

# teachyourselfmath issue 2

## EXPLAINER

This week we dive into **prime numbers and their distribution**.

Primes are the building blocks of integers: every number greater than 1 is either prime itself or can be written uniquely as a product of primes. This fact, known as the **Fundamental Theorem of Arithmetic**, underpins much of number theory.

Key ideas to explore:

- **Prime factorization:** every integer has a unique prime breakdown (order aside).
- **Divisibility properties:** if a prime divides a product, it divides at least one factor.
- **Canonical form:** integers can be expressed with primes raised to exponents (e.g.,  $360 = 2^3 \cdot 3^2 \cdot 5$ ).
- **Irrationality proofs:** primes play a role in showing numbers like  $\sqrt{2}$  are not rational.
- **Sieve of Eratosthenes:** an ancient yet efficient way to list primes up to  $n$ .
- **Infinitude of primes:** Euclid's elegant argument shows primes never run out.
- **Distribution mysteries:** gaps between primes, twin primes, Goldbach's conjecture (every even integer  $> 2$  is the sum of two primes).

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## EASY STUFF

- 1) Factorize 1260 into primes.

- 2) Write 504 in canonical form  $p_1^{k_1} p_2^{k_2} \dots$ .
  - 3) Show that 35 divides  $\binom{35}{2}$ .
  - 4) Verify that  $\sqrt{3}$  is irrational by mimicking the classic  $\sqrt{2}$  argument.
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### MEDIUM STUFF

- 1) Find all primes of the form  $n^2 - 4$  for integers  $n$ .
  - 2) Show that if  $p \geq 5$  is prime, then  $p^2 + 2$  is composite.
  - 3) Prove that every integer  $n > 11$  can be written as the sum of two composites.
  - 4) List all primes that divide  $50!$ .
  - 5) Find the prime factorization of 1234.
  - 6) Show that any integer  $n$  can be expressed uniquely as  $n = 2^k m$ , with  $m$  odd.
  - 7) Use the Sieve of Eratosthenes to list primes between 100 and 150.
  - 8) Show that any composite three-digit integer must have a prime factor  $\leq 31$ .
  - 9) Verify that 1949 and 1951 form a twin prime pair.
  - 10) Find all prime triplets of the form  $p, p + 2, p + 6$  with  $p < 50$ .
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### HARD STUFF

**Problem:** Prove that there are infinitely many primes of the form  $4n + 3$ .

*Sketch solution:*

- Assume only finitely many such primes  $q_1, q_2, \dots, q_s$ .
  - Consider  $N = 4q_1 q_2 \cdots q_s - 1$ .
  - $N$  is of the form  $4n + 3$ , so at least one prime factor of  $N$  must also be  $4n + 3$ .
  - But this new prime is not among  $q_1, \dots, q_s$  (since it divides  $N$  but not  $4q_1 \cdots q_s$ ).
  - Contradiction. Therefore infinitely many primes of the form  $4n + 3$  exist.
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“The primes are the jewels studding the vast expanse of numbers.” — *Richard Dedekind*