Chapter 2

Finite-Dimensional Vector Spaces

In the last chapter we learned about vector spaces. Linear algebra focuses not on arbitrary vector spaces, but on finite-dimensional vector spaces, which we introduce in this chapter.

We begin this chapter by considering linear combinations of lists of vectors. This leads us to the crucial concept of linear independence. The linear dependence lemma will become one of our most useful tools.

A list of vectors in a vector space that is small enough to be linearly independent and big enough so the linear combinations of the list fill up the vector space is called a basis of the vector space. We will see that every basis of a vector space has the same length, which will allow us to define the dimension of a vector space.

This chapter ends with a formula for the dimension of the sum of two subspaces.

standing assumptions for this chapter

- $\mathbf{F}$ denotes $\mathbf{R}$ or $\mathbf{C}$.
- V denotes a vector space over $\mathbf{F}$.

The main building of the Institute for Advanced Study, in Princeton, New Jersey. Paul Halmos (1916–2006) wrote the first modern linear algebra book in this building. Halmos's linear algebra book was published in 1942 (second edition published in 1958). The title of Halmos's book was the same as the title of this chapter.

2A Span and Linear Independence

We have been writing lists of numbers surrounded by parentheses, and we will continue to do so for elements of $\mathbf{F}^n$; for example, $(2, -7, 8) \in \mathbf{F}^3$. However, now we need to consider lists of vectors (which may be elements of $\mathbf{F}^n$ or of other vector spaces). To avoid confusion, we will usually write lists of vectors without surrounding parentheses. For example, $(4, 1, 6), (9, 5, 7)$ is a list of length two of vectors in $\mathbf{R}^3$.

2.1 notation: *list of vectors*

We will usually write lists of vectors without surrounding parentheses.

Linear Combinations and Span

A sum of scalar multiples of the vectors in a list is called a linear combination of the list. Here is the formal definition.

2.2 definition: *linear combination*

A *linear combination* of a list $v_1, \ldots, v_m$ of vectors in V is a vector of the form

$$a_1 v_1 + \cdots + a_m v_m,$$

where $a_1, \ldots, a_m \in \mathbf{F}$.

2.3 example: *linear combinations in* $\mathbf{R}^3$

- $(17, -4, 2)$ is a linear combination of $(2, 1, -3), (1, -2, 4)$, which is a list of length two of vectors in $\mathbf{R}^3$, because

$$(17, -4, 2) = 6(2, 1, -3) + 5(1, -2, 4).$$

- $(17, -4, 5)$ is not a linear combination of $(2, 1, -3), (1, -2, 4)$, which is a list of length two of vectors in $\mathbf{R}^3$, because there do not exist numbers $a_1, a_2 \in \mathbf{F}$ such that

$$(17, -4, 5) = a_1(2, 1, -3) + a_2(1, -2, 4).$$

  In other words, the system of equations

$$\begin{aligned} 17 &= 2a_1 + a_2 \\ -4 &= a_1 - 2a_2 \\ 5 &= -3a_1 + 4a_2 \end{aligned}$$

  has no solutions (as you should verify).

2.4 definition: ***span***

The set of all linear combinations of a list of vectors $v_1, \dots, v_m$ in V is called the *span* of $v_1, \dots, v_m$, denoted by $\operatorname{span}(v_1, \dots, v_m)$. In other words,

$$\operatorname{span}(v_1, \dots, v_m) = \{a_1 v_1 + \cdots + a_m v_m : a_1, \dots, a_m \in \mathbf{F}\}.$$

The span of the empty list $(\ )$ is defined to be $\{0\}$.

2.5 example: span

The previous example shows that in $\mathbf{F}^3$,

- $(17, -4, 2) \in \operatorname{span}\big((2, 1, -3), (1, -2, 4)\big)$;
- $(17, -4, 5) \notin \operatorname{span}\big((2, 1, -3), (1, -2, 4)\big)$.

Some mathematicians use the term ***linear span***, which means the same as span.

2.6 ***span is the smallest containing subspace***

The span of a list of vectors in V is the smallest subspace of V containing all vectors in the list.

Proof Suppose $v_1, \dots, v_m$ is a list of vectors in V.

First we show that $\operatorname{span}(v_1, \dots, v_m)$ is a subspace of V. The additive identity is in $\operatorname{span}(v_1, \dots, v_m)$ because

$$0 = 0v_1 + \cdots + 0v_m.$$

Also, $\operatorname{span}(v_1, \dots, v_m)$ is closed under addition because

$$(a_1 v_1 + \cdots + a_m v_m) + (c_1 v_1 + \cdots + c_m v_m) = (a_1 + c_1)v_1 + \cdots + (a_m + c_m)v_m.$$

Furthermore, $\operatorname{span}(v_1, \dots, v_m)$ is closed under scalar multiplication because

$$\lambda(a_1 v_1 + \cdots + a_m v_m) = \lambda a_1 v_1 + \cdots + \lambda a_m v_m.$$

Thus $\operatorname{span}(v_1, \dots, v_m)$ is a subspace of V (by 1.34).

Each v_k is a linear combination of $v_1, \dots, v_m$ (to show this, set $a_k = 1$ and let the other a's in 2.2 equal 0). Thus $\operatorname{span}(v_1, \dots, v_m)$ contains each v_k. Conversely, because subspaces are closed under scalar multiplication and addition, every subspace of V that contains each v_k contains $\operatorname{span}(v_1, \dots, v_m)$. Thus $\operatorname{span}(v_1, \dots, v_m)$ is the smallest subspace of V containing all the vectors $v_1, \dots, v_m$. ■

2.7 definition: ***spans***

If $\operatorname{span}(v_1, \dots, v_m)$ equals V, we say that the list $v_1, \dots, v_m$ *spans* V.

2.8 example: *a list that spans* $\mathbf{F}^n$

Suppose n is a positive integer. We want to show that

$$(1,0,\ldots,0),(0,1,0,\ldots,0),\ldots,(0,\ldots,0,1)$$

spans $\mathbf{F}^n$. Here the k^{th} vector in the list above has 1 in the k^{th} slot and 0 in all other slots.

Suppose $(x_1,\ldots,x_n)\in\mathbf{F}^n$. Then

$$(x_1,\ldots,x_n)=x_1(1,0,\ldots,0)+x_2(0,1,0,\ldots,0)+\cdots+x_n(0,\ldots,0,1).$$

Thus $(x_1,\ldots,x_n)\in\operatorname{span}\big((1,0,\ldots,0),(0,1,0,\ldots,0),\ldots,(0,\ldots,0,1)\big)$, as desired.

Now we can make one of the key definitions in linear algebra.

2.9 definition: *finite-dimensional vector space*

A vector space is called *finite-dimensional* if some list of vectors in it spans the space.

Example 2.8 above shows that $\mathbf{F}^n$ is a finite-dimensional vector space for every positive integer n.

Recall that by definition every list has finite length.

The definition of a polynomial is no doubt already familiar to you.

2.10 definition: *polynomial,* $\mathcal{P}(\mathbf{F})$

- A function $p\colon\mathbf{F}\to\mathbf{F}$ is called a *polynomial* with coefficients in $\mathbf{F}$ if there exist $a_0,\ldots,a_m\in\mathbf{F}$ such that

  $$p(z)=a_0+a_1z+a_2z^2+\cdots+a_mz^m$$

  for all $z\in\mathbf{F}$.
- $\mathcal{P}(\mathbf{F})$ is the set of all polynomials with coefficients in $\mathbf{F}$.

With the usual operations of addition and scalar multiplication, $\mathcal{P}(\mathbf{F})$ is a vector space over $\mathbf{F}$, as you should verify. Hence $\mathcal{P}(\mathbf{F})$ is a subspace of $\mathbf{F}^{\mathbf{F}}$, the vector space of functions from $\mathbf{F}$ to $\mathbf{F}$.

If a polynomial (thought of as a function from $\mathbf{F}$ to $\mathbf{F}$) is represented by two sets of coefficients, then subtracting one representation of the polynomial from the other produces a polynomial that is identically zero as a function on $\mathbf{F}$ and hence has all zero coefficients (if you are unfamiliar with this fact, just believe it for now; we will prove it later—see 4.8). **Conclusion:** the coefficients of a polynomial are uniquely determined by the polynomial. Thus the next definition uniquely defines the degree of a polynomial.

2.11 definition: *degree of a polynomial,* $\deg p$

- A polynomial $p \in \mathcal{P}(\mathbf{F})$ is said to have *degree* m if there exist scalars $a_0, a_1, \dots, a_m \in \mathbf{F}$ with $a_m \neq 0$ such that for every $z \in \mathbf{F}$, we have
$$p(z) = a_0 + a_1 z + \cdots + a_m z^m.$$
- The polynomial that is identically 0 is said to have degree $-\infty$.
- The degree of a polynomial p is denoted by $\deg p$.

In the next definition, we use the convention that $-\infty < m$, which means that the polynomial 0 is in $\mathcal{P}_m(\mathbf{F})$.

2.12 notation: $\mathcal{P}_m(\mathbf{F})$

For m a nonnegative integer, $\mathcal{P}_m(\mathbf{F})$ denotes the set of all polynomials with coefficients in $\mathbf{F}$ and degree at most m.

If m is a nonnegative integer, then $\mathcal{P}_m(\mathbf{F}) = \operatorname{span}(1, z, \dots, z^m)$ [here we slightly abuse notation by letting z^k denote a function]. Thus $\mathcal{P}_m(\mathbf{F})$ is a finite-dimensional vector space for each nonnegative integer m.

2.13 definition: *infinite-dimensional vector space*

A vector space is called *infinite-dimensional* if it is not finite-dimensional.

2.14 example: $\mathcal{P}(\mathbf{F})$ *is infinite-dimensional.*

Consider any list of elements of $\mathcal{P}(\mathbf{F})$. Let m denote the highest degree of the polynomials in this list. Then every polynomial in the span of this list has degree at most m. Thus z^{m+1} is not in the span of our list. Hence no list spans $\mathcal{P}(\mathbf{F})$. Thus $\mathcal{P}(\mathbf{F})$ is infinite-dimensional.

Linear Independence

Suppose $v_1, \dots, v_m \in V$ and $v \in \operatorname{span}(v_1, \dots, v_m)$. By the definition of span, there exist $a_1, \dots, a_m \in \mathbf{F}$ such that
$$v = a_1 v_1 + \cdots + a_m v_m.$$
Consider the question of whether the choice of scalars in the equation above is unique. Suppose $c_1, \dots, c_m$ is another set of scalars such that
$$v = c_1 v_1 + \cdots + c_m v_m.$$
Subtracting the last two equations, we have
$$0 = (a_1 - c_1) v_1 + \cdots + (a_m - c_m) v_m.$$

Thus we have written 0 as a linear combination of $(v_1, \dots, v_m)$. If the only way to do this is by using 0 for all the scalars in the linear combination, then each $a_k - c_k$ equals 0, which means that each a_k equals c_k (and thus the choice of scalars was indeed unique). This situation is so important that we give it a special name—linear independence—which we now define.

2.15 definition: *linearly independent*

- A list $v_1, \dots, v_m$ of vectors in V is called *linearly independent* if the only choice of $a_1, \dots, a_m \in \mathbf{F}$ that makes
$$a_1 v_1 + \cdots + a_m v_m = 0$$
is $a_1 = \cdots = a_m = 0$.
- The empty list $(\,)$ is also declared to be linearly independent.

The reasoning above shows that $v_1, \dots, v_m$ is linearly independent if and only if each vector in $\operatorname{span}(v_1, \dots, v_m)$ has only one representation as a linear combination of $v_1, \dots, v_m$.

2.16 example: *linearly independent lists*

(a) To see that the list $(1,0,0,0), (0,1,0,0), (0,0,1,0)$ is linearly independent in $\mathbf{F}^4$, suppose $a_1, a_2, a_3 \in \mathbf{F}$ and
$$a_1(1,0,0,0) + a_2(0,1,0,0) + a_3(0,0,1,0) = (0,0,0,0).$$
Thus
$$(a_1, a_2, a_3, 0) = (0,0,0,0).$$
Hence $a_1 = a_2 = a_3 = 0$. Thus the list $(1,0,0,0), (0,1,0,0), (0,0,1,0)$ is linearly independent in $\mathbf{F}^4$.

(b) Suppose m is a nonnegative integer. To see that the list $1, z, \dots, z^m$ is linearly independent in $\mathcal{P}(\mathbf{F})$, suppose $a_0, a_1, \dots, a_m \in \mathbf{F}$ and
$$a_0 + a_1 z + \cdots + a_m z^m = 0,$$
where we think of both sides as elements of $\mathcal{P}(\mathbf{F})$. Then
$$a_0 + a_1 z + \cdots + a_m z^m = 0$$
for all $z \in \mathbf{F}$. As discussed earlier (and as follows from 4.8), this implies that $a_0 = a_1 = \cdots = a_m = 0$. Thus $1, z, \dots, z^m$ is a linearly independent list in $\mathcal{P}(\mathbf{F})$.

(c) A list of length one in a vector space is linearly independent if and only if the vector in the list is not 0.

(d) A list of length two in a vector space is linearly independent if and only if neither of the two vectors in the list is a scalar multiple of the other.

If some vectors are removed from a linearly independent list, the remaining list is also linearly independent, as you should verify.

2.17 definition: *linearly dependent*

- A list of vectors in V is called *linearly dependent* if it is not linearly independent.
- In other words, a list $v_1, \dots, v_m$ of vectors in V is linearly dependent if there exist $a_1, \dots, a_m \in \mathbf{F}$, not all 0, such that $a_1 v_1 + \cdots + a_m v_m = 0$.

2.18 example: *linearly dependent lists*

- $(2,3,1), (1,-1,2), (7,3,8)$ is linearly dependent in $\mathbf{F}^3$ because
$$2(2,3,1) + 3(1,-1,2) + (-1)(7,3,8) = (0,0,0).$$
- The list $(2,3,1), (1,-1,2), (7,3,c)$ is linearly dependent in $\mathbf{F}^3$ if and only if $c = 8$, as you should verify.
- If some vector in a list of vectors in V is a linear combination of the other vectors, then the list is linearly dependent. (Proof: After writing one vector in the list as equal to a linear combination of the other vectors, move that vector to the other side of the equation, where it will be multiplied by -1.)
- Every list of vectors in V containing the 0 vector is linearly dependent. (This is a special case of the previous bullet point.)

The next lemma is a terrific tool. It states that given a linearly dependent list of vectors, one of the vectors is in the span of the previous ones. Furthermore, we can throw out that vector without changing the span of the original list.

2.19 *linear dependence lemma*

Suppose $v_1, \dots, v_m$ is a linearly dependent list in V. Then there exists $k \in \{1, 2, \dots, m\}$ such that
$$v_k \in \operatorname{span}(v_1, \dots, v_{k-1}).$$
Furthermore, if k satisfies the condition above and the k^{th} term is removed from $v_1, \dots, v_m$, then the span of the remaining list equals $\operatorname{span}(v_1, \dots, v_m)$.

Proof Because the list $v_1, \dots, v_m$ is linearly dependent, there exist numbers $a_1, \dots, a_m \in \mathbf{F}$, not all 0, such that
$$a_1 v_1 + \cdots + a_m v_m = 0.$$
Let k be the largest element of $\{1, \dots, m\}$ such that $a_k \neq 0$. Then
$$v_k = -\frac{a_1}{a_k} v_1 - \cdots - \frac{a_{k-1}}{a_k} v_{k-1},$$
which proves that $v_k \in \operatorname{span}(v_1, \dots, v_{k-1})$, as desired.

Now suppose k is any element of $\{1, \dots, m\}$ such that $v_k \in \operatorname{span}(v_1, \dots, v_{k-1})$. Let $b_1, \dots, b_{k-1} \in \mathbf{F}$ be such that

2.20
$$v_k = b_1 v_1 + \cdots + b_{k-1} v_{k-1}.$$

Suppose $u \in \operatorname{span}(v_1, \dots, v_m)$. Then there exist $c_1, \dots, c_m \in \mathbf{F}$ such that

$$u = c_1 v_1 + \cdots + c_m v_m.$$

In the equation above, we can replace v_k with the right side of 2.20, which shows that u is in the span of the list obtained by removing the k^{th} term from $v_1, \dots, v_m$. Thus removing the k^{th} term of the list $v_1, \dots, v_m$ does not change the span of the list. ■

If $k = 1$ in the linear dependence lemma, then $v_k \in \operatorname{span}(v_1, \dots, v_{k-1})$ means that $v_1 = 0$, because $\operatorname{span}(\,) = \{0\}$. Note also that parts of the proof of the linear dependence lemma need to be modified if $k = 1$. In general, the proofs in the rest of the book will not call attention to special cases that must be considered involving lists of length 0, the subspace $\{0\}$, or other trivial cases for which the result is true but needs a slightly different proof. Be sure to check these special cases yourself.

2.21 example: *smallest k in linear dependence lemma*

Consider the list

$$(1, 2, 3), (6, 5, 4), (15, 16, 17), (8, 9, 7)$$

in $\mathbf{R}^3$. This list of length four is linearly dependent, as we will soon see. Thus the linear dependence lemma implies that there exists $k \in \{1, 2, 3, 4\}$ such that the k^{th} vector in this list is a linear combination of the previous vectors in the list. Let's see how to find the smallest value of k that works.

Taking $k = 1$ in the linear dependence lemma works if and only if the first vector in the list equals 0. Because $(1, 2, 3)$ is not the 0 vector, we cannot take $k = 1$ for this list.

Taking $k = 2$ in the linear dependence lemma works if and only if the second vector in the list is a scalar multiple of the first vector. However, there does not exist $c \in \mathbf{R}$ such that $(6, 5, 4) = c(1, 2, 3)$. Thus we cannot take $k = 2$ for this list.

Taking $k = 3$ in the linear dependence lemma works if and only if the third vector in the list is a linear combination of the first two vectors. Thus for the list in this example, we want to know whether there exist $a, b \in \mathbf{R}$ such that

$$(15, 16, 17) = a(1, 2, 3) + b(6, 5, 4).$$

The equation above is equivalent to a system of three linear equations in the two unknowns a, b. Using Gaussian elimination or appropriate software, we find that $a = 3$, $b = 2$ is a solution of the equation above, as you can verify. Thus for the list in this example, taking $k = 3$ is the smallest value of k that works in the linear dependence lemma.

Now we come to a key result. It says that no linearly independent list in V is longer than a spanning list in V.

2.22 *length of linearly independent list $\leq$ length of spanning list*

In a finite-dimensional vector space, the length of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.

Proof Suppose that $u_1, \dots, u_m$ is linearly independent in V. Suppose also that $w_1, \dots, w_n$ spans V. We need to prove that $m \leq n$. We do so through the process described below with m steps; note that in each step we add one of the u's and remove one of the w's.

Step 1

Let B be the list $w_1, \dots, w_n$, which spans V. Adjoining u_1 at the beginning of this list produces a linearly dependent list (because u_1 can be written as a linear combination of $w_1, \dots, w_n$). In other words, the list

$$u_1, w_1, \dots, w_n$$

is linearly dependent.

Thus by the linear dependence lemma (2.19), one of the vectors in the list above is a linear combination of the previous vectors in the list. We know that $u_1 \neq 0$ because the list $u_1, \dots, u_m$ is linearly independent. Thus u_1 is not in the span of the previous vectors in the list above (because u_1 is not in $\{0\}$, which is the span of the empty list). Hence the linear dependence lemma implies that we can remove one of the w's so that the new list B (of length n) consisting of u_1 and the remaining w's spans V.

Step k, for $k = 2, \dots, m$

The list B (of length n) from step $k-1$ spans V. In particular, u_k is in the span of the list B. Thus the list of length $(n+1)$ obtained by adjoining u_k to B, placing it just after $u_1, \dots, u_{k-1}$, is linearly dependent. By the linear dependence lemma (2.19), one of the vectors in this list is in the span of the previous ones, and because $u_1, \dots, u_k$ is linearly independent, this vector cannot be one of the u's.

Hence there still must be at least one remaining w at this step. We can remove from our new list (after adjoining u_k in the proper place) a w that is a linear combination of the previous vectors in the list, so that the new list B (of length n) consisting of $u_1, \dots, u_k$ and the remaining w's spans V.

After step m, we have added all the u's and the process stops. At each step as we add a u to B, the linear dependence lemma implies that there is some w to remove. Thus there are at least as many w's as u's. ■

The next two examples show how the result above can be used to show, without any computations, that certain lists are not linearly independent and that certain lists do not span a given vector space.

2.23 example: *no list of length* 4 *is linearly independent in* $\mathbf{R}^3$

The list $(1,0,0),(0,1,0),(0,0,1)$, which has length three, spans $\mathbf{R}^3$. Thus no list of length larger than three is linearly independent in $\mathbf{R}^3$.

For example, we now know that $(1,2,3),(4,5,8),(9,6,7),(-3,2,8)$, which is a list of length four, is not linearly independent in $\mathbf{R}^3$.

2.24 example: *no list of length* 3 *spans* $\mathbf{R}^4$

The list $(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)$, which has length four, is linearly independent in $\mathbf{R}^4$. Thus no list of length less than four spans $\mathbf{R}^4$.

For example, we now know that $(1,2,3,-5),(4,5,8,3),(9,6,7,-1)$, which is a list of length three, does not span $\mathbf{R}^4$.

Our intuition suggests that every subspace of a finite-dimensional vector space should also be finite-dimensional. We now prove that this intuition is correct.

2.25 *finite-dimensional subspaces*

Every subspace of a finite-dimensional vector space is finite-dimensional.

Proof Suppose V is finite-dimensional and U is a subspace of V. We need to prove that U is finite-dimensional. We do this through the following multistep construction.

Step 1

If $U=\{0\}$, then U is finite-dimensional and we are done. If $U\neq\{0\}$, then choose a nonzero vector $u_1\in U$.

Step *k*

If $U=\operatorname{span}(u_1,\ldots,u_{k-1})$, then U is finite-dimensional and we are done. If $U\neq\operatorname{span}(u_1,\ldots,u_{k-1})$, then choose a vector $u_k\in U$ such that

$$u_k\notin\operatorname{span}(u_1,\ldots,u_{k-1}).$$

After each step, as long as the process continues, we have constructed a list of vectors such that no vector in this list is in the span of the previous vectors. Thus after each step we have constructed a linearly independent list, by the linear dependence lemma (2.19). This linearly independent list cannot be longer than any spanning list of V (by 2.22). Thus the process eventually terminates, which means that U is finite-dimensional. ∎

Exercises 2A

1 Find a list of four distinct vectors in $\mathbf{F}^3$ whose span equals

$$\{(x, y, z) \in \mathbf{F}^3 : x + y + z = 0\}.$$

2 Prove or give a counterexample: If v_1, v_2, v_3, v_4 spans V, then the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

also spans V.

3 Suppose $v_1, \dots, v_m$ is a list of vectors in V. For $k \in \{1, \dots, m\}$, let

$$w_k = v_1 + \cdots + v_k.$$

Show that $\operatorname{span}(v_1, \dots, v_m) = \operatorname{span}(w_1, \dots, w_m)$.

4 (a) Show that a list of length one in a vector space is linearly independent if and only if the vector in the list is not 0.

(b) Show that a list of length two in a vector space is linearly independent if and only if neither of the two vectors in the list is a scalar multiple of the other.

5 Find a number t such that

$$(3, 1, 4), (2, -3, 5), (5, 9, t)$$

is not linearly independent in $\mathbf{R}^3$.

6 Show that the list $(2, 3, 1), (1, -1, 2), (7, 3, c)$ is linearly dependent in $\mathbf{F}^3$ if and only if $c = 8$.

7 (a) Show that if we think of $\mathbf{C}$ as a vector space over $\mathbf{R}$, then the list $1 + i, 1 - i$ is linearly independent.

(b) Show that if we think of $\mathbf{C}$ as a vector space over $\mathbf{C}$, then the list $1 + i, 1 - i$ is linearly dependent.

8 Suppose v_1, v_2, v_3, v_4 is linearly independent in V. Prove that the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

is also linearly independent.

9 Prove or give a counterexample: If $v_1, v_2, \dots, v_m$ is a linearly independent list of vectors in V, then

$$5v_1 - 4v_2, v_2, v_3, \dots, v_m$$

is linearly independent.

10 Prove or give a counterexample: If $v_1, v_2, \ldots, v_m$ is a linearly independent list of vectors in V and $\lambda \in \mathbf{F}$ with $\lambda \neq 0$, then $\lambda v_1, \lambda v_2, \ldots, \lambda v_m$ is linearly independent.

11 Prove or give a counterexample: If $v_1, \ldots, v_m$ and $w_1, \ldots, w_m$ are linearly independent lists of vectors in V, then the list $v_1 + w_1, \ldots, v_m + w_m$ is linearly independent.

12 Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Prove that if $v_1 + w, \ldots, v_m + w$ is linearly dependent, then $w \in \operatorname{span}(v_1, \ldots, v_m)$.

13 Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Show that

$$v_1, \ldots, v_m, w \text{ is linearly independent} \iff w \notin \operatorname{span}(v_1, \ldots, v_m).$$

14 Suppose $v_1, \ldots, v_m$ is a list of vectors in V. For $k \in \{1, \ldots, m\}$, let

$$w_k = v_1 + \cdots + v_k.$$

Show that the list $v_1, \ldots, v_m$ is linearly independent if and only if the list $w_1, \ldots, w_m$ is linearly independent.

15 Explain why there does not exist a list of six polynomials that is linearly independent in $\mathcal{P}_4(\mathbf{F})$.

16 Explain why no list of four polynomials spans $\mathcal{P}_4(\mathbf{F})$.

17 Prove that V is infinite-dimensional if and only if there is a sequence $v_1, v_2, \ldots$ of vectors in V such that $v_1, \ldots, v_m$ is linearly independent for every positive integer m.

18 Prove that $\mathbf{F}^\infty$ is infinite-dimensional.

19 Prove that the real vector space of all continuous real-valued functions on the interval $[0, 1]$ is infinite-dimensional.

20 Suppose $p_0, p_1, \ldots, p_m$ are polynomials in $\mathcal{P}_m(\mathbf{F})$ such that $p_k(2) = 0$ for each $k \in \{0, \ldots, m\}$. Prove that $p_0, p_1, \ldots, p_m$ is not linearly independent in $\mathcal{P}_m(\mathbf{F})$.