

teachyourselfmath issue 2

EXPLAINER

This week we dive into **prime numbers and their distribution**.

Primes are the building blocks of integers: every number greater than 1 is either prime itself or can be written uniquely as a product of primes. This fact, known as the **Fundamental Theorem of Arithmetic**, underpins much of number theory.

Key ideas to explore:

- **Prime factorization:** every integer has a unique prime breakdown (order aside).
- **Divisibility properties:** if a prime divides a product, it divides at least one factor.
- **Canonical form:** integers can be expressed with primes raised to exponents (e.g., $360 = 2^3 \cdot 3^2 \cdot 5$).
- **Irrationality proofs:** primes play a role in showing numbers like $\sqrt{2}$ are not rational.
- **Sieve of Eratosthenes:** an ancient yet efficient way to list primes up to n .
- **Infinitude of primes:** Euclid's elegant argument shows primes never run out.
- **Distribution mysteries:** gaps between primes, twin primes, Goldbach's conjecture (every even integer > 2 is the sum of two primes).

EASY STUFF

- 1) Factorize 1260 into primes.

- 2) Write 504 in canonical form $p_1^{k_1} p_2^{k_2} \dots$.
 - 3) Show that 35 divides $\binom{35}{2}$.
 - 4) Verify that $\sqrt{3}$ is irrational by mimicking the classic $\sqrt{2}$ argument.
-

MEDIUM STUFF

- 1) Find all primes of the form $n^2 - 4$ for integers n .
 - 2) Show that if $p \geq 5$ is prime, then $p^2 + 2$ is composite.
 - 3) Prove that every integer $n > 11$ can be written as the sum of two composites.
 - 4) List all primes that divide $50!$.
 - 5) Find the prime factorization of 1234.
 - 6) Show that any integer n can be expressed uniquely as $n = 2^k m$, with m odd.
 - 7) Use the Sieve of Eratosthenes to list primes between 100 and 150.
 - 8) Show that any composite three-digit integer must have a prime factor ≤ 31 .
 - 9) Verify that 1949 and 1951 form a twin prime pair.
 - 10) Find all prime triplets of the form $p, p + 2, p + 6$ with $p < 50$.
-

HARD STUFF

Problem: Prove that there are infinitely many primes of the form $4n + 3$.

Sketch solution:

- Assume only finitely many such primes q_1, q_2, \dots, q_s .
 - Consider $N = 4q_1 q_2 \dots q_s - 1$.
 - N is of the form $4n + 3$, so at least one prime factor of N must also be $4n + 3$.
 - But this new prime is not among q_1, \dots, q_s (since it divides N but not $4q_1 \dots q_s$).
 - Contradiction. Therefore infinitely many primes of the form $4n + 3$ exist.
-

“The primes are the jewels studding the vast expanse of numbers.” — *Richard Dedekind*