Fractals

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Project Objective

- To learn about the basics of Fractals, their formation and their properties
- Approximating Fractals by plotting them upto n iterations
- Learn about the fractional nature of Dimension
- Find the dimensions of various fractals

Introduction

- Fractal - A pattern that the laws of nature repeats at different scales
- In the past, mathematics has been concerned largely with sets and functions to which the methods of classical calculus can be applied
- In recent years, it has been realised that these irregular sets represent some of the natural phenomenon in a much better way than the classical case

Triangular Fern

Motivating Examples

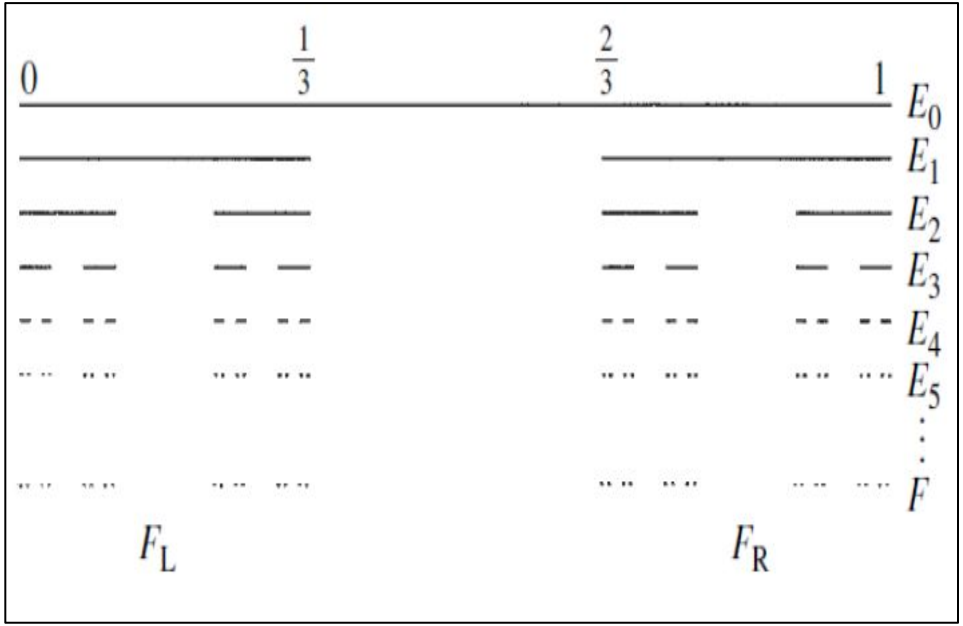
There are a lot of natural fractals which motivates for pursuing in this topic of study

- Triangular Fern
- Trees
- Roses
- River Deltas

Trees- the most natural example of Fractals

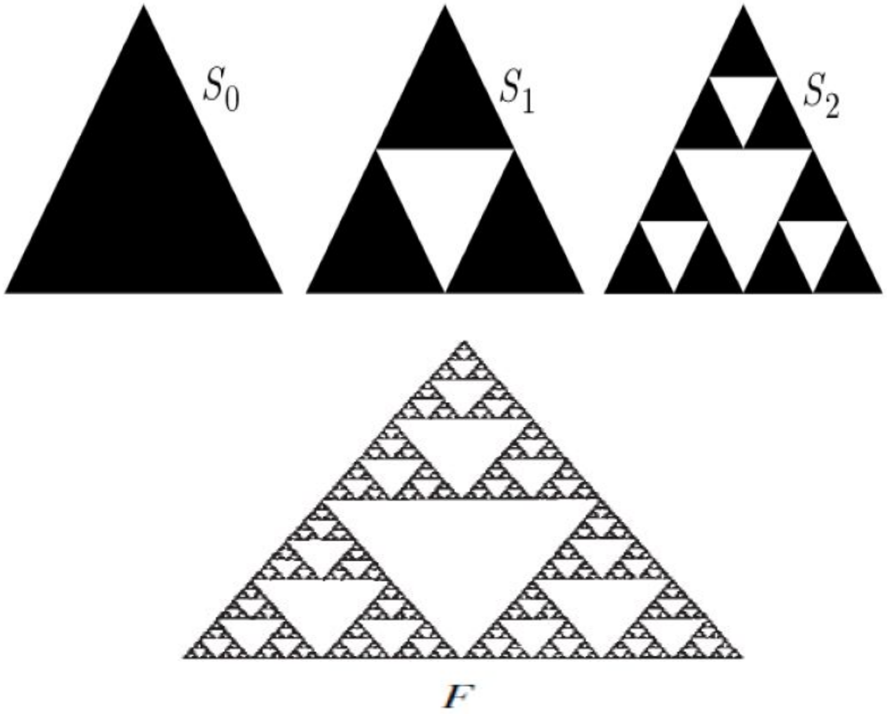
Middle Third Cantor Set

- One of the best known examples of Fractals
- Nth iteration set (E_n) is obtained by deleting middle one third of the intervals of the (n-1)th iteration set (E_{n-1})
- The final set is the intersection of all the iterative sets E_i
- If we begin the iterations using the unit interval [0,1] and apply above procedure, the sum of removed lengths equals unity
- Dimension less than 1!



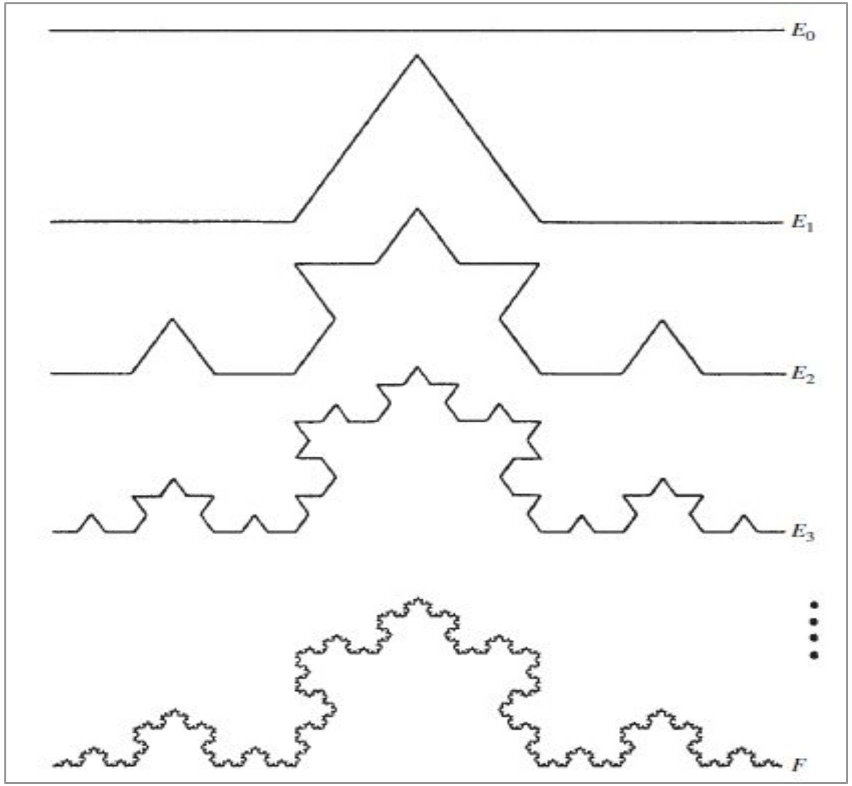
Sierpinski’s Triangle

- Initial set is an Equilateral Triangle
- Repeatedly we remove a smaller equilateral triangle from the middle with side length half of the original one
- At nth iteration, the number of equilateral triangles equal 3^{n-1}
- Sum of areas of all the removed triangles equal the area of original triangle
- Sum of length of the edges of the triangle shoot to infinity
- The dimension is somewhat in between 1 and 2!



Koch’s Curve

- Initial set is an Line Segment
- Repeatedly we remove the middle third of each line segment and replace it with two sides of equi. triangle based on removed segment.
- The size of line segment gets reduced to 1/3rd in each iteration
- The number of structures get 4 times in each iteration
- Koch Curve is the limiting curve of these



Iterated Function System : Definitions

- Let D be a closed subset of $\mathbb{R}^n$, often D is $\mathbb{R}^n$ itself. A mapping $S : D \to D$ is called a contraction on D if there exists a number r with **$0 < r < 1$** such that $|S(x) - S(y)| \leqslant r|x - y|$ for all $x, y \in D$
- A finite family of contractions $\{ S_1, S_2, \ldots, S_m \}$, with $m \geqslant 2$, is called an **iterated function system**
- A non-empty compact subset F of D is an **attractor** (or invariant set) for the IFS if it is made up of its images under the S, i.e. $F = \cup\ S_i(F)$
- If $\{S_1,\ldots,S_m\}$ is an IFS then there exists a **unique non-empty attractor F** of the IFS.
- An iterated function system realizing a **ratio list**$(r_1,r_2, \ldots, r_n)$ in a metric space is a list $(S_1, S_2, \ldots, S_n)$, where $S_i: \mathbb{R}^n \to \mathbb{R}^n$ is a similarity with ratio r_i, i.e $|S_i(x) - S_i(y)| = r_i\,|x - y| \quad (x, y \in \mathbb{R}^n)$
- The attractor of a collection of similarities is called a self-similar set
- The self-similarity dimension “s” satisfy : $\sum r_i^s = 1$

Plotting Fractals : Chaos Game

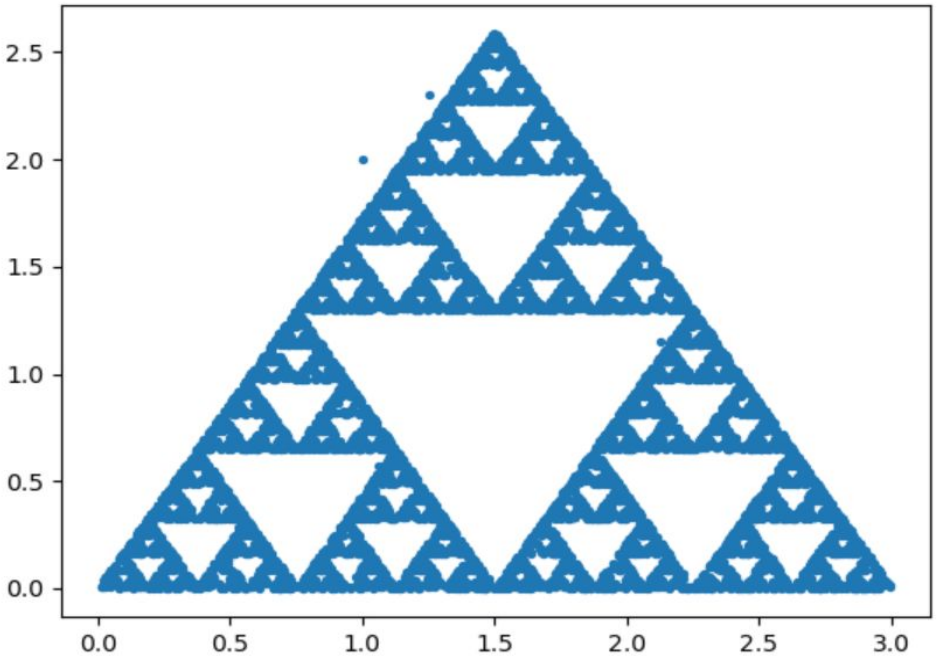
- Take a point inside the boundaries to be plotted
- Randomly operate any one of the Contraction maps on the point to find the next point
- The wanderer is **attracted** to the **Attractor of the IFS** and within a few thousand iterations, it starts following the attractor we try to plot

Plotting Sierpinski's Triangle

- IFS = $\{S_1, S_2, S_3\}$; A,B,C initial points
- $S_1(x) = (x+A)/2$
- $S_2(x) = (x+B)/2$
- $S_3(x) = (x+C)/2$
- Ratio list = [½,½,½]
- Similarity Dimension:

$$3*(½)^s = 1$$

$$s = (\log 3)/(\log 2)$$

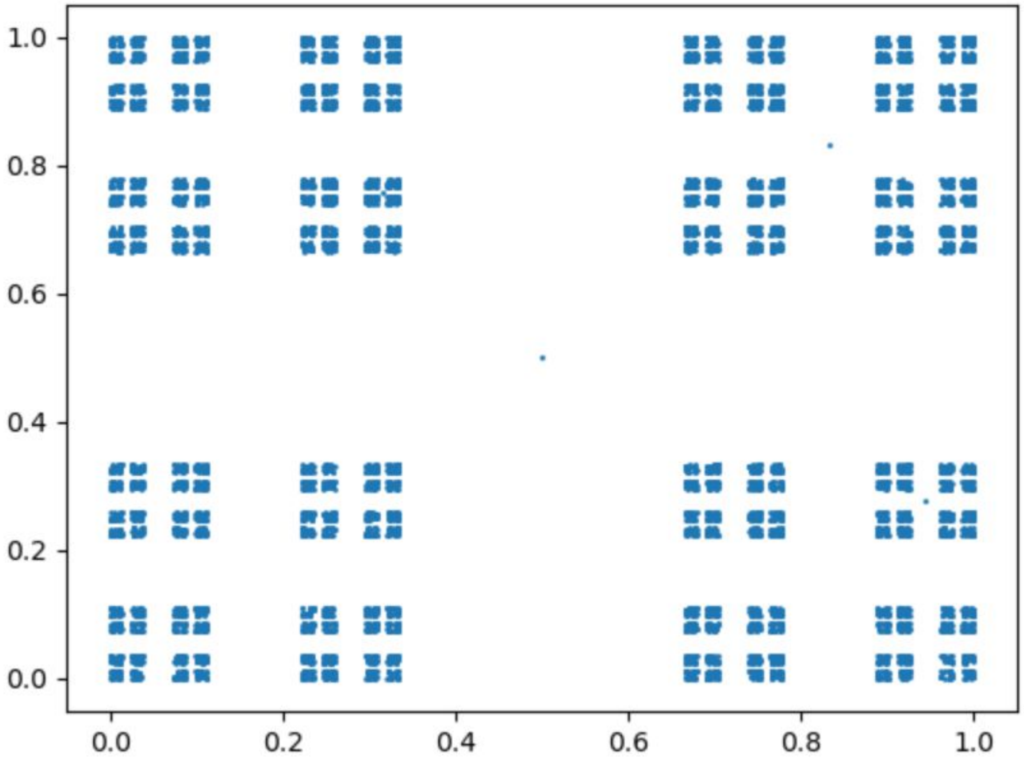


Plotting Cantor Dust

- IFS = $\{S_1, S_2, S_3, S_4\}$
- $S_1((x,y)) = (x/3, y/3)$
- $S_2((x,y)) = (x/3 + 2/3, y/3)$
- $S_3((x,y)) = (x/3, y/3 + 2/3)$
- $S_4((x,y)) = (x/3 + 2/3, y/3 + 2/3)$
- Ratio list = [⅓, ⅓, ⅓, ⅓]
- Similarity Dimension:

$$4*(⅓)^s = 1$$

$$s = (\log 4)/(\log 3)$$



Box Counting Dimension – Definition

- Given a subset F of the plane, for each $\delta > 0$, we find the smallest number of sets of diameter at most δ that can cover the set F and we call this number $N_\delta(F)$. The dimension of F reflects the way in which $N_\delta(F)$ grows as $\delta \to 0$. If $N_\delta(F)$ obeys, at least approximately, a power law $N_\delta(F) \simeq c\delta^{-s}$ for positive constants c and s, we say that F has box dimension s.
- Taking logarithm to solve yield $\log N_\delta(F) \simeq \log c - s \log \delta$

$$s = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}$$

Lower Box-counting dim : $\underline{\dim}_B F = \underline{\lim}_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}$

Upper Box-counting dim : $\overline{\dim}_B F = \overline{\lim}_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}$

If these are equal, we say common value is the Box-counting dimension of F, i.e.

$$\dim_B F = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}$$

Box Counting Dimension – Interpretation

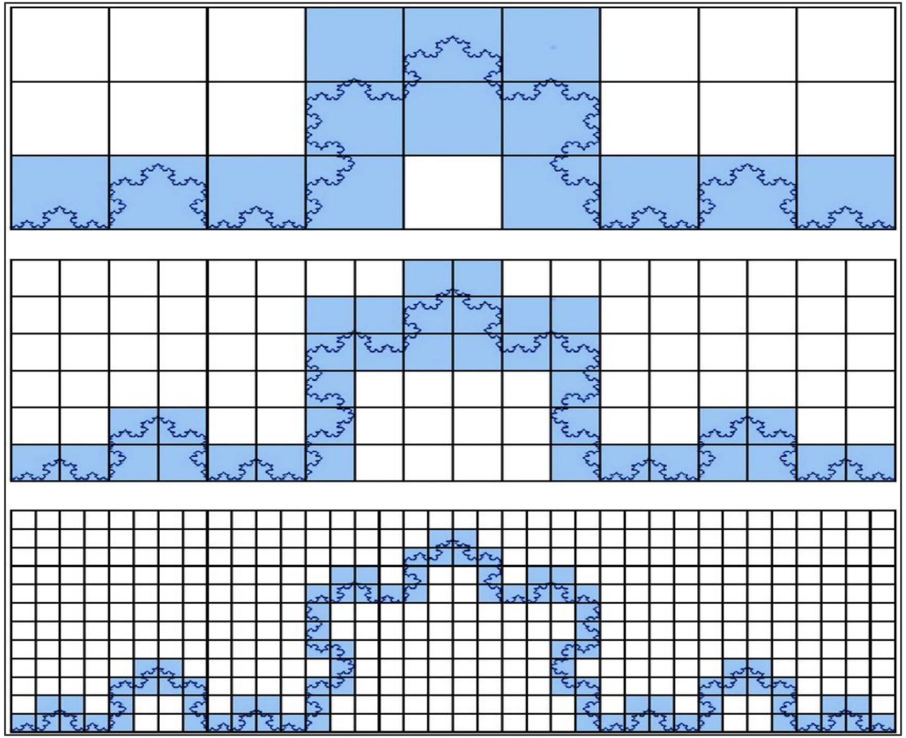
$N_{\delta}(F)$ can be interpreted as:

1. the smallest number of sets of diameter at most δ that cover F;

2. the smallest number of closed balls of radius δ that cover F;

3. the smallest number of cubes of side δ that cover F;

4. the number of δ-mesh cubes that intersect F



Box Counting Dimension (Theoretical) - Cantor Set

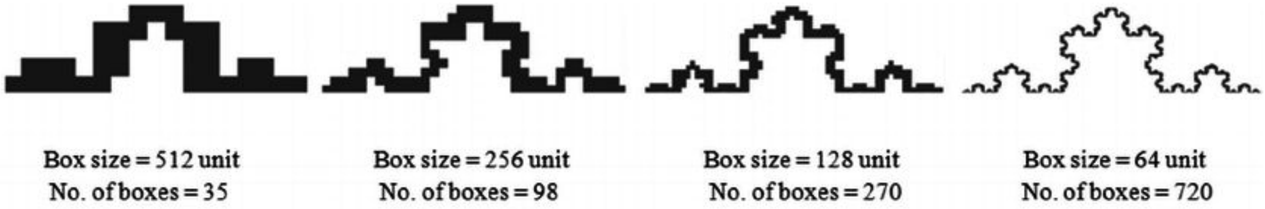
- If $3^{-k} < \delta \leqslant 3^{-k+1}$, then the 2^k level-k intervals of Ek of length 3^{-k} provide a δ-cover of F, so that $N_{\delta}(F) \leqslant 2^k$, where $N_{\delta}(F)$ is the least number of sets in a δ-cover of F.

$$\overline{\dim}_{\mathrm{B}} F = \overline{\lim_{\delta \to 0}} \frac{\log N_\delta(F)}{-\log \delta} \leqslant \overline{\lim_{k \to \infty}} \frac{\log 2^k}{-\log 3^{-k+1}} = \overline{\lim_{k \to \infty}} \frac{k \log 2}{(k-1) \log 3} = \frac{\log 2}{\log 3}.$$

- On the other hand, any interval of length δ with $3^{-k-1} \leqslant \delta < 3^{-k}$ intersects at most one of the level-k intervals of length 3^{-k} used in the construction of F (the gap between the level-k intervals is at least 3^{-k}). There are 2^k such intervals, all containing points of F, so at least 2^k intervals of length δ are required to cover F. Hence, $N_{\delta}(F) \geqslant 2^k$ so

$$\underline{\dim}_{\mathrm{B}} F = \underline{\lim}_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta} \geqslant \underline{\lim}_{k \to \infty} \frac{\log 2^k}{-\log 3^{-k-1}} = \underline{\lim}_{k \to \infty} \frac{k \log 2^k}{(k+1) \log 3} = \frac{\log 2}{\log 3}.$$

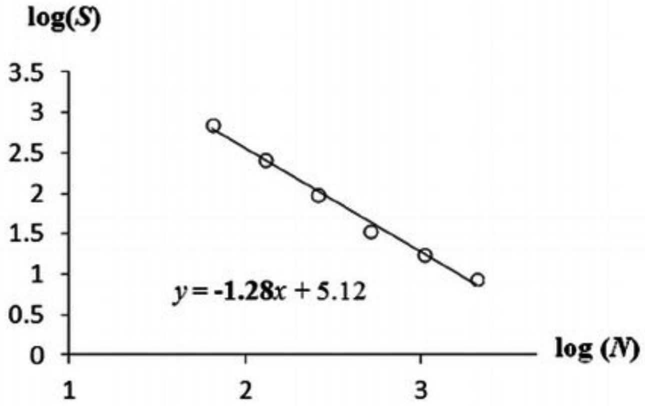
Box Counting Dimension (Practical) – Koch Curve



(a)

S	*N*	log (*S*)	log (*N*)
2048	9	3.31	0.95
1024	18	3.01	1.26
512	35	2.71	1.55
256	98	2.41	1.99
128	270	2.11	2.43
64	720	1.81	2.86

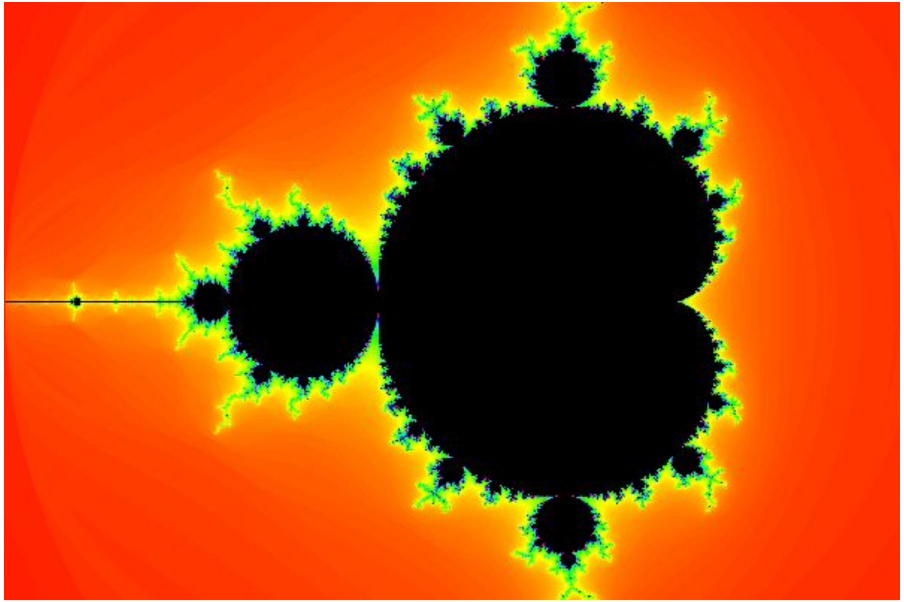
(b)



(c)

Conclusion and Further Work

- We have learnt about the basics of Fractals, their nature and some of their properties.
- We have been successful in finding the dimensions of various fractals along with plotting them to have a visualization of things going on in an approximation.
- Our further targets are to delve a little deeper into Mandelbrot Set and then go towards compression of Fractal images



References

1. Chaos Game : https://en.wikipedia.org/wiki/Chaos_game
2. Iterated Function System https://www.algorithm-archive.org/contents/IFS/IFS.html
3. Fractal Geometry by Kenneth Falconer, Third Edition, Wiley Publications
4. Measure, Topology and Fractal Geometry by Gerald Edgar, Second Edition, Springer Publications

Questions?

Thanks!!