3E Products and Quotients of Vector Spaces

Products of Vector Spaces

As usual when dealing with more than one vector space, all vector spaces in use should be over the same field.

3.87 definition: *product of vector spaces*

Suppose $V_1, \dots, V_m$ are vector spaces over $\mathbf{F}$.

- The *product* $V_1 \times \cdots \times V_m$ is defined by

$$V_1 \times \cdots \times V_m = \{(v_1, \dots, v_m) : v_1 \in V_1, \dots, v_m \in V_m\}.$$

- Addition on $V_1 \times \cdots \times V_m$ is defined by

$$(u_1, \dots, u_m) + (v_1, \dots, v_m) = (u_1 + v_1, \dots, u_m + v_m).$$

- Scalar multiplication on $V_1 \times \cdots \times V_m$ is defined by

$$\lambda(v_1, \dots, v_m) = (\lambda v_1, \dots, \lambda v_m).$$

3.88 example: *product of the vector spaces $\mathcal{P}_5(\mathbf{R})$ and $\mathbf{R}^3$*

Elements of $\mathcal{P}_5(\mathbf{R}) \times \mathbf{R}^3$ are lists of length two, with the first item in the list an element of $\mathcal{P}_5(\mathbf{R})$ and the second item in the list an element of $\mathbf{R}^3$.

For example, $\bigl(5 - 6x + 4x^2, (3, 8, 7)\bigr)$ and $\bigl(x + 9x^5, (2, 2, 2)\bigr)$ are elements of $\mathcal{P}_5(\mathbf{R}) \times \mathbf{R}^3$. Their sum is defined by

$$\begin{aligned}\bigl(5 - 6x + 4x^2, (3, 8, 7)\bigr) &+ \bigl(x + 9x^5, (2, 2, 2)\bigr) \\ &= \bigl(5 - 5x + 4x^2 + 9x^5, (5, 10, 9)\bigr).\end{aligned}$$

Also, $2\bigl(5 - 6x + 4x^2, (3, 8, 7)\bigr) = \bigl(10 - 12x + 8x^2, (6, 16, 14)\bigr)$.

The next result should be interpreted to mean that the product of vector spaces is a vector space with the operations of addition and scalar multiplication as defined by 3.87.

3.89 *product of vector spaces is a vector space*

Suppose $V_1, \dots, V_m$ are vector spaces over $\mathbf{F}$. Then $V_1 \times \cdots \times V_m$ is a vector space over $\mathbf{F}$.

The proof of the result above is left to the reader. Note that the additive identity of $V_1 \times \cdots \times V_m$ is $(0, \dots, 0)$, where the 0 in the k^{th} slot is the additive identity of V_k. The additive inverse of $(v_1, \dots, v_m) \in V_1 \times \cdots \times V_m$ is $(-v_1, \dots, -v_m)$.

3.90 example: $\mathbf{R}^2 \times \mathbf{R}^3 \neq \mathbf{R}^5$ *but* $\mathbf{R}^2 \times \mathbf{R}^3$ *is isomorphic to* $\mathbf{R}^5$

Elements of the vector space $\mathbf{R}^2 \times \mathbf{R}^3$ are lists

$$\big((x_1, x_2), (x_3, x_4, x_5)\big),$$

where $x_1, x_2, x_3, x_4, x_5 \in \mathbf{R}$. Elements of $\mathbf{R}^5$ are lists

$$(x_1, x_2, x_3, x_4, x_5),$$

where $x_1, x_2, x_3, x_4, x_5 \in \mathbf{R}$.

Although elements of $\mathbf{R}^2 \times \mathbf{R}^3$ and $\mathbf{R}^5$ look similar, they are not the same kind of object. Elements of $\mathbf{R}^2 \times \mathbf{R}^3$ are lists of length two (with the first item itself a list of length two and the second item a list of length three), and elements of $\mathbf{R}^5$ are lists of length five. Thus $\mathbf{R}^2 \times \mathbf{R}^3$ does not equal $\mathbf{R}^5$.

The linear map

$$\big((x_1, x_2), (x_3, x_4, x_5)\big) \mapsto (x_1, x_2, x_3, x_4, x_5)$$

is an isomorphism of the vector space $\mathbf{R}^2 \times \mathbf{R}^3$ onto the vector space $\mathbf{R}^5$. Thus these two vector spaces are isomorphic, although they are not equal.

This isomorphism is so natural that we should think of it as a relabeling. Some people informally say that $\mathbf{R}^2 \times \mathbf{R}^3$ *equals* $\mathbf{R}^5$*, which is not technically correct but which captures the spirit of identification via relabeling.*

The next example illustrates the idea that we will use in the proof of 3.92.

3.91 example: *a basis of* $\mathcal{P}_2(\mathbf{R}) \times \mathbf{R}^2$

Consider this list of length five of elements of $\mathcal{P}_2(\mathbf{R}) \times \mathbf{R}^2$:

$$\big(1, (0,0)\big), \big(x, (0,0)\big), \big(x^2, (0,0)\big), \big(0, (1,0)\big), \big(0, (0,1)\big).$$

The list above is linearly independent and it spans $\mathcal{P}_2(\mathbf{R}) \times \mathbf{R}^2$. Thus it is a basis of $\mathcal{P}_2(\mathbf{R}) \times \mathbf{R}^2$.

3.92 *dimension of a product is the sum of dimensions*

Suppose $V_1, \dots, V_m$ are finite-dimensional vector spaces. Then $V_1 \times \cdots \times V_m$ is finite-dimensional and

$$\dim(V_1 \times \cdots \times V_m) = \dim V_1 + \cdots + \dim V_m.$$

Proof Choose a basis of each V_k. For each basis vector of each V_k, consider the element of $V_1 \times \cdots \times V_m$ that equals the basis vector in the k^{th} slot and 0 in the other slots. The list of all such vectors is linearly independent and spans $V_1 \times \cdots \times V_m$. Thus it is a basis of $V_1 \times \cdots \times V_m$. The length of this basis is $\dim V_1 + \cdots + \dim V_m$, as desired. ■

In the next result, the map Γ is surjective by the definition of $V_1 + \cdots + V_m$. Thus the last word in the result below could be changed from "injective" to "invertible".

3.93 *products and direct sums*

Suppose that $V_1, \dots, V_m$ are subspaces of V. Define a linear map $\Gamma : V_1 \times \cdots \times V_m \to V_1 + \cdots + V_m$ by

$$\Gamma(v_1, \dots, v_m) = v_1 + \cdots + v_m.$$

Then $V_1 + \cdots + V_m$ is a direct sum if and only if Γ is injective.

Proof By 3.15, Γ is injective if and only if the only way to write 0 as a sum $v_1 + \cdots + v_m$, where each v_k is in V_k, is by taking each v_k equal to 0. Thus 1.45 shows that Γ is injective if and only if $V_1 + \cdots + V_m$ is a direct sum, as desired. ■

3.94 *a sum is a direct sum if and only if dimensions add up*

Suppose V is finite-dimensional and $V_1, \dots, V_m$ are subspaces of V. Then $V_1 + \cdots + V_m$ is a direct sum if and only if

$$\dim(V_1 + \cdots + V_m) = \dim V_1 + \cdots + \dim V_m.$$

Proof The map Γ in 3.93 is surjective. Thus by the fundamental theorem of linear maps (3.21), Γ is injective if and only if

$$\dim(V_1 + \cdots + V_m) = \dim(V_1 \times \cdots \times V_m).$$

Combining 3.93 and 3.92 now shows that $V_1 + \cdots + V_m$ is a direct sum if and only if

$$\dim(V_1 + \cdots + V_m) = \dim V_1 + \cdots + \dim V_m,$$

as desired. ■

In the special case $m = 2$, an alternative proof that $V_1 + V_2$ is a direct sum if and only if $\dim(V_1 + V_2) = \dim V_1 + \dim V_2$ can be obtained by combining 1.46 and 2.43.

Quotient Spaces

We begin our approach to quotient spaces by defining the sum of a vector and a subset.

3.95 notation: $v + U$

Suppose $v \in V$ and $U \subseteq V$. Then $v + U$ is the subset of V defined by

$$v + U = \{v + u : u \in U\}.$$

3.96 example: *sum of a vector and a one-dimensional subspace of* $\mathbf{R}^2$

Suppose

$$U = \{(x, 2x) \in \mathbf{R}^2 : x \in \mathbf{R}\}.$$

Hence U is the line in $\mathbf{R}^2$ through the origin with slope 2. Thus

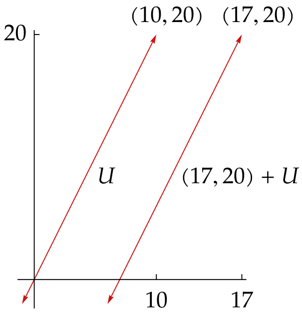
$$(17, 20) + U$$

is the line in $\mathbf{R}^2$ that contains the point $(17, 20)$ and has slope 2.

Because

$$(10, 20) \in U \quad \text{and} \quad (17, 20) \in (17, 20) + U,$$

we see that $(17, 20) + U$ is obtained by moving U to the right by 7 units.



$(17, 20) + U$ is parallel to the subspace U.

3.97 definition: *translate*

For $v \in V$ and U a subset of V, the set $v + U$ is said to be a *translate* of U.

3.98 example: *translates*

- If U is the line in $\mathbf{R}^2$ defined by $U = \{(x, 2x) \in \mathbf{R}^2 : x \in \mathbf{R}\}$, then all lines in $\mathbf{R}^2$ with slope 2 are translates of U. See Example 3.96 above for a drawing of U and one of its translates.
- More generally, if U is a line in $\mathbf{R}^2$, then the set of all translates of U is the set of all lines in $\mathbf{R}^2$ that are parallel to U.
- If $U = \{(x, y, 0) \in \mathbf{R}^3 : x, y \in \mathbf{R}\}$, then the translates of U are the planes in $\mathbf{R}^3$ that are parallel to the xy-plane U.
- More generally, if U is a plane in $\mathbf{R}^3$, then the set of all translates of U is the set of all planes in $\mathbf{R}^3$ that are parallel to U (see, for example, Exercise 7).

3.99 definition: *quotient space,* V/U

Suppose U is a subspace of V. Then the *quotient space* V/U is the set of all translates of U. Thus

$$V/U = \{v + U : v \in V\}.$$

3.100 example: *quotient spaces*

- If $U = \{(x, 2x) \in \mathbf{R}^2 : x \in \mathbf{R}\}$, then $\mathbf{R}^2/U$ is the set of all lines in $\mathbf{R}^2$ that have slope 2.
- If U is a line in $\mathbf{R}^3$ containing the origin, then $\mathbf{R}^3/U$ is the set of all lines in $\mathbf{R}^3$ parallel to U.
- If U is a plane in $\mathbf{R}^3$ containing the origin, then $\mathbf{R}^3/U$ is the set of all planes in $\mathbf{R}^3$ parallel to U.

Our next goal is to make V/U into a vector space. To do this, we will need the next result.

3.101 *two translates of a subspace are equal or disjoint*

Suppose U is a subspace of V and $v, w \in V$. Then

$$v - w \in U \iff v + U = w + U \iff (v + U) \cap (w + U) \neq \emptyset.$$

Proof First suppose $v - w \in U$. If $u \in U$, then

$$v + u = w + \big((v - w) + u\big) \in w + U.$$

Thus $v + U \subseteq w + U$. Similarly, $w + U \subseteq v + U$. Thus $v + U = w + U$, completing the proof that $v - w \in U$ implies $v + U = w + U$.

The equation $v + U = w + U$ implies that $(v + U) \cap (w + U) \neq \emptyset$.

Now suppose $(v + U) \cap (w + U) \neq \emptyset$. Thus there exist $u_1, u_2 \in U$ such that

$$v + u_1 = w + u_2.$$

Thus $v - w = u_2 - u_1$. Hence $v - w \in U$, showing that $(v + U) \cap (w + U) \neq \emptyset$ implies $v - w \in U$, which completes the proof. ■

Now we can define addition and scalar multiplication on V/U.

3.102 definition: *addition and scalar multiplication on V/U*

Suppose U is a subspace of V. Then *addition* and *scalar multiplication* are defined on V/U by

$$\begin{aligned}(v + U) + (w + U) &= (v + w) + U\\ \lambda(v + U) &= (\lambda v) + U\end{aligned}$$

for all $v, w \in V$ and all $\lambda \in \mathbf{F}$.

As part of the proof of the next result, we will show that the definitions above make sense.

3.103 *quotient space is a vector space*

Suppose U is a subspace of V. Then V/U, with the operations of addition and scalar multiplication as defined above, is a vector space.

Proof The potential problem with the definitions above of addition and scalar multiplication on V/U is that the representation of a translate of U is not unique. Specifically, suppose $v_1, v_2, w_1, w_2 \in V$ are such that

$$v_1 + U = v_2 + U \quad \text{and} \quad w_1 + U = w_2 + U.$$

To show that the definition of addition on V/U given above makes sense, we must show that $(v_1 + w_1) + U = (v_2 + w_2) + U$.

By 3.101, we have

$$v_1 - v_2 \in U \quad \text{and} \quad w_1 - w_2 \in U.$$

Because U is a subspace of V and thus is closed under addition, this implies that $(v_1 - v_2) + (w_1 - w_2) \in U$. Thus $(v_1 + w_1) - (v_2 + w_2) \in U$. Using 3.101 again, we see that

$$(v_1 + w_1) + U = (v_2 + w_2) + U,$$

as desired. Thus the definition of addition on V/U makes sense.

Similarly, suppose $\lambda \in \mathbf{F}$. We are still assuming that $v_1 + U = v_2 + U$. Because U is a subspace of V and thus is closed under scalar multiplication, we have $\lambda(v_1 - v_2) \in U$. Thus $\lambda v_1 - \lambda v_2 \in U$. Hence 3.101 implies that

$$(\lambda v_1) + U = (\lambda v_2) + U.$$

Thus the definition of scalar multiplication on V/U makes sense.

Now that addition and scalar multiplication have been defined on V/U, the verification that these operations make V/U into a vector space is straightforward and is left to the reader. Note that the additive identity of V/U is $0 + U$ (which equals U) and that the additive inverse of $v + U$ is $(-v) + U$. ■

The next concept will lead to a computation of the dimension of V/U.

3.104 definition: *quotient map, π*

Suppose U is a subspace of V. The *quotient map* $\pi \colon V \to V/U$ is the linear map defined by

$$\pi(v) = v + U$$

for each $v \in V$.

The reader should verify that π is indeed a linear map. Although π depends on U as well as V, these spaces are left out of the notation because they should be clear from the context.

3.105 *dimension of quotient space*

Suppose V is finite-dimensional and U is a subspace of V. Then

$$\dim V/U = \dim V - \dim U.$$

Proof Let π denote the quotient map from V to V/U. If $v \in V$, then $v+U = 0+U$ if and only if $v \in U$ (by 3.101), which implies that $\operatorname{null} \pi = U$. The definition of π implies $\operatorname{range} \pi = V/U$. The fundamental theorem of linear maps (3.21) now implies $\dim V = \dim U + \dim V/U$, which gives the desired result. ■

Each linear map T on V induces a linear map $\widetilde{T}$ on $V/(\operatorname{null} T)$, as defined below.

3.106 notation: $\widetilde{T}$

Suppose $T \in \mathcal{L}(V, W)$. Define $\widetilde{T} \colon V/(\operatorname{null} T) \to W$ by

$$\widetilde{T}(v + \operatorname{null} T) = Tv.$$

To show that the definition of $\widetilde{T}$ makes sense, suppose $u, v \in V$ are such that $u + \operatorname{null} T = v + \operatorname{null} T$. By 3.101, we have $u - v \in \operatorname{null} T$. Thus $T(u-v) = 0$. Hence $Tu = Tv$. Thus the definition of $\widetilde{T}$ indeed makes sense. The routine verification that $\widetilde{T}$ is a linear map from $V/(\operatorname{null} T)$ to W is left to the reader.

The next result shows that we can think of $\widetilde{T}$ as a modified version of T, with a domain that produces a one-to-one map.

3.107 *null space and range of* $\widetilde{T}$

Suppose $T \in \mathcal{L}(V, W)$. Then

(a) $\widetilde{T} \circ \pi = T$, where π is the quotient map of V onto $V/(\operatorname{null} T)$;

(b) $\widetilde{T}$ is injective;

(c) $\operatorname{range} \widetilde{T} = \operatorname{range} T$;

(d) $V/(\operatorname{null} T)$ and $\operatorname{range} T$ are isomorphic vector spaces.

Proof

(a) If $v \in V$, then $\big(\widetilde{T} \circ \pi\big)(v) = \widetilde{T}\big(\pi(v)\big) = \widetilde{T}(v + \operatorname{null} T) = Tv$, as desired.

(b) Suppose $v \in V$ and $\widetilde{T}(v + \operatorname{null} T) = 0$. Then $Tv = 0$. Thus $v \in \operatorname{null} T$. Hence 3.101 implies that $v + \operatorname{null} T = 0 + \operatorname{null} T$. This implies that $\operatorname{null} \widetilde{T} = \{0 + \operatorname{null} T\}$. Hence $\widetilde{T}$ is injective, as desired.

(c) The definition of $\widetilde{T}$ shows that $\operatorname{range} \widetilde{T} = \operatorname{range} T$.

(d) Now (b) and (c) imply that if we think of $\widetilde{T}$ as mapping into $\operatorname{range} T$, then $\widetilde{T}$ is an isomorphism from $V/(\operatorname{null} T)$ onto $\operatorname{range} T$. ■

Exercises 3E

1 Suppose T is a function from V to W. The *graph* of T is the subset of $V \times W$ defined by

$$\text{graph of } T = \{(v, Tv) \in V \times W : v \in V\}.$$

Prove that T is a linear map if and only if the graph of T is a subspace of $V \times W$.

Formally, a function T from V to W is a subset T of $V \times W$ such that for each $v \in V$, there exists exactly one element $(v, w) \in T$. In other words, formally a function is what is called above its graph. We do not usually think of functions in this formal manner. However, if we do become formal, then this exercise could be rephrased as follows: Prove that a function T from V to W is a linear map if and only if T is a subspace of $V \times W$.

2 Suppose that $V_1, \dots, V_m$ are vector spaces such that $V_1 \times \cdots \times V_m$ is finite-dimensional. Prove that V_k is finite-dimensional for each $k = 1, \dots, m$.

3 Suppose $V_1, \dots, V_m$ are vector spaces. Prove that $\mathcal{L}(V_1 \times \cdots \times V_m, W)$ and $\mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)$ are isomorphic vector spaces.

4 Suppose $W_1, \dots, W_m$ are vector spaces. Prove that $\mathcal{L}(V, W_1 \times \cdots \times W_m)$ and $\mathcal{L}(V, W_1) \times \cdots \times \mathcal{L}(V, W_m)$ are isomorphic vector spaces.

5 For m a positive integer, define V^m by

$$V^m = \underbrace{V \times \cdots \times V}_{m \text{ times}}.$$

Prove that V^m and $\mathcal{L}\big(\mathbf{F}^m, V\big)$ are isomorphic vector spaces.

6 Suppose that v, x are vectors in V and that U, W are subspaces of V such that $v + U = x + W$. Prove that $U = W$.

7 Let $U = \{(x, y, z) \in \mathbf{R}^3 : 2x + 3y + 5z = 0\}$. Suppose $A \subseteq \mathbf{R}^3$. Prove that A is a translate of U if and only if there exists $c \in \mathbf{R}$ such that

$$A = \{(x, y, z) \in \mathbf{R}^3 : 2x + 3y + 5z = c\}.$$

8 (a) Suppose $T \in \mathcal{L}(V, W)$ and $c \in W$. Prove that $\{x \in V : Tx = c\}$ is either the empty set or is a translate of null T.

(b) Explain why the set of solutions to a system of linear equations such as 3.27 is either the empty set or is a translate of some subspace of $\mathbf{F}^n$.

9 Prove that a nonempty subset A of V is a translate of some subspace of V if and only if $\lambda v + (1 - \lambda)w \in A$ for all $v, w \in A$ and all $\lambda \in \mathbf{F}$.

10 Suppose $A_1 = v + U_1$ and $A_2 = w + U_2$ for some $v, w \in V$ and some subspaces U_1, U_2 of V. Prove that the intersection $A_1 \cap A_2$ is either a translate of some subspace of V or is the empty set.

11 Suppose $U = \{(x_1, x_2, \dots) \in \mathbf{F}^\infty : x_k \neq 0 \text{ for only finitely many } k\}$.

(a) Show that U is a subspace of $\mathbf{F}^\infty$.

(b) Prove that $\mathbf{F}^\infty/U$ is infinite-dimensional.

12 Suppose $v_1, \dots, v_m \in V$. Let

$$A = \{\lambda_1 v_1 + \cdots + \lambda_m v_m : \lambda_1, \dots, \lambda_m \in \mathbf{F} \text{ and } \lambda_1 + \cdots + \lambda_m = 1\}.$$

(a) Prove that A is a translate of some subspace of V.

(b) Prove that if B is a translate of some subspace of V and $\{v_1, \dots, v_m\} \subseteq B$, then $A \subseteq B$.

(c) Prove that A is a translate of some subspace of V of dimension less than m.

13 Suppose U is a subspace of V such that V/U is finite-dimensional. Prove that V is isomorphic to $U \times (V/U)$.

14 Suppose U and W are subspaces of V and $V = U \oplus W$. Suppose $w_1, \dots, w_m$ is a basis of W. Prove that $w_1 + U, \dots, w_m + U$ is a basis of V/U.

15 Suppose U is a subspace of V and $v_1 + U, \dots, v_m + U$ is a basis of V/U and $u_1, \dots, u_n$ is a basis of U. Prove that $v_1, \dots, v_m, u_1, \dots, u_n$ is a basis of V.

16 Suppose $\varphi \in \mathcal{L}(V, \mathbf{F})$ and $\varphi \neq 0$. Prove that $\dim V/(\operatorname{null} \varphi) = 1$.

17 Suppose U is a subspace of V such that $\dim V/U = 1$. Prove that there exists $\varphi \in \mathcal{L}(V, \mathbf{F})$ such that $\operatorname{null} \varphi = U$.

18 Suppose that U is a subspace of V such that V/U is finite-dimensional.

(a) Show that if W is a finite-dimensional subspace of V and $V = U + W$, then $\dim W \geq \dim V/U$.

(b) Prove that there exists a finite-dimensional subspace W of V such that $\dim W = \dim V/U$ and $V = U \oplus W$.

19 Suppose $T \in \mathcal{L}(V, W)$ and U is a subspace of V. Let π denote the quotient map from V onto V/U. Prove that there exists $S \in \mathcal{L}(V/U, W)$ such that $T = S \circ \pi$ if and only if $U \subseteq \operatorname{null} T$.