Injective Colouring in Circular-Convex Bipartite Graphs

Course Project Presentation

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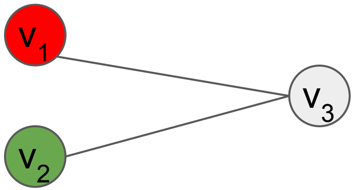
Problem Statement

Find injective coloring algorithm for circular convex bipartite graph

Injective Coloring

An ***Injective Colouring*** of a graph G is the vertex-colouring in which vertices having common neighbour gets different colours.

The minimum number of colours required for G to be injectively coloured is its ***injective chromatic number*** denoted by $\mathbf{X}_{i}\mathbf{(G)}$



v1 and v2 share a common neighbor v3 and hence colored different

<u>Note</u>: *(v1 ,v3) or (v2, v3) can have same colors.*

Decide Injective Colouring Problem

Given a graph G and an integer K,
Is it possible to Injectively color the Graphs G using K-colours.

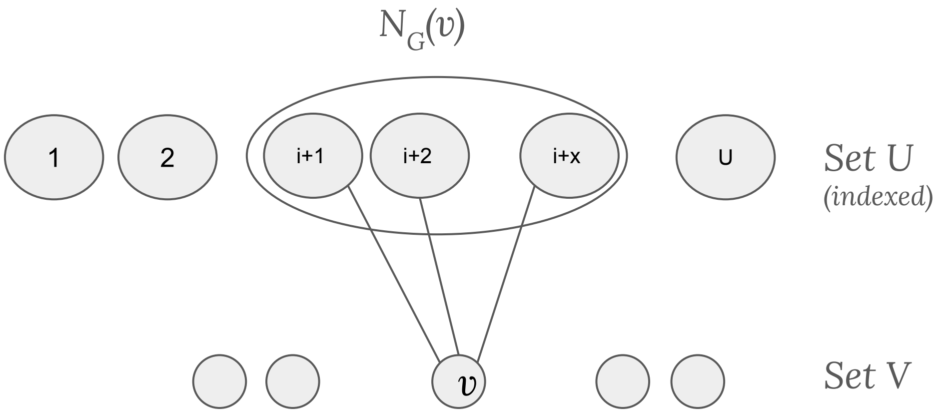
This problem is Np-Complete in general graphs and also in bipartite graphs.

For any general graph $\Delta \leq X_i(G) \leq \Delta^2 - \Delta + 1$

Convex Bipartite Graph

Let $G = (U \cup V, E)$ be a bipartite graph with vertex partitions U and V.
Let $N_G(v)$ denote the neighbourhood of the vertex $v \in V$.

The graph ***G is said to be convex over the vertex set U*** if the vertices in U can be indexed from 1 to $|U|$ such that, for all $v \in V$ the vertices in the $N_G(v)$ are adjacent with respect to the indexing i.e.,
if $d(v) = x$ then all $v \in N_G(v)$ have continuous indices i.e. $i+1, i+2, \ldots, i+x$.



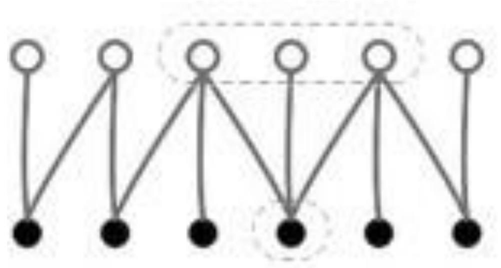


Fig 1: *Example of a Convex bipartite graph, convex over the set of white vertices*

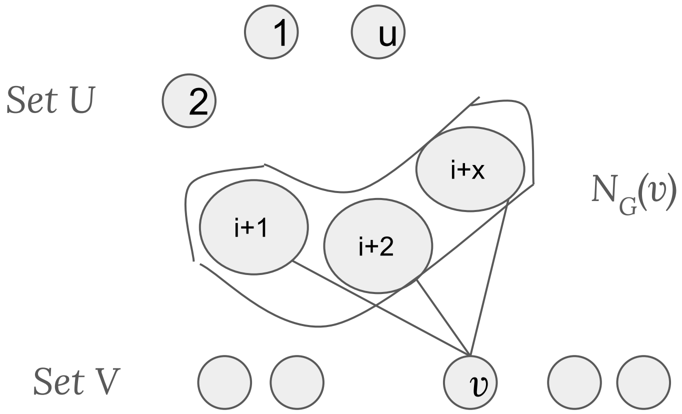
Problem Statement

Find injective coloring algorithm for circular convex bipartite graph

Circular Convex Bipartite Graph

A Bipartite Graph G = (U ∪ V, *E*), is said to be circular convex over U if there exists a circular ordering in U such that for every $v \in$ V the neighbourhood of v forms a (continuous)circular arc

A convex bipartite graph is a circular convex bipartite, but converse is not true.



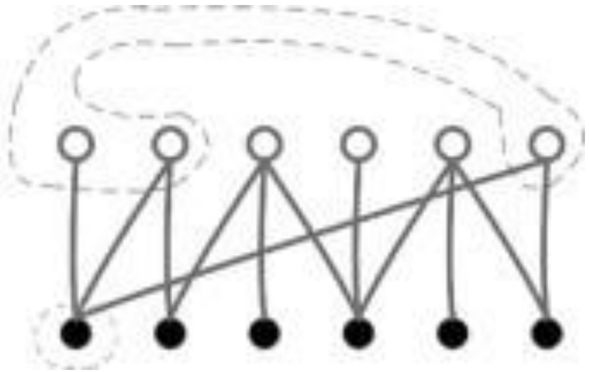


Fig 2: *Example of a Circular Convex bipartite graph, convex over the set of white vertices*

Literature: Algorithm for Convex Bipartite Graph

Lex-Convex Ordering: Let G = (X U Y, E) be a convex bipartite graph, then

An ordering $\sigma = (x_1, x_2, \ldots, x_{nx}, y_1, y_2, \ldots, y_{ny})$ is said to be a lexicographic convex(lex-convex) ordering if:

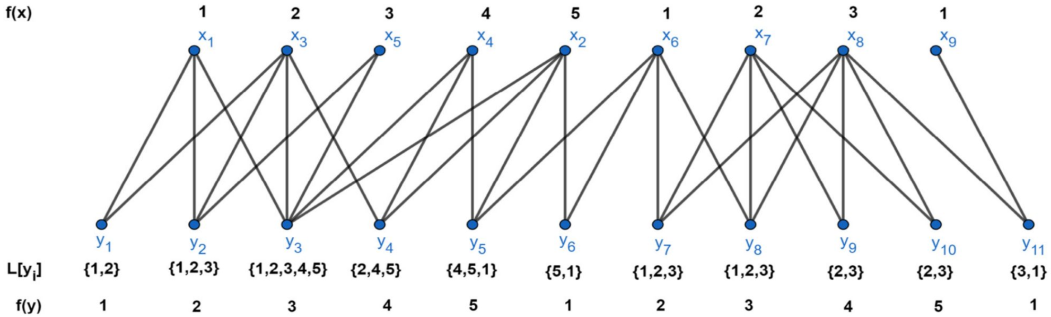
a. *σ is a convex ordering* &

b. *for x_i and x_j with $i < j$, the lowest index in $N_G(x_i) \leq N_G(x_j)$. Also in the case for the highest index.*

- Lex-convex ordering of a graph G that is convex over Y can be determined in O(|X|log|X|) time
- Given a lex-convex ordering of a convex bipartite graph G, this below polynomial time algorithm computes an optimal injective coloring of G
- The algorithm maintains an array $L[y_i]$ of size Δ for each vertex y_i in Y, $1 \leq i \leq n_y$, that stores the set of colors used by its neighbors

Algorithm 2 Convex Bipartite Graph

Input: A convex bipartite graph $G = (X, Y, E)$ with maximum degree Δ and lex-convex ordering of G, $\sigma = (x_1, x_2, \ldots, x_{n_x}, y_1, y_2, \ldots, y_{n_y})$.
Output: An injective coloring, f of G.
begin
  Initialize $S = \emptyset$ and $L[y_i] = \emptyset$, for each $1 \leq i \leq n_y$;
  for $i = 1$ *to* n_y **do**
    if $(i \bmod \Delta \neq 0)$ **then**
      $f(y_i) = i \bmod \Delta$;
    else
      $f(y_i) = \Delta$;
  $i = 1$;
  while $i \leq n_y$ **do**
    if (y_i *has a neighbor x, taken in the order* $(x_1, x_2, \ldots, x_{n_x})$*, which is not yet colored*) **then**
      Let $\{y'_1, \ldots, y'_{d_G(x)}\}$ be the neighbors of x;
      $S = L[y'_1] \cup \ldots \cup L[y'_{d_G(x)}]$;
      Assign $f(x) = \min \{c \in \{1, \ldots, n_x\} \setminus S\}$
      for $r = 1$ *to* $d_G(x)$ **do**
        $L[y'_r] = f(x) \cup L[y'_r]$;
    else
      $i = i + 1$;
  return f;



Approach 1

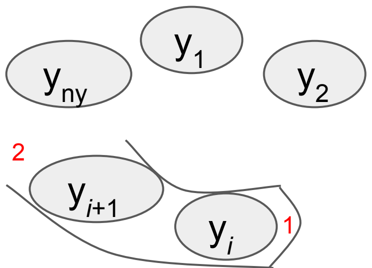
- Similar to previous algorithm

Let $y_1 \rightarrow y_2 \rightarrow \dots \rightarrow y_{ny} \rightarrow y_1$ be the circular convex ordering of the partition Y of G.

1. Initially find x_i that has maximum degree, its neighbours forms an arc A in Y.
2. Starting with one end point of A, colour towards other in the fashion $1,2,\dots,\Delta,1,2,\dots$.(Y done)
3. Later on x_i's are coloured following the while loop of the Algorithm 2.

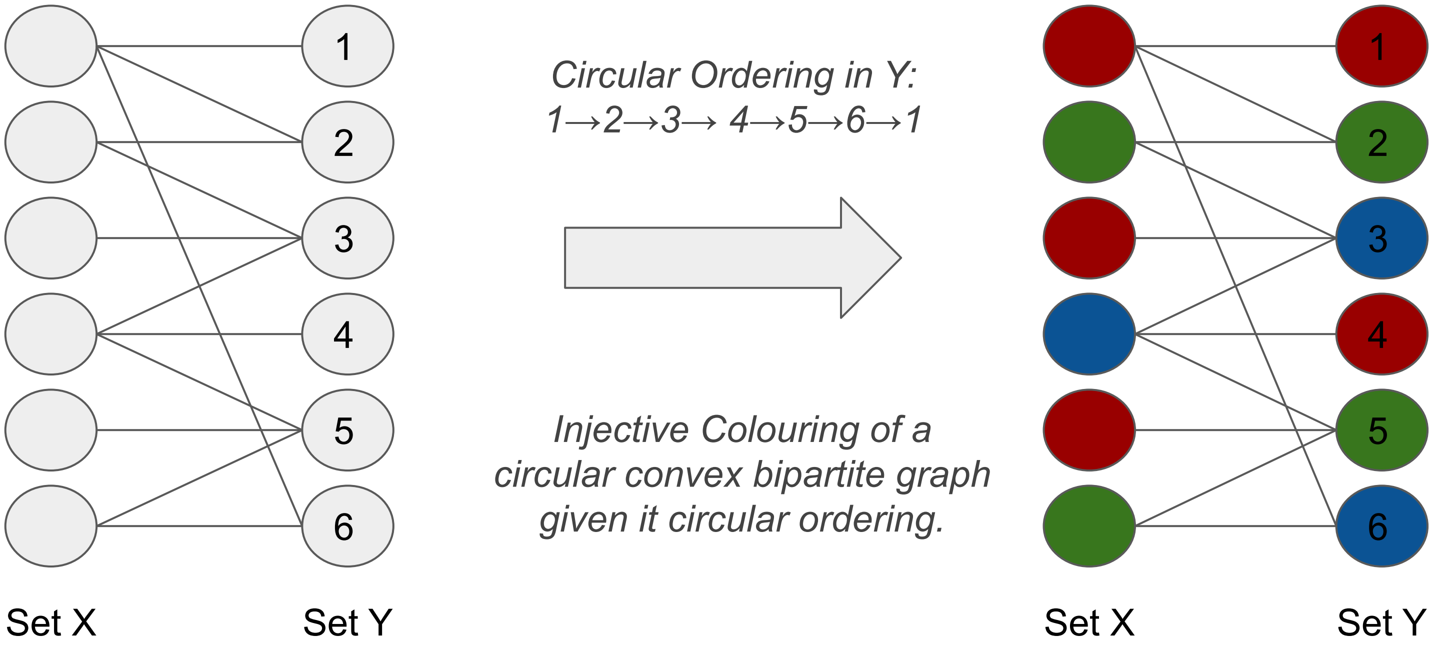
- The only difference is that the neighbours of any particular 'x' may be in the order something like, $\{y_i, y_{i+1}, \dots, y_{ny}, y_1, \dots, y_{d(x)-ny-i-1}\}$, i.e., overshooting y_{ny}.

Complexity of the algorithm is O(mn), where m = |vertex set| and n = |edge set| of G.



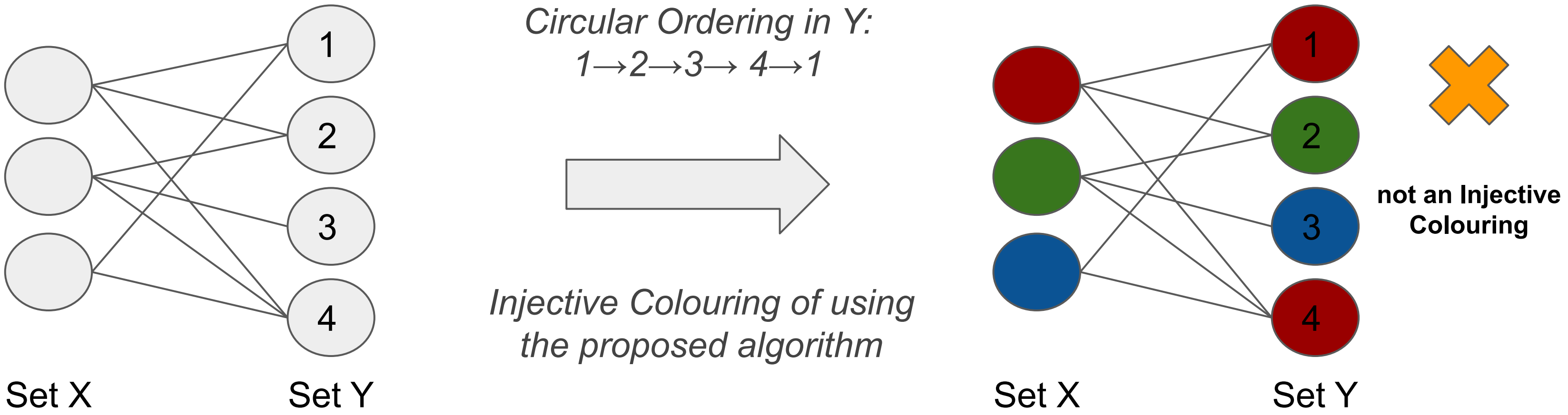
Result 1 (Approach 1)

Result 1: A circular convex bipartite graph with Δ = 3



Result 2: *Failed!*

Result 2: A circular convex bipartite graph with Δ = 3, but cannot be coloured using above algorithm



A circular convex bipartite graph cannot always be coloured using Δ colors injectively.

Even though with Δ = 3, every vertex in Y are adjacent to each other requiring |Y| = 4 colours.

Approach 2- Modification in Approach 1

Let G = (X ∪ Y, *E*) be a circular convex bipartite graph with convexity over Y.

Let $y_1 \rightarrow y_2 \rightarrow \ldots \rightarrow y_{ny} \rightarrow y_1$ be its circular convex ordering of vertex set Y.

The algorithm maintains two set of arrays $L[x_i]$, 1<= i <= nx and $L[y_j]$, 1<= j <= ny.

These arrays stores the colors used by their respective neighbours.

1. Now assign each ***x*** a color **min{ {1,2,...nx}\ ∪ L[y_i'] }**, where "y_i'" $1 <= i <= d_G(x)$ are the neighbours of x.
2. Similarly colour each y in Y.

Time Complexity = O(mn)

m = number of vertices in G

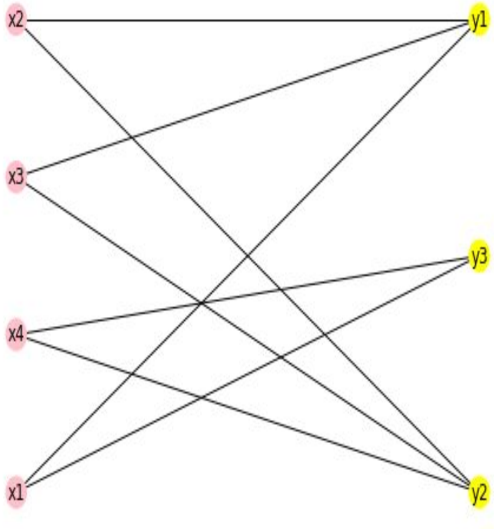
n = number of edges in G

Algorithm implemented in python

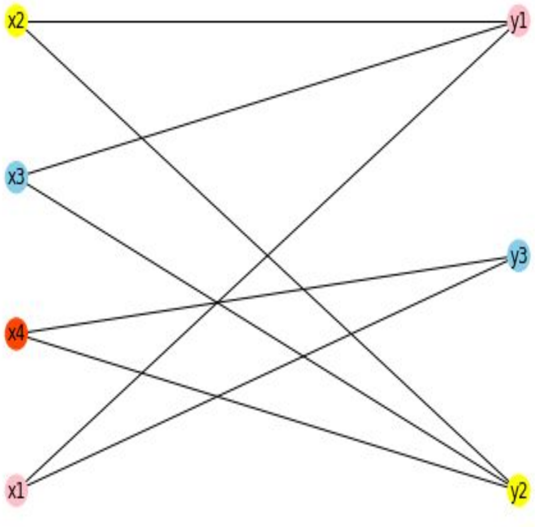
```
# Coloring X
for x in range(1,len(X)+1):
    if(colorX[x-1] == -1):
        S = set()
        for _y in X[x-1]:
            for _x in Y[_y-1]:
                if (colorX[_x-1] != -1): S.add(colorX[_x-1])

        for i in range(1,len(X)+1):
            if i not in S:
                colorX[x-1] = i
                break

# Coloring Y
for y in range(1,len(Y)+1):
    if(colorY[y-1] == -1):
        S = set()
        for _x in Y[y-1]:
            for _y in X[_x-1]:
                if (colorY[_y-1] != -1): S.add(colorY[_y-1])

        for i in range(1,len(Y)+1):
            if i not in S:
                colorY[y-1] = i
                break
```

Result 1 (*previous Counterexample*)

Chromatic No : 4



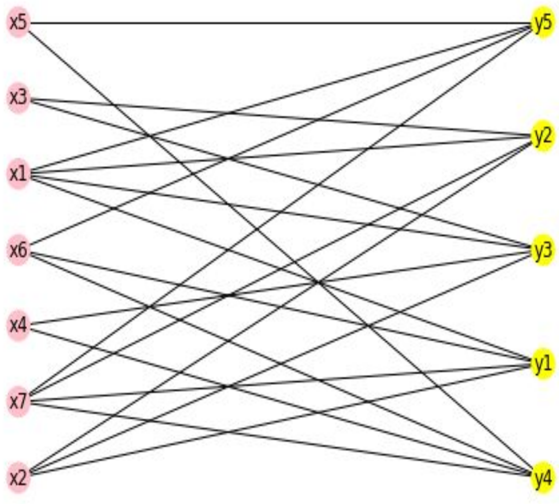
Input



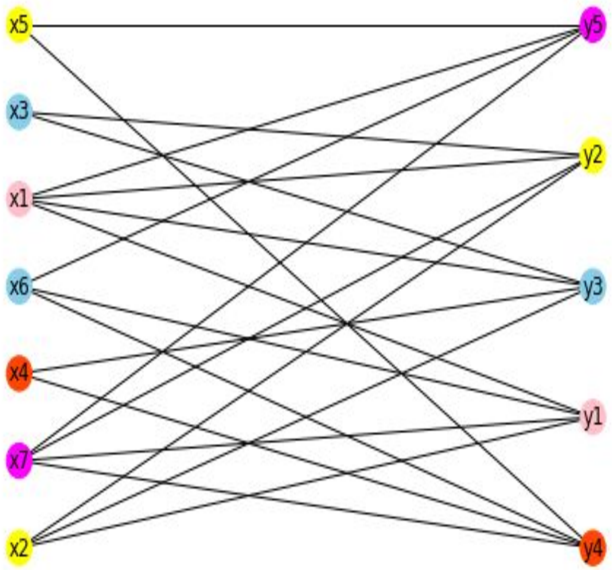
Output

Results 2

Chromatic No : 5



Input



Output

Conclusion and Further Works

- A circular convex bipartite graph may have injective chromatic no >= Δ
- Approach 1 failed as it uses exactly Δ colors
- Approach 2 works for any bipartite graph
- Complexity of Approach 2 is $O(mn)$

Optimality of the Algorithm in the approach 2 needs to be proved.

REFERENCES

1. Fig 1 & 2: https://www.sciencedirect.com/science/article/pii/S0304397514003508
2. Algorithm 1: https://www.sciencedirect.com/science/article/pii/S0166218X20305217
3. Algorithm 2: https://www.sciencedirect.com/science/article/pii/S0166218X20305217
4. Fig 3 & 4: https://www.sciencedirect.com/science/article/pii/S0166218X20305217

THANK YOU