamics can be written as:

$$p^T Y = I\ddot{\theta} + b\dot{\theta} + C\sin\theta = \tau$$

This **linear-in-parameters** structure is key for adaptive control.

Error System Representation

Define error states: $x_1 = e = \theta_d - \theta$, $x_2 = \dot{e}$

Error Dynamics

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -k_p x_1 - k_d x_2 + I^{-1}(p - \hat{p})^T Y$$

In matrix form:

$$\dot{x} = Ax + bw$$

where:

$$A = \begin{bmatrix} 0 & 1 \\ -k_p & -k_d \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad w = I^{-1}(p - \hat{p})^T Y$$

Lyapunov Candidate

$$V(x, \tilde{p}) = \frac{1}{2}x^T Qx + \frac{1}{2\gamma}\tilde{p}^T\tilde{p}$$

Where:

- $\tilde{p} = p - \hat{p}$: parameter error
- $Q > 0$: symmetric positive definite matrix
- $\gamma > 0$: adaptation gain

Note: Q is the Lyapunov matrix (distinct from sliding surface S).

Taking the time derivative:

$$\dot{V} = x^T Q \dot{x} - \frac{1}{\gamma} \tilde{p}^T \dot{\hat{p}}$$

$$= x^T Q (Ax + bw) - \frac{1}{\gamma} \tilde{p}^T \dot{\hat{p}}$$

$$= x^T QAx + x^T Qb \cdot I^{-1} \tilde{p}^T Y - \frac{1}{\gamma} \tilde{p}^T \dot{\hat{p}}$$

We need to choose $\dot{\hat{p}}$ to cancel the cross term.

To ensure $\dot{V} < 0$, we want:

$$x^T Qb \cdot I^{-1} \tilde{p}^T Y = \frac{1}{\gamma} \tilde{p}^T \dot{\hat{p}}$$

Adaptation Law

$$\dot{\hat{p}} = \gamma I^{-1} (x^T Qb) Y$$

This cancels the indefinite term, leaving:

$$\dot{V} = x^T QAx < 0 \quad \text{for } x \neq 0$$

Control Law

$$\tau = \hat{I}[\ddot{\theta}_d + k_d \dot{e} + k_p e] + \hat{b}\dot{\theta} + \hat{C} \sin \theta$$

Parameter Adaptation

$$\dot{\hat{p}} = \gamma I^{-1}(x^T Qb)Y$$

Or in integral form:

$$\hat{p}(t) = \hat{p}(0) + \gamma \int_0^t I^{-1}(x^T Qb)Y \, d\tau$$

Key Methods

- **Feedback Linearization**: Lie brackets for nonlinear controllability; transform to linear system
- **Robust Control**: Handle bounded uncertainties via Lyapunov-based design
- **Sliding Mode**: Discontinuous control for robustness; fast convergence
- **Adaptive Control**: Online parameter estimation for unknown systems

Next Lecture

- Model Predictive Control (MPC)
- Control Barrier Functions (CBF)
- Safety-critical control

These methods build on the Lyapunov stability concepts we've developed.

End of Lecture 6

Questions?