Fractals

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MTD421: Btech Project (BTP) Report

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Primary Citations :

1. Fractal Geometry by Kenneth Falconer, Third Edition, Wiley Publications
2. Measure, Topology and Fractal Geometry by Gerald Edgar, Second Edition, Springer Publications
3. Prem Melville, Vinhthuy Phan (2000), “ Fractal Compression”, The University of Memphis

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1. Introduction

1.1. Project Objective

Pre-MidTerm

- Learn about the basics of Fractals, their formation and their properties.
- Approximating Fractals up to n iterations of formation
- Learn about the fractional nature of dimension in case of fractals
- Find the dimensions of various Fractals

Post-MidTerm

- Explore the theoretical and practical aspects of the Mandelbrot and Julia Sets
- Study and Implement Fractal Image compression

1.2. Abstract

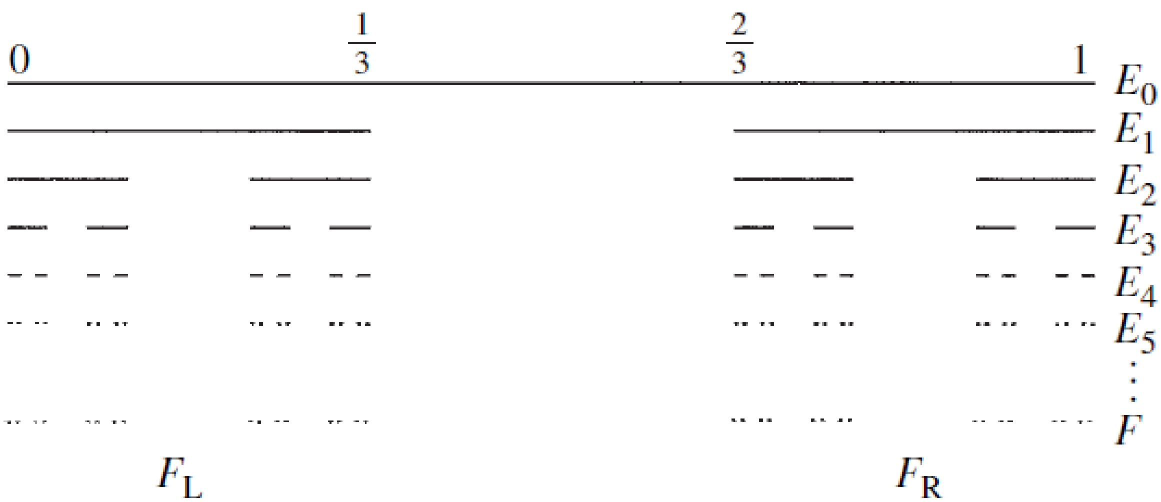


Fig : Middle Third Cantor Set

In the past, mathematics has been concerned largely with sets and functions to which the methods of classical calculus can be applied. Sets or functions that are not sufficiently smooth or regular have tended to be ignored as 'pathological' and not worthy of study.

In recent years, It has been realised that a great deal can be said, and is worth saying, about the mathematics of non-smooth objects. Moreover, irregular sets provide a much better representation of many natural phenomena than do the figures of classical geometry. Fractal geometry provides a general framework for the study of such irregular sets.

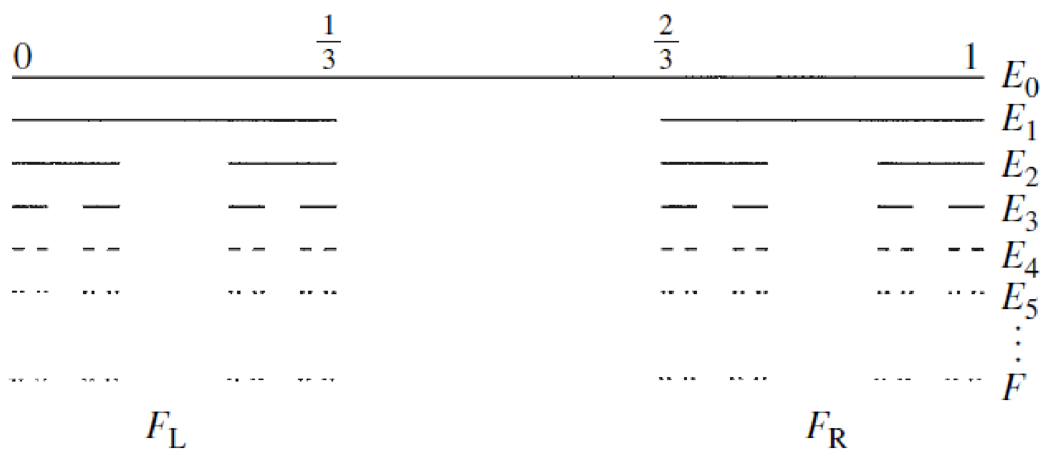
Motivation - Fractals in Nature

A fractal is a pattern that the laws of nature repeat at different scales. Examples are everywhere in the forest. Trees are natural fractals, patterns that repeat smaller and smaller copies of themselves to create the biodiversity of a forest. Each tree branch, from the trunk to the tips, is a copy of the one that came before it. This is a **basic principle** that we see over and over again in the fractal structure of organic life forms throughout the natural world.

Not only just the trees, but a lot of natural objects that we see around have a fractal structure. For example, the flowers around us, like the rose and even deltas of rivers, show fractal structure. They are self-repeating structures of the same pattern again and again on zooming in/out.

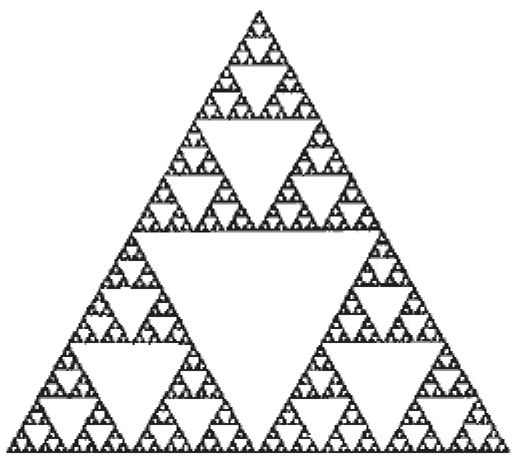
Looking mathematically, let's glance through some of the examples of fractals and then look at the basic general properties each of them shows.

Middle Third Cantor Set



The middle third Cantor set is one of the best known and most easily constructed fractals; nevertheless, it displays many typical fractal characteristics. It is constructed from a unit interval by a sequence of deletion operations (see above figure). Let E_0 be the interval [0, 1]. (Recall that [a, b] denotes the set of real numbers x such that $a \leqslant x \leqslant b$.) Let E_1 be the set obtained by deleting the middle third of E_0 so that E_1 consists of the two intervals [0, ⅓] and [⅔,1]. Deleting the middle thirds [of these intervals gives E_2; thus, E_2 comprises the four intervals [0, 1/9], [2/9,1/3], [2/3,7/9], [8/9,1]. We continue in this way, with E_k obtained by deleting the middle third of each interval in E_{k-1}. Thus, E_k consists of 2^k intervals, each of length 3^{-k}. The middle third cantor set is the intersection of all of these E_k. We can look at the length of this Set, and it comes out equal to zero! Hence it's dimension is something less than 1.

Sierpinski's Gasket



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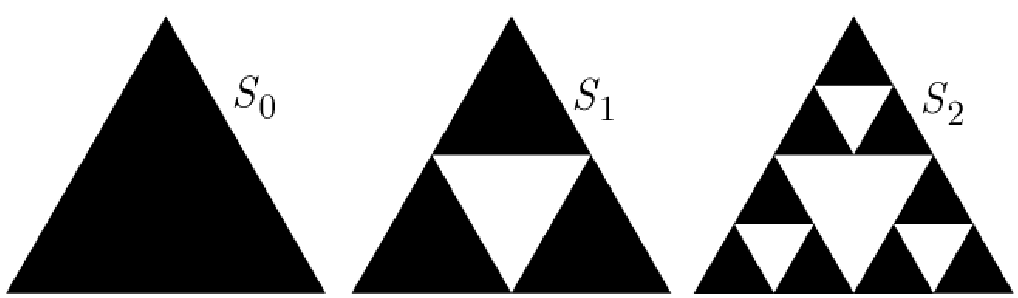
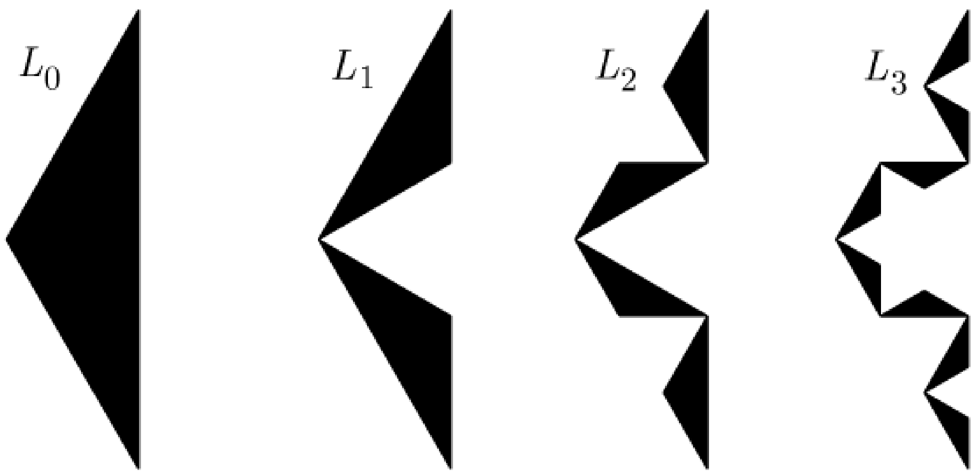
The Sierpinski triangle or gasket is obtained by repeatedly removing (inverted) equilateral triangles from an initial equilateral triangle of unit side length. (For many purposes, it is better to think of this procedure as repeatedly replacing an equilateral triangle by three triangles of half the height). Similar to the last example, we begin its construction from an equilateral triangle of unit side and remove the middle equilateral triangle of side length ½. We call it S_1. Next, we apply the same procedure on the three triangles and get a total of 9 triangles which we call S_2.



Going in the same fashion and taking the intersection, we get the Sierpinski's Gasket. We can look for the total area of this gasket which comes out equal to zero! Hence it's dimension is something less than 2. But if we try to find the sum of the length of the boundaries on the gasket, it shoots to infinity. It makes us believe that its dimension is somewhat lying between One and Two!

Koch Curve



For the construction of the Koch Curve, we begin with an isosceles triangle with angles = 120°, 30°, 30°. At each step, we remove an equilateral triangle from the triangle such that the remaining two triangles from the triangle are isosceles with the same angles as the original one. Like, L_0 is the base triangle, and after removal of the equilateral triangle, we get L_1. Similarly, after removals from L_1, we get L_2. The Koch Curve is the intersection of all the L_s.

We can look for the total area of this curve which comes out equal to zero! Hence it's dimension is something less than 2. But if we try to find the sum of the length of the boundaries, it shoots to infinity. It makes us believe that its dimension is somewhat lying between One and Two!

Observations from the above examples:

Looking at the above examples, when we refer to a set F as a fractal, we will typically have the following in mind.

1. F has a fine structure, that is, detail on arbitrarily small scales.

2. F is too irregular to be described in traditional geometrical language, both locally and globally.
3. Often F has some form of self-similarity, perhaps approximate or statistical.
4. Usually, the '**fractal dimension**' of F (defined in some way) is greater than its topological dimension.
5. In most cases of interest, F is defined in a very simple way, perhaps recursively.

All of the examples we saw led us to these observations about fractals. With this basic idea of fractals, we can define our objectives of the project.

2. Prerequisites

2.1. Metric Space

A metric space is a set S together with a function $d : S \times S \rightarrow [0,\infty)$ satisfying,

(a) $d(x,y) = 0 \iff x = y$;
(b) $d(x,y) = d(y,x)$;
(c) $d(x,z) \leq d(x,y) + d(y,z)$;

Theorem : Union of open sets are open

Contraction Mapping Theorem: *A contraction mapping f on a complete non-empty metric space S has a unique fixed point.*

Proof. First, there is at most one fixed point. If x and y are both fixed points, then $d(x, y) = d(f(x), f(y)) \leq rd(x, y)$. But $0 \leq r < 1$, so this is impossible if $d(x, y) > 0$. So $d(x, y) = 0$, hence $x = y$.

Now let x_0 be any point of S. Then define recursively

$$x_{n+1} = f(x_n) \text{ for } n \geq 0.$$

We claim that (x_n) is a Cauchy sequence. Write $a = (x_0, x_1)$. It follows by induction that $d(x_{n+1}, x_n) \leq ar^n$. But then, if $m < n$ we have

$$d(x_m, x_n) \leq \sum_{j=m}^{n-1} d(x_{j+1}, x_j) \leq \sum_{j=m}^{n-1} ar^n = (ar^m - ar^n) / (1-r)$$

$$= ar^m (1 - ar^{n-m}) / (1-r) \leq ar^m / (1-r).$$

Therefore, if $\varepsilon > 0$ is given, choose N large enough that $ar^N /(1 - r) < \varepsilon$. Then, for $n,m \geq N$, we have $d(x_m, x_n) < \varepsilon$. Now S is complete, and (x_n) is a Cauchy sequence, so it converges. Let x be the limit. Now f is continuous, so from $x_n \to x$ follows also $f(x_n) \to f(x)$. But $f(x_n) = x_{n+1}$, so $f(x_n) \to$ x. Therefore the two limits are equal, $x = f(x)$, so x is a fixed point.

Sequentially Compact: A metric space S is called sequentially compact iff every sequence in S has at least one cluster point (in S).

Countably Compact: A metric space S is called countably compact iff every infinite subset of S has at least one accumulation point (in S).

Bicompact: A metric space S is (temporarily) called bi-compact iff every family of closed sets with the finite intersection property has a nonempty intersection.

Theorem: Let S be a metric space. The following are equivalent:

(a) S is sequentially compact,
(b) S is countably compact,
(c) S is bicompact.

Compact: A metric space S will be called compact iff it has one (and therefore all) of the properties of the above theorem.

Compact** (for $\mathbb{R}^n$):** Let A $\subseteq \mathbb{R}^n$ (Euclidean Space). Then A is compact if and only if A is closed and bounded.

***Theorem*:** Let $f: S \to T$ be continuous. Let $A \subseteq S$ be compact. Then $f[A]$ is compact.

Uniform Continuity : f is uniformly continuous iff, for every $\varepsilon > 0$, there exists $\delta > 0$ so that for every $x \in S$, we have $f[B_\delta(x)] \subseteq B_\varepsilon(f(x))$.

Note*: In continuity, δ is allowed to depend on both ε and x, but in uniform continuity, δ does not depend on x.*

Uniform Convergence: Let S and T be two metric spaces. We will be considering functions $f: S \to T$. Let fn be a sequence of functions from S to T and let f be another function from S to T. The sequence f_n converges uniformly (on S) to the function f iff for every $\varepsilon > 0$, there exists $N \in N$ so that for all $x \in S$ and all $n \geq N$, we have $d(f_n(x), f(x)) < \varepsilon$.

2.2. Measure theory

We call μ a measure on $\mathbb{R}^n$ if μ assigns a non-negative number, possibly ∞, to each subset of $\mathbb{R}^n$ such that

(a) $\mu(\emptyset) = 0$;

(b) $\mu(A) \leqslant \mu(B)$ if $A \subset B$;

(c) if A1, A2,. . . . . is a countable (or finite) sequence of sets, then

$$\mu\left(\bigcup_{i=1}^{\infty}\right) \leqslant \sum_{i=1}^{\infty} \mu(A_i)$$

with equality if the A_i are disjoint Borel sets.

Lebesgue measure on $\mathbb{R}^n$***:*** We call a set of the form $A = \{(x_1,..., x_n) \in \mathbb{R}^n : a_i \leqslant x_i \leqslant b_i\}$ a coordinate parallelepiped in $\mathbb{R}^n$, its n-dimensional volume of A is given by

$$vol^n (A) = (b_1 - a_1)(b_2 - a_2). . . .(b_n - a_n)$$

For $\mathbb{R}$, it is length, for $\mathbb{R}^2$ it is the area.

Hausdorff metric (D): If A is a set and r > 0, then the *open r-neighbourhood* of A is

$$N_r(A) = \{ y : (x,y) < r \text{ for some } x \in A \} .$$

The definition of the Hausdorff metric D for bounded and non-empty set A and B

$$D(A,B) = \inf \{ r > 0 : A \subseteq N_r(B) \text{ and } B \subseteq N_r(A) \} .$$

By convention, $\inf \varnothing = \infty$.

3. Fractal Dimension and Iterated Function System (IFS)

This section is dedicated to the dimensions of the fractals and ways to find them. The familiar notion of Dimension is that a straight line/smooth curve is 1-Dimensional, a surface is 2-Dimensional and so on. The surprising feature for the fractal dimensions is that they need not be integers: they can be fractions.

Roughly, a dimension indicates, in some way, how much space a set occupies near each of its points. Fundamental to most definitions of dimension is the idea of 'measurement of a set at scale δ'. For each δ, we measure a set in a way that detects irregularities of size delta δ, and we see how these measurements behave as $\delta \to 0$.

3.1. Box Counting Dimension

Given a subset F of the plane, for each $\delta > 0$, we find the smallest number of sets of diameter at most δ that can cover the set F and we call this number $N_\delta(F)$ indicating the number of 'clumps' of size about δ into which F may be divided. The dimension of F reflects the way in which $N_\delta(F)$ grows as $\delta \to 0$. If $N_\delta(F)$ obeys, at least approximately, a power law

$$N_\delta(F) \simeq c\delta^{-s}$$

for positive constants c and s, we say that F has box dimension s. To solve for s, we take logarithms.

$$\log N_\delta(F) \simeq \log c - s \log \delta,$$

and we might hope to obtain as

$$s = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}$$

with the second term disappearing in the limit.

Let F be any non-empty bounded subset of $\mathbb{R}^n$ and let $N_\delta(F)$ be the least number of sets of diameter at most δ which can cover F, that is, the least number of sets in any δ-cover of F. The lower and upper box-counting dimensions of F, respectively, are defined as

$$\overline{dim}_B F = \lim_{\delta \to 0} sup \frac{\log N_\delta(F)}{-\log \delta}$$

$$\underline{dim}_B F = \lim_{\delta \to 0} inf \frac{\log N_\delta(F)}{-\log \delta}$$

Clearly, Lower Box Dimension ≤ Upper Box Dimension, and if these are equal, we refer to the common value as the box-counting dimension or box dimension of F. Also, to avoid problems with 'log 0' or 'log ∞', we generally consider box dimension only for non-empty bounded sets.

Equivalent Definitions:

The lower and upper box-counting dimensions of a subset F of $\mathbb{R}^n$ are given by

$$\overline{dim}_B F = \lim_{\delta \to 0} sup \frac{log\ N_\delta(F)}{-\ log\ \delta}$$

$$\underline{dim}_B F = \lim_{\delta \to 0} inf \frac{log\ N_\delta(F)}{-\ log\ \delta}$$

and the box-counting dimension of F by

$$dim_B F = \lim_{\delta \to 0} inf \frac{log\ N_\delta(F)}{-\ log\ \delta}$$

(if this limit exists), where $N_\delta(F)$ is any of the following:

1. the smallest number of sets of diameter at most δ that cover F;
2. the smallest number of closed balls of radius δ that cover F;
3. the smallest number of cubes of side δ that cover F;
4. the number of δ-mesh cubes that intersect F.

Using the above definitions, we are in a position to calculate the box-counting dimensions of some of the fractals we have seen earlier.

Middle Third Cantor Set:

Let F be the middle third Cantor set. Then the box-counting dimension equals log 2 / log 3.
Calculation:

We shall calculate it by showing that the lower box-counting dimension equals the upper box-counting dimension, which equals log 2 / log 3.

If $3^{-k} < \delta \leqslant 3^{-k+1}$, then the 2^k level-k intervals of E_k of length 3^{-k} provide a δ-cover of F, so that $N_\delta(F) \leqslant 2^k$, where $N_\delta(F)$ is the least number of sets in a δ-cover of F.

$$\overline{dim}_B F = \lim_{\delta \to 0} sup \frac{log\ N_\delta(F)}{-\ log\ \delta} \leq \lim_{k \to \infty} sup \frac{log\ 2^k}{-log\ 3^{(-k+1)}} = \lim_{k \to \infty} sup \frac{k\ log\ 2}{(k-1)\ log\ 3} = \frac{log\ 2}{log\ 3}$$

$$\underline{dim}_B F = \lim_{\delta \to 0} inf \frac{log\ N_\delta(F)}{-\ log\ \delta} \geq \lim_{k \to \infty} inf \frac{log\ 2^k}{-log\ 3^{(-k-1)}} = \lim_{k \to \infty} inf \frac{k\ log\ 2}{(k+1)\ log\ 3} = \frac{log\ 2}{log\ 3}$$

Hence the box-Counting dimension for the Middle Third Cantor Set equals $\frac{log\ 2}{log\ 3}$.

Note that the dimension came out less than 1!

Sierpinski's Triangle:

Let F be the Sierpinski triangle. Then the box Counting Dimension equals $\frac{\log 3}{\log 2}$.

Calculation:

We assume that the initial side length was 1. Then the basic geometric observation is that in the construction of F, the kth stage of the construction consists of 3^k equilateral triangles of side length and diameter 2^{-k}. Thus, if $2^{-k} < \delta \leqslant 2^{-k+1}$, the 3^k triangles of E_k give a δ cover of F, so $N_\delta(F) \leqslant 3^k$.

$$\overline{dim}_B F = \lim_{\delta \to 0} sup \frac{\log N_\delta(F)}{-\log \delta} \leq \lim_{k \to \infty} sup \frac{\log 3^k}{-\log 2^{(-k+1)}} = \frac{\log 3}{\log 2}$$

$$\underline{dim}_B F = \lim_{\delta \to 0} inf \frac{\log N_\delta(F)}{-\log \delta} \geq \lim_{k \to \infty} inf \frac{\log 3^{k-1}}{-\log 2^{(-k-1)}} = \frac{\log 3}{\log 2}$$

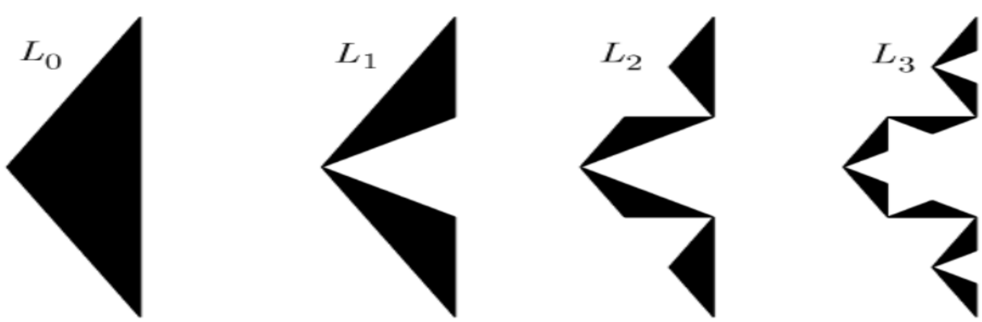
Hence the box-counting dimension of F is $\frac{\log 3}{\log 2}$.

Note that the box-counting dimension is greater than one and less than 2.

Note : *More generally, a set F made up of m similar disjoint copies of itself at scale r has the box-counting dimension of log m / − log r.*

Koch Curve:

For Koch's curve, we shall apply the above general statement taking advantage of the self-similarity in the curve.



Carefully looking at the Koch Curve's formation, we see that after every two steps, the scale is reduced to one-third of the original, and the number of copies is multiplied by a factor of 4. And hence the box dimension shall equal

$$\frac{log(4^{k})}{-log(3^{-k})} = \frac{log\ 4}{log\ 3}$$

3.2. Iterated Function System (IFS)

Contraction Map: Let D be a closed subset of $\mathbb{R}^n$, often D is $\mathbb{R}^n$ itself. A mapping $S : D \to D$ is called a contraction on D if there exists a number r with **0 < r < 1** such that $|S(x) - S(y)| \leqslant r|x - y|$ *for all* $x, y \in D$.

Iterated Function System (IFS) : A finite family of contractions $\{ S_1, S_2, \ldots, S_m \}$, with $m \geqslant 2$, is called an iterated function system.

***Attractor*:** A non-empty compact subset F of D is an attractor (or invariant set) for the IFS if it is made up of its images under the S, i.e.

$$F = \bigcup_{i=1}^{m} S_i(F)$$

***Theorem* :** *Let* $\{ S_1, S_2, \ldots, S_m \}$ *be an IFS of contractions on a closed set* $D \subset \mathbb{R}^n$*, so that*

$$|S_i(x) - S_i(y)| \leqslant r_i|x - y| \qquad (x, y \in D) \tag{3.2.1}$$

with $r_i < 1$ *for each i. Then the system has a unique attractor F, that is, a unique non-empty compact set such that*

$$F = \bigcup_{i=1}^{m} S_i(F) \tag{3.2.2}$$

Moreover,

$$F = \bigcap_{k=0}^{\infty} S^k(E) \tag{3.3.3}$$

for every non-empty compact set $E \in \mathfrak{R}$ *such that* $S_i(E) \subset E$ *for all i, where* $\mathfrak{R}$*denote the class of all non-empty compact subsets of D.*

<u>*Proof*</u>. If A, B $\in \mathfrak{R}$, then

$$d(S(A), S(B)) = d\left(\bigcup_{i=1}^{m} S_i(A), \bigcup_{i=1}^{m} S_i(B)\right) \leqslant max_{1<=i<=m}\, d(S_i(A), S_i(B))$$

using the definition of the metric d and noting that if the δ-neighbourhood $(S_i(A))_\delta$ contains $S_i(B)$ for all i, then $\left(\bigcup_{i=1}^{m} S_i(A)\right)_\delta$ contains $\bigcup_{i=1}^{m} S_i(B)$, and vice versa.

By (3.3.1), $$d(S(A), S(B)) \leqslant (max_{1<=i<=m}\, r_i)\, d(A, B)$$
(3.3.4)

It may be shown that d is a complete metric, that is, every Cauchy sequence of sets in is convergent to a set in. Since $0 < max_{1<=i<=m}\, r_i < 1$, (3.3.4) means that S is a contraction on the complete metric space ($\mathfrak{R}$, d). By Banach's contraction mapping theorem, S has a unique fixed point, that is, there is a unique set F $\in \mathfrak{R}$, such that S(F) = F, satisfying (3.3.2), and moreover that $d(S^k(E), F) \to 0$ as $k \to \infty$. In particular, if $S_i(E) \subset E$ for all i, then S(E) $\subset$ E, so that $S^k(E)$ is a decreasing sequence of non-empty compact sets containing F with intersection $\bigcap_{k=0}^{\infty} S^k(E)$ which must equal F.

3.3. Self Similarity Dimension

Ratio List : A ratio list is a finite list of positive numbers, $(r_1, r_2, \ldots, r_n)$.

Similarity : An iterated function system realizing a ratio list$(r_1, r_2, \ldots, r_n)$ in a metric space is a list $(S_1, S_2, \ldots, S_n)$, where $S_i : \mathbb{R}^n \to \mathbb{R}^n$ is a similarity with ratio r_i, i.e.

$$|S_i(x) - S_i(y)| = r_i\, |x - y| \qquad (x, y \in \mathbb{R}^n)$$

***Self Similar Set*:** The attractor of a collection of similarities is called a self-similar set, being a union of a number of smaller similar copies of itself.

Self Similarity Dimension: The self-similarity dimension *"s"* satisfy : $\sum_{i=1}^{m} r_i^s = 1$

Open Set Condition: We say that the S_i satisfy the open set condition if there exists a non-empty bounded open set V such that $\bigcup_{i=1}^{m} S_i(V) \subset V$ with this union disjoint.

Note : In particular, the open set condition holds if the images of the attractor $S_1(F), \ldots, S_m(F)$ are disjoint.

Theorem : S*uppose that the open set condition holds for the similarities S_i on $\mathbb{R}^n$ with ratios $0 < r_i < 1$ for $1 \leqslant i \leqslant m$. If F is the attractor of the IFS $\{S_1, S_2, \ldots, S_m\}$, that is,*

$$F = \bigcup_{i=1}^{m} S_i(F) ,$$

($\dim_B F$: Box - Counting Dimension)

then $\dim_H F = \dim_B F = s$, where s is given by

($\dim_H F$: Hausdorff Dimension)

$$\sum_{i=1}^{m} r_i^s = 1 .$$

Moreover, for this value of s, $0 < \mathcal{H}^S(F) < \infty$.

Example: The self-similarity dimension of the Cantor set is log(2)/log(3).

<u>*Proof*</u>: For the middle-third Cantor Set, the attractor is given by:

$$S_1(x) = \frac{x}{3}$$

$$S_2(x) = \frac{x}{3} + \frac{2}{3}$$

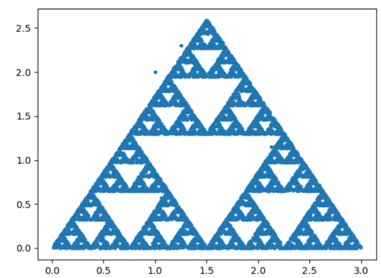
S_1 and S_2 represent the left and right halves of the Cantor Set. The ratios are nothing but ⅓,⅓ respectively and $F = S_1(F) \cup S_2(F)$. Thus F is an attractor of the IFS consisting of the contractions $\{S_1, S_2\}$. Now using the above theorem

$$2 * (\frac{1}{3})^s = 1 \Rightarrow s = \frac{\log 2}{\log 3}$$

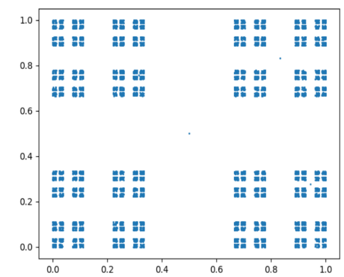
Which is the same as the box-counting dimension of the Cantor Set.

3.4. Plotting Fractals

1. Choose a point inside the Fractal boundaries
2. Choose one of the Contraction maps randomly to operate on the point and update the point as your new point
3. Plot the points



Sierpinski's Triangle



Cantor Dust

Plotting The Fractals

Input : Fractal Boundary Points, IFS functions, Max Points
Output : Image I
begin
  Initialise start Point P inside the Fractal Boundary Points
  for i in range(Max Points):
    Store the Point P
    Choose an IFS randomly (say f)
    Set $P = f(P)$
  Plot the stored Points
  return I
end

4. Julia and Mandelbrot Sets

This section is dedicated to some highly intricate sets generated using a very simple process. Functions of the form $f(z) = z^n + c$, can give rise to fractals of exotic appearance. These sets result from iteration of a function of a complex variable f

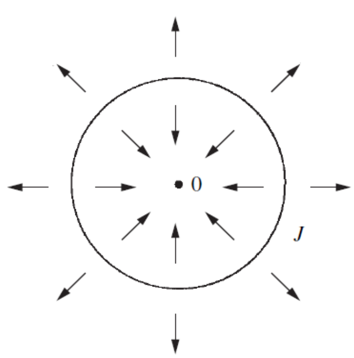
4.1. Julia Sets

Take $f: C \to C$to be a polynomial of degree $n \geq 2$ with complex coefficients. We write f^k for the k-fold composition $f \circ \cdots \circ f$ of the function f, so that $f^k(z)$ is the k[th] iterate $f\,(f\,(\cdots(f\,(z))\cdots))$ of z. A general Julia set is defined in the behavior of these iterates for large k values. Before the Julia Sets, we define the *filled-in* Julia Set of the polynomial f.

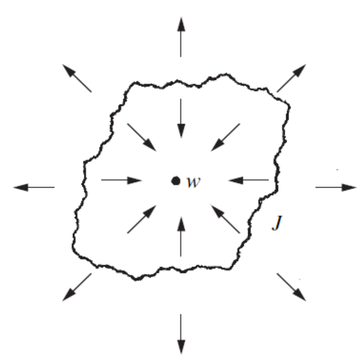
$$K(f) = \{z \in C: f^k(z) \nrightarrow \infty\}$$

The Julia set of f is the boundary of the filled-in Julia set, $J(f) = \partial K(f)$. Thus a point belonging to Julia set has neighbourhood points w and v such that $f^k(w) \to \infty$ and $f^k(v)$remains finite for high values of k.

For the simplest example, let $f\,(z) = z^2$, so that $f^k(z) = z^{2^k}$. Clearly, $f^k(z) \to 0$ as $k \to \infty$ if $|z| < 1$and $f^k(z) \to \infty$ if $|z| > 1$, but with $f^k(z)$ remaining on the circle $|z| = 1\ \forall\ k$ if $|z| = 1$. Thus, the filled-in Julia set k is the unit disc $|z| \leqslant 1$, and the Julia set J is its boundary, the unit circle, $|z| = 1$. The Julia set J is the boundary between the sets of points that iterate to 0 and ∞. Of course, in this special case, J is not a fractal.



$f\,(z) = z^2$



$f\,(z) = z^2 + c$

But, if we modify this example a little bit, taking $f\ (z)\ =\ z^2 + c$ where is a small complex number, we find that $f^k(z) \to w$ if $z$is small where $w$is a fixed point of f close to 0, and $f^k(z) \to \infty$if z is large. Again, the Julia set is the boundary between these two types of behaviour, but it turns out that now J is a fractal curve.

4.2. Quadratic Functions

We study the Julia sets of the form $f_c(z)\ = z^n + c$ with $n\ =\ 2$ where c is a complex constant. Later on, for fixed value of c we will look at the Julia Sets for various n.

We define the **Mandelbrot set *M*** to be the set of parameters c for which the Julia set of f_c is connected.

$$M\ =\ \{c \in \mathbb{C} : J(f_c)\ is\ connected\}$$

Theorem: For $c \in \mathbb{C}$, the Julia set $J(f_c)$ is connected if the sequence of iterates $\{f_c^k(0)\}_{k=1}^{\infty}$ is bounded and is totally disconnected otherwise. Thus:

$$M\ =\ \{c \in \mathbb{C} : J(f_c)\ is\ connected\}$$

$$=\ \{c \in \mathbb{C} : \{f_c^k(0)\}_{k=1}^{\infty}\ is\ bounded\}$$

$$=\ \{c \in \mathbb{C} : f_c^k(0)\ <\ \infty\ as\ k \to \infty\}$$

This theorem allows us to theoretically claim some results about the set M.

Results

- If $|c|\ \leq\ 1/4$ and $|z|\ \leq\ 1/2$, then $f_c(z)\ \leq\ 1/2$, deducing that $B(0, ¼)\ \subset\ M$.

  Proof: Given $|c|\ \leq\ 1/4$ and $|z|\ \leq\ 1/2$, applying Δinequality

$$|f_c(z)| \le |z^2 + c| \le |z|^2 + |c| \le (1/2)^2 + 1/4 = 1/2$$

Applying inductively, $|f_c^k(0)| \le 1/2 \; \forall \; k \in Z^+$. Thus $B(0, ¼) \subset M$.

- If $|c + 1| \le 1/20$ and $|z| \le 1/10$, then $f_c(f_c(z)) \le 1/10$, deducing that $B(-1, 1/20) \subset M$.
- If $\epsilon > 0$ and if $|z| > max(2 + \epsilon, |c|)$, then $|f_c(z)| \ge |z|(1 + \epsilon)$ deducing that if $|c| > 2$ then $c \notin M$.

  Proof:
  $|f_c(z)| \ge |z^2 + c| \ge |z|^2 - |c| = |z|(|z| - \frac{|c|}{|z|}) \ge |z|(2 + \epsilon - 1) \ge |z|(1 + \epsilon)$ using $|z| > max(2 + \epsilon, |c|)$

  If $|c| > 2$ we can choose $\epsilon > 0$ such that $|c| > 2 + \epsilon$, so $f_c(0) = c$and applying the above estimate inductively, $|f_c^k(0)| \ge (1 + \epsilon)^k \to \infty$.
  Thus $c \notin M$.

- $M^C = \{c \in \mathbb{C} : f_c^k(0) > 2 \, for \, some \, k\}$

  Proof: Let c be such that $f_c^k(0) > 2 \, for \, some \, k$.

  1. $|c| > 2 \Rightarrow c \notin M$ by previous result.
  2. $|c| \le 2 \Rightarrow \exists \, \epsilon > 0$such that $|f_c^k(0)| > 2 + \epsilon > |c|$
     $\Rightarrow |f_c^{k+n}(0)| = |f_c^n(f_c^k(0))| \ge (1 + \epsilon)^n |f_c^k(0)| \to \infty$. So $c \notin M$.

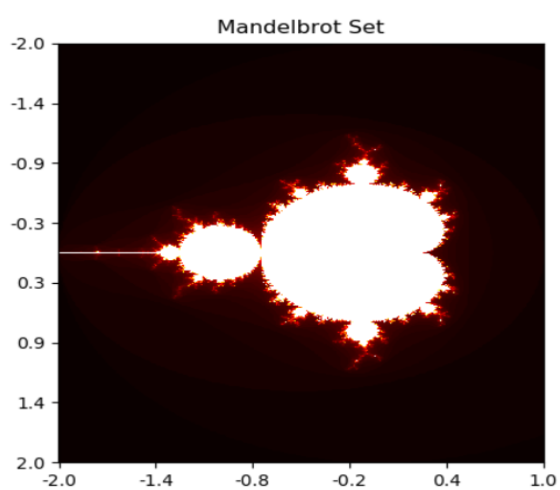
  Conversely, if $c \notin M \Rightarrow |f_c^k(0)| \to \infty$, so $|f_c^k(0)| > 2$for some k.

The above theorem gives an idea to plot the Mandelbrot set. The bounds we get using the above results.

Idea/Approach*:* Choose numbers $r \geq 2$ and k_0 of the order of, say, 1000. For each$c$, compute successive terms of the sequence $\{f_c^k(0)\}$ until either $|f_c^k(0)| > r$ in which case $c \notin M$ or $k = k_0$in which case we deem c to be in M.

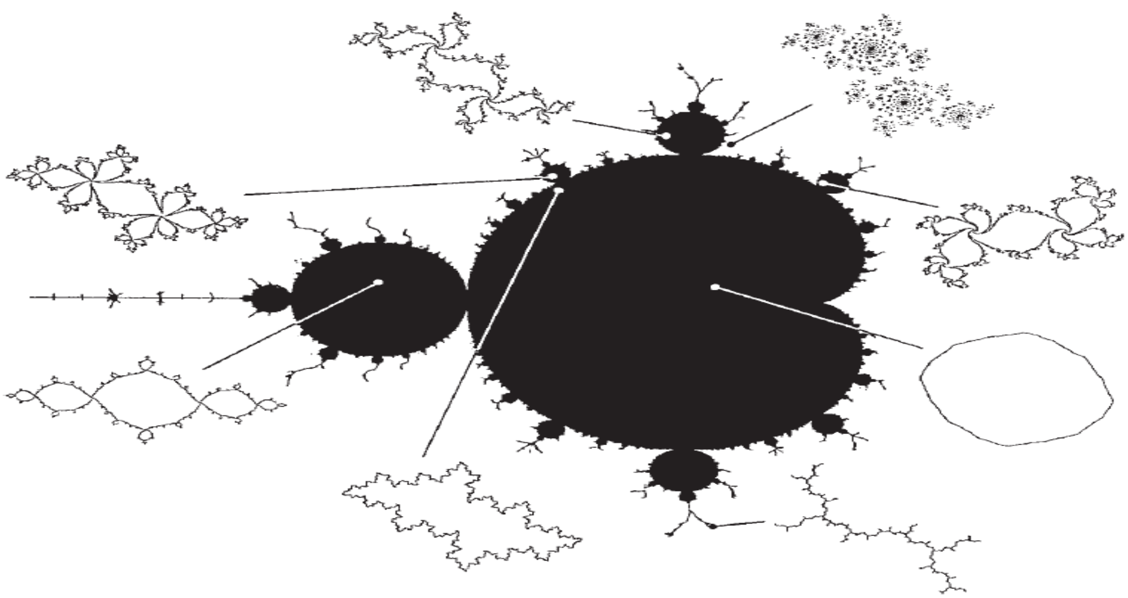
Plotting The Mandelbrot Set

Input : Max Number of Iterations, Absolute Limit
Output : Image I
begin
    for each pixel (Px, Py) on the screen:
        Scale the x and y coordinates of pixel
        Initialise the complex number z = 0
        Initialise the complex constant c = (x, y)
        for i in range(maxIter) and z < Absolute_Limit:
            $z = z^2 + c$
        Set color of pixel based on the number of iterations loop was run
    Plot I
    return I
end



The above we obtained when we started iterating from $z = 0$at different values of c for the quadratic function $f_c(z) = z^2 + c$. But our actual discussion began from having a fixed c and variating the z parameter. So, next we work on how the structure of Julia Sets $J(f_c)$ varies according to the parameter c in the complex plane.

Julia Sets explore the various parts of the Mandelbrot Set in detail. The following figure contains the structure of Julia Sets when we vary the complex constant c in the various parts of the Mandelbrot Set. Clearly, if c lies outside the Mandelbrot Set, f_c has no attractive periodic points and by definition, the Julia Set is not connected.



To look at the nature of the Julia Sets for various complex constants (c), we plotted the Julia Sets for the quadratic polynomial $f_c(z) = z^2 + c$. The idea to plot the Julia Sets was a similar one which we used to plot the Mandelbrot Set. The difference being, this time with a fixed c, we variate z instead of beginning z from 0 like in the Mandelbrot Case.

Julia Set Plotting

Input : Complex Constant c, Max Number of Iterations, Absolute Limit
Output : Image I
begin
    for each pixel (Px, Py) on the screen:
        Scale the x and y coordinates of pixel
        Initialise the complex number z to x, y
        for i in range(maxIter) and z < Absolute_Limit:

$$z = z^2 + c$$

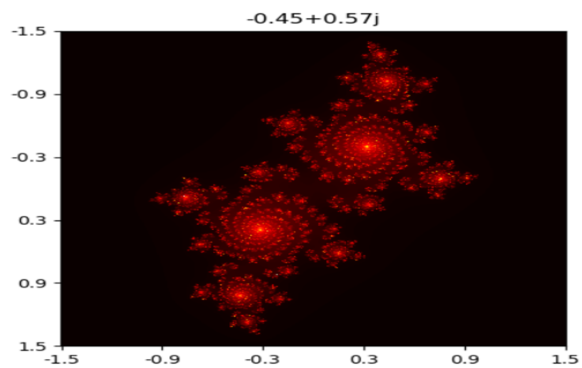
Set color of pixel based on the number of iterations loop was run

Plot I

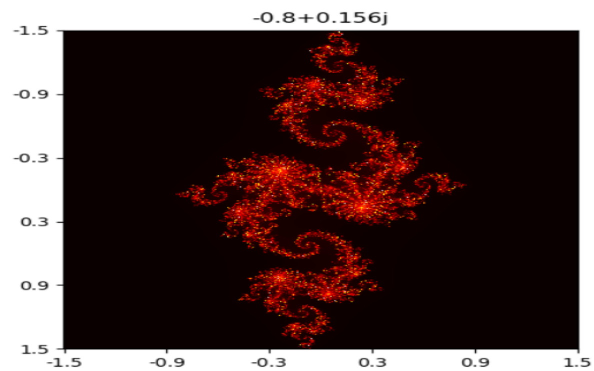
return I

end

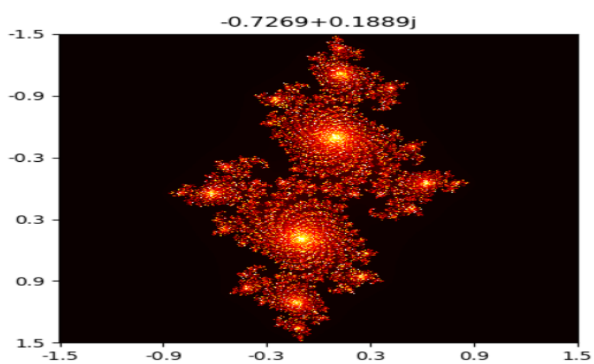
Results



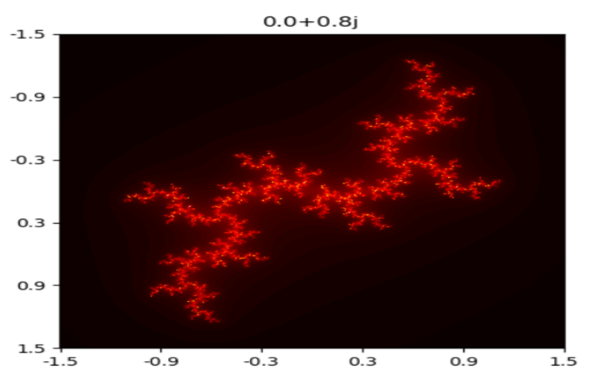
$c = -0.45 + 0.57i$



$c = -0.8 + 0.156i$



$c = -0.7269 + 0.1889i$



$c = 0.8i$

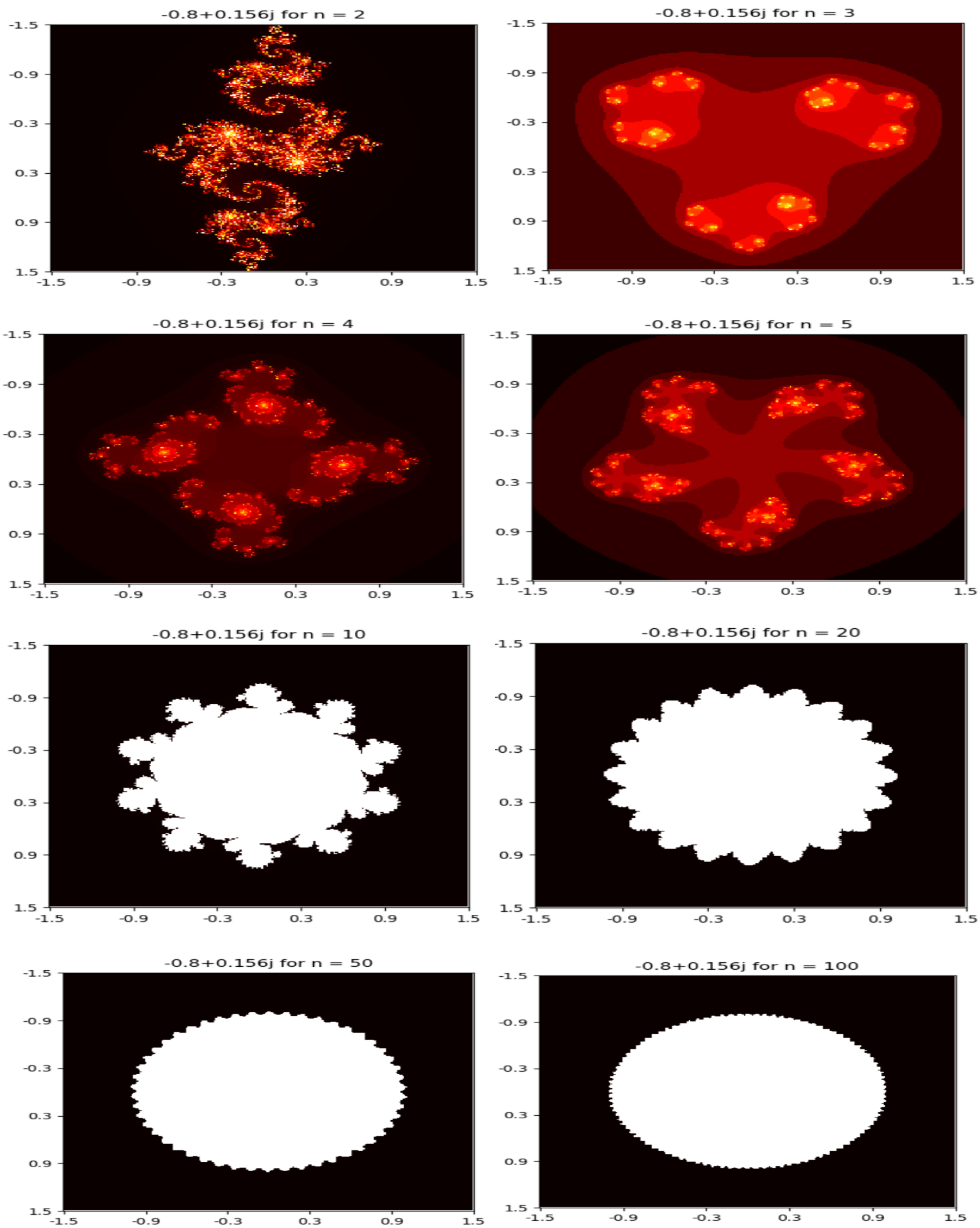
Julia Sets Plotted for various c values

We observed that we are finding two copies of a repeating pattern in every Julia Set. This leaded to a thought of how our Julia Set is affected when we fix a c and plot for various polynomials of form $f_c(z) = z^n + c$ by variating over various n

4.3. Higher Order Julia Sets

We set $c = -0.8 + 0.156i$ and plot the Julia Sets for $f_c(z) = z^n + (-0.8 + 0.156i)$.

Results:



Higher Order Julia Sets for various n

4.4. Observations

- The number of copies of the pattern depend on the degree of our polynomial.
- As $n \to \infty$, the Julia Set converges to a Unit Circle.

We tried with various complex constants and came to the conclusion that the value of c does not affect the above observations.

5. Fractal Image Compression

Fractal image compression is a lossy compression which uses the concept of contraction mapping and unique fixed point to store image in the form of a set of encoded functions. This method has not gained widespread acceptance for use on the internet due to its slow encoding and lossy nature. However it offers a higher compression ratio than other lossless compression like JPEG or GIF methods for some types of images. It has its limits in encoding but decoding is much faster.

5.1. Theory of Fractal Compression

Idea : Given an Image x, if we are able to find a contractive mapping f such that its fixed point is x then we can store f which takes less no. of bits and we can obtain x by applying f iteratively on any random image y since $f^n(y) = x$ for $n \to \infty$.

The Inverse Problem : We know that every contractive function has one unique fixed point. But given a point x (image), does there exist a contractive function f such that x is its fixed point? Since we allow incomplete spaces, the problem is to find a contractive function f with fixed point x_o that is the best approximation to x. This problem has been shown to be **NP-Hard**, and no polynomial time approximation to it has been proposed. A different approach by virtue of the Collage theorem allows one to set an upper bound for the closeness of x and x_o.

Collage Theorem **:** *Let (X, d) be a complete metric space and $f : D \to X, D \subset X,$ a contractive mapping with contractivity factor s, and a unique fixed point x_o. Then,*

$$d(x, x_o) \leq \frac{d(x, f(x))}{1-s}$$

<u>*Proof*</u> *:* Note that,

$$d(f^{m-1}(x), f^m(x)) \leq s \,.\, d(f^{m-2}(x), f^{m-1}(x)) \quad \text{(since } f \text{ has a contractivity factor } s\text{)}$$
$$\leq s^2 \,.\, d(f^{m-3}(x), f^{m-2}(x))$$
$$\vdots$$

$$\leq s^{m-1} . d(x, f(x))$$

Then,

$$d(x, x_o) = d(x, lim_{n\to\infty} f^n(x)) = lim_{n\to\infty} d(x, f^n(x))$$
$$\leq lim_{n\to\infty} d(x, f(x)) + d(f(x), f^2(x)) + .. + d(f^{n-1}(x), f^n(x))$$
$$\leq lim_{n\to\infty} d(x, f(x)) (1 + s + s^2 + . . + s^{n-1})$$
$$\leq \frac{d(x, f(x))}{1-s}$$

Note **:** When *(X, d)* is incomplete then x_o may not belong to *X* but we have ,

$$d(x, f^n(x)) \leq \frac{d(x, f(x))}{1-s} \text{ when } n \text{ is large enough}$$

This theorem tells us that if we find a contraction *f* such that *f*(*x*) is near to *x* then we are sure that the fixed point of *f* is also near to *x*

5.2. Problem Formulation and Algorithm

Local IFS *:* A Local IFS is an IFS with the property that all domains D_i's of f_i's are strictly contained in the space *X.*

Given an image $I \in R^3$, suppose that $f = \{ f_1, f_1, . . f_n \}$ is a local IFS with the following properties:

1. $f_i : R^3 \to R^3$, with domain D_i and range R_i
2. $R_i \cap R_j = \phi$ if $i \neq j$
3. $f_i(I) \cap f_j(I) = \phi$ if $i \neq j$
4. $I = \bigcup_{i=1}^{n} R_i$ (Ranges creates partition on image I)

Domain blocks

D_1	D_2	D_3	D_4
D_5	D_6	D_7	D_8
D_9	D_{10}	D_{11}	D_{12}
D_{13}	D_{14}	D_{15}	D_{16}

Range blocks

R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8
R_9	R_{10}	R_{11}	R_{12}	R_{13}	R_{14}	R_{15}	R_{16}
R_{17}	R_{18}	R_{19}	R_{20}	R_{21}	R_{22}	R_{23}	R_{24}
R_{25}	R_{26}	R_{27}	R_{28}	R_{29}	R_{30}	R_{31}	R_{32}
R_{33}	R_{34}	R_{35}	R_{36}	R_{37}	R_{38}	R_{39}	R_{40}
R_{41}	R_{42}	R_{43}	R_{44}	R_{45}	R_{46}	R_{47}	R_{48}
R_{49}	R_{50}	R_{51}	R_{52}	R_{53}	R_{54}	R_{55}	R_{56}
R_{57}	R_{58}	R_{59}	R_{60}	R_{61}	R_{62}	R_{63}	R_{64}

Also define metric $d(x, y) = \sqrt{\sum_{i=1}^{h} \sum_{j=1}^{w} (x_{ij} - y_{ij})^2}$ where x and y are hxw grayscale matrices and d is nothing but frobenius norm.
So if $d(R_i , f_i (D_i)) < \epsilon \ \forall \ i$ then,

$$d(I, f(I)) = d(\bigcup_{i=1}^{n} R_i , \bigcup_{i=1}^{n} f_i (D_i)) = \sum_{i=1}^{n} d(R_i , f_i (D_i) < n\epsilon$$

Using collage theorem will imply:

$$d(I, I_o) \leq \frac{d(I, f(I))}{1-s} = \frac{n\epsilon}{1-s} \text{ where } I_o \text{ is the fixed point of IFS } f.$$

Compression **:** So to compress an image $I = \bigcup_{i=1}^{n} R_i$ we search for a $f = \bigcup_{i=1}^{n} f_i (D_i)$ such that $d(R_i , f_i (D_i))$ is small enough in a pool of functions f's and domain D_i's. The function f is the compressed image of I ; s and ϵ determine the lossiness of I.

Decompression **:** To decompress an image given $f = \bigcup_{i=1}^{n} f_i (D_i)$, iterate f on any image I' (as a consequence of the fixed point theorem):

$$f^n(I') = I_o \text{ and since } d(I, I_o) < \frac{n\epsilon}{1-s} \Rightarrow d(I, f^n(I')) < \frac{n\epsilon}{1-s} \Rightarrow f^n(I') = I \text{ for large } n.$$

5.2.2. The Affine Transformation : Pool of IFS

*Affine Transformation : An affine transformation is **any transformation that preserves collinearity** (i.e., all points lying on a line initially still lie on a line after transformation) and ratios of distances (e.g., the midpoint of a line segment remains the midpoint after transformation).*

We consider affine transformation as our IFS and hence finding IFS becomes finding parameters of this transformation of form:

$$w_i \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_i & b_i & 0 \\ c_i & d_i & 0 \\ 0 & 0 & \alpha_i \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} e_i \\ f_i \\ \beta_i \end{pmatrix}$$

Where,

1. $a_i , b_i , c_i , d_i \in \{ 0, 0.5 \}$ generates different contraction mapping with ratio 0.5 or 0.25
2. α_i and β_i are contrast and brightness rescale of z which is grayscale value at (x,y)

3. e_i and f_i are translation parameters

Claim: Above defined affine transformation is contractive if $\alpha_i < 1$.

This claim has been proven in paper *"Fractal Compression"* by P Melville and V Phan

Note : Experiments shows that t when $\alpha_i \leq 1.2$*, the contraction of* $f = \bigcup_{i=1}^{n} f_i\,(D_i)$ *is guaranteed*

Encoding of Contraction Map

To reduce complexity we have fixed the contraction ratio to be 0.25 ($D_i = 8x8$ & $R_i = 4x4$)

We have added flipping {1, -1} and right angle rotation {0, 90, 180, 270}

- 1 bit for flip + 2 bit for rotation
- 5 bit for contrast ($-1.2 \leq \alpha_i \leq 1.2$) + 8 bit for brightness ($0 \leq \beta_i \leq 256$)
- 8 bit for each translation parameter (coordinate from 0 to 256)

Total : 1+2+5+8+8+8 = 32 bit for each transformation

5.2.3. Algorithm

Compression Algorithm

Input : Image I, Domain size : 8x8, Range size : 4x4

Output : Encoded transformation

```
begin
    Segment I in Domain of 8x8
    Partition I in Range of 4x4

    Encoded transformation = []
    for D in Domain pool:
        Generate all possible transformations using :
            f (D) = contrast * rotate( flip ( reduce(D))) + brightness

        for R in Range pool:
            find f with minimum d( f(D), R )
            Encode f  and insert it Encoded transformation
    return Encoded transformation
end
```

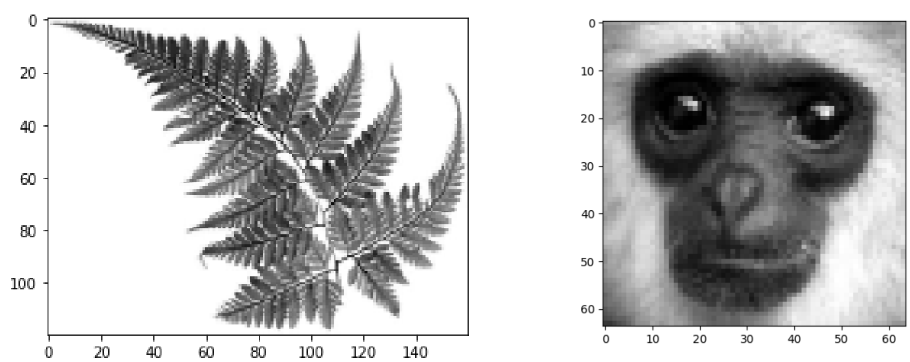
Decompression Algorithm

Input : Encoded Transformations, No of steps
Output : Image I
begin
    Decode the Encoded transformations
    Take any initial random image I
    for i in range (No of steps):
        for *f* in decoded transformations:
            I = f (I)
    return I
end

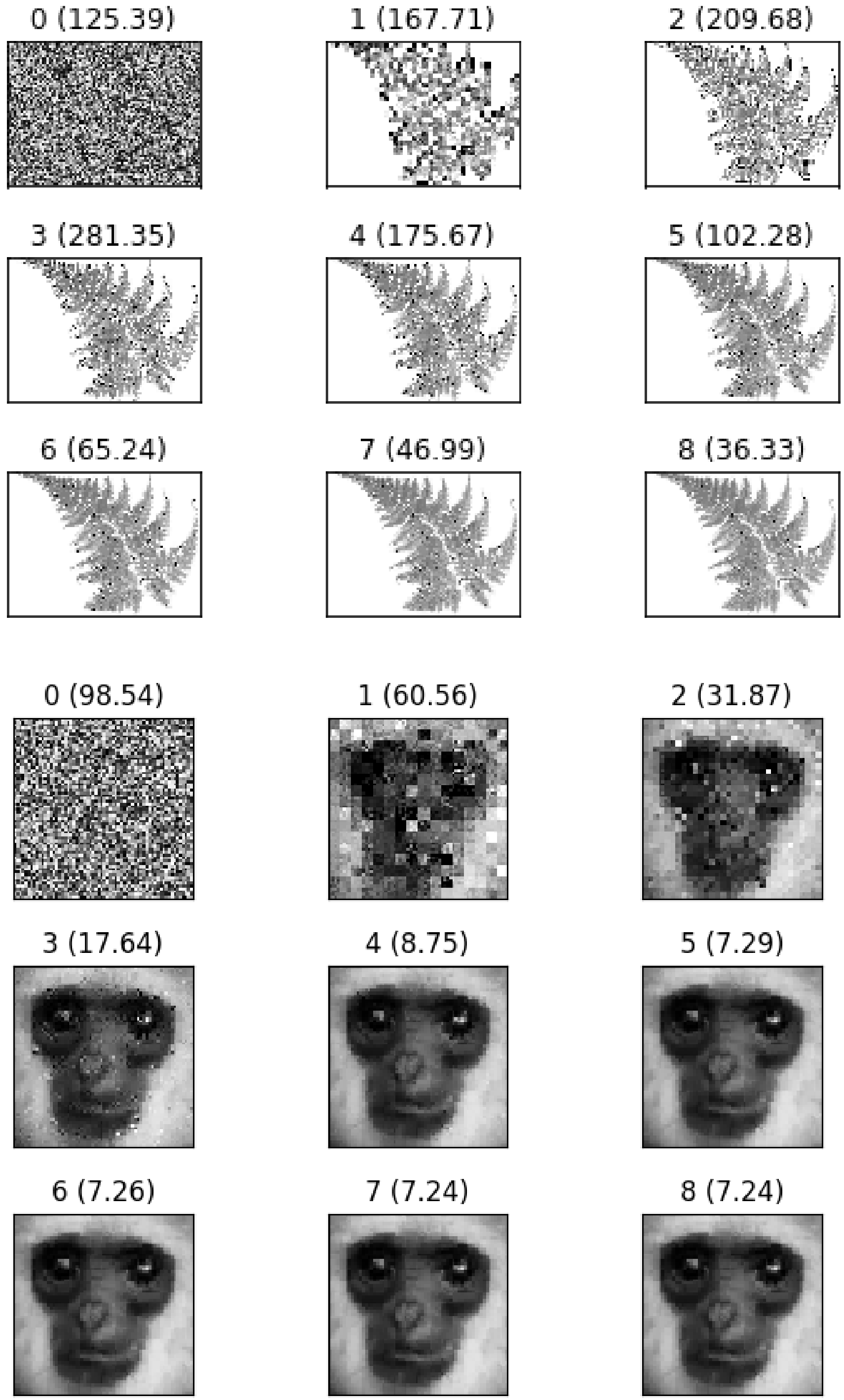
5.3. Results and Discussion

5.3.1. Results of Fractal Compression

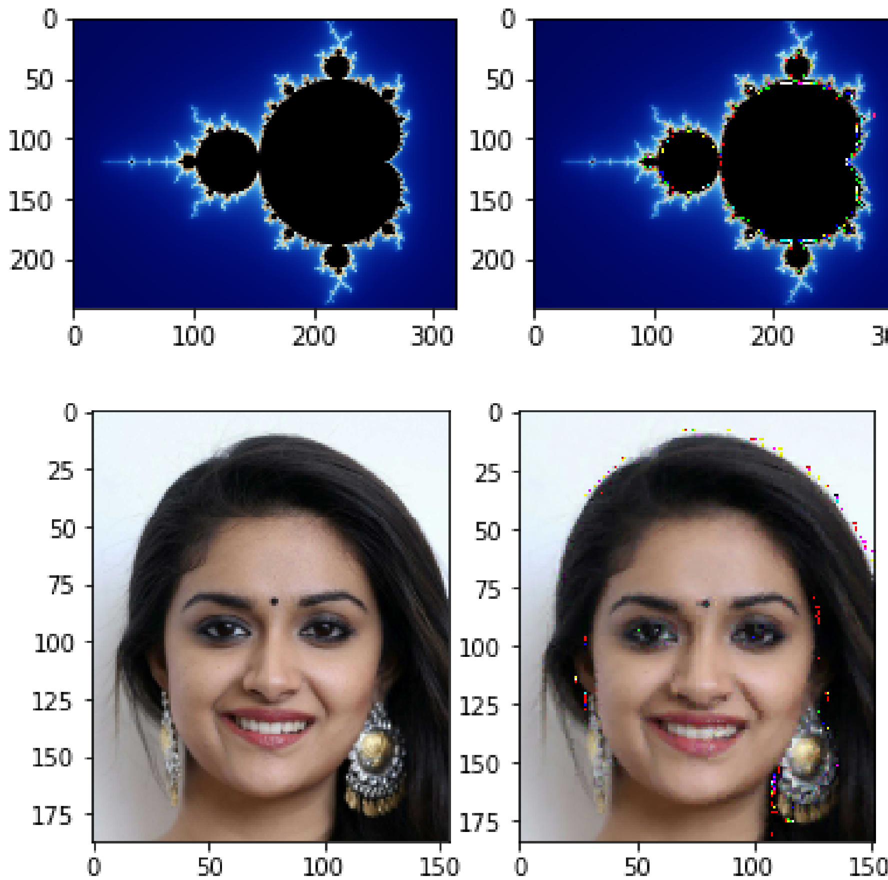
We have tested it for both grayscale and rgb images.



Input Image



Decompressing original image from random image in each iteration (loss)



Observation

- Fractal image compression is very slow. We have found that it can take from 1 min up to several hours in compressing an image ranging from 64x64 to 1080x1080. This observation is evident from the fact that solving *the inverse problem* is NP-hard.
- The lossy nature of this technique becomes very significant if the input image has sharp transitions. *FERN* and *MANDELBROT* images look very different from their input in the regions with sharp transitions even after 16 epochs while *MONKEY* and *KEERTHY* images are very close in just 5 to 8 epochs.

5.3.2. Fractal Compression and Other Compression

PNG	***JPEG***	***Fractal Compression***
Lossless	Lossy	Lossy
Low Compression Ratio	High Compression Ratio	High Compression Ratio
Uses DEFLATE algorithm and Huffman Coding, it supports palette based grayscale and RGB images	Uses visual effect that human are more sensitive to, like luminance over chrominance	Uses the fact that in certain images, parts of the image resemble other parts of the same image
Better Choice for images with text, line art or sharp transitions	It work better for all types of images	Better choice for images with repetitive patterns and natural scene
Quick compression and decompression	Quick compression and decompression	Compression is very slow and even decompression can be slow for specific type of images
Widely used	New version of JPEG are widely used	It is yet in a development phase

6. Conclusion

We began learning about Fractals from their nature and their appearances in various places. We then learnt about their mathematical definitions and some of the most important and simple Fractals like the Cantor Set and the Sierpinski's Gasket. The nature of these brought the concept of fractional nature of dimensions and hence we delved to learn the various ways of representing dimensions in Fractals. The major topics we explored in this aspect were the Box Counting Dimension and the Self-similarity Dimension. With the basics of Fractals in our hand, we begin working towards the plotting of these structures. The important topics under this were the Iterated Function Systems (IFS) and the Chaos Game. With these concepts in mind, we ended the midterm with plots of various fractals including the CantorSet, the Cantor dust, the Sierpinski's Gasket.

After the mid term we focused mainly on the application part of Fractal in the Mandelbrot & Julia Set and Fractal Image Compression. We studied the theory, solved some problems related to the bounds and then went onto exploring first the Mandelbrot Set and then a variety of Julia Sets for various constants by plotting them. Observing dependency on the order of our polynomial, we explored the changes based on the order of the polynomial for Julia Sets. In Fractal Image Compression, we started with the basic theory of fractal compression and a very important theorem, *Collage Theorem,* which guaranteed the feasibility of fractal compression. We learnt about the affine transformation, encoding of transformations and the algorithm to implement and find the results. We also compared a few other image compression techniques based on existing research papers.

7. Appendix - Codes and Plots

All the codes and additional results of the file have been made available at the following link:
https://drive.google.com/drive/folders/1IED6mqvK54IEFLFiw6qiyKYeDQmwfnEL?usp=sharing

8. References and Bibliography

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