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Master Thesis

A Review of CMS Swap Pricing Approaches

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Abstract

Evaluating Constant Maturity Swap (CMS) derivatives is a lot more complex than plain vanilla interest rate swaps, because of the unnatural schedule of their payments. Their pricing requires either a convexity adjustment or the use of a model. Hence multiple approaches have been proposed. We selected the most common ones in the marketplace, to analyse and compare them by examining their characteristics. We found that the most accurate prices are generated by Monte Carlo simulations with BGM model for forward rates, while the continuous swaption replication method offers prices consistent with the other instruments of the trading book.

La valorisation des produits dérivés CMS se révèle bien plus complexe que celle des swaps vanilles, en raison de leurs calendriers de paiements non naturels. Les évaluer nécessite soit un ajustement de convexité, soit l'utilisation d'un modèle. De nombreuses approches ont été proposées. Nous avons sélectionné les plus fréquemment utilisées sur les marchés pour les analyser et les comparer, en examinant leurs caractéristiques. Les prix les plus précis sont générés par la méthode de simulation de Monte-Carlo avec le modèle BGM pour les taux forward. D'autre part, la méthode de réplication continue par des swaptions donne des prix cohérents avec ceux des autres instruments du portefeuille du desk de trading.

Contents

1 Introduction

Unlike usual interest rate swap, Constant Maturity Swap (CMS) derivatives are financial instruments whose valuation is not trivial. Because of their unnatural payment schedule, valuing them requires models. Several pricing approaches have been proposed in the financial literature. In this paper, we propose to review and compare some of the most common pricing approaches, highlighting the characteristics of each of them, thereby offering a good insight to practitioners into their valuation method choice.

As many other interest rate derivatives pricing techniques, CMS pricing can be divided into two groups: either by calculating a convexity adjustment or by trying to model the evolution of the term structure of interest rates, using a no-arbitrage model.

The pricing methods from the first group try to add a convexity adjustment to a naive swap pricing. Indeed, the CMS swap price is well known to be the difference of the net present value of the fixed or LIBOR leg and of the net present value of the CMS leg. The value of the LIBOR leg is easily computed, with forward interest rates along the LIBOR yield curve, as for vanilla interest rate swaps. However, the value of the CMS leg depends on the CMS rates, which are not equal to the implied forward swap rates. They need to be corrected for convexity. Intuitively, this adjustment arises from the fact that the CMS rate is exchanged only once instead of being an annuity, which leads to an unnatural time schedule.

Pelsser (2003) showed that the convexity adjustment can also be interpreted as a side-effect of a change of probability measure: implied forward swap rates are the expected CMS rates under the swap measure while CMS swaps are valued under a zero-coupon bond measure.

The simplest way of determining convexity correction is a Taylor expansion of the forward par swap rate. This method is formalised by Benhamou (2000) for any instrument that provides a payoff function of a bond yield, and by Hull (2000) for the specific case of CMS rates.

Another adjustment formula, which is the most common in the marketplace, is derived by Hagan (2003). It consists in continuous replication of CMS caplets, floorlets and swaplets by swaptions, from which the call-put parity is verified. This replication method provides exact results (if swaption prices are also exact) and allows to hedge a CMS position with swaptions. But its computation is complex and slow. Hence, Hagan (2003) also gives an alternative method using approximations from "bond math" and Taylor expansions.

Other corrections can be added to convexity, such as timing adjustment for lagged payments or quanto adjustment for multi-currency CMS swaps, both examined by Brigo and Mercurio (2001). However, while it is worth mentioning their existence, this paper will not study these cases.

The second category of pricing methods are based on no-arbitrage models of the term structure, known as market models: directly observable variables are modelled through a lognormal distribution. The Brace-Gatarek-Musiela (BGM) model is the most frequently used.

The forward LIBOR rates modelled with BGM are then used to price CMS, as Lu and Neftci (2003), and then Gatarek (2003) showed. After calibrating the model (which is the challenging part) and determining the dynamics of the forward rates, they obtain the forward LIBOR rates under a single measure, through a Monte Carlo simulation. Implied CMS rate is then computed, since swap rates are a function of forward rates. This method has the advantage of not needing any convexity adjustment, but it is not model independent any more.

Clearly, all these variants have different properties. This paper aims to study them: we want to compare their accuracy, ease of use, or any other advantages that each of them can have, by deriving their formulas and then understanding which factors affect them.

To do so, this study is organised as follows. After starting with some reminders on the nature of CMS derivatives and on their pertinence for some specific uses, we introduce several valuation methods. We start with the simplest way of determining the value of convexity adjustment: with a bond-price yield convexity approximation.

Then we derive the convexity adjustment through replication of CMS swaptions. We study this second method deeper than any other, since it is the most used by practitioners today. Third, we propose an analytical formula of the previous approach. Finally, we present pricing of CMS swaps with a Monte Carlo algorithm, using BGM models for the dynamic of forward rates.

As expected, we find that the choice of any pricing method is a trade-off between accuracy and ease of use. However, the swaption replication method may still be standing out: while still bearing high complexity, resulting CMS prices are very accurate. In addition, they are consistent with other instrument prices of the book, since the price comes from quoted swaption prices. Last but not least, despite convexity not being a tradeable asset, this approach provides a hedge against it, which can be a very interesting feature from a sell-side trader's point of view.

Throughout this report, we make some assumptions to keep it as clear as possible, namely:

- The nominal amount of every interest rate derivative is set to 1, and then can be omitted.
- Payment periods are assumed to be full-year multiples. The period accrual factors are then all equal to 1 and are omitted.
- Fixed and floating legs of any swap have the same day-count conventions, with payment dates occurring at the same time.
- Swaps are all set in arrears, not in advance (despite it is the market standard). Therefore, there is no timing adjustment.
- All swaps use the same currency (i.e. there is no cross-currency swap). Hence there is no quanto adjustment needed.
- The environment is default-free, and there is no credit risk.
- Market frictions are inexistent.
- The market is arbitrage-free and complete.
- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}})$ is the filtered probability space that we will use.

2 Defining CMS swaps concepts

A Constant Maturity Swap (CMS) rate is a point on the swap yield curve, i.e. the term structure of par swap rates. Therefore a CMS rate is an implied par swap rate. Since the swap yield curve is mostly used for long-term interest rates, CMS rates are also long-term rates.

Because of the par swap rate calculation method, a CMS rate can also be seen as the weighted average of forward rates of a reference index, such as the LIBOR index.

CMS derivatives are contingent claims whose payoffs are function of on CMS rates. The one studied in this paper is the CMS swap. It is a swap with one of its two legs indexed on a CMS rate. The other leg can either be fixed or floating (LIBOR based). In this second case, a spread, referred to as a CMS spread, is added to the rates such that the CMS swap value is null at initiation. An example of a common CMS swap is a three-year quarterly reset swap, with one leg paying the five-year CMS rate and receiving the LIBOR rate.

In a plain vanilla swap, the maturity of the floating rate is more or less equal to the time between two resets. This is called a fixed maturity. A swap leg is said to have constant maturity when its floating rates have a much longer maturity than the time between two resets. This simple difference gives an unnatural schedule to CMS rates, which makes CMS swap pricing dependent of a model, or at least of a convexity adjustment.

2.1 Naive pricing

The most common method to evaluate a European fixed income derivative is to calculate its discounted expected payoff by assuming that the future expected underlying rates are equal to the forward rates. This is used to evaluate FRAs or vanilla interest rate swaps.

$$V_f(0) = Z_0(t) \times f[F_0(t,T)] \tag{1}$$

where $V_f(0)$ is today price of a derivative, with the payoff function f. $Z_0(t)$ is today value of a zero-coupon bond of maturity t, here used as the discount factor. The underlying rate $F_0(t,T)$ is today forward $(T-t)$-year rate starting at time t.

In a similar fashion, one could try to price a CMS swap by taking the difference between the two legs discounted future cash flows, with the assumption that forward par swap rates are equal to future expected par swap rates.

However this naive pricing would be wrong: under the risk-neutral framework, forward par swap rates are not equal to future expected par swap rates i.e. forward CMS rates. This is due to the fact that the par swap rate and the discounted payoff of a CMS derivative have different natural martingale measures: the measure of the par swap rate is the swap measure with numerical duration as the numeraire, while the measure of the discounted payoff of a CMS derivative is the forward measure of the payment date. Adjusting forward rates for convexity would correct this bias.

An easy way of observing this would be to try to hedge a swaplet (a single payment of a swap) of a CMS swap with a vanilla swap of maturity equal to the CMS rate of this swaplet. The P&L would not be null at the date of payment.

For example, let us say that we pay a 10-year CMS rate, and that we try to static hedge it by entering into a 10-year payer vanilla interest rate swap:

- if the 10-year rate increases, we lose on the CMS swap but we win on the hedging swap. However the amount earned on the hedging swap is smaller than the one lost on the CMS swap, because of the positive convexity of the swap.
- if the 10-year rate decreases, we win on the CMS swap but we lose on the hedging swap. Here again the amount lost on the hedging swap is smaller than the one earned on the CMS swap.

2.2 Use of CMS swaps

Pension funds, insurers or governments, among others, may have long-term liabilities, which are then naturally strongly correlated to long-term interest rates. CMS swaps can be used by these institutions to hedge their positions with short-term exposure. The CMS swap will pay a long-term rate in exchange for a short-term one, such as LIBOR.

Other market participants are willing to become counterparts of these institutions, and convert their short-term rates in long-term ones. Such a position would let them take advantage of changes in the shape of the term structure of interest rates, by being exposed to long-term rates. Thus, CMS swaps must have some properties making their price dependent of the yield curve movements.

To better understand these properties, we simulated the effects of different yield curve movements on the net receipt value of a CMS swaplet.

The net receipt (also called margin) of a CMS swap is the difference between the short-term LIBOR rate and the long-term par swap rate of the CMS swap at initiation, such that its value is null.

The CMS swaplet used here is exchanging a 3M LIBOR rate against a 5-year annual CMS rate.

This net receipt is calculated for different yield curves scenario, described in table 1 and in figure 1.

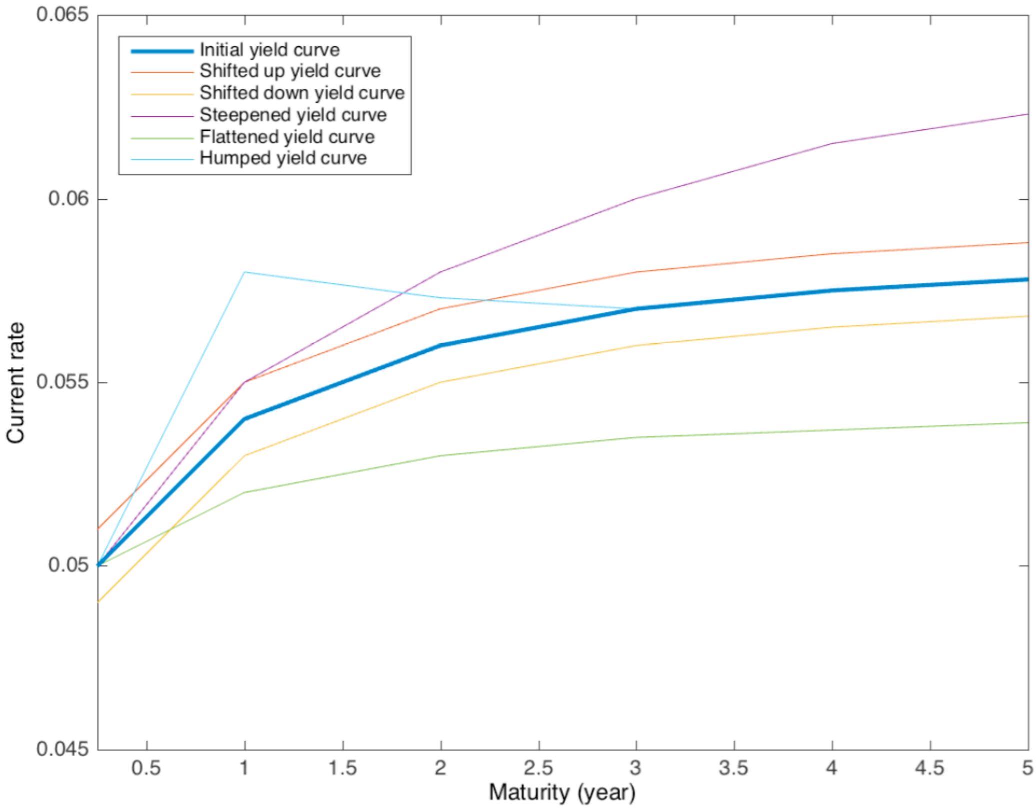

Figure 1: Different yield curves scenarios

Maturity (year)	0.25	1	2	3	4	5
Initial yield curve	5%	5.4%	5.6%	5.7%	5.75%	5.78%
Shifted up yield curve	5.1%	5.5%	5.7%	5.8%	5.85%	5.88%
Shifted down yield curve	4.9%	5.3%	5.5%	5.6%	5.65%	5.68%
Steepened yield curve	5%	5.5%	5.8%	6%	6.15%	6.23%
Flattened yield curve	5%	5.2%	5.3%	5.35%	5.37%	5.39%
Humped yield curve	5%	5.8%	5.73%	5.7%	5.75%	5.78%

Table 1: Rates of each yield curve scenario

The LIBOR rate is already known for each scenario. Only the par swap rate must be calculated. To do so, we use this basic swap valuation formula:

$$S_0(0,T) = \frac{1 - Z_0(T)}{\sum_{t=1}^{T} Z_0(t)} \tag{2}$$

where $S_t(t_1, t_n)$ is the par swap rate at time t of an interest rate vanilla swap, starting at t_1 and ending at t_n. T is the maturity of the swap, and $Z_0(t)$ the discount factor between today and time t.

A summary of the results is presented in table 2.

	5Y par swap rate	LIBOR rate	Net receipt	% change from initial yield net receipt
Initial yield curve	5.7674%	5%	0.7674%	0%
Shifted up yield curve	5.8672%	5.1%	0.7672%	0.0283%
Shifted down yield curve	5.6676%	4.9%	0.7676%	-0.0283%
Steepened yield curve	6.1981%	5%	1.1981%	-35.952%
Flattened yield curve	5.3839%	5%	0.3839%	99.880%
Humped yield curve	5.7752%	5%	0.7752%	-1.016%

Table 2: Net receipts of the CMS swaplet for each yield curves scenario

These results show that the slope of the yield curve (steepening or flattening) is the strongest factor influencing the price of a CMS swap. The shape of the curve has small effects, but parallel shifts have barely not influenced the net receipt.

This effect is even stronger than the one put in light here, because we tested it only with one CMS swaplet. In practice, CMS swaps are priced with the forward interest rate curve. A steep curve is even steeper when using its forward values, while a flat one is even flatter.

Therefore CMS swaps can be very useful to investors predicting a steepening or flattening of the yield curve. In the steepening scenario (i.e. the future swap rates rise above the forward swap rates) one should receive the CMS rate for a profit, while in an expected flattening scenario (i.e. swap rates stay below what is implied by the forward curve) one should pay the CMS rate.

3 CMS convexity adjustment as an approximation from bond price-yield convexity

Because bonds have a convex price-yield relationship, when a bond expected price is equal to its forward price as it should be in a forward risk neutral environment, its expected yield is not equal to its forward yield.

This can be easily observed by placing three points on the price-yield curve, with one of them being the average price of the other two. The expected price (average of the three prices) and forward price have the same value, but the expected yield (average of the three yields) and forward yield (corresponding to the forward price) are not equal. This is obviously due to the convexity of the relationship.

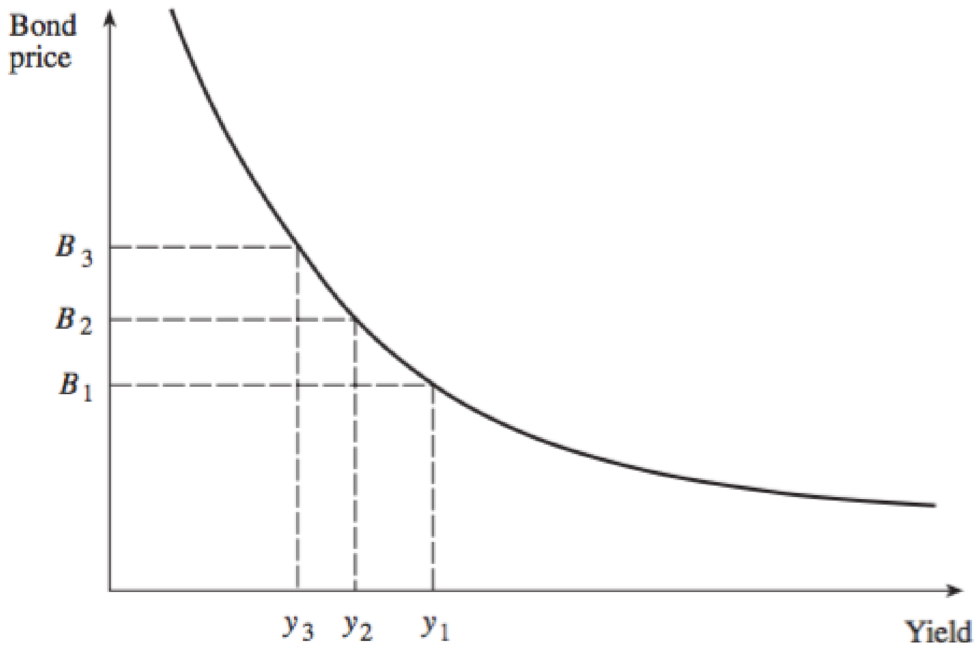

Figure 2: Bond price-yield relationship

One of the most common pricing methods of bond yield derivatives is to discount its expected payoff. However, this expected payoff depends on future values of bond yields, and only the forward yields are known (from the observable forward prices). An adjustment is needed to make these forward yields equal to the expected future yields.

This bond yield derivatives argument is also valid for CMS swaps, because they can be seen as a particular kind of bond yield derivatives, by making the assumption that a swap rate is a yield on a bond. The payoff of each of the swaplet of the CMS swap is indeed dependent from a swap rate observed at the time of the payoff. Because a swap rate is a par yield, the N-year swap rate can be approximated as the yield on an N-year bond, which pays a coupon equal to today forward swap rate.

To calculate the convexity adjustment, let B_T be the bond price at maturity $t = T$, y_T the bond yield at $t = T$, and G the function giving the relationship between them.

$$B_T = G(y_T) \tag{3}$$

Let also y_0 be the forward bond yield at $t = 0$ for a forward contract with maturity T.

The second order Taylor expansion of $G(y_T)$ about $y_T = y_0$ is

$$B_T = G(y_T) \approx G(y_0) + G'(y_0) \times (y_T - y_0) + \frac{1}{2} G''(y_0) \times (y_T - y_0)^2 \tag{4}$$

with $G'(x)$ and $G''(x)$ the first and second derivatives of G.

Taking expectations, in the forward risk neutral framework,

$$E_T[B_T] = G(y_0) + G'(y_0) \times E_T[y_T - y_0] + \frac{1}{2} G''(y_0) \times E_T[(y_T - y_0)^2] \tag{5}$$

$G(y_0)$ is the forward bond price by definition, and in our forward risk neutral framework $E_T[B_T]$ is also the forward bond price. Therefore,

$$G'(y_0) \times E_T[y_T - y_0] + \frac{1}{2} G''(y_0) \times E_T[(y_T - y_0)^2] = 0 \tag{6}$$

By rearranging the terms of this expression,

$$E_T[y_T] = y_0 - \frac{1}{2} \frac{G''(y_0)}{G'(y_0)} \times E_T[(y_T - y_0)^2] \tag{7}$$

From the variance definition, $E_T[(y_T - y_0)^2] \approx \sigma_y^2 y_0^2 T$.

which gives us the convexity adjustment formula for a single bond:

$$E_T[y_T] = y_0 - \frac{1}{2} y_0^2 \sigma_y^2 T \frac{G''(y_0)}{G'(y_0)} \tag{8}$$

As previously stated, in a CMS swap the CMS rate of every swaplet is comparable to a single bond's yield. Hence the convexity adjustment formula can be adapted for CMS swaps to take into account several cash flows. For a single CMS swaplet:

$$E_{t_i}[S_{t_i}(t_i, T_i)] = S_0(t_i, T_i) - \frac{1}{2} S_0(t_i, T_i)^2 \sigma_i^2 (T_i - t_i) \frac{G_i''[S_0(t_i, T_i)]}{G_i'[S_0(t_i, T_i)]} \tag{9}$$

with

- $E_{t_i}[S_{t_i}(t_i, T_i)]$ the expected spot par swap rate at future time t_i, which is the expected CMS rate of the swaplet i, paid at t_i.
- $S_0(t_i, T_i)$ today forward par swap rate of the CMS swaplet i, starting at t_i and of tenor $T_i - t_i$.
- σ_i the volatility of $S_0(t_i, T_i)$. It can be implied from swaptions prices.
- $T_i - t_i$ the CMS tenor of the swaplet i, paid at t_i.
- $G_i(x)$ the price at t_i of a bond, as a function of its yield x. This bond pays coupons at rate $S_0(t_i, T_i)$ with the same maturity and frequency as the swap from which the CMS rate is implied. $G_i'(x)$ and $G_i''(x)$ are its first and second derivatives with respect to x.

The value of the CMS swap is just the sum of all its convexity adjusted CMS swaplet values.

This method is the simplest way to obtain an approximation of a CMS swap value. However it is done in a single curve context: the yield curve is assumed to keep this level and shape forever. This assumption is somewhat over-simplistic, and because of it most of the time this formula fails to give prices close to the ones observed in the marketplace.

4 CMS swaps continuously replicated by swaptions

This valuation method consists in replicating CMS swaps with vanilla swaptions. A basic financial notion is that the value of an instrument is equal to the value of its hedge, so valuing the hedge of a CMS swap would provide its price. Without the convexity adjustment, hedging a CMS swaplet would be obvious: by entering into a forward starting swap. However the convexity cannot be ignored, that is why a precise hedging position requires swaptions, whose optionality brings the convexity needed.

The replication can be achieved by using a model of the yield curve movements to simulate any possible scenario, and to hedge any of these scenarios with a swaption of matching strike rate. That is why this replication is said to be continuous.

4.1 Forward par swap rate

As CMS rates are implied from points on the swap yield curve, the first step into valuing a CMS swap is to study the forward par swap rate of a vanilla swap.

Let $Z_t(T)$ be the value or a zero-coupon bond of maturity T at time t. Then $Z_0(t)$ is today discount factor for maturity t.

The value of a swap V_{swap} is the difference of its floating leg value V_{float} and its fixed leg value V_{fix}.

At any time t, for an N-year swap between its start date t_0 and its end date t_N:

$$V_{float}(t) = Z_t(t_0) - Z_t(t_N) \tag{10}$$

and

$$V_{fix}(t) = R_{fix} \times \sum_{j=1}^{N} Z_t(t_j) \tag{11}$$

with R_{fix} the fixed rate of the swap.

The value of the swap is

$$\begin{aligned} V_{swap}(t) &= V_{float}(t) - V_{fix}(t) \\ &= Z_t(t_0) - Z_t(t_N) - R_{fix} \times \sum_{j=1}^{N} Z_t(t_j) \end{aligned} \tag{12}$$

If the numerical duration of the swap (also called DV01 or level of the swap) is denoted by $L(t)$, defined as

$$L(t) = \frac{\partial V_{swap}(t)}{\partial R_{fix}} = \sum_{j=1}^{N} Z_t(t_j) \tag{13}$$

the swap value is

$$V_{swap}(t) = [S_t(t, t_N) - R_{fix}] \times L(t) \tag{14}$$

where

$$S_t(t_0, t_N) = \frac{Z_t(t_0) - Z_t(t_N)}{L(t)} \tag{15}$$

Because the swap value is 0 when $S_t(t_0, t_N) = R_{fix}$, $S_t(t_0, t_N)$ is the par swap rate at time t.

Under this swap measure, i.e. with $L(t)$ as the numeraire, the swap rate process S is a martingale, since it is only made of zero-coupon bonds:

$$S_t(t_0, t_N) = E_T\Big[S_T(t_0, t_N)\Big|\mathcal{F}_t\Big] \quad \forall t \leqslant T \tag{16}$$

Therefore it can be assumed to follow a model of rates, such as Black's with the form

$$S_t(t_0, t_N) = S_0(t_0, t_N)e^{-\frac{\sigma^2}{2}t+\sigma\sqrt{t}x} \tag{17}$$

where x is a normally distributed random variable of mean 0 and of variance 1.

Other models could be used instead, such as Heston or SABR models.

The expected value of $S_T(t_0, t_N)$ can be determined with this formula. The forward par swap rates are now known, and can be used to value a number of par-swap rate derivatives, such as swaptions.

However to value CMS swaps, one needs expected future par swap rates, which are not equal to par swap rates, as aforementioned: a convexity adjustment is still needed.

4.2 Price of a swaption with numerical duration as numeraire

In the following steps, the value of a European vanilla swaption will be needed.

Let us start with a payer swaption. It is a call option on a vanilla swap. Here, the swap starts at time t_p and ends at time t_N. The strike rate is K. Then, from equation 14, the price of the swaption at exercise date τ (i.e. the payoff) is

$$V^{payer}_{swaption}(\tau, K) = [S_\tau(t_p, t_N) - R_{fix}]^+ L(\tau) \tag{18}$$

The numerical duration $L(t)$ is a collection of zero-coupon bonds, which are freely tradeable instruments. It can then be set as our numeraire. With this numeraire, the value $V(t)$ of any tradeable instrument that is a martingale is

$$V(t) = L(t) \times E\left[\frac{V(\tau)}{L(\tau)}\middle|\mathcal{F}_t\right] \quad \forall t < \tau \tag{19}$$

Because the swaption is a martingale, its value with $L(t)$ as our numeraire at $t < \tau$ is

$$\begin{aligned} V^{payer}_{swaption}(t, K) &= L(t) \times E\left[\frac{V^{payer}_{swaption}(\tau, K)}{L(\tau)}\middle|\mathcal{F}_t\right] \\ &= L(t) \times E\Big\{[S_\tau(t_p, t_N) - K]^+\Big|\mathcal{F}_t\Big\} \end{aligned} \tag{20}$$

Today, at time $t = 0$

$$V^{payer}_{swaption}(0, K) = L(0) \times E\Big\{[S_\tau(t_p, t_N) - K]^+\Big|\mathcal{F}_0\Big\} \tag{21}$$

Similarly for a receiver swaption:

$$V^{receiver}_{swaption}(0, K) = L(0) \times E\Big\{[K - S_\tau(t_p, t_N)]^+\Big|\mathcal{F}_0\Big\} \tag{22}$$

4.3 Price of a CMS caplet

Recall the call-put parity:

$$V_{swap}(t) = V_{cap}(t) - V_{floor}(t) \tag{23}$$

This parity is also true for CMS derivatives, and can therefore be written as

$$V^{CMS}_{swap}(t) = V^{CMS}_{cap}(t) - V^{CMS}_{floor}(t) \tag{24}$$

Hence to evaluate a CMS swap, we can use the values of a CMS cap and a CMS floor. Let us start with caps.

The payoff of a CMS caplet paid at τ with strike K, and with the underlying vanilla swap starting date t_p and end date t_N:

$$V_{caplet}^{CMS}(\tau) = [S_\tau(t_p, t_N) - K]^+ Z_\tau(t_p) \tag{25}$$

The discount factor $Z_\tau(t_p)$ is needed since the payment date and the rate fixing date are different.

With the numerical duration $L(t)$ as its numeraire, the value of the caplet at any time $t < \tau$ is

$$V_{caplet}^{CMS}(t) = L(t) \times E\left[\frac{[S_\tau(t_p, t_N) - K]^+ Z_\tau(t_p)}{L(\tau)}\middle|\mathcal{F}_t\right] \tag{26}$$

And today

$$V_{caplet}^{CMS}(0) = L(0) \times E\left[\frac{[S_\tau(t_p, t_N) - K]^+ Z_\tau(t_p)}{L(\tau)}\middle|\mathcal{F}_0\right] \tag{27}$$

Notice that the ratio $\frac{Z_\tau(t_p)}{L(\tau)}$ is also a martingale:

$$E\left[\frac{Z_\tau(t_p)}{L(\tau)}\middle|\mathcal{F}_0\right] = \frac{Z_0(t_p)}{L(0)} \tag{28}$$

Hence $Z_0(t_p)/L(0)$ is its mean value.

By dividing the expectation in equation 27 by this mean, we obtain

$$V_{caplet}^{CMS}(0) = L(0)\frac{Z_0(t_p)}{L(0)} \times E\left\{[S_\tau(t_p, t_N) - K]^+ \times \frac{Z_\tau(t_p)/L(\tau)}{Z_0(t_p)/L(0)}\middle|\mathcal{F}_0\right\} \tag{29}$$

With the identity $E(xy) = E(x) \times E(y) + Cov(x, y)$, and because $E\left(\frac{Z_\tau(t_p)/L(\tau)}{Z_0(t_p)/L(0)}\right) = 1$ this equation can be rewritten as

$$\begin{aligned} V_{caplet}^{CMS}(0) &= Z_0(t_p) \times E\{[S_\tau(t_p, t_N) - K]^+|\mathcal{F}_0\} \\ &\quad + Z_0(t_p) \times E\left\{[S_\tau(t_p, t_N) - K]^+ \times \left(\frac{Z_\tau(t_p)/L(\tau)}{Z_0(t_p)/L(0)} - 1\right)\middle|\mathcal{F}_0\right\} \end{aligned} \tag{30}$$

Equation 21 shows that the first term $Z_0(t_p) \times E\{[S_\tau(t_p, t_N) - K]^+|\mathcal{F}_0\}$ is exactly the price of a swaption of strike rate K, if the notional is $Z_0(t_p)/L(0)$.

The second term is the convexity adjustment ca.

$$ca = Z_0(t_p) \times E\left\{[S_\tau(t_p, t_N) - K]^+ \times \left(\frac{Z_\tau(t_p)/L(\tau)}{Z_0(t_p)/L(0)} - 1\right)\middle|\mathcal{F}_0\right\} \tag{31}$$

$$V_{caplet}^{CMS}(0) = \frac{Z_0(t_p)}{L(0)} \times V_{swaption}^{payer}(0, K) + ca \tag{32}$$

4.4 Convexity adjustment valuation

To evaluate the convexity adjustment, the yield curve movements will be modelled such that the zero-coupon bond price $Z_\tau(t_p)$ and the numerical duration $L(\tau)$ can be written as a function G of the par swap rate $S_\tau(t_p, t_N)$.

$$\frac{Z_\tau(t_p)}{L(\tau)} = G[S_\tau(t_p, t_N)] \tag{33}$$

Then the convexity adjustment ca at time $t = 0$ is

$$ca = Z_0(t_p) \times E\left\{[S_\tau(t_p, t_N) - K]^+ \times \left(\frac{G[S_\tau(t_p, t_N)]}{G[S_0(t_p, t_N)]} - 1\right)\middle|\mathcal{F}_0\right\} \tag{34}$$

To find this expected value, a yield curve model must be chosen in order to have an expression for the function G. A few of them are presented in Table 3, with increasing accuracy but also complexity.

Name	$\mathbf{G(S)}$[1]	Characteristics
Street-standard	$\frac{S}{(1+S/q)^{\Delta}} \times \frac{1}{1-\frac{1}{(1+S/q)^n}}$ with q the frequency of resets and Δ the fraction of period between start t_0 and date of payment t_p $\Delta = \frac{t_p - t_0}{t_1 - t_0}$	- Approximate schedule and day count of the reference swaption - Flat yield curve over the swaption tenor - Perfect correlation of rates of different maturities
Exact yield	$\frac{S}{(1+\alpha_1 S)^{\Delta}} \times \frac{1}{1-\prod_{k=1}^{n}\frac{1}{1+\alpha_j S}}$ with α_j the year fraction of the j^{th} period	- Exact schedule and day count of the reference swaption - Flat yield curve - Only allows parallel shifts
Parallel shifts	$\frac{Se^{-(t_p-t_0)x}}{1-\frac{Z_0(t_N)}{Z_0(t_0)}e^{-(t_N-t_0)x}}$ with x the parallel shift amount	- Start from initial yield curve shape - Only allows parallel shifts - Perfect correlation between long and short term rates
Hull-White (Non-parallel shifts)	$\frac{Se^{-[(h(t_p)-h(t_0)]x}}{1-\frac{Z_0(t_N)}{Z_0(t_0)}e^{-[(h(t_N)-h(t_0)]x}}$ with $h(t)$ the effect of the shift on maturity t. This function is often assumed to be mean-reverting	- Start from initial yield curve shape - Allows non-parallel shifts

Table 3: Models of the yield curve movements

[1]In this table, the notation $S_\tau(t_p, t_N)$ is shortened to S for clarity.

4.5 Caplet, floorlet and swaplet replications with swaptions

The payoff term $[S_\tau(t_p, t_N) - K]^+$ in the convexity adjustment expression (equation 34) can be replicated with payer swaptions.

Let us introduce a smooth function f, with $f(S_\tau(t_p, t_N)) \in \mathcal{C}^2$. Using Fourier transforms, Carr and Madan (2002) showed that such functions can be written as

$$\begin{aligned} f(S_\tau(t_p,t_N)) &= f(K) + f'(K)([S_\tau(t_p,t_N) - K]^+ - [K - S_\tau(t_p,t_N)]^+) \\ &\quad + \int_0^K [x - S_\tau(t_p,t_N)]^+ f''(x)dx + \int_K^{+\infty} [S_\tau(t_p,t_N) - x]^+ f''(x)dx \end{aligned} \tag{35}$$

If f is such that $f(K) = 0$ then

$$\begin{aligned} &f'(K)[S_\tau(t_p,t_N) - K]^+ + \int_K^{+\infty} [S_\tau(t_p,t_N) - x]^+ f''(x)dx \\ &= \begin{cases} f(S_\tau(t_p,t_N)) & \text{for} \quad S_\tau(t_p,t_N) > K \\ 0 & \text{for} \quad S_\tau(t_p,t_N) < K \end{cases} \end{aligned} \tag{36}$$

By choosing

$$f[S_\tau(t_p,t_N)] = (S_\tau(t_p,t_N) - K)\left(\frac{G(S_\tau(t_p,t_N))}{G(S_0(t_p,t_N))} - 1\right) \tag{37}$$

equation 34 can be rewritten as

$$\begin{aligned} ca &= Z_0(t_p) \times E[f(S_\tau(t_p,t_N))|\mathcal{F}_0] \\ &= Z_0(t_p) \times E\left\{ f'(K)[S_\tau(t_p,t_N) - K]^+ + \int_K^{+\infty} [S_\tau(t_p,t_N) - x]^+ f''(x)dx \middle| \mathcal{F}_0 \right\} \\ &= Z_0(t_p)\left\{ f'(K)E\Big([S_\tau(t_p,t_N) - K]^+|\mathcal{F}_0\Big) + \int_K^{+\infty} f''(x)E([S_\tau(t_p,t_N) - x]^+|\mathcal{F}_0)dx \right\} \end{aligned} \tag{38}$$

The convexity adjustment can be expressed in terms of payer swaptions, by substituting equation 21 in equation 38:

$$ca = \frac{Z_0(t_p)}{L(0)} \times \left[f'(K)V^{payer}_{swaption}(0,K) + \int_K^{+\infty} f''(x)V^{payer}_{swaption}(0,x)dx \right] \tag{39}$$

After adding the first term of equation 32, the value of a CMS caplet with strike rate K at time $t = 0$ is

$$V^{CMS}_{caplet}(0,K) = \frac{Z_0(t_p)}{L(0)} \times \left\{ [1 + f'(K)]V^{payer}_{swaption}(0,K) + \int_K^{+\infty} f''(x)V^{payer}_{swaption}(0,x)dx \right\} \tag{40}$$

This formula shows how to replicate a CMS caplet with swaptions of different strikes. In practice, the integral is computed through an integration by part, using the sum of available swaptions prices with strikes spaced by 10 basis points, for example. The swaptions prices are easily computed with parameters implied from the vanilla book.

An identical argument can be applied for CMS floorlets. They can be replicated with receiver swaptions, with the following formula:

$$V_{floorlet}^{CMS}(0,K) = \frac{Z_0(t_p)}{L(0)} \times \left\{ [1+f'(K)]V_{swaption}^{receiver}(0,K) + \int_{-\infty}^{K} f''(x)V_{swaption}^{receiver}(0,x)dx \right\} \tag{41}$$

The replication of a CMS swaplet is then easily derived from the call-put parity (equation 24) and the two replication formulas 40 and 41.

$$\begin{aligned} V_{swaplet}^{CMS}(0,S_0(t_p,t_N)) &= Z_0(t_p)S_0(t_p,t_N) \\ &+ \frac{Z_0(t_p)}{L(0)} \times \left\{ \int_{S_0(t_p,t_N)}^{+\infty} f''(x)V_{swaption}^{payer}(0,x)dx + \int_{-\infty}^{S_0(t_p,t_N)} f''(x)V_{swaption}^{receiver}(0,x)dx \right\} \end{aligned} \tag{42}$$

with the same function f as for caplets and floorlets, where the strike is not K any more but the par swap rate $S_0(t_p,t_N)$.

Otherwise, this result could have been directly derived by repeating previous CMS caplet arguments with the CMS swaplet payoff.

These formulas replicate every scenario generated by the chosen yield curve model, with one swaption of different strike for each scenario.

This pricing through replication is a very accurate method, and is consistent with other prices in the vanilla book of the desk, including its skew adjustment. However the main drawback is its high computing intensiveness. To remedy this, the following section presents analytical formulas, less precise but fast and easy to compute.

In addition, these formulas can be reversed to get implied swap volatility smile from quoted CMS swaps prices. Most of the time, implied swap volatility is directly derived through swaptions quotes. But some strikes are too far away-from-the-money and there is not any swaption price available for these strikes. Theoretical prices of the needed swaptions can be derived from CMS swaps value with these formulas.

5 Analytical Formulas From Swaption Replication Method

Because of the previously cited drawbacks, an analytical form of the previous results would be useful. This section presents an approximation of these results to find these analytical formulas.

5.1 General analytical formulas

G is a smooth function, for any chosen yield curve model. In addition, the most probable swap rates $S_\tau(t_p, t_N)$ are close to $S_0(t_p, t_N)$. Then a first order Taylor expansion of $G(x)$ about $S_0(t_p, t_N)$ should not cause a large accuracy loss. For more accurate formulas, a higher order expansion should be used instead.

$$G(x) = G[S_0(t_p, t_N)] + G'[S_0(t_p, t_N)] \times [x - S_0(t_p, t_N)] \tag{43}$$

Using this approximated form in f (equation 37):

$$f(x) = (x - K)[x - S_0(t_p, t_N)]\frac{G'[S_0(t_p, t_N)]}{G[S_0(t_p, t_N)]} \tag{44}$$

f first and second derivatives are:

$$f'(x) = [2x - K - S_0(t_p, t_N)]\frac{G'[S_0(t_p, t_N)]}{G[S_0(t_p, t_N)]} \tag{45}$$

$$f''(x) = 2 \times \frac{G'[S_0(t_p, t_N)]}{G[S_0(t_p, t_N)]} \tag{46}$$

From equation 33

$$G[S_0(t_p, t_N)] = \frac{Z_0(t_p)}{L(0)} \tag{47}$$

Using this expression of $G[S_0(t_p, t_N)]$ in the CMS caplet replication equation (40), with our new approximation of function f (equations 44, 45 and 46) we have:

$$\begin{aligned} V_{caplet}^{CMS}(0) = {} & \frac{Z_0(t_p)}{L(0)} \times V_{swaption}^{payer}(0, K) \\ & + G'[S_0(t_p, t_N)]\left\{[K - S_0(t_p, t_N)]V_{swaption}^{payer}(0, K) + 2\int_K^{+\infty} V_{swaption}^{payer}(0, x)dx\right\} \end{aligned} \tag{48}$$

By substituting the payer swaption formula (equation 21) in the integral, and by resolving it, we obtain

$$\begin{aligned}\int_K^{+\infty} V_{swaption}^{payer}(0,x)dx &= L(0)E\left\{\int_K^{+\infty}[S_\tau(t_p,t_N)-x]^+dx\middle|\mathcal{F}_0\right\} \\ &= \frac{1}{2}L(0)E\{([S_\tau(t_p,t_N)-K]^+)^2|\mathcal{F}_0\}\end{aligned} \tag{49}$$

Putting it back in equation 48, along with equation 21, we have

$$\begin{aligned}V_{caplet}^{CMS}(0) = &\frac{Z_0(t_p)}{L(0)}V_{swaption}^{payer}(0,K) \\ &+ G'[S_0(t_p,t_N)] \times L(0) \times E\{[S_\tau(t_p,t_N)-K]^+|\mathcal{F}_0\} \\ &\times \left\{E\left[(S_\tau(t_p,t_N)-K)^+\middle|\mathcal{F}_0\right] + K - S_0(t_p,t_N)\right\}\end{aligned} \tag{50}$$

$$\begin{aligned}V_{caplet}^{CMS}(0) = &\frac{Z_0(t_p)}{L(0)}V_{swaption}^{payer}(0,K) \\ &+ G'[S_0(t_p,t_N)] \times L(0) \times E\{[S_\tau(t_p,t_N)-K]^+[S_\tau(t_p,t_N)-S_0(t_p,t_N)]|\mathcal{F}_0\}\end{aligned} \tag{51}$$

Notice that the convexity correction has the form of an expected quadratic payoff.

By applying the same reasoning to floorlets we obtain

$$\begin{aligned}V_{floorlet}^{CMS}(0) = &\frac{Z_0(t_p)}{L(0)}V_{swaption}^{receiver}(0,K) \\ &- G'[S_0(t_p,t_N)] \times L(0) \times E\{[S_\tau(t_p,t_N)-K]^+[S_0(t_p,t_N)-S_\tau(t_p,t_N)]|\mathcal{F}_0\}\end{aligned} \tag{52}$$

And with swaplets:

$$V_{swaplet}^{CMS}(0) = Z_0(t_p)S_0(t_p,t_N) + G'[S_0(t_p,t_N)] \times L(0) \times E\{[S_\tau(t_p,t_N)-S_0(t_p,t_N)]^2|\mathcal{F}_0\} \tag{53}$$

5.2 Approximation for the specific case of the street-standard yield curve model

We will now replace L and G in the previous formula by choosing the "street-standard" model for the function G (see table 3).

For simplicity, we make the assumption that the frequency of resets q is 1, and that the payment date of the underlying swap t_p is equal to t_0, so $\Delta = 0$.

We have:

$$G[S_0(t_0, t_N)] = \frac{S_0(t_0, t_N)[1 + S_0(t_0, t_N)]^N}{[1 + S_0(t_0, t_N)]^N - 1} \tag{54}$$

We can easily determine the expressions of $G'[S_0(t_0, t_N)]$ and of $L(0)$ with this G function.

$$G'[S_0(t_0, t_N)] = \frac{[1 + S_0(t_0, t_N)]^{2N} - (1 + S_0(t_0, t_N))^N - NS_0(t_0, t_N)(1 + S_0(t_0, t_N))^{N-1})}{([1 + S_0(t_0, t_N)]^N - 1)^2} \tag{55}$$

and from equation 33:

$$\begin{aligned} L(0) &= \frac{Z_0(t_0)}{G[S_0(t_0, t_N)]} = \frac{1}{G[S_0(t_0, t_N)]} \\ &= \frac{[1 + S_0(t_0, t_N)]^N - 1}{S_0(t_0, t_N)[1 + S_0(t_0, t_N)]^N} \end{aligned} \tag{56}$$

By replacing their values into equation 53, we obtain:

$$\begin{aligned} V^{CMS}_{swaplet}(0) = S_0(t_0, t_N)\Bigg\{1 + \left[1 - \frac{NS_0(t_0, t_N)}{(1 + S_0(t_0, t_N))[(1 + S_0(t_0, t_N))^N - 1]}\right] \\ \times E\left[\frac{[S_\tau(t_0, t_N) - S_0(t_0, t_N)]^2}{S_0(t_0, t_N)^2}\middle|\mathcal{F}_0\right]\Bigg\} \end{aligned} \tag{57}$$

To calculate the expectation, we need to invoke a model for the swap rate $S_\tau(t_p, t_N)$. With Black's model (equation 17), the expectation value is $e^{\sigma^2\tau} - 1$, where σ is the volatility of $S_\tau(t_0, t_N)$, and the CMS swaplet value becomes:

$$V^{CMS}_{swaplet}(0) = S_0(t_0, t_N)\left\{1 + \left[1 - \frac{NS_0(t_0, t_N)}{(1 + S_0(t_0, t_N))[(1 + S_0(t_0, t_N))^N - 1]}\right] \times (e^{\sigma^2\tau} - 1)\right\} \tag{58}$$

If $S_0(t_0, t_N)$ is small (which is often the case in practice, because it is shrinked by the reset period factor, that we omitted in this paper), $(1 + S_0(t_0, t_N))^N - 1 \approx N S_0(t_0, t_N)$, and equation 58 simplifies to

$$V^{CMS}_{swaplet}(0) = S_0(t_0, t_N) + \frac{S_0(t_0, t_N)^2}{1 + S_0(t_0, t_N)} \times (e^{\sigma^2 \tau} - 1) \tag{59}$$

Of course, we still need to remove the fixed (or LIBOR) leg value, since this is only the value of the CMS leg of the swaplet.

Figure 3 presents the value of the CMS swap rate according to this formula, as a function of its underlying swap rate $S_0(t_0, t_N)$ and its volatility σ (for a fixed swaplet payment date $\tau = 5$).

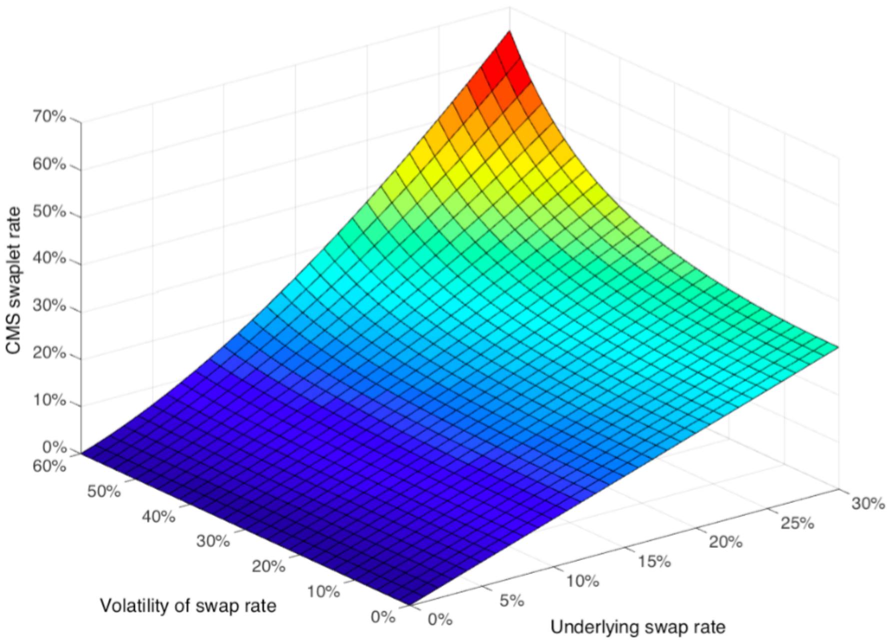

Figure 3: CMS swaplet rate as a function of $S_0(t_0, t_N)$ and σ

The second term of the sum is the convexity adjustment. Its accuracy depends on three factors:

- The absolute value of $S_0(t_0, t_N)$, that needs to be small to keep an accurate approximation.
- The shape of the realised volatility $\sigma(t)$, as Black Model assumes a constant volatility: if the volatility of volatility is high, the approximation will deteriorate.

- The shape of the yield curve. The chosen model assumes a flat yield curve. An accurate approximation can only be reached with a curve that does not steepen too much.

CMS swaps can be valued in a very simple way with this last formula. However one should keep in mind that this analytical form is less accurate than the continuous replication provided in equation 42.

Otherwise, a more accurate analytical formula can be obtained by increasing the order of the Taylor expansion of function f, or another G expression can be used when choosing a yield curve model in table 3.

Black's model suffers many critics, such as not taking into account the skew and assuming that the future swap rate distribution is log normal. Also the volatility of rates can be delicate to determine. Hence more complex models (such as SABR or Heston's) should also be considered.

Finally, a major drawback raised when using these analytical formulas is that the convexity adjustment is not a tradeable asset any more, unlike with the swaption replication method. It can vary over time, as forward rates move. Traders cannot hedge their CMS positions against convexity with other market instruments.

6 Monte Carlo Simulation with Brace–Gatarek–Musiela Model

Any derivative can be valued by Monte Carlo simulations if the payoff function is known and if there is a realistic model for the process of the underlying asset. CMS swaps are no exception.

The first challenge to using Monte Carlo for valuing a CMS swap is to find its payoff function. In this section, we will prove that these derivatives are functions of forward rates under a single measure, since vanilla swap rates are themselves function of forward rates. Then, by using the Brace–Gatarek–Musiela Model for simulating paths of forward rates, we will show how to price CMS swaps with Monte Carlo simulations.

6.1 Expressing price of CMS swaps with forward rates

For simplicity, we will work on this pricing method with the example of a two-period CMS swap. Of course, the argument would be true for any CMS swap, but the formulas would become very complex.

Consider the two-period CMS swap exchanging cash flows at t_1 and t_2, whose CMS rates have maturities of two periods. The other leg pays fixed rates. Let X_{t_0} be its par CMS swap rate at t_0, i.e. the fixed rate that cancels out the value of the CMS swap V_{swap}^{CMS}.

At t_2, the cash flows exchanged are the fixed rate X_{t_0} and the CMS rate $S_{t_1}(t_1, t_3)$, which is the par rate of a vanilla swap fix versus LIBOR L_{t_i}, between t_1 and t_3, with two payments at t_2 and t_3. Similarly, at t_3 the rates used are X_{t_0} and $S_{t_2}(t_2, t_4)$.

At t_0, the value of the CMS swap is:

$$V_{swap}^{CMS} = \sum_{i=1}^{2} Z_0(t_i) \times (X_{t_0} - E[S_0(t_i, t_{i+1})]) = 0 \tag{60}$$

where $Z_0(t_i)$ is the discount factor of maturity t_i and $S_0(t_i, t_{i+1})$ is the par swap rate at $t = 0$ of a forward starting (at time t_i) vanilla swap of maturity 1.

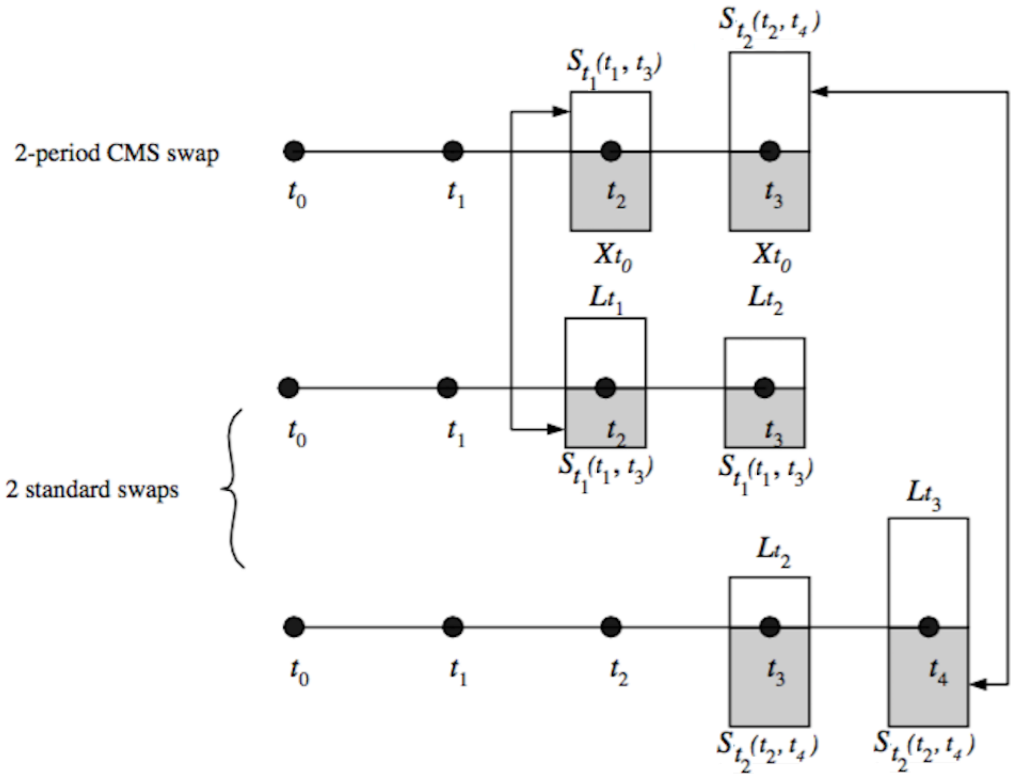

Figure 4: A two-period CMS swap

By making X_{t_0} the subject of equation 60:

$$X_{t_0} = E\left[\frac{Z_0(t_2) \times S_{t_1}(t_1, t_3) + Z_0(t_3) \times S_{t_2}(t_2, t_4)}{Z_0(t_2) + Z_0(t_3)}\right] \tag{61}$$

With forward rates instead of zero-coupon rates, the discount factor formula is:

$$Z_0(t_N) = \frac{1}{(1 + L_{t_0}) \times \prod_{i=1}^{N}[1 + F_{t_0}(t_i, t_{i+1})]} \tag{62}$$

where $F_{t_0}(t_i, t_{i+1})$ is today forward LIBOR rate between t_i the starting time of period i, and its end time t_{i+1}.

The par CMS swap rate expression (equation 61) can be rewritten:

$$X_{t_0} = \frac{E\left[\frac{S_{t_1}(t_1,t_3)}{(1+L_{t_0})(1+F_{t_0}(t_1,t_2))} + \frac{S_{t_2}(t_2,t_4)}{(1+L_{t_0})(1+F_{t_0}(t_1,t_2))(1+F_{t_0}(t_2,t_3))}\right]}{E\left[\frac{1}{(1+L_{t_0})(1+F_{t_0}(t_1,t_2))} + \frac{1}{(1+L_{t_0})(1+F_{t_0}(t_1,t_2))(1+F_{t_0}(t_2,t_3))}\right]} \tag{63}$$

The valuation formula of the par swap rate of forward vanilla interest swaps can be applied to $S_{t_1}(t_1, t_3)$ and $S_{t_2}(t_2, t_4)$:

$$\begin{aligned} S_{t_1}(t_1, t_3) &= \frac{Z_0(t_2) \times F_{t_0}(t_1, t_2) + Z_0(t_3) \times F_{t_0}(t_2, t_3)}{Z_0(t_2) + Z_0(t_3)} \\ &= \frac{\frac{F_{t_0}(t_1,t_2)}{(1+L_{t_0})(1+F_{t_0}(t_1,t_2))} + \frac{F_{t_0}(t_2,t_3)}{(1+L_{t_0})(1+F_{t_0}(t_1,t_2))(1+F_{t_0}(t_2,t_3))}}{\frac{1}{(1+L_{t_0})(1+F_{t_0}(t_1,t_2))} + \frac{1}{(1+L_{t_0})(1+F_{t_0}(t_1,t_2))(1+F_{t_0}(t_2,t_3))}} \end{aligned} \quad (64)$$

and

$$\begin{aligned} S_{t_2}(t_2, t_4) &= \frac{Z_0(t_3) \times F_{t_0}(t_2, t_3) + Z_0(t_4) \times F_{t_0}(t_3, t_4)}{Z_0(t_3) + Z_0(t_4)} \\ &= \frac{\frac{F_{t_0}(t_2,t_3)}{(1+L_{t_0})(1+F_{t_0}(t_1,t_2))(1+F_{t_0}(t_2,t_3))} + \frac{F_{t_0}(t_3,t_4)}{(1+L_{t_0})(1+F_{t_0}(t_1,t_2))(1+F_{t_0}(t_2,t_3))(1+F_{t_0}(t_3,t_4))}}{\frac{1}{(1+L_{t_0})(1+F_{t_0}(t_1,t_2))(1+F_{t_0}(t_2,t_3))} + \frac{1}{(1+L_{t_0})(1+F_{t_0}(t_1,t_2))(1+F_{t_0}(t_2,t_3))(1+F_{t_0}(t_3,t_4))}} \end{aligned} \quad (65)$$

These expressions show that forward par swap rates are weighted averages of forward rates. So $S_{t_1}(t_1, t_3)$ and $S_{t_2}(t_2, t_4)$ can be written as

$$S_{t_1}(t_1, t_3) = \omega_1^{t_1} F_{t_0}(t_1, t_2) + \omega_2^{t_1} F_{t_0}(t_2, t_3) \quad (66)$$

and

$$S_{t_2}(t_2, t_4) = \omega_1^{t_2} F_{t_0}(t_2, t_3) + \omega_2^{t_2} F_{t_0}(t_3, t_4) \quad (67)$$

The weight factors ω_i are not constants, but also functions of forward rates. However, they should be far less volatile than forward rates.

Notice that in equation 63, the par CMS swap rate X_{t_0} is a function of forward rates $F_{t_0}(t_1, t_2)$, $F_{t_0}(t_2, t_3)$ and $F_{t_0}(t_3, t_4)$. Hence, modelling the dynamics of these three LIBOR forward rates would let us find terminal values for the CMS swap rate, under the same forward measure.

Then, the value of the CMS swap rate can be easily found with a Monte Carlo simulation.

6.2 Stochastic Differential Equation for forward rates

In this section, we present a relevant stochastic differential equation (SDE) for forward swap rates, which will then be used in the aforementioned Monte Carlo simulation.

In a one-factor model, the forward rates $F_{t_0}(t_i, t_{i+1})$ have the following dynamics:

$$dF_{t_0}(t_i, t_{i+1}) = \mu_i F_{t_0}(t_i, t_{i+1})dt + \sigma_i F_{t_0}(t_i, t_{i+1})dW_t \tag{68}$$

with μ_i the forward rate drift, σ_i the forward rate volatility and W_t a Brownian motion (under the real world probability measure). σ_i can be estimated from the volatility of i-year caps.

Also, the forward rate beginning at t_i and of maturity t_{i+1}, from $t = 0$ is defined as

$$F_{t_0}(t_i, t_{i+1}) = \frac{Z_0(t_i)}{Z_0(t_{i+1})} - 1 \tag{69}$$

Hence, because it is made of zero-coupon bonds which are martingales, $1 + F_{t_0}(t_i, t_{i+1})$ is also a martingale under the $P^{t_{i+1}}$ forward measure. μ_i, the drift term of the SDE is then null. The SDE can be rewritten as:

$$dF_{t_0}(t_i, t_{i+1}) = \sigma_i F_{t_0}(t_i, t_{i+1})dW_t \tag{70}$$

This is the Brace–Gatarek–Musiela Model, also known as the LIBOR Market Model.

This expression will be very easy to use with a Monte Carlo simulation. However, the equation 63 is non-linear: several forward rates must be used simultaneously, under the same forward measure $P^{t_{i+1}}$. Then an adjusted dynamic must be used for the lagged forward rate $dF_{t_0}(t_{i-1}, t_i)$ under the $P^{t_{i+1}}$ forward measure, since its natural forward measure is P^{t_i}.

After applying the measure change[2], the adjusted dynamic expression under the $P^{t_{i+1}}$ forward measure is:

$$dF_{t_0}(t_{i-1}, t_i) = -\sigma_{i-1} F_{t_0}(t_{i-1}, t_i) \frac{\sigma_i F_{t_0}(t_i, t_{i+1})}{1 + F_{t_0}(t_i, t_{i+1})} dt + \sigma_{i-1} F_{t_0}(t_{i-1}, t_i)dW_t \tag{71}$$

The two SDE 70 and 71, can be used in the par CMS swap rate X_{t_0} expression (equation 63) to perform the Monte Carlo simulations.

[2] See Lu and Neftci (2003)'s appendix for computation details.

7 Conclusion

In this paper, after defining CMS swap concepts and proving their utility for holding positions on the yield curve steepness, we examined the main approaches for valuing them. Unlike for most other interest rate derivatives, because of their unnatural payment schedule (i.e. a long maturity swap rate is exchanged at shorter time intervals), forward rates involved in their pricing are not equal to their future expected rate. Then, pricing CMS swaps requires a model of forward rates, or at least a convexity adjustment. We reviewed four different methods to evaluate their prices.

The first one, which offers a very simple convexity adjustment formula for expected future swap rates, is obtained by assuming that bonds price-yield convexity is similar to CMS swaps convexity. Then the bond formula just needs to be adapted to CMS swaps. This pricing method is the easiest, but simplicity comes at a price: the results are so inaccurate that it is very uncommon to encounter it in the industry.

Next, we presented the swaption replication method, which consists in using optionality of swaptions to reproduce the convexity of the CMS derivatives. To derive it, we identified the swaption price equation in the CMS swap valuation formula. To do so, we used the call-put parity. Hence caplet and floorlet formulas were also derived. To replicate CMS swaps prices, we hedge them with swaptions. Hence every yield curve scenario need to be simulated. We presented several models of yield curve movements that can be used for this purpose. This method is very precise, and have many benefits. As swaptions prices are available in the marketplace, the price of CMS derivatives will be consistent with other instruments, in particular the implied volatilities of the rates of its underlying swaps, including the skew.

Since the swaption replication method is highly computing intensive, we proposed an analytical form, based on approximations of the models of the yield curve. After applying it to the street-standard yield curve model, and with Black model for valuing the future expected swap rates, we obtained a simple formula for the convexity adjustment. Here again, we should keep in mind that this is based on many assumptions, and that the prices generated by this method can be imprecise. In addition, we renounced to the interesting property of having existing market instruments to hedge the convexity of CMS.

Finally, the last presented approach is a Monte Carlo simulation. To perform it, a payoff function is needed. We proved that a CMS swap payoff is a function of several forward rates, because underlying swap rates are themselves function of forward rates under a single measure. We could not use Monte Carlo for the other pricing methods, because it can only generate paths with a single forward measure, which was not the case before, because the pricing was done with numerical duration as numeraire. The stochastic differential equation used for forward rates is the one of the Brace–Gatarek–Musiela model, which has the required property of estimating the dynamic of forward rates under a single measure.

Each approach has pros and cons: accuracy, ease of use, or allowing a hedge with existing instruments. To determine the method having the best accuracy while still being not too complex, we could programmatically compare them: the Monte Carlo method is extremely accurate, and could be used as a benchmark for testing the other approaches. Unfortunately such tests could not be performed here, due to a lack of historical data, such as swaption prices and forward rates.

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8 Bibliography

Benhamou, E. (2000): "A martingale result for convexity adjustment in the Black pricing model", Working Paper, London School of Economics

Black, F. (1976): "The pricing of commodity contracts", The Journal of Financial Economics, 3, 167-179

Brace, A., D. Gatarek and M. Musiela (1997): "The Market Model of Interest Rate Dynamics", Mathematical Finance, 7(2), 127-154.

Brigo, D. and F. Mercurio (2001): "Interest rate models - Theory and practice", 2nd ed., Springer Finance

Carr, P. and D. Madan (2002): "Towards a Theory of Volatility Trading", RISK Publications

Gatarek, D. (2003): "Constant Maturity Swaps, Forward Measure and LIBOR market model", Working Paper, Deloitte & Touche CE

Hagan, P. (2003): "CMS Conundrums: Pricing CMS Swaps, Caps, and Floors", Wilmott Magazine, 38-44

Hull, J.C. (2012): "Options, Futures and other Derivatives", 8th ed., Prentice Hall

Lu, Y. and S. Neftci (2003): "Convexity adjustment and forward LIBOR model: case of constant maturity swaps", FINRISK - Working Paper Series

Mercurio, F. and A. Pallavicini (2006): "Smiling at convexity: bridging swaption skews and CMS adjustments", Working paper, Banca IMI

Pelsser, A. (2003): "Mathematical foundation of convexity correction", Quantitative Finance, 3, 59-65

Selic, N. (2003): "An examination of the convexity adjustment technique in the pricing of constant maturity swaps", Advanced Mathematics of Finance Honours Project