Lyapunov Stability Analysis of Lithium-Ion Batteries

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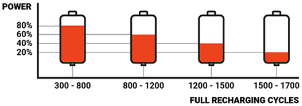
Motivation & Significance

- Market for Lithium-ion battery is exploding
  - Electric Vehicles, IoT Devices, ESS ....
- Some of the major technical concerns include :

(a) Faster Charging

(b) Range per Charge



(c) Battery Depreciation

- These are highly related to **Battery Management System (BMS).**

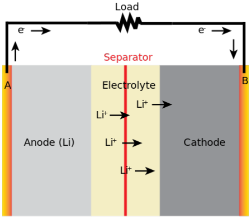
Motivation & Significance

What makes the BMS problem difficult?

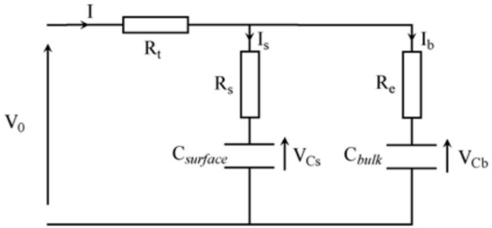
- Cannot fully analyze electrochemical behavior of a battery
  - We instead use equivalent model to describe it.
- We only can observe nominal voltage, V_0, of a battery
  - We would have to estimate rest of the states.
- Depreciation does not have a physical model
  - It depends on how you use it.

Model: Lithium-Ion Battery

- Model is formulated using an **Electrical Equivalent Model** for an electrochemical battery.
- As batteries age, parameters change in a way that battery no longer can hold an electric potential
- C_{bulk}, the bulk capacitance, is the most sensitive and important among parameters



A/B: Current collectors; negative (A), positive (B)



Dynamic System $\dot{x} = f(x, u, t)$

- States: $x = [V_{C_b}, V_{C_s}, V_0, \alpha]^T$
  - V_{C_b} : Voltage across bulk capacitor
  - V_{C_s} : Voltage across surface capacitor
  - V_0 : Nominal voltage
  - $\alpha = \frac{1}{C_{bulk}}$, C_{bulk} is bulk capacitance and here we adopt dynamic equation from Ref [**2**]:

$$C_{bulk} = \left\{ \begin{array}{cc} C_0 & t \leq t_0 \\ -k_1 I(t - t_0) - k_2(t - t_0) + C_0 & t > t_0 \end{array} \right\}$$

    $k_1 = 0.00039$ and $k_2 = -0.0025$.
  - $x \in D \triangleq (\mathbb{R}^+)^4$, since batteries cannot have negative voltage
- Input : $u = I$
  - I : charging/discharging current

State Space Equation $\dot{x} = f(x, u, t)$

$$\dot{x} = f(x, u, t) \;,\; y = [0\ 0\ 1\ 0]x$$

$$\begin{bmatrix} \dot{V}_{C_b} \\ \dot{V}_{C_s} \\ \dot{V}_0 \\ \dot{\alpha} \end{bmatrix} = \begin{bmatrix} \frac{(V_{C_s} - V_{C_b})\alpha}{(R_e + R_s)} + \frac{IR_s\alpha}{R_e + R_s} \\ \frac{1}{C_{surface}}\left(\frac{V_{C_b} - V_{C_s}}{R_e + R_s} + \frac{IR_e}{R_e + R_s}\right) \\ V_{C_b} f_1 + V_0 f_2 + I f_3 \\ (k_1 I + k_2)\alpha^2 \end{bmatrix}$$

where,

$f_1 = -\frac{R_s\alpha}{R_e(R_e + R_s)} + \frac{1}{C_{surface}(R_e + R_s)}$

$f_2 = -f_1$

$f_3 =$

$\frac{R_e^2}{C_{surface}(R_e + R_s)^2} - \frac{R_s R_t \alpha}{R_e(R_e + R_s)} + \frac{R_t}{C_{surface}(R_e + R_s)} + \frac{R_e R_s}{C_{surface}(R_e + R_s)}$

Existence/Uniqueness

- Note that $\alpha < \alpha_0 = \frac{1}{C_0}$ since $\dot{\alpha} < 0$
- The Jacobian of $f(x,u,t)$ is as follows:

$$\frac{\partial f(x,u,t)}{\partial x}\Big|_{u(t)=0} = \begin{bmatrix} -\frac{\alpha}{(R_e+R_x)} & \frac{\alpha}{(R_e+R_x)} & 0 & F(1,4) \\ \frac{1}{C_s(R_e+R_s)} & -\frac{1}{C_s(R_e+R_s)} & 0 & 0 \\ F(3,1) & 0 & F(3,3) & F(3,4) \\ 0 & 0 & 0 & 2\alpha k_2 \end{bmatrix}$$

$$F(1,4) = -\frac{V_{C_b}+V_{C_s}}{(R_e+R_s)}$$
$$F(3,1) = -F(3,3) = -\frac{R_s\alpha}{R_e(R_e+R_s)} + \frac{1}{C_s(R_e+R_s)}$$
$$F(3,4) = -\frac{V_{C_b}R_s}{(R_e+R_s)^2} - \frac{V_{C_b}R_s^2}{R_e(R_e+R_s)} + \frac{V_0 R_s}{R_e(R_e+R_s)}$$

Existence/Uniqueness

- Note that $f(x, u, t)$ is continuous in its physical domain $D = (\mathbb{R}^+)^4$
- Elements in Jacobian are either constants or a linear combination of states x, and hence the Jacobian is continuous in D
- It is globally Lipschitz on domain D with boundary L

$$||\frac{\partial f(x, u, t)}{\partial x}||_\infty \leq L$$

$$\begin{aligned} L = \max(&|-\frac{\alpha}{(R_e + R_x)}| + |\frac{\alpha}{(R_e + R_x)}| + |F(1,4)|), \\ &|\frac{1}{C_s(R_e + R_s)}| + |-\frac{1}{C_s(R_e + R_s)}|, \\ &|F(3,1)| + |F(3,3)| + |F(3,4)|, \\ &|2\alpha k_2|) \end{aligned} \tag{1}$$

Existence/Uniqueness

- This is true for $x \in \mathcal{B}_r \triangleq \{x : ||x - x_s|| < r\}$
- r can be chosen arbitrarily large
- Solution exists, and is unique for all $x \in D$
- This is intuitively true - batteries with same design parameters and same initial conditions will show identical behaviors.

Using the Lyapunov's Direct method, define

$$V(x) = \frac{1}{2}V_{C_b}^2 + \frac{1}{2}V_{C_s}^2 C_s \alpha + \alpha^2$$

This is valid Lyapunov candidate since:

- $V(x) = 0$ at $x = 0$
- $V(x) > 0, \forall x \in D - \{0\}$

Also, assert that $V(x)$ is radially unbounded:

- We need to consider $V_0 \to \infty$, with $V_{C_b}, V_{C_s}, \alpha = 0$
- Since $V_0 = \frac{V_{C_b} R_s}{R_e + R_s} + \frac{V_{C_s} R_e}{R_e + R_s}$,
  $(V_0 \to \infty) \Rightarrow ((V_{C_b} \to \infty) \vee (V_{C_s} \to \infty)) \Rightarrow (V(x) \to \infty)$

Stability Analysis Cont'd...

$$\dot{V}(x) = V_{C_b}\left(-\frac{V_{C_b}\alpha}{R_e+R_s}+\frac{V_{C_s}\alpha}{R_e+R_s}\right)+V_{C_s}C_s\alpha\left(\frac{V_{C_b}}{R_e+R_s}-\frac{V_{C_s}}{R_e+R_s}\right) + \left(2\alpha+\frac{1}{2}V_{C_s}^2 C_s\right)\left(\frac{k_2}{k_2^2(t-t_0)^2-2k_2C_0(t-t_0)+C_0^2}\right) \quad (2)$$

$$\dot{V}(x) = -\frac{\alpha(V_{C_b}-V_{C_s})^2}{R_e+R_s}+ \left(2\alpha+\frac{1}{2}V_{C_s}^2 C_s\right)\left(\frac{k_2}{k_2^2(t-t_0)^2-2k_2C_0(t-t_0)+C_0^2}\right) \quad (3)$$

We observe that, $-\frac{\alpha(V_{C_b}-V_{C_s})^2}{R_e+R_s} \leq 0$, because α, R_e, and $R_s>0$

Stability Analysis Cont'd...

Also, $(2\alpha + \frac{1}{2}V_{C_s}^2 C_s)(\frac{k_2}{k_2^2(t-t_0)^2 - 2k_2C_0(t-t_0) + C_0^2}) < 0$, because

- $k_2^2(t-t_0)^2 - 2k_2C_0(t-t_0) + C_0^2 > 0$
- $k_2 < 0$
- $2\alpha + \frac{1}{2}V_{C_s}^2 C_s > 0$

Since $\dot{V}(x) < 0 \ \forall x \in D$, and $V(x)$ is radially unbounded, the system is globally asymptotically stable.

Summary/Conclusion

- The electrical equivalent model is non-linear and non-autonomous.
- The chosen system has a unique solution.
- The chosen system is **Globally Asymptotically Stable (GAS)** as per stability analysis from Lyapunov's direct method.

References

1. B. S. Bhangu, P. Bentley, D. A. Stone and C.M. Bingham, **Nonlinear Observers for Predicting State-of-Charge and State-of-Health for Lead-Acid Batteries for Hybrid-Electric Vehicles**, *IEEE Transactions on Vehicular Technology*, Vol. 54, No. 3, pp. 783-794, May 2005.

2. H. Saberi and F.R. Salmasi, **Genetic Optimization of Charging Current for Lead-Acid Batteries in Hybrid Electric Vehicles**, in *Proceeding of International Conference on Electrical Machines and Systems*, Seoul, Korea, Oct. 2007, pp. 2028-2032.

Model Reference Adaptive Control (MRAC) for Lithium-Ion Batteries

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$November\ 3^{rd}, 2021$

System model selection

- We selected a system model from Ref [1] that uses a lumped thermal model for battery. Uncertainties in thermal model parameters occur as the battery ages.

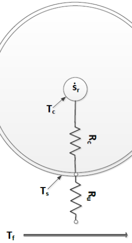


Figure 1: Lumped Thermal Model of a Cylindrical Cell

where, $\mathbf{T_c}$ is the core temperature, $\mathbf{T_s}$ is the surface temperature, $\mathbf{T_f}$ is the coolant temperature $\mathbf{R_c}, \mathbf{R_u}$ are internal resistances & $\dot{\mathbf{s}}_\mathbf{r}$ is heat generation rate.

Dynamic System $\dot{x} = f(x, u)$

Dividing the state-space equation into linear and non-linear parts and further linearly parameterizing the non-linear part, we have:

$$\dot{\mathbf{x}}(\mathbf{t}) = \mathbf{A}\mathbf{x}(\mathbf{t}) + \mathbf{B}\boldsymbol{\Lambda}(\mathbf{u}(\mathbf{t}) + \mathbf{W}^{\mathbf{T}}\boldsymbol{\Phi}(\mathbf{x}))$$

where

$$A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix},\ B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},\ \Lambda = \begin{bmatrix} 4/5 & 0 \\ 0 & 4/5 \end{bmatrix}$$

$$W = \begin{bmatrix} 0.2 & 0 \\ 0 & -0.1 \end{bmatrix}\ ,\ \Phi(x) = \begin{bmatrix} T_c^2 \\ T_s^2 \end{bmatrix}$$

$$x = [T_c\ T_s]^T\ \&\ u = [I^2\ 0]^T$$

Reference Model & Matching Condition

The reference model dynamics are defined as follows:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -4 & -2 \end{bmatrix} \mathbf{x_m} + \begin{bmatrix} 0 \\ 4 \end{bmatrix} \mathbf{r}$$

Through matching conditions:

$$\mathbf{A} + \mathbf{B\Lambda}(\mathbf{k_x}^*)^{\mathbf{T}} = \mathbf{A_m}$$

$$\mathbf{B\Lambda}(\mathbf{k_r}^*)^{\mathbf{T}} = \mathbf{B_m}$$

$$\mathbf{k_x}^* = \begin{bmatrix} 2.5000 & 1.2500 \\ -3.7500 & -1.2500 \end{bmatrix} \& \ \mathbf{k_r}^* = \begin{bmatrix} -5 \\ 5 \end{bmatrix}$$

Model Reference Adaptive Control (MRAC)

Plant

$$\dot{\mathbf{x}}(\mathbf{t}) = \mathbf{A}\mathbf{x}(\mathbf{t}) + \mathbf{B}\boldsymbol{\Lambda}(\mathbf{u}(\mathbf{t}) + \mathbf{W}^{\mathbf{T}}\boldsymbol{\Phi}(\mathbf{x}))$$

Control

$$\mathbf{u}(\mathbf{t}) = \mathbf{k_x}(\mathbf{t})^{\mathbf{T}}\mathbf{x}(\mathbf{t}) + \mathbf{k_r}(\mathbf{t})^{\mathbf{T}}\mathbf{r}(\mathbf{t}) - \hat{\mathbf{W}}(\mathbf{t})^{\mathbf{T}}\boldsymbol{\Phi}(\mathbf{x})$$

Adaptation Laws

$$\dot{\mathbf{k}}_{\mathbf{x}}(\mathbf{t}) = -\boldsymbol{\Gamma}_{\mathbf{x}}\mathbf{x}(\mathbf{t})\mathbf{e}^{\mathbf{T}}(\mathbf{t})\mathbf{P}\mathbf{B}\mathbf{sgn}(\boldsymbol{\Lambda}),\ \mathbf{k_x}(\mathbf{0}) = \mathbf{0_{2x2}}$$

$$\dot{\mathbf{k}}_{\mathbf{r}}(\mathbf{t}) = -\boldsymbol{\Gamma}_{\mathbf{r}}\mathbf{r}(\mathbf{t})\mathbf{e}^{\mathbf{T}}(\mathbf{t})\mathbf{P}\mathbf{B}\mathbf{sgn}(\boldsymbol{\Lambda}),\ \mathbf{k_r}(\mathbf{0}) = \mathbf{0_{2x1}}$$

$$\dot{\hat{\mathbf{W}}}(\mathbf{t}) = \boldsymbol{\Gamma}_{\mathbf{W}}\boldsymbol{\Phi}(\mathbf{x}(\mathbf{t}))\mathbf{e}^{\mathbf{T}}(\mathbf{t})\mathbf{P}\mathbf{B}\mathbf{sgn}(\boldsymbol{\Lambda}),\ \hat{\mathbf{W}}(\mathbf{0}) = \mathbf{0_{2x2}}$$

$r(t) = Step(10)$ & $(\Gamma_x, \ \Gamma_r, \ \Gamma_W) = I_2$

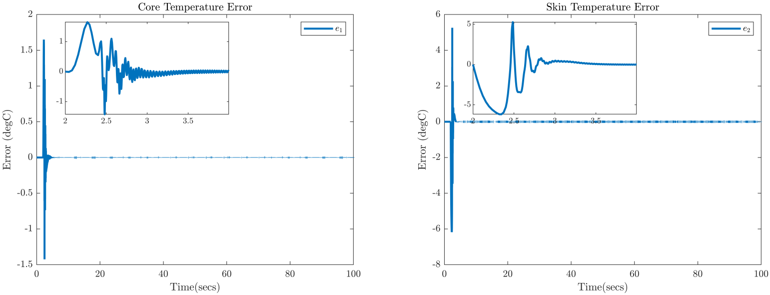


Figure 2: Error $x - x_m$ for constant reference inputs.

Observations

Error converges asymptotically.

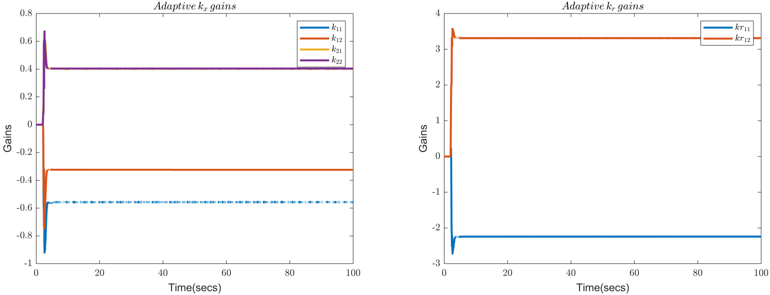


Figure 3: Adaptive gains k_x & k_r for constant reference inputs

Observations

Parameter errors are bounded.

$r(t) = Step(10)$ & $(\Gamma_x,\ \Gamma_r,\ \Gamma_W) = 10I_2$

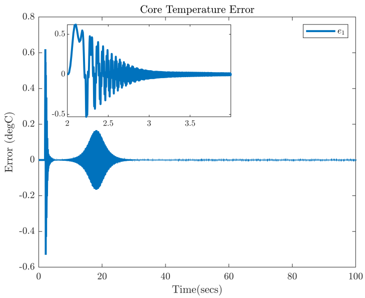
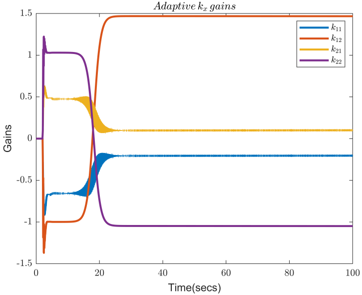


Figure 4: Error $x - x_m$ for constant reference inputs

Observations

Error converges asymptotically and tracking performance is better as compared to the case with smaller gains .

$r(t) = Step(10)$ & $(\Gamma_x,\ \Gamma_r,\ \Gamma_W) = 10I_2$



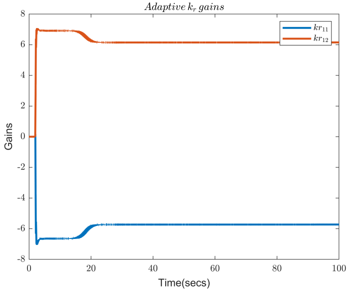


Figure 5: Adaptive gains k_x & k_r for constant reference inputs

Observations

Parameter errors are bounded.

Robustness comparison for lower and higher gain cases

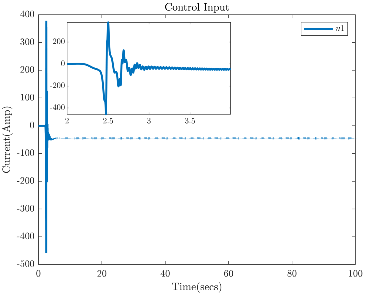


Figure 6: Controller inputs u_1 for lower and higher gains respectively.

Observations

Control inputs oscillation frequency and magnitude increases with higher adaptation gains.Higher gains may help in better tracking performance but at the expense of robustness.

Exciting $r(t)$ with $(\Gamma_x,\ \Gamma_r,\ \Gamma_W) = 10I_2$

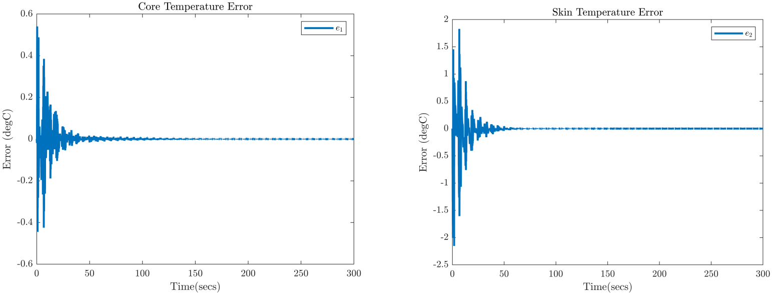
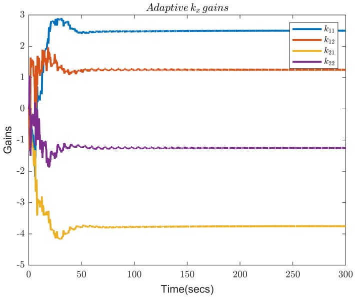


Figure 7: Error $x - x_m$ for persistently exciting reference inputs.

Observations

Asymptotic Error Convergence.

Exciting $r(t)$ *with* $(\Gamma_x,\ \Gamma_r,\ \Gamma_W) = 10I_2$



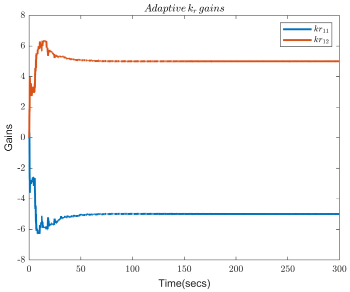


Figure 8: Adaptive gains for for persistently exciting reference inputs

Observations

Parameter errors remain bounded and converges to $\mathbf{k}_\mathbf{x}^*$ & $\mathbf{k}_\mathbf{r}^*$.
Generic persistently exciting (PE) condition is hard to verify for non-linear systems.

Exciting $r(t)$ *with* $(\Gamma_x,\ \Gamma_r,\ \Gamma_W) = 10I_2$.

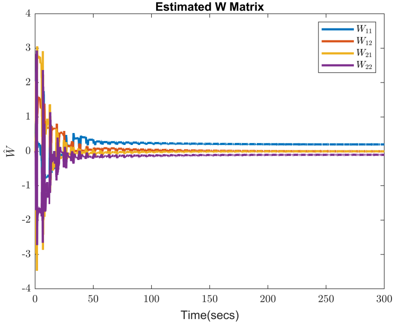


Figure 9: Estimated $\hat{W}$ for persistently exciting reference inputs

Observations

Error in estimating $\hat{\mathbf{W}}$ remains bounded and converges to the nominal $\mathbf{W}$.

Multiple step input $r(t)$ with $(\Gamma_x, \Gamma_r, \Gamma_W) = I_2$

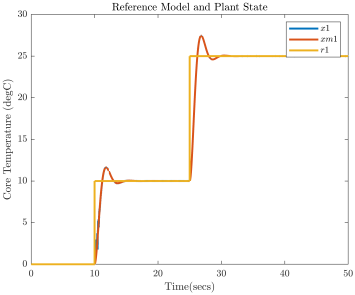


Figure 10: x_2 & xm_2 for multi-step reference inputs.

Observations

Tracking performance is not consistent for various reference signals. Re-tuning is necessary for different reference inputs.

Effect of disturbance on gains (Parameter Drift)

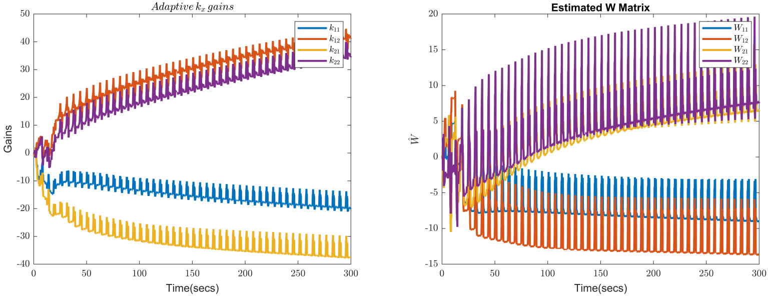


Figure 11: k_x & $\hat{W}$ with disturbances.

Observations

Exploding gains. Control gain and parametric estimates are drifting.

Summary

- Pros
  - Error converges asymptotically & parameter error boundedness is guaranteed with MRAC even in the presence of uncertainties.
- Cons
  - MRAC can be applied to only a special class of systems where the non-linearities are linearly parameterizable.
  - No transient performance guarantees on the system with uncertainties.
  - Retuning of parameter gains is necessary for different reference inputs. No one size fits all.
  - Higher adaptation gains result in better tracking but comes with a trade off with robustness.

References

1. Arasaratnam, J. Tjong, R. Ahmed, M. El-Sayed and S. Habibi, **Adaptive temperature monitoring for battery thermal management,** 2013 *IEEE Transportation Electrification Conference and Expo (ITEC)*, 2013, pp. 1-6, doi: 10.1109/ITEC.2013.6574504.
2. Babaei, Naser & Salamci, Metin. (2013). **State dependent riccati equation based model reference adaptive control design for nonlinear systems.** 2013 24th *International Conference on Information, Communication and Automation Technologies*, ICAT 2013. 1-8. 10.1109/ICAT.2013.6684058.

$\mathcal{L}_1$-Adaptive Controller Design for Lithium-Ion Batteries

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December $8^{th}, 2021$

System Model Selection

State space representation of lumped thermal capacitance model of a battery :

$$\dot{x}(t) = Ax(t) + b\lambda(u(t) + f(x) + d)$$

$$y = cx$$

- $f(x)$: unstructured, unknown uncertainty
- $\lambda \in [0, 1]$: scaling uncertainty in the controller
- d : disturbance

$$f(x) = 5sin(2x_1(t)) + cos(4x_2(t)) + exp(-x_1(t)^2 - x_2(t)^2)$$

$$A = \begin{bmatrix} 0 & 1 \\ -0.01 & 0.02 \end{bmatrix}, \; b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; c = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$A_m = \begin{bmatrix} 0 & 1 \\ -1 & -1.2 \end{bmatrix}, \quad \lambda = 0.75$$

Control Architecture

The control architecture and parameter bounds are shown below:

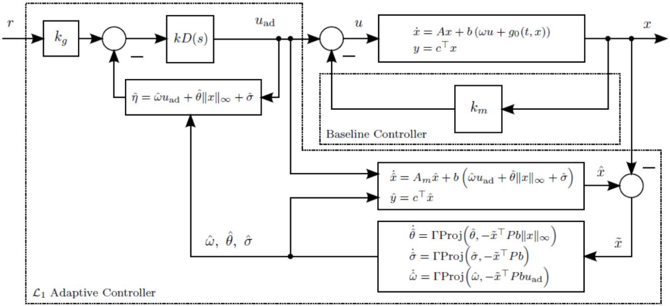

Figure 1: Closed loop adaptive system.

Parameter Bounds

$\Theta = [-5, 5]\ , \Sigma = [-10, 10]\ \&\ \Omega = [0.1, 2]$

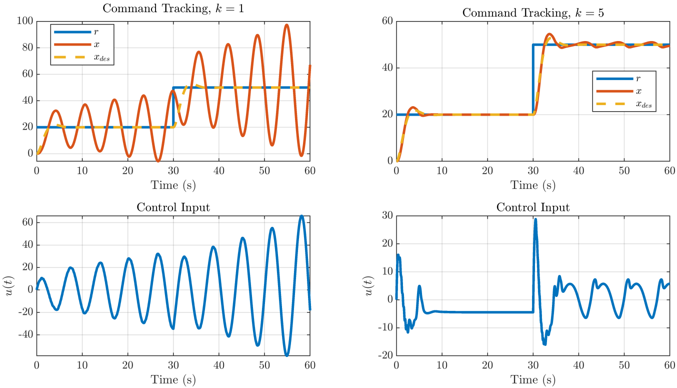

Figure 2: $\mathcal{L}_1$ Performance with $k = 1, 5$ & $\Gamma = 10000$

Notes & Observations

$k = 5$ is close to the minimum value that satisfies $\mathcal{L}_1$-norm condition $||G(s)||_{\mathcal{L}_1} L < 1$. Does not track when $k = 1$.

$\mathcal{L}_1$ vs. MRAC in Presence of d $= sin(\pi) + sin(5\pi/7)$

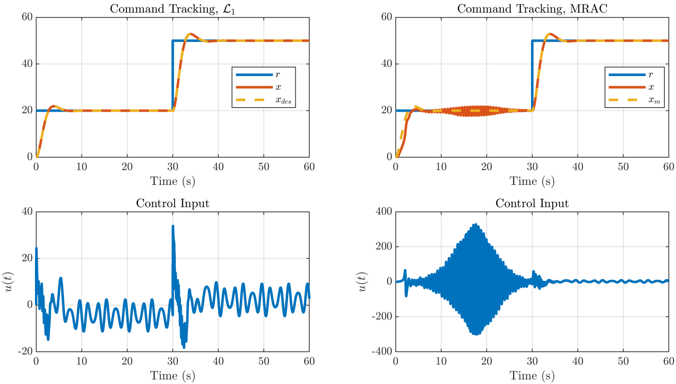

Figure 3: Tracking and input with disturbance for $k = 100$.

Observations

$\mathcal{L}_1$ compensates for disturbance by scaling control input, whereas MRAC does not.

Performance Comparison for $D(s) = \frac{1}{s}$ vs $D(s) = \frac{s+15}{s(s+60)}$

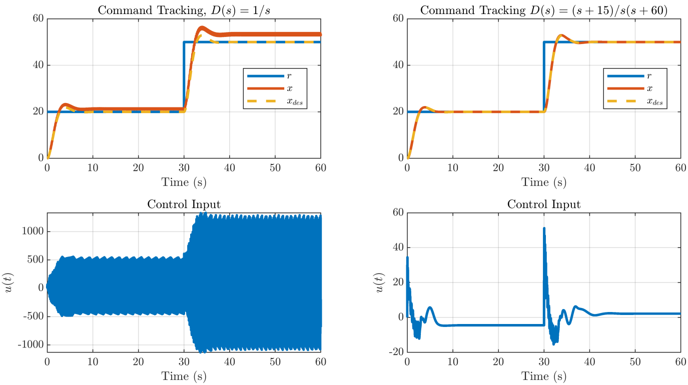

Figure 4: Performance with $t_d = 0.03$s & $k = 100$.

Observations

$D(s) = \frac{s+15}{s(s+60)}$ has larger time delay margin.

$\mathcal{L}_1$ Time Delay Margin Analysis

- $k = 5, 100, 500$ has TDM of $0.3, 0.025, 0.007(s)$, respectively

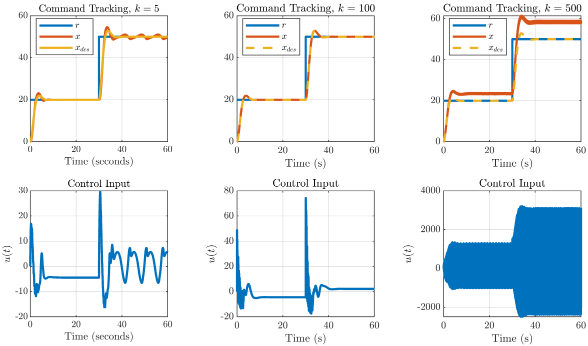

Figure 5: Responses with different k and $t_d = 0.02$

Observations

Higher k allows higher frequency signal, and has tracking performance. This comes at the cost of reduced time delay margin.

MRAC : Adaptation Parameter Tuning

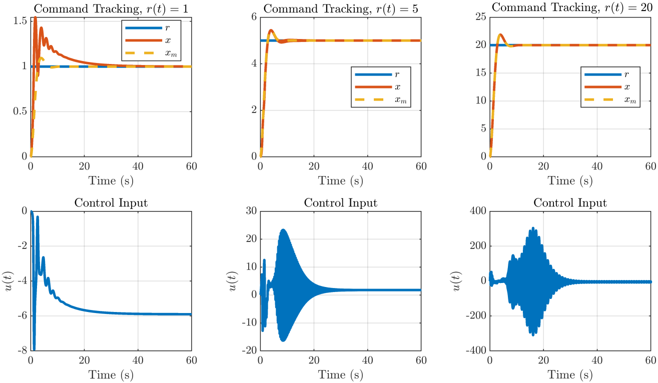

Figure 6: MRAC responses to different reference with $\Gamma = 1$

Observations

Re-tuning of adaptation parameters is required for different reference input.

$\mathcal{L}_1$: Adaptation Parameter Tuning

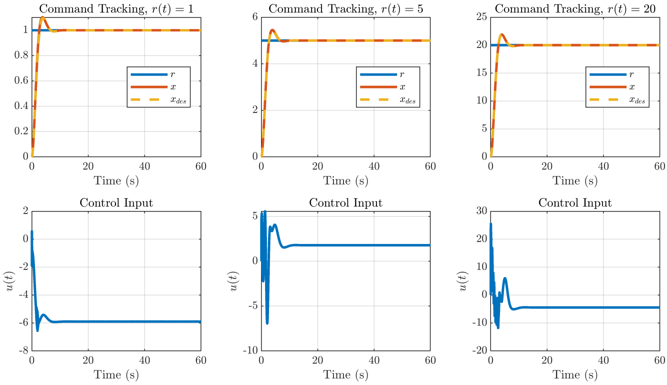

Figure 7: $\mathcal{L}_1$ with constant $\Gamma = 10000$ & $k = 100$

Observations

Without additional tuning, $\mathcal{L}_1$ shows good performance across different reference inputs.

$\mathcal{L}_1$ vs. MRAC : Effect of Shifting Plant Parameters

- A is shifted from $\begin{bmatrix} 0 & 1 \\ -0.01 & 0.02 \end{bmatrix}$ to $\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$ at $t = 20$

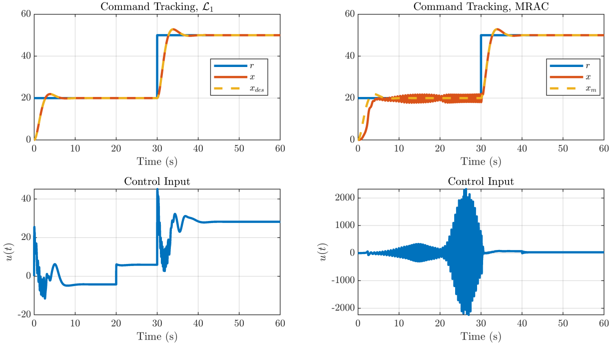

Figure 8: $\mathcal{L}_1$ and MRAC for shifting plant parameters

Observations

$\mathcal{L}_1$ can handle shifting plant parameters, but MRAC does not.

Summary

- $\mathcal{L}_1$ control framework provides **uniform and guaranteed performance**, even during the transient period.
- $\mathcal{L}_1$ **decouples adaptation from robustness** in contrast to the trade-off in MRAC.
- In the presence of disturbance, $\mathcal{L}_1$ guarantees **asymptotic convergence**, whereas MRAC only guarantees ultimate boundedness.
- For different reference inputs, MRAC requires re-tuning of adaptation parameters, while for $\mathcal{L}_1$ retuning is not required.
- $\mathcal{L}_1$ can handle shifting plant parameters, but MRAC cannot.

References

1. Hovakimyan, Naira, and Chengyu Cao. $\mathcal{L}_1$ adaptive control theory: Guaranteed robustness with fast adaptation. Society for Industrial and Applied Mathematics, 2010.
2. Nguyen, Nhan T. Model-Reference Adaptive Control. Springer, Cham, 2018. 83-123.