NYU TANDON SCHOOL OF ENGINEERING

Autonomous Mobile Robots

Lecture 5: Lyapunov Stability and Nonlinear Systems

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- Nonlinear Systems Fundamentals
- Phase Portrait Analysis
- Eigenvalue-Based Stability Classification
- Lyapunov Stability Theory
- Lyapunov Function Examples
- Linearization for Stability Analysis
- Lyapunov-Based Controller Design

Course Topics

- Introduction & Physical Constraints
- Vehicle Kinematics & EoM
- Control & Perception Fundamentals
- Nonlinear Control Methods
- **NMPC + Lyapunov Stability** → **Today's Focus**
- Motion Planning Algorithms
- Learning-Based Planning
- Industry Standards & Safety

Autonomous System (time-independent):

$$\dot{X} = f(X)$$

This is a **closed-loop equation** where dynamics depend only on the current state.

The system evolves independently of explicit time t.

Non-Autonomous System (with control input):

$$\dot{X} = f(X, u)$$

With state feedback control $u = g(x)$:

$$\dot{X} = f(X, g(X))$$

This transforms the non-autonomous system into an autonomous closed-loop system.

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Fundamental Difference

For nonlinear systems, we study behavior **from equilibrium point to equilibrium point**.

Unlike linear systems with typically one equilibrium, nonlinear systems can have:

- Multiple equilibrium points
- Different stability properties at each equilibrium
- Complex behaviors between equilibria

Linear Systems

- Single equilibrium point
- Predictable behavior
- Global stability analysis
- Superposition applies

Nonlinear Systems

- Multiple equilibrium points
- Chaos, bifurcations
- Local stability only
- No superposition

Important Limitation of Linearization

Critical Note

Linearization gives only **local information** at each equilibrium point $\dot{x} = 0$.

It tells us nothing about:

- Global behavior of the system
- Behavior far from the equilibrium
- Interactions between multiple equilibria

Phase Portrait Analysis

Plotting trajectories in state space helps visualize system behavior.

- Shows how states evolve over time
- Reveals equilibrium points and their stability
- Identifies limit cycles and attractors

Limitation: Can be used at most for 2nd-order differential equations (2D state space).

General Linear Oscillator:

$$\ddot{y} + 2\varepsilon\omega_n\dot{y} + \omega_n^2 y = 0$$

where:

- ε = damping ratio
- ω_n = natural frequency

Define state variables:

$$X_1 = y \qquad X_2 = \dot{y} \qquad X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

State equations:

$$\dot{X}_1 = X_2$$

$$\dot{X}_2 = -\omega_n^2 X_1 - 2\varepsilon\omega_n X_2$$

State-space representation:

$$\dot{X} = AX$$

$$A = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\varepsilon\omega_n \end{bmatrix}$$

Equilibrium condition:

$$\dot{X} = 0 \quad \Rightarrow \quad X^* = 0$$

For this linear system, **the origin is the only equilibrium point**.

The stability of this equilibrium depends on the eigenvalues of matrix A.

System behavior is determined by eigenvalues λ_1, λ_2 of matrix A.

Key factors:

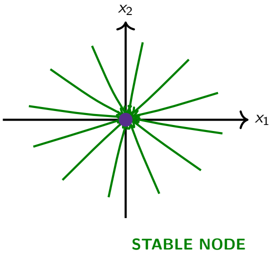
- Real part: $\text{Re}\{\lambda\}$ determines growth/decay
- Imaginary part: $\text{Im}\{\lambda\}$ determines oscillation

Let's examine each case with phase portraits.

Eigenvalue Conditions:

$$\text{Re}\{\lambda_1, \lambda_2\} < 0$$

$$\text{Im}\{\lambda_1, \lambda_2\} = 0$$



Both eigenvalues are **real and negative**.

$\Rightarrow$ **Stable Equilibrium**

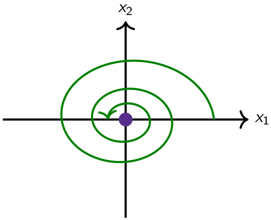
Eigenvalue Conditions:

$$\text{Re}\{\lambda_1, \lambda_2\} < 0$$

$$\text{Im}\{\lambda_1, \lambda_2\} \neq 0$$

Complex conjugate eigenvalues with negative real part.

$\Rightarrow$ **Stable Spiral**



STABLE FOCUS

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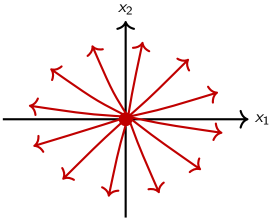
Eigenvalue Conditions:

$$\text{Re}\{\lambda_1, \lambda_2\} < 0$$

$$\text{Im}\{\lambda_1, \lambda_2\} = 0$$

Note: Despite Re < 0 shown, if signs were positive, trajectories diverge.

This is an **Unstable Equilibrium** — stable only at exactly $x^* = 0$.



UNSTABLE NODE

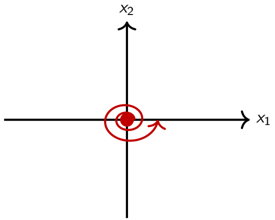
Eigenvalue Conditions:

$$\text{Re}\{\lambda_1, \lambda_2\} > 0$$

$$\text{Im}\{\lambda_1, \lambda_2\} = 0$$

Both eigenvalues are **real and positive**.

$\Rightarrow$ **Unstable Equilibrium**



UNSTABLE FOCUS

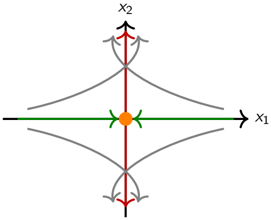
Eigenvalue Conditions:

$$\mathrm{Re}\{\lambda_1\} < 0$$

$$\mathrm{Re}\{\lambda_2\} > 0$$

One stable direction, one unstable direction.

$\Rightarrow$ **Saddle Point (Unstable)**



SADDLE POINT

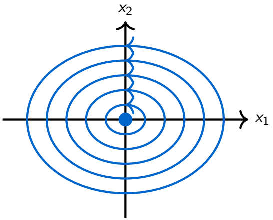
Eigenvalue Conditions:

$$\mathrm{Re}\{\lambda_1, \lambda_2\} = 0$$

$$\mathrm{Im}\{\lambda_1, \lambda_2\} \neq 0$$

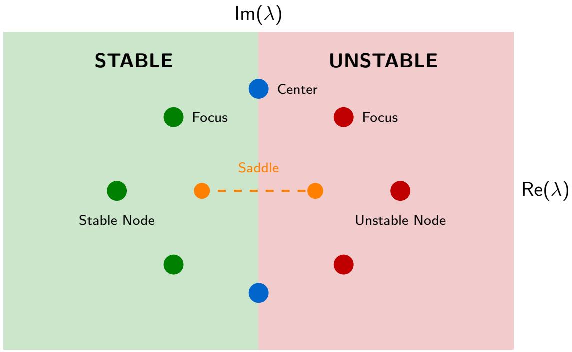
Purely imaginary eigenvalues.

Example: Spring-damper with no damping.



CENTER

Summary: Eigenvalue Stability Classification

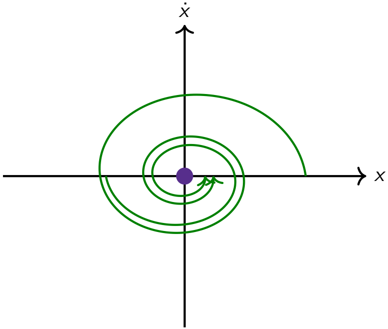


Cubic Damping System:

$$\ddot{x} + \alpha\dot{x} + x^3 = 0$$

- Nonlinear restoring force: x^3
- Linear damping: $\alpha\dot{x}$
- Stiffness increases with displacement

Nonlinear System: Phase Portrait



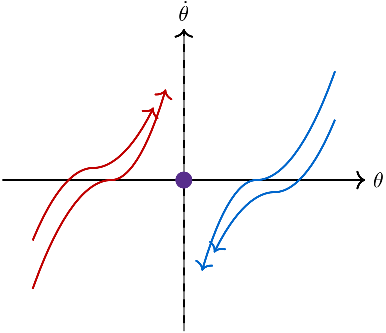
Nonlinear stable system with $\alpha > 0$

Satellite Attitude Control (Bang-Bang):

$$\ddot{\theta} = K \cdot \mathrm{sign}(-\theta)$$

- Used in spacecraft attitude control
- Discontinuous control input
- Can achieve finite-time convergence

Switching Model: Phase Portrait



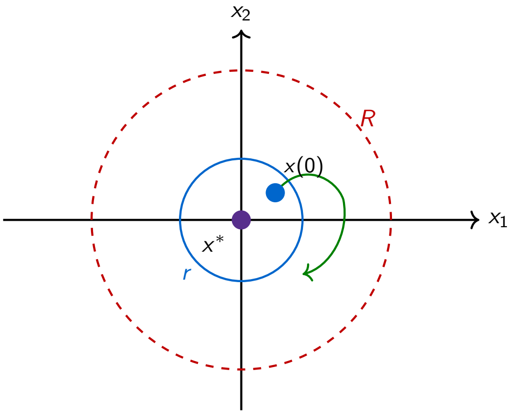
Bang-bang control with switching at $\theta = 0$

Lyapunov Stability

An equilibrium $x^* = 0$ is **Lyapunov stable** if:

$$\forall R > 0, \ \exists r > 0 : \quad \|x(0)\| < r \ \Rightarrow \ \|x(t)\| < R \quad \forall t \geq 0$$

Intuition: Starting close to equilibrium means staying close.


x_2
R
$x(0)$
x_1
x^*
r

Asymptotic Stability

An equilibrium is **asymptotically stable** if it is Lyapunov stable AND:

$$x(t) \to 0 \quad \text{as} \quad t \to \infty$$

Key Insight: Asymptotic stability is **stronger** than Lyapunov stability.

- Lyapunov stable: stays bounded
- Asymptotically stable: converges to equilibrium

For system $\dot{x} = f(x)$ with equilibrium $x^* \Rightarrow \dot{x}^* = 0$:

Lyapunov's Direct Method

If there exists a function $V(x)$ such that:

1. $V(x) > 0$ for $x \neq 0$ and $V(0) = 0$ (positive definite)
2. $\dot{V}(x) \leq 0$ for $x \neq 0$ (non-increasing)

Then the equilibrium is **Lyapunov stable**.

If additionally:

- $\dot{V}(x) = 0$ **only at** $x = 0$

Then $x(t) \to 0$ as $t \to \infty \Rightarrow$ **Asymptotic stability**

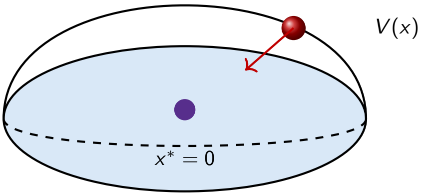
Global Stability Condition

For **global asymptotic stability**, we additionally need:

$$V(x) \to \infty \quad \text{as} \quad \|x\| \to \infty$$

This is called the **radially unbounded** condition.

It ensures stability holds for *any* initial condition, not just those near the equilibrium.



$V(x)$ acts like “energy” — if energy always decreases, system approaches minimum.

System:

$$\ddot{X} + |\dot{X}|\dot{X} + X^3 = 0$$

State Variables:

$$X_1 = X \qquad X_2 = \dot{X}$$

We choose:

$$V(x) = \frac{1}{2}\dot{x}^2 + \frac{1}{4}x^4$$

Verification:

- $V(0) = 0 \quad \checkmark$
- $V(x) > 0$ for $x \neq 0 \quad \checkmark \quad$ (sum of positive terms)

This is a valid **Lyapunov function candidate**.

Derivative calculation:

$$\dot{V}(x) = \ddot{X} \cdot \dot{X} + \dot{X} \cdot X^3 = \dot{X}(\ddot{X} + X^3)$$

Substitute from original equation: $\ddot{X} = -|\dot{X}|\dot{X} - X^3$

$$\dot{V}(x) = \dot{X}(-|\dot{X}|\dot{X} - X^3 + X^3) = -|\dot{X}|^3$$

Result:

$$\dot{V}(x) = -|\dot{X}|^3 \leq 0$$

Since $\dot{V} \leq 0$, the system is **Lyapunov stable**.

Since $V(x) \to \infty$ as $\|x\| \to \infty$, it is **globally stable**.

Stability Analysis Summary

Semi-negative definite $\dot{V} \Rightarrow$ Lyapunov stable and globally stable

Nonlinear Equation of Motion:

$$\ddot{x} + |\dot{x}|\dot{x} + x + x^3 = 0$$

State Variables:

$$x_1 = x, \quad x_2 = \dot{x}$$

Goal: Analyze stability at equilibrium $x^* = 0$.

System in state-space form:

$$\dot{X} = f(X) \quad \text{with} \quad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T$$

$$f_1 : \quad \dot{x}_1 = x_2$$

$$f_2 : \quad \dot{x}_2 = -|x_2|x_2 - x_1 - x_1^3$$

Jacobian at equilibrium:

$$A = \left.\frac{\partial f}{\partial x}\right|_{(0,0)}$$

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

General linearization formula:

$$f(x,u) = f(0,0) + \left.\frac{\partial f}{\partial x}\right|_{x=0,u=0} \cdot x + \left.\frac{\partial f}{\partial u}\right|_{x=0,u=0} \cdot u + \text{H.O.T.}$$

At equilibrium $(0,0)$:

$$\dot{x}_2 = -x_1 - 0 \cdot x_2 = -x_1$$

Characteristic equation:

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda & -1 \\ 1 & \lambda \end{bmatrix} = \lambda^2 + 1 = 0$$

Eigenvalues:

$$\lambda_{1,2} = \pm i$$

Result: $\lambda_{1,2} = \pm i$ (purely imaginary)

Important Caveat

Linearization at a **center** is inconclusive!

- Linear analysis suggests center (marginally stable)
- Nonlinear terms determine actual stability
- Must use Lyapunov analysis for definitive answer

1. **Energy Interpretation:**

Any scalar function that mirrors system energy.

2. **Sum of Squares Method:**

$$V(x) = F(x)^T F(x)$$

3. **Quadratic Form (Linear Systems):**

$$V(x) = x^T P x$$

where P is positive definite.

General quadratic form:

$$V(x) = \frac{1}{2}x^T x$$

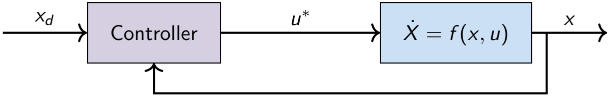
For linear systems $\dot{x} = Ax$:

$$V = x^T P x$$

where P solves the Lyapunov equation: $A^T P + PA = -Q$ for positive definite Q.

Problem: Given $\dot{X} = f(x, u)$

Goal: Find $u^*(t)$ such that $x(0) \rightarrow x_d$ as $t \rightarrow \infty$



Lyapunov-Based Control Design Steps:

1. Choose a Lyapunov candidate $V(x)$
2. Compute $\dot{V}(x, u)$
3. Design u to make $\dot{V} \leq 0$ (or < 0)

System:

$$\ddot{x} + \dot{x}^2 + x^3 = u$$

Goal: Design u such that $x \to 0$ as $t \to \infty$

Choose Lyapunov function:

$$V(x) = \frac{1}{2}\dot{x}^2 + \frac{1}{4}x^4$$

Time derivative:

$$\dot{V}(x) = \ddot{x} \cdot \dot{x} + \dot{x} \cdot x^3$$
$$= \dot{x}(\ddot{x} + x^3)$$
$$= \dot{x}(u - \dot{x}^2)$$

Now we need to choose u to make $\dot{V} < 0$.

Choose:

$$u^* = \dot{x}^2 + x^2 - \dot{x}$$

Then:

$$\dot{V} = \dot{x}(\dot{x}^2 + x^2 - \dot{x} - \dot{x}^2) = \dot{x} \cdot x^2 - \dot{x}^2$$

With control $u^* = \dot{x}^2 + x^2 - \dot{x}$, the closed-loop system becomes:

$$\ddot{x} + \dot{x} + x^3 = 0$$

Result

This system is **globally asymptotically stable** by Lyapunov analysis!

Euler-Lagrange Equations:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau$$

where:

- $M(q)$: inertia matrix (positive definite)
- $C(q,\dot{q})$: Coriolis/centrifugal matrix
- $G(q)$: gravity vector

Important Property:

$\dot{M} - 2C$ is **skew-symmetric**:

$$x^T(\dot{M} - 2C)x = 0 \quad \forall x$$

This property is crucial for Lyapunov-based stability proofs in robot control.

Control Law:

$$\tau = -K_p q - K_d \dot{q} + G(q)$$

where K_p, K_d are diagonal positive-definite gain matrices.

Choose:

$$V = \frac{1}{2}\dot{q}^T M(q)\dot{q} + \frac{1}{2}q^T K_p q$$

Verification:

- $V(0,0) = 0 \quad \checkmark$
- $V > 0$ for $(q, \dot{q}) \neq 0 \quad \checkmark$

(Since M and K_p are positive definite)

Computing $\dot{V}$:

$$\dot{V} = \frac{1}{2}\dot{q}^T \dot{M}\dot{q} + \dot{q}^T M\ddot{q} + \dot{q}^T K_p q$$

After substituting closed-loop dynamics:

$$\dot{V} = \frac{1}{2}\dot{q}^T(\dot{M} - 2C)\dot{q} - \dot{q}^T K_d \dot{q}$$

Using skew-symmetry of $(\dot{M} - 2C)$:

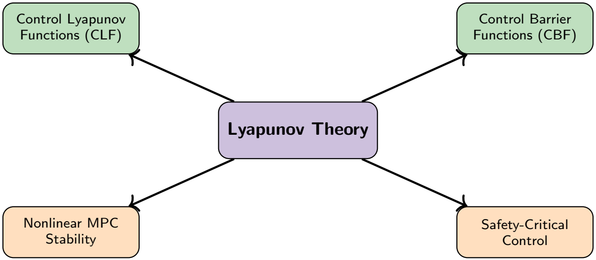
$$\boxed{\dot{V} = -\dot{q}^T K_d \dot{q} \leq 0}$$

Conclusion

The robot system is **globally asymptotically stable**: $q, \dot{q} \to 0$

- Nonlinear systems have multiple equilibria with local stability
- Phase portraits visualize 2D system behavior
- Eigenvalues classify stability: $\text{Re}(\lambda) < 0 \Rightarrow$ stable
- Lyapunov: $V > 0$, $\dot{V} \leq 0 \Rightarrow$ stable
- $\dot{V} < 0 \Rightarrow$ asymptotically stable
- Lyapunov functions enable controller design

Connection to Next Lectures



Next: NMPC with safety constraints via Control Barrier Functions.

End of Lecture 5

Questions?

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