Supplemental Material for "Measuring noncommutativity and clustering quantum observables with a quantum switch"

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I. PROPERTIES OF THE MED AND ITS GENERALIZATION

Here we establish the main properties of the MED, including symmetry, nonnegativity, faithfullness, robustness to noise, being a metric for von Neumann measurements, and reaching its maximum value for mutually unbiased bases. Some of the properties will be established for a generalized version of the MED, which can also be experimentally accessed using the quantum switch.

In the following, we will often consider the *generalized MED*

$$\mathrm{MED}_\rho(\mathscr{A},\mathscr{B}) := \sqrt{1-\sum_{i,j} \mathrm{Re}\,\mathrm{Tr}[\rho P_i Q_j P_i Q_j]}\,. \qquad (1)$$

Note that, by definition, one has

$$\mathrm{MED}_{\frac{I}{d}}(\mathscr{A},\mathscr{B}) = \sqrt{1-\frac{1}{d}\sum_{i,j} \mathrm{Tr}[P_i Q_j P_i Q_j]} \equiv \mathrm{MED}(\mathscr{A},\mathscr{B})\,, \qquad (2)$$

for every pair of observables A and B. In the following we establish a number of properties of MED_ρ and MED. As it turns out, some of the properties of MED_ρ require the density matrix ρ to have some properties, such as being invertible, or commuting with the projectors onto the eigenspaces of A and B. All these properties are automatically satisfied by the choice $\rho = I/d$.

Symmetry. By definition, one has

$$\begin{aligned}\mathrm{MED}_\rho(\mathscr{A},\mathscr{B}) &= \sqrt{1-\sum_{i,j} \mathrm{Re}\,\mathrm{Tr}[\rho P_i Q_j P_i Q_j]}\\ &= \sqrt{1-\sum_{i,j} \mathrm{Re}\overline{\mathrm{Tr}[\rho P_i Q_j P_i Q_j]}}\\ &= \sqrt{1-\sum_{i,j} \mathrm{Re}\,\mathrm{Tr}\left[(\rho P_i Q_j P_i Q_j)^\dagger\right]}\\ &= \sqrt{1-\sum_{i,j} \mathrm{Re}\,\mathrm{Tr}[Q_j P_i Q_j P_i \rho]}\\ &= \sqrt{1-\sum_{i,j} \mathrm{Re}\,\mathrm{Tr}[\rho Q_j P_i Q_j P_i]}\\ &= \mathrm{MED}_\rho(\mathscr{B},\mathscr{A})\,. \end{aligned} \qquad (3)$$

Here the fourth equality follows from the property $(XY)^\dagger = Y^\dagger X^\dagger$, valid for arbitrary operators X and Y, along with the fact that the operators ρ, P_i, and Q_j are self-adjoint. The fifth equality follows from the cyclic property of the trace.

Nonnegativity. The generalized MED can be equivalently expressed as

$$\mathrm{MED}_\rho(\mathscr{A},\mathscr{B}) = \sqrt{\frac{\sum_{i,j}\mathrm{Tr}\left\{[P_i,Q_j]\rho\,[P_i,Q_j]^\dagger\right\}}{2}} \qquad (4)$$

The right-hand side is non-negative, because each of the operators $[P_i,Q_j]\rho\,[P_i,Q_j]^\dagger$ is non-negative, and therefore has a non-negative trace.

Faithfulness. We now show that the generalized MED is faithful for every invertible state ρ: for two arbitrary dephasing channels $\mathscr{A}$ and $\mathscr{B}$, $\mathrm{MED}_\rho(\mathscr{A},\mathscr{B}) > 0$ if and only if the observables A and B are incompatible.

The proof is based on Eq. (A4), which shows that $\mathrm{MED}_\rho(\mathscr{A},\mathscr{B})$ is non-negative and equals to zero if and only if $\mathrm{Tr}\left\{[P_i,Q_j]\rho\,[P_i,Q_j]^\dagger\right\} = 0$ for every i and every j. By the cyclicity of the trace, this condition holds if and only if $\mathrm{Tr}\left\{\sqrt{\rho}\,[P_i,Q_j]^\dagger[P_i,Q_j]\sqrt{\rho}\right\} = 0$ for every i and every j. Moreover, since each operator $\sqrt{\rho}\,[P_i,Q_j]^\dagger[P_i,Q_j]\sqrt{\rho}$ is positive, its trace is zero if and only if the operator itself is zero. Finally, since ρ is invertible, the condition $\sqrt{\rho}\,[P_i,Q_j]^\dagger[P_i,Q_j]\sqrt{\rho} = 0$ holds if and only if $[P_i,Q_j]^\dagger[P_i,Q_j] = 0$, or equivalently, if and only if $[P_i,Q_j] = 0$. Summarizing, the condition $\mathrm{MED}_\rho(\mathscr{A},\mathscr{B}) = 0$ is equivalent to the condition $[P_i,Q_j] = 0\,, \forall i,j$, which is equivalent to the compatibility of the observables A and B.

Monotonicity under coarse-graining. Here we show that $\mathrm{MED}_\rho(\mathscr{A},\mathscr{B})$ is non-increasing under coarse-graining, provided that the density matrix ρ commutes with the measurement that is being coarse-grained:

Proposition 1 *If the state ρ commutes with the observable A, then one has $\mathrm{MED}_\rho(\mathscr{A},\mathscr{B}) \ge \mathrm{MED}_\rho(\mathscr{A}',\mathscr{B})$ whenever $\mathscr{A}'$ is the dephasing channel associated to a coarse-grained measurement of A.*

Proof. Let $\mathbf{P}' := (P'_k)_{k=1}^{o'}$ be a coarse-graining of the measurement $\mathbf{P} = (P_i)_{i=1}^{o}$, meaning that there is a surjective function $f : \{1, \dots, o\} \rightarrow \{1, \dots, o'\}$ such that

$$P'_k = \sum_{i:f(i)=k} P_i \,. \tag{5}$$

Note that, by definition, one has the operator inequality

$$P_i \leq P'_{f(i)} \,. \tag{6}$$

Now, suppose that the density matrix ρ commutes with the projectors $(P_i)_{i=1}^{o}$. In this case, one has the relation

$$\begin{aligned} \mathrm{Tr}[\rho P_i Q_j P_{i'} Q_j] &= \mathrm{Tr}[(\sqrt{\rho} P_i \sqrt{\rho})\,(Q_j P_{i'} Q_j)] \\ &\geq 0 \,, \end{aligned} \tag{7}$$

valid for every pair of indices i and i'. Here, the last inequality follows from the fact that the operators $\sqrt{\rho}P_i\sqrt{\rho}$ and $Q_jP_{i'}Q_j$ are positive semidefinite, and therefore the trace of their product is non-negative.

In particular, Eq. (A7) implies that $\mathrm{Tr}[\rho P_i Q_j P_i Q_j]$ is a real number. Hence, the real part in the definition of the generalized MED can be dropped, and one has

$$\mathrm{MED}_\rho(\mathscr{A}, \mathscr{B}) = 1 - \sum_{i,j} \mathrm{Tr}[\rho P_i Q_j P_i Q_j] \,. \tag{8}$$

Since ρ commmutes also with the projectors $(P'_k)_{k=1}^{o'}$, we also have the relation

$$\mathrm{MED}_\rho(\mathscr{A}', \mathscr{B}) = 1 - \sum_{k,j} \mathrm{Tr}[\rho P'_k Q_j P'_k Q_j] \,, \tag{9}$$

where $\mathscr{A}'$ is the dephasing channel associated to the coarse-grained measurement $(P'_k)_{k=1}^{o'}$

At this point, it is easy to see that $\mathrm{MED}_\rho(\mathscr{A}', \mathscr{B}) \leq \mathrm{MED}_\rho(\mathscr{A}, \mathscr{B})$. Indeed, one has the bound

$$\begin{aligned} \sum_{k,j} \mathrm{Tr}[\rho\, P'_k\, Q_j P'_k Q_j] &= \sum_{k,j} \sum_{\substack{i:f(i)=k \\ i':f(i')=k}} \mathrm{Tr}[\rho\, P_i\, Q_j P_{i'} Q_j] \\ &= \sum_{i,j} \mathrm{Tr}[\rho\, P_i\, Q_j P_i Q_j] \\ &\quad + \sum_{k,j} \sum_{\substack{i:f(i)=k \\ i':f(i')=k \\ i\neq i'}} \mathrm{Tr}[\rho\, P_i\, Q_j P_{i'} Q_j] \\ &\geq \sum_{i,j} \mathrm{Tr}[\rho\, P_i\, Q_j P_i Q_j], \end{aligned} \tag{10}$$

where the last inequality follows from Eq. (A7). The inequality $\mathrm{MED}_\rho(\mathscr{A}', \mathscr{B}) \leq \mathrm{MED}_\rho(\mathscr{A}, \mathscr{B})$ then follows by combining Eq. (A10) with Eqs. (A8) and (A9). ■

Metric on von Neumann measurements. We now show that the generalized MED based on an invertible state is a metric on the set of rank-one projective measurements.

Lemma 1 *For every pair of dephasing channels $\mathscr{A}$ and $\mathscr{B}$ associated to rank-one measurements, one has the expression*

$$\mathrm{MED}_\rho(\mathscr{A}, \mathscr{B}) = \frac{1}{\sqrt{2}} \,\|(\tau_{\mathscr{A}} - \tau_{\mathscr{B}})\,(\sqrt{\rho} \otimes I)\|_2 \,, \tag{11}$$

where $\|X\|_2 = \sqrt{\mathrm{Tr}[X^\dagger X]}$ is the Hilbert-Schmidt norm, and $\tau_{\mathscr{A}}$ and $\tau_{\mathscr{B}}$ are the Choi operators of $\mathscr{A}$ and $\mathscr{B}$, given by

$$\tau_{\mathscr{A}} := \sum_{m,n} \mathscr{A}(|m\rangle\langle n|) \otimes |m\rangle\langle n| = \sum_i P_i \otimes \overline{P}_i, \tag{12}$$

$$\tau_{\mathscr{B}} := \sum_{m,n} \mathscr{B}(|m\rangle\langle n|) \otimes |m\rangle\langle n| = \sum_j Q_j \otimes \overline{Q}_j \,, \tag{13}$$

with $\overline{X}$ denoting the complex conjugate of a matrix X.

Proof. Let us define the product

$$\langle X, Y\rangle_\rho := \mathrm{Tr}[(\rho \otimes I)\, X^\dagger Y] \,. \tag{14}$$

For every self-adjoint operator τ, we have the relation

$$\begin{aligned} \|\tau\,(\sqrt{\rho} \otimes I)\,\|_2^2 &= \mathrm{Tr}[(\sqrt{\rho} \otimes I)\tau^2(\sqrt{\rho} \otimes I)] \\ &= \mathrm{Tr}[(\rho \otimes I)\tau^2] \\ &= \langle \tau, \tau\rangle_\rho \,. \end{aligned} \tag{15}$$

Using the two equations above, we obtain

$$\begin{aligned} \|(\tau_{\mathscr{A}} - \tau_{\mathscr{B}})\,(\sqrt{\rho} \otimes I)\,\|_2^2 =& \langle(\tau_{\mathscr{A}} - \tau_{\mathscr{B}}), (\tau_{\mathscr{A}} - \tau_{\mathscr{B}})\rangle_\rho \\ =& \langle \tau_{\mathscr{A}}, \tau_{\mathscr{A}}\rangle_\rho + \langle \tau_{\mathscr{B}}, \tau_{\mathscr{B}}\rangle_\rho \\ & - \langle \tau_{\mathscr{A}}, \tau_{\mathscr{B}}\rangle_\rho - \langle \tau_{\mathscr{B}}, \tau_{\mathscr{A}}\rangle_\rho \,. \end{aligned} \tag{16}$$

Note that one has

$$\begin{aligned} \langle \tau_{\mathscr{A}}, \tau_{\mathscr{A}}\rangle_\rho &= \mathrm{Tr}[(\rho \otimes I)\,\tau_{\mathscr{A}}^2] \\ &= \mathrm{Tr}[(\rho \otimes I)\,\tau_{\mathscr{A}}] \\ &= \sum_i \mathrm{Tr}[\rho\, P_i] \times \mathrm{Tr}[\overline{P}_i] \\ &= \sum_i \mathrm{Tr}[\rho\, P_i] \\ &= 1 \,, \end{aligned} \tag{17}$$

where the second equality follows from the fact that $\tau_{\mathscr{A}}$ is a projector, the third equality follows from Eq. (A12), the fourth equality follows from the fact that the measurement is rank-one (and therefore $\mathrm{Tr}[\overline{P}_i] = 1$ for every i), and the fifth equality follows from the normalization condition $\sum_i P_i = I$. Similarly, we have

$$\begin{aligned} \langle \tau_{\mathscr{B}}, \tau_{\mathscr{B}}\rangle_\rho &= \mathrm{Tr}[(\rho \otimes I)\,\tau_{\mathscr{B}}^2] \\ &= \mathrm{Tr}[(\rho \otimes I)\,\tau_{\mathscr{B}}] \\ &= \sum_j \mathrm{Tr}[\rho\, Q_j] \times \mathrm{Tr}[\overline{Q}_j] \\ &= \sum_j \mathrm{Tr}[\rho\, Q_j] \\ &= 1 \,. \end{aligned} \tag{18}$$

For the remaining terms, we have

$$\begin{aligned}\langle \tau_{\mathscr{A}}, \tau_{\mathscr{B}} \rangle_\rho &= \mathrm{Tr}[(\rho \otimes I)\, \tau_{\mathscr{A}}^\dagger \tau_{\mathscr{B}}] \\ &= \sum_{i,j} \mathrm{Tr}[\rho\, P_i Q_j] \times \mathrm{Tr}[\overline{P}_i\, \overline{Q}_j] \\ &= \sum_{i,j} \mathrm{Tr}[\rho\, P_i Q_j] \times \mathrm{Tr}[P_i\, Q_j] \\ &= \sum_{i,j} \mathrm{Tr}[\rho\, P_i Q_j P_i Q_j]\,, \end{aligned} \tag{19}$$

where the last equality follows from the fact that the operators P_i and Q_j are rank-one.

Similarly, we have

$$\begin{aligned}\langle \tau_{\mathscr{B}}, \tau_{\mathscr{A}} \rangle_\rho &= \mathrm{Tr}[(\rho \otimes I)\, \tau_{\mathscr{B}}^\dagger \tau_{\mathscr{A}}] \\ &= \sum_{i,j} \mathrm{Tr}[\rho\, Q_j P_i] \times \mathrm{Tr}[\overline{Q}_j \overline{P}_i] \\ &= \sum_{i,j} \mathrm{Tr}[\rho\, Q_j P_i] \times \mathrm{Tr}[Q_j P_i] \\ &= \sum_{i,j} \mathrm{Tr}[\rho\, Q_j P_i Q_j P_i] \\ &= \sum_{i,j} \overline{\mathrm{Tr}[\rho\, P_i Q_j P_i Q_j]}. \end{aligned} \tag{20}$$

Hence,

$$\begin{aligned}||\tau_{\mathscr{A}} - \tau_{\mathscr{B}}||^2 &= 2 - 2\mathrm{Re} \sum_{i,j} \mathrm{Tr}[\rho P_i Q_j P_i Q_j] \\ &= 2\, \left[\mathrm{MED}_\rho(\mathscr{A}, \mathscr{B})\right]^2 . \end{aligned} \tag{21}$$

Eq. (A21) shows that $\mathrm{MED}_\rho(\mathscr{A}, \mathscr{B})$ coincides with the Hilbert-Schmidt norm $||(\tau_{\mathscr{A}} - \tau_{\mathscr{B}})(\sqrt{\rho} \otimes I)||_2$ up to a constant factor. ■

Lemma 2 *For every invertible density matrix ρ, MED_ρ is a metric on the space of von Neumann measurements.*

Proof. To prove that MED_ρ is a metric, we need to show that

1. $\mathrm{MED}_\rho(\mathscr{A}, \mathscr{B}) = \mathrm{MED}_\rho(\mathscr{B}, \mathscr{A})$ for every $\mathscr{A}$ and $\mathscr{B}$ (symmetry)
2. $\mathrm{MED}_\rho(\mathscr{A}, \mathscr{B}) \geq 0$ for every $\mathscr{A}$ and $\mathscr{B}$, with the equality if and only if $\mathscr{A} = \mathscr{B}$ (nonnegativity and identity of indiscernibles)
3. $\mathrm{MED}_\rho(\mathscr{A}, \mathscr{C}) \leq \mathrm{MED}_\rho(\mathscr{A}, \mathscr{B}) + \mathrm{MED}_\rho(\mathscr{B}, \mathscr{C})$ (triangle inequality).

Symmetry was established at the beginning of this Supplemental Material, in Eq. (A3).

Nonegativity of the generalized MED was also established earlier in this Supplemental Material in the demonstration of faithfulness of MED_ρ. When ρ is invertible, we also showed that $\mathrm{MED}(\mathscr{A}, \mathscr{B}) = 0$ implies $[P_i, Q_j] = 0$ for every i and j. For von Neumann measurements, $(P_i)_i$ and $(Q_j)_j$ are two maximal sets of rank-one projectors, and the commutation condition means that there exists a permutation $\pi : \{1, \dots, d\} \to \{1, \dots, d\}$ such that $P_i = Q_{\pi(i)}$. In this case, one has $\mathscr{A}(X) = \sum_i P_i X P_i = \sum_i Q_{\pi(i)} X Q_{\pi(i)} = \sum_j Q_j X Q_j = \mathscr{B}(X)$ for every $d \times d$ matrix X.

Finally, the triangle inequality can be deduced from Eq. (A11) of Lemma 1 and from the triangle inequality of the Hilbert-Schmidt norm. ■

Robustness to noise. We have seen that the generalized MED based on an invertible state ρ is a faithful measure of noncommutativity. We now show that this faithfulness property is preserved even when the ideal projective measurements of A and B are replaced by noisy measurements. Precisely, we consider the noisy scenario where the canonical channels $\mathscr{A}$ and $\mathscr{B}$ are replaced by quantum channels $\mathscr{A}'$ and $\mathscr{B}'$ of the form $\mathscr{A}' = \sum_{i,k} A_{i,k} \rho A_{i,k}^\dagger$ and $\mathscr{B}'(\rho) = \sum_{j,l} B_{j,l} \rho B_{j,l}^\dagger$, with

$$\sum_k A_{i,k}^\dagger A_{i,k} = (1 - \lambda)\, P_i + \lambda p_i\, I \tag{22}$$

and

$$\sum_l B_{j,l}^\dagger B_{j,l} = (1 - \mu)\, Q_j + \mu\, q_j\, I, \tag{23}$$

with suitable probabilities $\lambda, \mu \in [0, 1]$ and suitable probability distributions $\mathbf{p}$ and $\mathbf{q}$. In the following, we show robustness under the assumption that at least one of the two channels $\mathscr{A}'$ and $\mathscr{B}'$ is self-adjoint (recall that a linear map $\mathscr{M}$ is self-adjoint if, for every pair of $d \times d$ matrices X and Y, one has $\mathrm{Tr}[X \mathscr{M}(Y)] = \mathrm{Tr}[\mathscr{M}(X) Y]$).

In the noisy case, we consider the noncommutativity

$$\mathrm{NCOM}_\rho(\mathscr{A}', \mathscr{B}') = \sqrt{\frac{\sum_{i,j,k,l} \mathrm{Tr}\left(\rho\, \left|[A_{i,k}, B_{j,l}]\right|^2\right)}{2}}\,. \tag{24}$$

If this quantity is zero, then each of the commutators $[A_{i,k}, B_{j,l}]$ must vanish, i.e. one must have the relations

$$[A_{i,k}, B_{j,l}] = 0 \qquad \forall i, j, k, l \tag{25}$$

and

$$[A_{i,k}^\dagger, B_{j,l}^\dagger] = 0 \qquad \forall i, j, k, l\,, \tag{26}$$

where the second relation is obtained from the first by taking the adjoint on both sides of the equality sign.

Note that the above relations must hold for every possible Kraus decomposition of the channels $\mathscr{A}'$ and $\mathscr{B}'$. If channel $\mathscr{A}'$ is self-adjoint, the operators $(A_{i,k})_{i,k}$ also form a Kraus representation, and therefore one must have the relation

$$[A_{i,k}^\dagger, B_{j,l}] = 0 \qquad \forall i, j, k, l\,, \tag{27}$$

and, taking the adjoint on both sides of the equality

$$[A_{i,k}, B_{j,l}^\dagger] = 0 \qquad \forall i,j,k,l\,. \tag{28}$$

Similarly, if channel $\mathscr{B}'$ is self-adjoint, the above relations must hold. Using Eqs. (A22), (A23, (A25), (A26), (A27), and (A28), we then obtain

$$(1-\lambda)(1-\mu)\,[P_i, Q_j] = \sum_{k,l}[A_{i,k}^\dagger A_{i,k}, B_{j,l}^\dagger B_{j,l}] = 0 \qquad \forall i,j\,. \tag{29}$$

Hence, if the noncommutativity of the channels $\mathscr{A}'$ and $\mathscr{B}'$ is zero, then the ideal measurements $(P_i)_i$ and $(Q_j)_j$ must be compatible, for every value of the noise parameters λ and μ except in the trivial case $\lambda = 1$ or $\mu = 1$, in which the original measurements are replaced by white noise.

Maximality for maximally complementary observables. We now show the inequality $\mathrm{MED}(\mathscr{A},\mathscr{B}) \leq \sqrt{1-1/\min\{k_A,k_B\}}$, where k_A (k_B) is the number of projectors in the spectral decomposition of the observable A (B). The maximum value is given by $\mathrm{MED}(\mathscr{A},\mathscr{B}) = \sqrt{1-1/d}$ and attained if and only if A and B are maximally complementary [1] or in other words, their POVM operators are rank-one projectors onto the basis vectors of two mutually unbiased bases.

The proof uses a series of lemmas.

Lemma 3 *Let A, B be $n\times n$ Hermitian matrices: If A and B are positive semi-definite, then AB is diagonalizable and has non-negative eigenvalues.*

Proof. The proof can be found in Corollary 7.6.2(b) of Ref. [2]. ■

Lemma 4 *For every pair of observables A and B, one has the bound $\mathrm{MED}(\mathscr{A},\mathscr{B}) \leq \sqrt{1-\frac{1}{k_B}}$ where k_B is the number of distinct eigenvalues of B. The equality holds only if $\mathrm{Tr}[P_iQ_j]$ is independent of j.*

Proof. Using the definitions (A1) and (A2), we focus on a term $\sum_j \mathrm{Tr}[P_iQ_jP_iQ_j]$, fixing thereby the index i. If P_i is a projector of rank r_i, then

$$\mathrm{rank}(P_iQ_j) \leq r_i. \tag{30}$$

Both P_i and Q_j are orthogonal projectors, they are Hermitian and positive semi-definite operators. Using Lemma 3, we can diagonalize the operator P_iQ_j as

$$P_iQ_j = X_{ij}\Delta_{ij}X_{ij}^{-1},$$

where Δ_{ij} is a diagonal matrix with non-negative entries. Let r_{ij} be the rank of Δ_{ij} and assume that the first r_{ij} diagonal entries of Δ_{ij}, denoted by $\lambda_{ij,1},\cdots\lambda_{ij,r_i}$, are non-zero. Then, we can write

$$\mathrm{Tr}[P_iQ_j] = \sum_{s=1}^{r_{ij}} \lambda_{ij,s}. \tag{31}$$

Taking into account the relation $\sum_{j=1}^{k_B}\mathrm{Tr}[P_iQ_j] = \mathrm{Tr}[P_i] =: r_i$, we find

$$\sum_{j=1}^{k_B}\sum_{s=1}^{r_{ij}} \lambda_{ij,s} = r_i. \tag{32}$$

On the other hand,

$$\sum_{j=1}^{k_B}\mathrm{Tr}[P_iQ_jP_iQ_j] = \sum_{j=1}^{k_B}\sum_{s=1}^{r_{ij}} (\lambda_{ij,s})^2. \tag{33}$$

The minimum of (A33) under the constraint (A32) can be computed with the method of Lagrange multipliers. The coefficients that minimize (A33) are given by

$$\lambda_{ij,s}^{\min} = \frac{r_i}{\sum_{l=1}^{k_B} r_{il}} \qquad \forall j\in\{1,\dots,k_B\},\ \forall s\in\{1\dots,r_{ij}\}\,. \tag{34}$$

Plugging the optimal coefficients into the right hand side of Eq. (A33) we then obtain

$$\sum_{j=1}^{k_B}\mathrm{Tr}[P_iQ_jP_iQ_j] \geq \frac{r_i^2}{\sum_{l=1}^{k_B} r_{il}}\,. \tag{35}$$

Now, recall that $r_{il} = \mathrm{rank}(P_iQ_l) \leq \mathrm{rank}(P_i) = r_i$. Plugging this relation into the previous inequality, we obtain

$$\sum_{j=1}^{k_B}\mathrm{Tr}[P_iQ_jP_iQ_j] \geq \frac{r_i}{k_B}\,. \tag{36}$$

Finally, summing over i yields the lower bound

$$\sum_{i=1}^{k_A}\sum_{j=1}^{k_B}\mathrm{Tr}[P_iQ_jP_iQ_j] \geq \frac{d}{k_B}\,. \tag{37}$$

Hence, we obtained

$$\begin{aligned}\mathrm{MED}(\mathscr{A},\mathscr{B}) &= \sqrt{1-\frac{\sum_{j=1}^{k_B}\mathrm{Tr}[P_iQ_jP_iQ_j]}{d}}\\ &\leq \sqrt{1-\frac{1}{k_B}}\,.\end{aligned} \tag{38}$$

A necessary condition for achieving the equality is that the eigenv0alues $\lambda_{ij,s}$ depend only on i (and not on j and s), cf. Eq. (A34). This condition implies in particular that the sum $\sum_s \lambda_{ij,s} = \mathrm{Tr}[P_iQ_j]$ is independent of j. ■

Lemma 5 *For every two observables A and B, one has the bound $\mathrm{MED}(\mathscr{A},\mathscr{B}) \leq \sqrt{1-\frac{1}{\min\{k_A,k_B\}}}$. The equality is achieved only if*

$$\mathrm{Tr}[P_iQ_j] = \frac{d}{k_Ak_B} \qquad \forall i\in\{1,\dots,k_A\}\,,\forall j\in\{1,\dots,k_B\}\,. \tag{39}$$

Proof. Lemma 4 implies the upper bound $\mathrm{MED}(\mathscr{A},\mathscr{B}) \leq \sqrt{1-\frac{1}{k_B}}$. Moreover, the symmetry of the MED yields the condition $\mathrm{MED}(\mathscr{A},\mathscr{B}) = \mathrm{MED}(\mathscr{B},\mathscr{A}) \leq \sqrt{1-\frac{1}{k_A}}$ (the inequality following again from Lemma 4). Hence, one has the bound $\mathrm{MED}(\mathscr{A},\mathscr{B}) \leq \sqrt{1-\frac{1}{\min\{k_A,k_B\}}}$. By Lemma 4, the equality holds only if $\mathrm{Tr}[P_iQ_j]$ is independent of j, and only $\mathrm{Tr}[Q_jP_i] \equiv \mathrm{Tr}[P_iQ_j]$ is independent of i. In summary, it is necessary that $\mathrm{Tr}[P_iQ_j]$ is constant, say $\mathrm{Tr}[P_iQ_j] = c$ for some constant c and for every i and j. The value of the constant can be obtained from the condition

$$\begin{aligned} d &= \mathrm{Tr}[I] \\ &= \sum_{i=1}^{k_A}\sum_{j=1}^{k_B} \mathrm{Tr}[P_iQ_j] \\ &= \sum_{i=1}^{k_A}\sum_{j=1}^{k_B} c \\ &= k_A\, k_B\, c\,, \end{aligned} \tag{40}$$

which implies $c = d/(k_Ak_B)$. ■

Lemma 6 *For every two observables A and B, one has the bound $\mathrm{MED}(\mathscr{A},\mathscr{B}) \leq \sqrt{1-\frac{1}{d}}$ and the equality holds if and only if A and B are non-degenerate and their eigenvectors form two mutually unbiased bases.*

Proof. The inequality $\mathrm{MED}(\mathscr{A},\mathscr{B}) \leq \sqrt{1-\frac{1}{d}}$ is immediate from Lemma 5 and from the fact that $\min\{k_A,k_B\}$ is at most d. Let us now determine when the equality sign is attained. A first necessary condition is that $\min\{k_A,k_B\} = d$, which implies $k_A = k_B = d$, that is, both observables A and B are nondegenerate. The necessary condition in Lemma 5 then becomes $\mathrm{Tr}[P_iQ_j] = 1/d\,, \forall i,j$. Hence, P_i and Q_j must be projectors on two vectors from two mutually unbiased bases.

Conversely, suppose that $(|\alpha_i\rangle)_{i=1}^d$ and $(|\beta_j\rangle)_{j=1}^d$ are two mutually unbiased bases, and that $(P_i)_{i=1}^d$ and $(Q_j)_{j=1}^d$ are the corresponding rank-1 projectors. Then, we have

$$\mathrm{Tr}[P_iQ_jP_iQ_j] = \left|\langle\alpha_i|\beta_j\rangle\right|^4 = \frac{1}{d^2} \qquad \forall i,j\,. \tag{41}$$

Hence, the MED of the corresponding observables is

$$\begin{aligned} \mathrm{MED}(\mathscr{A},\mathscr{B}) &= \sqrt{1-\frac{1}{d}\sum_{i,j}\mathrm{Tr}[P_iQ_jP_iQ_j]} \\ &= \sqrt{1-\frac{1}{d}}\,. \end{aligned} \tag{42}$$

In summary, the MED reaches the maximum value $\mathrm{MED}(\mathscr{A},\mathscr{B}) = \sqrt{1-\frac{1}{d}}$ if and only if A and B are nondegenerate observables associated to mutually unbiased bases. ■

II. NONCOMMUTATIVITY BETWEEN ENTANGLED AND PRODUCT MEASUREMENTS

The mathematical description of a measurement process, including both the measurement statistics and the post-measurement states, is provided by a quantum instrument, that is, a collection of completely positive, trace non-increasing maps $(\mathscr{C}_i)_{i=1}^k$ such that the sum $\sum_{i=1}^k \mathscr{C}_i$ is trace-preserving. Here the index i labels the possible measurement outcomes, which occur on an input state ρ with probability $p(i|\rho) = \mathrm{Tr}[\mathscr{C}_i(\rho)]$, leaving the system in the post-measurement state $\rho_i = \mathscr{C}_i(\rho)/\mathrm{Tr}[\mathscr{C}_i(\rho)]$. The average evolution due to the measurement is then given by the quantum channel $\mathscr{C} := \sum_{i=1}^k \mathscr{C}_i$.

We define the noncommutativity between two instruments as the noncommutativity of the corresponding average evolutions:

$$\mathrm{NCOM}_\rho\left((\mathscr{C}_i)_{i=1}^{k_A}\,,\,(\mathscr{D}_j)_{j=1}^{k_B}\right) := \mathrm{NCOM}_\rho(\mathscr{C},\mathscr{D})\,, \tag{43}$$

with $\mathscr{C} := \sum_{i=1}^{k_A}\mathscr{C}_i$ and $\mathscr{D} := \sum_{j=1}^{k_B}\mathscr{D}_j$.

We now provide a lower bound on the noncommutativity between a maximally entangled measurement and a product measurement. Consider the case of a bipartite system, consisting of two subsystems S_1 and S_2, of dimensions $2 \leq d_1 \leq d_2$. By “maximally entangled measurement" we mean a quantum instrument $(\mathscr{C}_i)_{i=1}^{k_A}$ of the form $\mathscr{C}_i(\rho) = C_i\rho C_i^\dagger$ with $C_i = \lambda_i|\Psi_i\rangle\langle\Phi_i|$ where $|\Phi_i\rangle$ and $|\Psi_i\rangle$ are maximally entangled states, and $0 \leq \lambda_i \leq 1$ is a suitable coefficient. By a “product measurement”, we mean an instrument $(\mathscr{D}_j)_{j=1}^{k_B}$ of the form $\mathscr{D}_j(\rho) = D_j\rho D_j^\dagger$ with $D_j = \mu_j\,|\gamma_j\rangle\langle\alpha_j| \otimes |\delta_j\rangle\langle\beta_j|$, where $|\alpha_j\rangle$ and $|\gamma_j\rangle$ ($|\beta\rangle$ and $|\delta_j\rangle$) are pure states of system S_1 (S_2), and $0 \leq \mu_j \leq 1$ is a suitable coefficient. (Note that we are not assuming that the instrument $(\mathscr{D}_j)_{j=1}^{k_B}$ can be realized by local operations and classical communication. In other words, our notion of product measurement corresponds to fine-grained separable instruments [3], which are not necessarily realizable through local operations and classical communication).

The noncommutativity of the instruments $(\mathscr{C}_i)_{i=1}^{k_A}$ and $(\mathscr{D}_j)_{j=1}^{k_B}$ is given by

$$\begin{aligned} \mathrm{NCOM}_\rho\left((\mathscr{C}_i)_{i=1}^{k_A},(\mathscr{D}_j)_{j=1}^{k_B}\right) &= \sqrt{\frac{\sum_{i,j}\mathrm{Tr}\left(\rho\left|[C_i,D_j]\right|^2\right)}{2}} \\ &= \sqrt{1-\mathrm{Re}\left[\sum_{i,j}\mathrm{Tr}[C_i^\dagger D_j^\dagger C_iD_j\rho]\right]}\,. \end{aligned} \tag{44}$$

Here we consider the case where the state ρ is maximally mixed, namely $\rho = I_1/d_1 \otimes I_2/d_2$. In this case, one has the bound

$$\mathrm{Re}\left[\sum_{i,j}\mathrm{Tr}[C_i^\dagger D_j^\dagger C_i D_j]\right] \le \sum_{i,j}\left|\mathrm{Tr}[C_i^\dagger D_j^\dagger C_i D_j]\right| = \sum_{i,j}\lambda_i^2 \left|\langle\Phi_i|D_j|\Phi_i\rangle\,\langle\Psi_i|D_j^\dagger|\Psi_i\rangle\right| \le \frac{1}{d_1}\sum_{i,j}\lambda_i^2\,\mu_j\left|\langle\Phi_i|D_j|\Phi_i\rangle\right|, \quad (45)$$

the last bound following form the expression $D_j = \mu_j\,|\gamma_j\rangle\langle\alpha_j| \otimes |\delta_j\rangle\langle\beta_j|$ and from the fact that the state $|\Psi_i\rangle$ is maximally entangled.

Using the polar decomposition $D_j = U|D_j|$, we then have the bound

$$\sum_i \lambda_i^2\left|\langle\Phi_i|D_j|\Phi_i\rangle\right| = \sum_i \lambda_i^2\left|\langle\Phi_i|\,U|D_j|\,|\Phi_i\rangle\right| \le \sqrt{\sum_i \lambda_i^2\,\langle\Phi_i|\,|D_j|\,|\Phi_i\rangle} \times \sqrt{\sum_i \lambda_i^2\,\langle\Phi_i|U|D_j|U^\dagger|\Phi_i\rangle} = \mathrm{Tr}[|D_j|] = \mu_j\,, \quad (46)$$

the second to last equality following from the completeness relation $\sum_{i=1}^{k_A}\lambda_i^2|\Phi_i\rangle\langle\Phi_i| = I_1\otimes I_2$, implied by the normalization of the instrument $(\mathscr{C}_i)_{i=1}^{k_A}$.

Summarizing, we obtained the bound

$$\mathrm{Re}\left[\sum_{i,j}\mathrm{Tr}[C_i^\dagger D_j^\dagger C_i D_j]\right] \le \frac{1}{d_1}\sum_j \mu_j^2 = \frac{1}{d_1}\,(d_1 d_2)\,, \quad (47)$$

where the last equality follows by taking the trace on both sides of the completeness relation $\sum_{j=1}^{k_B}\mu_j^2|\alpha_j\rangle\langle\alpha_j|\otimes|\beta_j\rangle\langle\beta_j| = I_1\otimes I_2$, implied by the normalization of the instrument $(\mathscr{D}_j)_{j=1}^{k_B}$.

Hence, the noncommutativity is lower bounded as

$$\mathrm{NCOM}_{\frac{I_1\otimes I_2}{d_1 d_2}}\left((\mathscr{C}_i)_{i=1}^{k_A},(\mathscr{D}_j)_{j=1}^{k_B}\right) = \sqrt{1-\mathrm{Re}\left[\sum_{i,j}\frac{1}{d_1 d_2}\mathrm{Tr}[C_i^\dagger D_j^\dagger C_i D_j]\right]} \ge \sqrt{1-\frac{1}{d_1}}\,. \quad (48)$$

The above bound can be immediately extended to the multipartite case, by considering a bipartition of the system into the lowest dimensional sybsystem and all the remaining ones.

III. SAMPLE COMPLEXITY OF (GENERALIZED) MED ESTIMATION

Our protocol provides a direct estimate of the MED in terms of the probability that a Fourier basis measurement on the control system yields outcome "−". The probability can be estimated from the frequency of the outcome "−" in a number of repetitions of the experiment. We now show that the number of repetitions is independent of the system's dimension and scales inverse polynomially with the desired level of accuracy.

The proof is standard, and is provided here just for completeness. The result of the Fourier basis measurement defines a Bernoulli variable with outcomes $k \in \{0,1\}$. Here, outcome 1 corresponds to the outcome corresponding to the basis vector $|-\rangle$, while outcome 0 corresponds to the basis vector $|0\rangle$. The probability mass function of this Bernoulli variable is $f(k;p_-)$ defined by

$$f(-;p_-) = p_- \qquad \text{and} \qquad f(+;p_-) = 1-p_-\,. \quad (49)$$

If the experiment is repeated for n times, the outcomes are a sequence of independent Bernoulli variables $X_1, X_2, \ldots, X_n$, with each variable distributed according to the probability mass function $f(k;p_-)$. The sum of these variables, denoted by $S = \sum_i X_i$, is distributed according to the binomial distribution $B(s;n,p_-) := p_-^s(1-p_-)^{n-s}\binom{n}{s}$. Then, Hoeffding's inequality [4] implies that the empirical frequency S/n is close to p_- with high probability, namely

$$\forall\epsilon > 0, P\left(\left|\frac{S}{n}-p\right| < \epsilon\right) \ge 1-2e^{-2\epsilon^2 n},$$

Hence, the minimum number of repetitions of the experiment needed to guarantee that the empirical frequency has probability at most δ to deviate from the frequency by at most ϵ is

$$n(\epsilon,\delta) = \left\lceil -\frac{1}{2\epsilon^2}\log\frac{\delta}{2}\right\rceil\,. \quad (50)$$

This expression shows that the sample complexity is independent of the dimension of the system under consideration. In particular, the sample complexity does not increase when the MED is measured on multiparticle systems, in contrast with the sample complexity of process tomography which increases exponentially with the number of particles.

IV. COMPARISON WITH OTHER EXPERIMENTAL SCHEMES

A naïve way to estimate the MED is to characterize the channels $\mathscr{A}$ and $\mathscr{B}$ via process tomography [5–10], or the projective measurements **P** and **Q** via measurement tomography [11], and then use Eq. (A2). However, process and measurement tomography requires a number of experimental settings that grows polynomially with the dimension, and therefore exponentially in the number of particles for multi-particle quantum systems.

To avoid the exponential complexity of tomography, one needs a direct measurement protocol whose sample complexity is independent of the system's dimension, and, therefore, also on the number of particles. A possible approach is to use the operational scheme that motivated the definition of the MED. This scheme, described in the main text, involves the initialization of the system in a random eigenstate of observable A. Practically, this can be achieved by preparing the maximally mixed state, and then applying a measurement of the observable A. The estimation of the MED would then proceed by performing a measurement of the observable B, and finally, another measurement of the observable A. The total probability that the two measurements of A give equal outcomes is equal to $\mathrm{Prob}(A,\mathscr{B}) = 1-\mathrm{MED}(\mathscr{A},\mathscr{B})^2$. In this case, the complexity of estimating the MED does not grow exponentially with the system's size.

The main difference between the above scheme and the scheme using the quantum switch is that the above scheme requires access to the outcomes of two measurements of the observable A. The ability to estimate the incompatibility without access to the outcomes offers an advantage in situations where the experimenter wants to discover the incompatibility of two observables measured by two parties in their local laboratories. In this case, the quantum switch scheme allows the experimenter to estimate the incompatiblity/noncommutativity in a black box fashion, by sending input states to the two laboratories and observing their outputs states, without any access to the outcomes generated inside the laboratories.

V. PROOF OF EQ. (11) IN THE MAIN TEXT

From Eq. (9) in the main text, we have

$$\begin{aligned}\mathrm{NCOM}_\rho(\mathscr{C},\mathscr{D}) &= \sqrt{\frac{\sum_{i,j}\mathrm{Tr}\left(\rho\left|[C_i,D_j]\right|^2\right)}{2}}\\ &= \sqrt{1-\mathrm{Re}\left[\sum_{i,j}\mathrm{Tr}\left(C_iD_j\rho C_i^\dagger D_j^\dagger\right)\right]}\\ &= \sqrt{1-\mathrm{Re}\left[\sum_j\mathrm{Tr}\left(\mathscr{C}(D_j\rho)\,D_j^\dagger\right)\right]}\\ &= \sqrt{1-\mathrm{Re}\left[\sum_j\langle\!\langle D_j\,|\,\mathscr{C}(D_j\rho)\rangle\!\rangle\right]}\,,\end{aligned} \tag{51}$$

where we used the double-ket notation $|X\rangle\!\rangle := \sum_{m,n}\langle m|X|n\rangle\,|m\rangle\otimes|n\rangle$ for an arbitrary operator X, and the property $\langle\!\langle X|Y\rangle\!\rangle = \mathrm{Tr}[X^\dagger Y] = \mathrm{Tr}[YX^\dagger]$, valid for arbitrary X and Y (in our case, $X=D_j$ and $Y=\mathscr{C}(D_j\rho)$).

Now, consider the operator $\check{C}$ defined through the relation [12]

$$\check{C}|X\rangle\!\rangle := |\mathscr{C}(X)\rangle\!\rangle\,,\qquad \forall X\,. \tag{52}$$

Using this definition, the noncommutativity can be expressed as

$$\begin{aligned}\mathrm{NCOM}_\rho(\mathscr{C},\mathscr{D}) &= \sqrt{1-\mathrm{Re}\left[\sum_j\langle\!\langle D_j\,|\,\check{C}\,|D_j\rho\rangle\!\rangle\right]}\\ &= \sqrt{1-\mathrm{Re}\left[\sum_j\langle\!\langle D_j\,|\,\check{C}\,(I\otimes\rho^T)\,|D_j\rangle\!\rangle\right]}\\ &= \sqrt{1-\mathrm{Re}\,\mathrm{Tr}\left[\left(\sum_j|D_j\rangle\!\rangle\langle\!\langle D_j|\right)\check{C}\,(I\otimes\rho^T)\right]}\\ &= \sqrt{1-\mathrm{Re}\,\mathrm{Tr}\left[D\,\check{C}\,(I\otimes\rho^T)\right]}\,,\end{aligned} \tag{53}$$

where $D := \sum_j|D_j\rangle\!\rangle\langle\!\langle D_j| = (\mathscr{D}\otimes\mathscr{I})(|I\rangle\!\rangle\langle\!\langle I|)$ is the Choi operator of $\mathscr{D}$, with $\mathscr{I}$ being identity channel.

In particular, when the state ρ is maximally mixed, the noncommutativity takes the simple expression

$$\mathrm{NCOM}_{\frac{I}{d}}(\mathscr{C},\mathscr{D}) = \sqrt{1-\frac{\mathrm{Re}\,\mathrm{Tr}\left[D\,\check{C}\right]}{d}}\,. \tag{54}$$

The operator $\check{C}$ is in one-to-one correspondence with the map $\mathscr{C}$. Explicitly, Eq. (E2) implies the explicit expression

$$\check{C} = \sum_i C_i\otimes\overline{C}_i\,, \tag{55}$$

where $\overline{C}_i$ is the complex conjugate of the matrix C_i. The operator $\check{C}$ can be equivalently expressed in terms of its Choi operator via the relation

$$\check{C} = \left(I_{\mathrm{out},\overline{\mathrm{out}}}\otimes\langle\!\langle I_{\mathrm{in},\overline{\mathrm{in}}}|\right)(C_{\mathrm{out,in}}\otimes I_{\overline{\mathrm{out}},\overline{\mathrm{in}}})\left(|I_{\mathrm{out},\overline{\mathrm{out}}}\rangle\!\rangle\otimes\mathrm{SWAP}_{\mathrm{in},\overline{\mathrm{in}}}\right)\,, \tag{56}$$

where "in" and "out" label the input and output systems of channel $\mathscr{C}$, respectively, $\overline{\mathrm{in}}$ and $\overline{\mathrm{out}}$ are auxiliary systems of the same dimensions of in and out, respectively, and $\mathrm{SWAP}_{\mathrm{in},\overline{\mathrm{in}}}$ is the swap operator, defined by the relation $\mathrm{SWAP}_{\mathrm{in},\overline{\mathrm{in}}}|\varphi\rangle|\psi\rangle = |\psi\rangle|\varphi\rangle$ for every pair of vectors $|\varphi\rangle$ and $|\psi\rangle$. Eq. (E6) can be verified explicitly using the expressions $\check{C}=\sum_i C_i\otimes\overline{C}_i$ and $C=\sum_i|C_i\rangle\!\rangle\langle\!\langle C_i|$.

VI. EXPRESSION OF THE NONCOMMUTATIVITY OF TWO CHANNELS IN TERMS OF THEIR UNITARY REALIZATIONS

Here we provide an alternative expression of the noncommutativity (and therefore of the MED) in terms of the unitary realizations of the two channels $\mathscr{C}$ and $\mathscr{D}$. Consider two unitary realizations, of the form

$$\begin{aligned}\mathscr{C}(\rho) &= \mathrm{Tr}_E\left[U_{SE}\,(\rho_S\otimes|\eta\rangle\langle\eta|_E)\,U_{SE}^\dagger\right]\\ \mathscr{D}(\rho) &= \mathrm{Tr}_F\left[V_{SF}\,(\rho_S\otimes|\phi\rangle\langle\phi|_F)\,V_{SF}^\dagger\right]\,,\end{aligned} \tag{57}$$

where E and F are two suitable quantum systems, serving as the environments, $|\eta\rangle$ and $|\phi\rangle$ are pure states of E and F, respectively, and U_{SE} and V_{SF} are two unitary evolutions between the target system (denoted by S) and the environments E and F, respectively.

To compute the action of the channel $\mathscr{S}(\mathscr{C},\mathscr{D})$, we consider the quantum switch of the unitary gates $U_{SE}\otimes I_F$ and $V_{SF}\otimes I_E$, and then take the partial trace over the environments. The quantum switch of the unitary gates $U_{SE}\otimes I_F$ and $V_{SF}\otimes I_E$ yields the controlled unitary gate

$$\begin{aligned} W &= (U_{SE}\otimes I_F)(V_{SF}\otimes I_E)\otimes|0\rangle\langle 0|_C \\ &\quad + (V_{SF}\otimes I_E)(U_{SE}\otimes I_F)\otimes|1\rangle\langle 1|_C\,, \end{aligned} \tag{58}$$

where the subscript C denotes the control system, and it is implicitly understood that the Hilbert spaces in the tensor product are suitably arranged according to the subscripts of the corresponding systems.

Hence, the action of the channel $\mathscr{S}_{\mathscr{C},\mathscr{D}}$ on the state $\rho\otimes|+\rangle\langle+|$ is given by

$$\begin{aligned} \mathscr{S}_{\mathscr{C},\mathscr{D}}\left(\rho_S\otimes|+\rangle\langle+|_C\right) &= \mathrm{Tr}_{EF}[W\,(\rho_S\otimes|\eta\rangle\langle\eta|_E \\ &\quad \otimes|\phi\rangle\langle\phi|_F\otimes|+\rangle\langle+|_C)\,W^\dagger]\,. \end{aligned} \tag{59}$$

To reduce the above expression to the case where the input state is pure, we take a purification of ρ, given by a pure state $|\Psi\rangle\in\mathcal{H}_S\otimes\mathcal{H}_R$, where R is a suitable purifying system. Let us define the state

$$\begin{aligned} |\Gamma\rangle_{SREFC} &:= (W\otimes I_R)\,|\Psi\rangle_{SR}\otimes|\eta\rangle_E\otimes|\phi\rangle_F\otimes|+\rangle_C \\ &= \frac{|\Lambda_0\rangle_{SREF}\otimes|0\rangle_C+|\Lambda_1\rangle_{SREF}\otimes|1\rangle_C}{\sqrt{2}}\,, \end{aligned} \tag{60}$$

with

$$|\Lambda_0\rangle_{SREF} := \Big[(U_{SE}\otimes I_F)(V_{SF}\otimes I_E)\otimes I_R\Big]\,|\Psi\rangle_{SR}\otimes|\eta\rangle_E\otimes|\phi\rangle_F\,, \tag{61}$$

$$|\Lambda_1\rangle_{SREF} := \Big[(V_{SF}\otimes I_E)(U_{SE}\otimes I_F)\otimes I_R\Big]\,|\Psi\rangle_{SR}\otimes|\eta\rangle_E\otimes|\phi\rangle_F\,. \tag{62}$$

The probability of the outcome - when the output state is measured on the Fourier basis is then given by

$$\begin{aligned} p_- &= \langle-|\,\mathrm{Tr}_S[\mathscr{S}_{\mathscr{C},\mathscr{D}}\left(\rho_S\otimes|+\rangle\langle+|_C\right)]\,|-\rangle \\ &= \langle-|\,\mathrm{Tr}_{SREF}[|\Gamma\rangle\langle\Gamma|]\,|-\rangle \\ &= \|(I_{SREF}\otimes\langle-|_C)\,|\Gamma\rangle_{SREFC}\,\|^2 \\ &= \frac{1}{4}\,\||\Lambda_0\rangle-|\Lambda_1\rangle\,\|^2 \\ &= \frac{1-\mathrm{Re}\langle\Lambda_0|\Lambda_1\rangle}{2}\,. \end{aligned} \tag{63}$$

Using the relation, $\mathrm{NCOM}_\rho(\mathscr{C},\mathscr{D})=\sqrt{2p_-}$, we finally obtain

$$\mathrm{NCOM}_\rho(\mathscr{C},\mathscr{D})=\sqrt{1-\mathrm{Re}\langle\Lambda_0|\Lambda_1\rangle}\,. \tag{64}$$

The evaluation of the noncommutativity is reduced to the evaluation of the overlap between the states $|\Lambda_0\rangle$ and $|\Lambda_1\rangle$. When the system S consists of a large number N of particles, its Hilbert space has exponentially large dimension in N, and therefore the evaluation of the overlap may be computationally demanding. However, there exist many relevant situations in which the overlap between two states of exponentially large dimension can nevertheless be computed efficiently, *i.e.* in a polynomial number of steps. This is the case, for example, if the states $|\Lambda_0\rangle$ and $|\Lambda_1\rangle$ are matrix product states [13–15] or MERA states. Physically, these cases correspond to the situation in which the pure state $|\Psi\rangle$ and the unitary evolutions U_{SE} and V_{SF} have an appropriate tensor network structure, generated by sequences of local interactions.

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