NYU TANDON SCHOOL OF ENGINEERING

Autonomous Mobile Robots

Lecture 2: Vehicle Kinematics & Equations of Motion

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Last Lecture: Introduction to Autonomous Mobile Robots

This Lecture: Vehicle Kinematics & Equations of Motion

- Reference frames and coordinate transformations
- Differential-drive kinematics
- Bicycle model dynamics (Ackermann steering)
- Omni-directional robots
- Lagrangian dynamics

Next Lecture: Control & Perception Fundamentals

Motivation:

- Each wheel generates forces and imposes motion constraints
- Wheels are rigidly attached to the chassis
- We need a **common reference frame** to combine all wheel effects

Two key frames:

- **Global (Inertial) Frame**: Fixed to the world
- **Robot (Body) Frame**: Attached to the robot, moves with it

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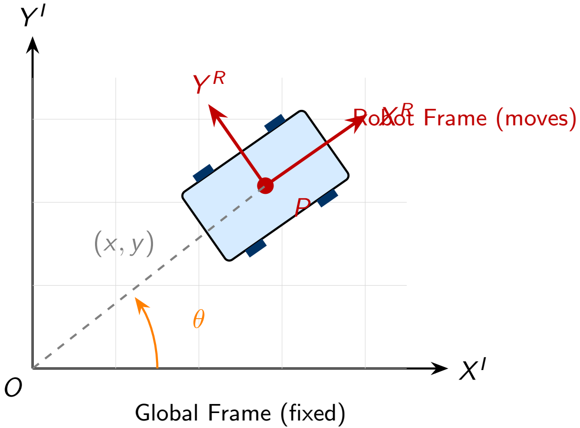


Global (Inertial) Frame (X^I, Y^I):

- Fixed to the world — does not move
- Used to describe:
  - Maps and environments
  - Goal positions
  - Trajectories
- Origin O is a fixed reference point

Robot (Body) Frame (X^R, Y^R):

- Attached to the robot at reference point P
- P is often placed at the center of the wheel axle
- X^R points forward (heading direction)
- Y^R points to the left of the robot
- Moves and rotates with the robot

Robot Pose

The **pose** of the robot is defined by three quantities:

- x: position along global X^I axis
- y: position along global Y^I axis
- θ: orientation of X^R with respect to X^I

Pose Vector

$$\xi^I = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$$

The relationship between frames is captured by the **rotation matrix**:

Rotation Matrix $R(\theta)$

$$R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This matrix transforms vectors from the **global frame** to the **robot frame**.

Key Properties:

- $R(\theta)^{-1} = R(\theta)^T = R(-\theta)$
- The matrix is **orthogonal**: $R^T R = I$
- $\det(R) = 1$ (proper rotation, no reflection)
- The third row/column handles θ, which is the same in both frames

Robot velocities can be expressed in either frame:

Global frame velocities:

$$\dot{\xi}^I = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix}$$

Robot frame velocities:

$$\dot{\xi}^R = \begin{bmatrix} v_x \\ v_y \\ \omega \end{bmatrix}$$

where v_x = forward, v_y = lateral, ω = angular velocity.

Velocity Mapping

$$\boxed{\dot{\xi}^R = R(\theta)\,\dot{\xi}^I}$$

Key insight: This mapping is **pose-dependent**.

It depends on the current orientation θ of the robot.

Inverse transformation (robot $\rightarrow$ global):

$$\dot{\xi}^I = R(\theta)^{-1}\dot{\xi}^R = R(-\theta)\dot{\xi}^R$$

When the robot faces the Y^I direction ($\theta = \pi/2$):

$$R\left(\frac{\pi}{2}\right) = \begin{bmatrix} \cos\frac{\pi}{2} & \sin\frac{\pi}{2} & 0 \\ -\sin\frac{\pi}{2} & \cos\frac{\pi}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Velocity mapping:

$$\dot{\boldsymbol{\xi}}^R = R\left(\frac{\pi}{2}\right)\dot{\boldsymbol{\xi}}^I = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{y} \\ -\dot{x} \\ \dot{\theta} \end{bmatrix}$$

Interpretation:

- Robot's forward velocity: $v_x = \dot{y}$
- Robot's lateral velocity: $v_y = -\dot{x}$
- Angular velocity: $\omega = \dot{\theta}$ (same in both frames)

Key constraint: Wheeled robots cannot move sideways.

This means no lateral slip — the wheels roll, they don't skid sideways.

In the robot frame, this constraint is:

No Side-Slip Constraint

$$\boxed{v_y = 0}$$

This is called a **nonholonomic constraint**.

Nonholonomic = restricts **velocity** but not **position**.

- The robot **can** reach any position (x, y) and orientation θ
- The robot **cannot** move instantaneously sideways
- It must maneuver (like parallel parking a car)

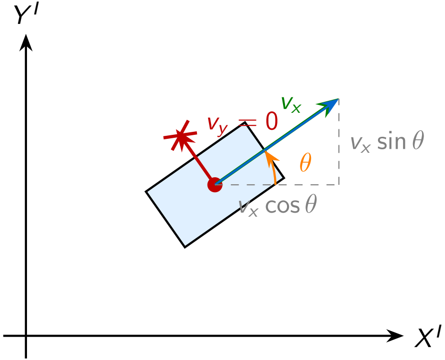
Contrast with holonomic: A holonomic robot (like omni-drive) can move in any direction instantly.

With $v_y = 0$, the robot kinematics simplify to:

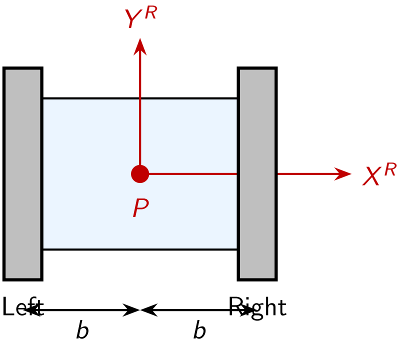
Unicycle Kinematic Equations

$$\dot{x} = v_x \cos\theta$$
$$\dot{y} = v_x \sin\theta$$
$$\dot{\theta} = \omega$$

This is the **unicycle model —** a planar rigid body with no lateral motion.



The robot moves in the direction of its heading θ.



Y^R
X^R
P
Left
Right
b
b

Parameters:

- r = wheel radius
- b = half the axle length (distance from center to wheel)
- $\dot{\phi}_1$ = right wheel angular speed (rad/s)
- $\dot{\phi}_2$ = left wheel angular speed (rad/s)

Sign conventions:

- Forward motion $(+)$: $v_x > 0$
- Counter-clockwise yaw $(+)$: $\omega > 0$
- Right wheel is at position $+b$
- Left wheel is at position $-b$

Key relationship: Each wheel's linear velocity is $v = r \cdot \dot{\phi}$

Each wheel's linear velocity combines forward motion and rotation.

Right wheel (at position $+b$):

$$v_{\text{right}} = v_x + b \cdot \omega$$

Left wheel (at position $-b$):

$$v_{\text{left}} = v_x - b \cdot \omega$$

Since linear velocity $v = r\dot{\phi}$:

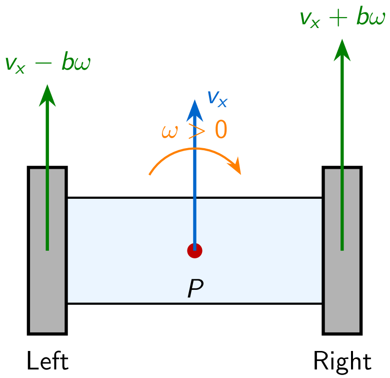
Right Wheel

$$r\dot{\phi}_1 = v_x + b\omega$$

Left Wheel

$$r\dot{\phi}_2 = v_x - b\omega$$

Physical Interpretation



When $\omega > 0$ (turning left):

- Right wheel moves faster than left wheel

When $\omega = 0$ (going straight):

- Both wheels have the same speed

When $v_x = 0$ and $\omega \neq 0$:

- Robot spins in place

Add the two wheel equations:

$$r\dot{\phi}_1 + r\dot{\phi}_2 = 2v_x$$

Forward Velocity

$$v_x = \frac{r}{2}(\dot{\phi}_1 + \dot{\phi}_2)$$

Subtract the two wheel equations:

$$r\dot{\phi}_1 - r\dot{\phi}_2 = 2b\omega$$

Angular Velocity

$$\omega = \frac{r}{2b}(\dot{\phi}_1 - \dot{\phi}_2)$$

Jacobian Mapping (Robot Frame)

$$\begin{bmatrix} v_x \\ v_y \\ \omega \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{r}{2} & \frac{r}{2} \\ 0 & 0 \\ \frac{r}{2b} & -\frac{r}{2b} \end{bmatrix}}_{J^R} \begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix}$$

The Jacobian J^R maps wheel angular velocities to body velocities.

Note: $v_y = 0$ always (nonholonomic constraint).

To navigate, we need motion in **global coordinates**.

Transform from robot frame to global frame:

Global Frame Kinematics

$$\dot{\xi}^I = R(-\theta)\begin{bmatrix} v_x \\ 0 \\ \omega \end{bmatrix} = \begin{bmatrix} v_x \cos\theta \\ v_x \sin\theta \\ \omega \end{bmatrix}$$

Expanded form:

$$\dot{x} = \frac{r}{2}(\dot{\phi}_1 + \dot{\phi}_2)\cos\theta$$

$$\dot{y} = \frac{r}{2}(\dot{\phi}_1 + \dot{\phi}_2)\sin\theta$$

$$\dot{\theta} = \frac{r}{2b}(\dot{\phi}_1 - \dot{\phi}_2)$$

These equations describe how wheel speeds produce robot motion.

Kinematics only explains how velocity inputs produce motion.

For car-like robots (Ackermann steering):

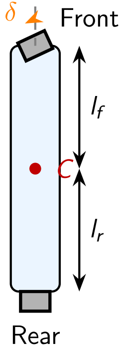
- Forces and inertia matter
- Need to model turning, drifting, braking

Newton's Laws:

$$\sum \boldsymbol{F} = m\boldsymbol{a}, \quad \sum M = I\dot{\omega}$$

Simplification:

- 4-wheel vehicle $\rightarrow$ 2-wheel equivalent
- Front wheel can steer (angle δ)
- Rear wheel is fixed
- Reference: Center of Mass (COM)



Geometric parameters:

- l_f: distance from COM to front axle
- l_r: distance from COM to rear axle
- δ: steering angle of front wheel

Velocities at COM:

- V_x: longitudinal velocity
- V_y: lateral velocity
- ω: yaw rate

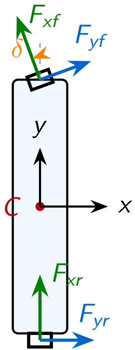
Forces:

- F_{xf}, F_{yf}: forces at front wheel
- F_{xr}, F_{yr}: forces at rear wheel

Governing principle:

$$\sum \boldsymbol{F} = m\boldsymbol{a}$$

$$\sum M = I\dot{\omega}$$



Applying Newton's laws in the body frame:

Longitudinal (x-axis):

$$m\dot{V}_x - mV_y\omega = F_{xf}\cos\delta - F_{yf}\sin\delta + F_{xr}$$

The term $mV_y\omega$ is a **Coriolis effect** from the rotating frame.

Lateral (y-axis):

$$m\dot{V}_y + mV_x\omega = F_{yf}\cos\delta + F_{xf}\sin\delta + F_{yr}$$

The term $mV_x\omega$ is also a **Coriolis/centrifugal effect**.

Yaw (rotation about z-axis):

$$I_{zz}\dot{\omega} = l_f(F_{yf}\cos\delta + F_{xf}\sin\delta) - l_r F_{yr}$$

This describes how torques from tire forces create yaw acceleration.

Define state vector: $\boldsymbol{q} = [V_x, V_y, \omega]^T$

Mass/Inertia Matrix:

$$M = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$

Coriolis/Centrifugal terms:

$$C\dot{\boldsymbol{q}} = \begin{bmatrix} -mV_y\omega \\ mV_x\omega \\ 0 \end{bmatrix}$$

This can be written as:

$$C = \begin{bmatrix} 0 & -m\omega & 0 \\ m\omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Vehicle Dynamics

$$\boxed{M\ddot{\boldsymbol{q}} + C\dot{\boldsymbol{q}} + G = B_y F_y + B_x F_x}$$

where:

- M: mass/inertia matrix
- C: Coriolis matrix
- $G = 0$ on flat ground
- B_y, B_x: input matrices for tire forces

Lateral force input:

$$B_y = \begin{bmatrix} -\sin\delta & 0 \\ \cos\delta & 1 \\ l_f \cos\delta & -l_r \end{bmatrix}, \quad F_y = \begin{bmatrix} F_{yf} \\ F_{yr} \end{bmatrix}$$

Longitudinal force input:

$$B_x = \begin{bmatrix} \cos\delta & 1 \\ \sin\delta & 0 \\ l_f \sin\delta & 0 \end{bmatrix}, \quad F_x = \begin{bmatrix} F_{xf} \\ F_{xr} \end{bmatrix}$$

Applications of Bicycle Model

This model is the foundation for:

- Vehicle stability control
- Yaw stability and skid avoidance
- Path tracking and trajectory control
- Autonomous vehicle motion planning

Why omni-drive?

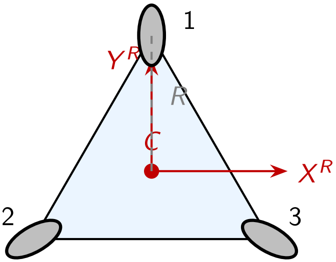
- 3 omni-wheels placed 120° apart
- Full **3-DOF control**: (x, y, θ)
- Robot can move in **any direction** instantly
- Can rotate **without reorientation**

Holonomic system:

- 3 actuators $\leftrightarrow$ 3 DOF
- No velocity constraints
- Can move sideways instantly

Contrast with differential drive:

- Differential drive has constraint $v_y = 0$
- Cannot move sideways without rotating



120° spacing between wheels

Robot frame: (x, y, θ) at center of mass

Global frame: (X, Y) fixed to world

Wheel positions:

- Equally spaced at angles $\alpha_1, \alpha_2, \alpha_3$
- Each wheel at radius R from center
- Individual wheel radius: r

Typical: $\alpha_1 = 90$, $\alpha_2 = 210$, $\alpha_3 = 330$

For wheel i at angle α_i:

Wheel Velocity

$$r\dot{\phi}_i = -v_x \sin\alpha_i + v_y \cos\alpha_i + R\omega$$

Matrix form:

$$r \begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{\phi}_3 \end{bmatrix} = \begin{bmatrix} -\sin\alpha_1 & \cos\alpha_1 & R \\ -\sin\alpha_2 & \cos\alpha_2 & R \\ -\sin\alpha_3 & \cos\alpha_3 & R \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ \omega \end{bmatrix}$$

Omni-Drive: Force Balance

Apply Newton's laws: $\sum \boldsymbol{F} = m\boldsymbol{a}$

X-direction:

$$F_1 \cos\alpha_1 + F_2 \cos\alpha_2 + F_3 \cos\alpha_3 = m\ddot{x}$$

Y-direction:

$$F_1 \sin\alpha_1 + F_2 \sin\alpha_2 + F_3 \sin\alpha_3 = m\ddot{y}$$

Rotation:

$$R(F_1 + F_2 + F_3) = I_{zz}\ddot{\theta}$$

Compact Form

$$M\ddot{\boldsymbol{\xi}} = B\boldsymbol{F}$$

Mass matrix:

$$M = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$

Input matrix:

$$B = \begin{bmatrix} \cos\alpha_1 & \cos\alpha_2 & \cos\alpha_3 \\ \sin\alpha_1 & \sin\alpha_2 & \sin\alpha_3 \\ R & R & R \end{bmatrix}$$

Key property: $\det(B) \neq 0$

Therefore:

- B^{-1} exists
- Forces and accelerations are **uniquely related**
- Full controllability in all 3 DOF

Goal: Derive dynamic models using energy methods.

Generalized coordinates:

$$\boldsymbol{q} = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}, \quad \dot{\boldsymbol{q}} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix}$$

Lagrangian:

$$L = K - V$$

where K = kinetic energy, V = potential energy.

For planar mobile robots on flat ground:

- Potential energy: $V = 0$ (no gravity in plane)
- Lagrangian simplifies to: $L = K$

Euler-Lagrange Equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = F_i$$

where F_i are the generalized forces.

Translational:

$$K_t = \frac{1}{2}m(v_x^2 + v_y^2) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

Rotational:

$$K_r = \frac{1}{2}I_{zz}\omega^2 = \frac{1}{2}I_{zz}\dot{\theta}^2$$

Total:

$$K = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I_{zz}\dot{\theta}^2$$

Lagrangian (with $V = 0$):

$$L = K = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I_{zz}\dot{\theta}^2$$

Partial derivatives:

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}, \quad \frac{\partial L}{\partial x} = 0$$

$$\frac{\partial L}{\partial \dot{y}} = m\dot{y}, \quad \frac{\partial L}{\partial y} = 0$$

$$\frac{\partial L}{\partial \dot{\theta}} = I_{zz}\dot{\theta}, \quad \frac{\partial L}{\partial \theta} = 0$$

Applying Euler-Lagrange:

For x:

$$\frac{d}{dt}(m\dot{x}) = F_x \quad \Rightarrow \quad m\ddot{x} = F_x$$

For y:

$$\frac{d}{dt}(m\dot{y}) = F_y \quad \Rightarrow \quad m\ddot{y} = F_y$$

For θ:

$$\frac{d}{dt}(I_{zz}\dot{\theta}) = \tau \quad \Rightarrow \quad I_{zz}\ddot{\theta} = \tau$$

Equations of Motion

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} F_x \\ F_y \\ \tau \end{bmatrix}$$

The generalized forces (F_x, F_y, τ) come from wheel forces mapped to the global frame.

- Global (inertial) vs. Robot (body) frames
- Rotation matrix $R(\theta)$ transforms between frames
- Velocity mapping: $\dot{\boldsymbol{\xi}}^R = R(\theta)\dot{\boldsymbol{\xi}}^I$

- Nonholonomic constraint: $v_y = 0$
- Unicycle model: $\dot{x} = v_x \cos\theta$, $\dot{y} = v_x \sin\theta$
- Jacobian maps wheel velocities to body velocities
- $v_x = \frac{r}{2}(\dot{\phi}_1 + \dot{\phi}_2)$, $\omega = \frac{r}{2b}(\dot{\phi}_1 - \dot{\phi}_2)$

- Dynamics include forces, inertia, Coriolis terms
- Compact form: $M\ddot{\boldsymbol{q}} + C\dot{\boldsymbol{q}} = B_y F_y + B_x F_x$
- Foundation for vehicle control and stability

Omni-Drive:

- Holonomic: full 3-DOF control
- $M\ddot{\xi} = B\boldsymbol{F}$

Lagrangian Dynamics:

- Energy-based derivation: $L = K - V$
- Euler-Lagrange equations give equations of motion

End of Lecture 2

Vehicle Kinematics & Equations of Motion

Questions?