Chapter 5

Eigenvalues and Eigenvectors

Linear maps from one vector space to another vector space were the objects of study in Chapter 3. Now we begin our investigation of operators, which are linear maps from a vector space to itself. Their study constitutes the most important part of linear algebra.

To learn about an operator, we might try restricting it to a smaller subspace. Asking for that restriction to be an operator will lead us to the notion of invariant subspaces. Each one-dimensional invariant subspace arises from a vector that the operator maps into a scalar multiple of the vector. This path will lead us to eigenvectors and eigenvalues.

We will then prove one of the most important results in linear algebra: every operator on a finite-dimensional nonzero complex vector space has an eigenvalue. This result will allow us to show that for each operator on a finite-dimensional complex vector space, there is a basis of the vector space with respect to which the matrix of the operator has at least almost half its entries equal to 0.

standing assumptions for this chapter

- $\mathbf{F}$ denotes $\mathbf{R}$ or $\mathbf{C}$.
- V denotes a vector space over $\mathbf{F}$.

Statue of Leonardo of Pisa (1170–1250, approximate dates), also known as Fibonacci. Exercise 21 in Section 5D shows how linear algebra can be used to find the explicit formula for the Fibonacci sequence shown on the front cover.

5A *Invariant Subspaces*

Eigenvalues

5.1 definition: *operator*

A linear map from a vector space to itself is called an *operator*.

Suppose $T \in \mathcal{L}(V)$. If $m \geq 2$ and

$$V = V_1 \oplus \cdots \oplus V_m,$$

Recall that we defined the notation $\mathcal{L}(V)$ to mean $\mathcal{L}(V, V)$.

where each V_k is a nonzero subspace of V, then to understand the behavior of T we only need to understand the behavior of each $T|_{V_k}$; here $T|_{V_k}$ denotes the restriction of T to the smaller domain V_k. Dealing with $T|_{V_k}$ should be easier than dealing with T because V_k is a smaller vector space than V.

However, if we intend to apply tools useful in the study of operators (such as taking powers), then we have a problem: $T|_{V_k}$ may not map V_k into itself; in other words, $T|_{V_k}$ may not be an operator on V_k. Thus we are led to consider only decompositions of V of the form above in which T maps each V_k into itself. Hence we now give a name to subspaces of V that get mapped into themselves by T.

5.2 definition: *invariant subspace*

Suppose $T \in \mathcal{L}(V)$. A subspace U of V is called *invariant* under T if $Tu \in U$ for every $u \in U$.

Thus U is invariant under T if $T|_U$ is an operator on U.

5.3 example: *subspace invariant under differentiation operator*

Suppose that $T \in \mathcal{L}\bigl(\mathcal{P}(\mathbf{R})\bigr)$ is defined by $Tp = p'$. Then $\mathcal{P}_4(\mathbf{R})$, which is a subspace of $\mathcal{P}(\mathbf{R})$, is invariant under T because if $p \in \mathcal{P}(\mathbf{R})$ has degree at most 4, then p' also has degree at most 4.

5.4 example: *four invariant subspaces, not necessarily all different*

If $T \in \mathcal{L}(V)$, then the following subspaces of V are all invariant under T.

- $\{0\}$ The subspace $\{0\}$ is invariant under T because if $u \in \{0\}$, then $u = 0$ and hence $Tu = 0 \in \{0\}$.
- V The subspace V is invariant under T because if $u \in V$, then $Tu \in V$.
- $\operatorname{null} T$ The subspace $\operatorname{null} T$ is invariant under T because if $u \in \operatorname{null} T$, then $Tu = 0$, and hence $Tu \in \operatorname{null} T$.
- $\operatorname{range} T$ The subspace $\operatorname{range} T$ is invariant under T because if $u \in \operatorname{range} T$, then $Tu \in \operatorname{range} T$.

Must an operator $T \in \mathcal{L}(V)$ have any invariant subspaces other than $\{0\}$ and V? Later we will see that this question has an affirmative answer if V is finite-dimensional and $\dim V > 1$ (for $\mathbf{F} = \mathbf{C}$) or $\dim V > 2$ (for $\mathbf{F} = \mathbf{R}$); see 5.19 and Exercise 29 in Section 5B.

The previous example noted that $\operatorname{null} T$ and $\operatorname{range} T$ are invariant under T. However, these subspaces do not necessarily provide easy answers to the question above about the existence of invariant subspaces other than $\{0\}$ and V, because $\operatorname{null} T$ may equal $\{0\}$ and $\operatorname{range} T$ may equal V (this happens when T is invertible).

We will return later to a deeper study of invariant subspaces. Now we turn to an investigation of the simplest possible nontrivial invariant subspaces—invariant subspaces of dimension one.

Take any $v \in V$ with $v \neq 0$ and let U equal the set of all scalar multiples of v:

$$U = \{\lambda v : \lambda \in \mathbf{F}\} = \operatorname{span}(v).$$

Then U is a one-dimensional subspace of V (and every one-dimensional subspace of V is of this form for an appropriate choice of v). If U is invariant under an operator $T \in \mathcal{L}(V)$, then $Tv \in U$, and hence there is a scalar $\lambda \in \mathbf{F}$ such that

$$Tv = \lambda v.$$

Conversely, if $Tv = \lambda v$ for some $\lambda \in \mathbf{F}$, then $\operatorname{span}(v)$ is a one-dimensional subspace of V invariant under T.

The equation $Tv = \lambda v$, which we have just seen is intimately connected with one-dimensional invariant subspaces, is important enough that the scalars λ and vectors v satisfying it are given special names.

5.5 definition: *eigenvalue*

Suppose $T \in \mathcal{L}(V)$. A number $\lambda \in \mathbf{F}$ is called an *eigenvalue* of T if there exists $v \in V$ such that $v \neq 0$ and $Tv = \lambda v$.

In the definition above, we require that $v \neq 0$ because every scalar $\lambda \in \mathbf{F}$ satisfies $T0 = \lambda 0$.

The word ***eigenvalue*** *is half-German, half-English. The German prefix* ***eigen*** *means "own" in the sense of characterizing an intrinsic property.*

The comments above show that V has a one-dimensional subspace invariant under T if and only if T has an eigenvalue.

5.6 example: *eigenvalue*

Define an operator $T \in \mathcal{L}(\mathbf{F}^3)$ by

$$T(x, y, z) = (7x + 3z, 3x + 6y + 9z, -6y)$$

for $(x, y, z) \in \mathbf{F}^3$. Then $T(3, 1, -1) = (18, 6, -6) = 6(3, 1, -1)$. Thus 6 is an eigenvalue of T.

The equivalences in the next result, along with many deep results in linear algebra, are valid only in the context of finite-dimensional vector spaces.

5.7 *equivalent conditions to be an eigenvalue*

Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbf{F}$. Then the following are equivalent.

(a) λ is an eigenvalue of T.

(b) $T - \lambda I$ is not injective.

(c) $T - \lambda I$ is not surjective.

(d) $T - \lambda I$ is not invertible.

Reminder: $I \in \mathcal{L}(V)$ is the identity operator. Thus $Iv = v$ for all $v \in V$.

Proof Conditions (a) and (b) are equivalent because the equation $Tv = \lambda v$ is equivalent to the equation $(T - \lambda I)v = 0$. Conditions (b), (c), and (d) are equivalent by 3.65. ■

5.8 definition: *eigenvector*

Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbf{F}$ is an eigenvalue of T. A vector $v \in V$ is called an *eigenvector* of T corresponding to λ if $v \neq 0$ and $Tv = \lambda v$.

In other words, a nonzero vector $v \in V$ is an eigenvector of an operator $T \in \mathcal{L}(V)$ if and only if Tv is a scalar multiple of v. Because $Tv = \lambda v$ if and only if $(T - \lambda I)v = 0$, a vector $v \in V$ with $v \neq 0$ is an eigenvector of T corresponding to λ if and only if $v \in \operatorname{null}(T - \lambda I)$.

5.9 example: *eigenvalues and eigenvectors*

Suppose $T \in \mathcal{L}(\mathbf{F}^2)$ is defined by $T(w, z) = (-z, w)$.

(a) First consider the case $\mathbf{F} = \mathbf{R}$. Then T is a counterclockwise rotation by 90° about the origin in $\mathbf{R}^2$. An operator has an eigenvalue if and only if there exists a nonzero vector in its domain that gets sent by the operator to a scalar multiple of itself. A 90° counterclockwise rotation of a nonzero vector in $\mathbf{R}^2$ cannot equal a scalar multiple of itself. Conclusion: if $\mathbf{F} = \mathbf{R}$, then T has no eigenvalues (and thus has no eigenvectors).

(b) Now consider the case $\mathbf{F} = \mathbf{C}$. To find eigenvalues of T, we must find the scalars λ such that $T(w, z) = \lambda(w, z)$ has some solution other than $w = z = 0$. The equation $T(w, z) = \lambda(w, z)$ is equivalent to the simultaneous equations

5.10 $$-z = \lambda w, \qquad w = \lambda z.$$

Substituting the value for w given by the second equation into the first equation gives

$$-z = \lambda^2 z.$$

Now z cannot equal 0 [otherwise 5.10 implies that $w = 0$; we are looking for solutions to 5.10 such that (w, z) is not the 0 vector], so the equation above leads to the equation

$$-1 = \lambda^2.$$

The solutions to this equation are $\lambda = i$ and $\lambda = -i$.

You can verify that i and $-i$ are eigenvalues of T. Indeed, the eigenvectors corresponding to the eigenvalue i are the vectors of the form $(w, -wi)$, with $w \in \mathbf{C}$ and $w \neq 0$. Furthermore, the eigenvectors corresponding to the eigenvalue $-i$ are the vectors of the form (w, wi), with $w \in \mathbf{C}$ and $w \neq 0$.

In the next proof, we again use the equivalence

$$Tv = \lambda v \iff (T - \lambda I)v = 0.$$

5.11 *linearly independent eigenvectors*

Suppose $T \in \mathcal{L}(V)$. Then every list of eigenvectors of T corresponding to distinct eigenvalues of T is linearly independent.

Proof Suppose the desired result is false. Then there exists a smallest positive integer m such that there exists a linearly dependent list $v_1, \dots, v_m$ of eigenvectors of T corresponding to distinct eigenvalues $\lambda_1, \dots, \lambda_m$ of T (note that $m \geq 2$ because an eigenvector is, by definition, nonzero). Thus there exist $a_1, \dots, a_m \in \mathbf{F}$, none of which are 0 (because of the minimality of m), such that

$$a_1 v_1 + \cdots + a_m v_m = 0.$$

Apply $T - \lambda_m I$ to both sides of the equation above, getting

$$a_1(\lambda_1 - \lambda_m)v_1 + \cdots + a_{m-1}(\lambda_{m-1} - \lambda_m)v_{m-1} = 0.$$

Because the eigenvalues $\lambda_1, \dots, \lambda_m$ are distinct, none of the coefficients above equal 0. Thus $v_1, \dots, v_{m-1}$ is a linearly dependent list of $m - 1$ eigenvectors of T corresponding to distinct eigenvalues, contradicting the minimality of m. This contradiction completes the proof. ■

The result above leads to a short proof of the result below, which puts an upper bound on the number of distinct eigenvalues that an operator can have.

5.12 *operator cannot have more eigenvalues than dimension of vector space*

Suppose V is finite-dimensional. Then each operator on V has at most $\dim V$ distinct eigenvalues.

Proof Let $T \in \mathcal{L}(V)$. Suppose $\lambda_1, \dots, \lambda_m$ are distinct eigenvalues of T. Let $v_1, \dots, v_m$ be corresponding eigenvectors. Then 5.11 implies that the list $v_1, \dots, v_m$ is linearly independent. Thus $m \leq \dim V$ (see 2.22), as desired. ■

Polynomials Applied to Operators

The main reason that a richer theory exists for operators (which map a vector space into itself) than for more general linear maps is that operators can be raised to powers. In this subsection we define that notion and the concept of applying a polynomial to an operator. This concept will be the key tool that we use in the next section when we prove that every operator on a nonzero finite-dimensional complex vector space has an eigenvalue.

If T is an operator, then TT makes sense (see 3.7) and is also an operator on the same vector space as T. We usually write T^2 instead of TT. More generally, we have the following definition of T^m.

5.13 notation: T^m

Suppose $T \in \mathcal{L}(V)$ and m is a positive integer.

- $T^m \in \mathcal{L}(V)$ is defined by $T^m = \underbrace{T \cdots T}_{m \text{ times}}$.
- T^0 is defined to be the identity operator I on V.
- If T is invertible with inverse T^{-1}, then $T^{-m} \in \mathcal{L}(V)$ is defined by

$$T^{-m} = \left(T^{-1}\right)^m.$$

You should verify that if T is an operator, then

$$T^m T^n = T^{m+n} \quad \text{and} \quad \left(T^m\right)^n = T^{mn},$$

where m and n are arbitrary integers if T is invertible and are nonnegative integers if T is not invertible.

Having defined powers of an operator, we can now define what it means to apply a polynomial to an operator.

5.14 notation: $p(T)$

Suppose $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(\mathbf{F})$ is a polynomial given by

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$$

for all $z \in \mathbf{F}$. Then $p(T)$ is the operator on V defined by

$$p(T) = a_0 I + a_1 T + a_2 T^2 + \cdots + a_m T^m.$$

This is a new use of the symbol p because we are applying p to operators, not just elements of $\mathbf{F}$. The idea here is that to evaluate $p(T)$, we simply replace z with T in the expression defining p. Note that the constant term a_0 in $p(z)$ becomes the operator $a_0 I$ (which is a reasonable choice because $a_0 = a_0 z^0$ and thus we should replace a_0 with $a_0 T^0$, which equals $a_0 I$).

5.15 example: *a polynomial applied to the differentiation operator*

Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ is the differentiation operator defined by $Dq = q'$ and p is the polynomial defined by $p(x) = 7 - 3x + 5x^2$. Then $p(D) = 7I - 3D + 5D^2$. Thus

$$\big(p(D)\big)q = 7q - 3q' + 5q''$$

for every $q \in \mathcal{P}(\mathbf{R})$.

If we fix an operator $T \in \mathcal{L}(V)$, then the function from $\mathcal{P}(\mathbf{F})$ to $\mathcal{L}(V)$ given by $p \mapsto p(T)$ is linear, as you should verify.

5.16 definition: *product of polynomials*

If $p, q \in \mathcal{P}(\mathbf{F})$, then $pq \in \mathcal{P}(\mathbf{F})$ is the polynomial defined by

$$(pq)(z) = p(z)q(z)$$

for all $z \in \mathbf{F}$.

The order does not matter in taking products of polynomials of a single operator, as shown by (b) in the next result.

5.17 *multiplicative properties*

Suppose $p, q \in \mathcal{P}(\mathbf{F})$ and $T \in \mathcal{L}(V)$. Then

(a) $(pq)(T) = p(T)q(T)$;

(b) $p(T)q(T) = q(T)p(T)$.

Informal proof: When a product of polynomials is expanded using the distributive property, it does not matter whether the symbol is z or T.

Proof

(a) Suppose $p(z) = \sum_{j=0}^{m} a_j z^j$ and $q(z) = \sum_{k=0}^{n} b_k z^k$ for all $z \in \mathbf{F}$. Then

$$(pq)(z) = \sum_{j=0}^{m} \sum_{k=0}^{n} a_j b_k z^{j+k}.$$

Thus

$$\begin{aligned}(pq)(T) &= \sum_{j=0}^{m} \sum_{k=0}^{n} a_j b_k T^{j+k} \\ &= \Big(\sum_{j=0}^{m} a_j T^j\Big)\Big(\sum_{k=0}^{n} b_k T^k\Big) \\ &= p(T)q(T).\end{aligned}$$

(b) Using (a) twice, we have $p(T)q(T) = (pq)(T) = (qp)(T) = q(T)p(T)$. ■

We observed earlier that if $T \in \mathcal{L}(V)$, then the subspaces $\operatorname{null} T$ and $\operatorname{range} T$ are invariant under T (see 5.4). Now we show that the null space and the range of every polynomial of T are also invariant under T.

5.18 *null space and range of $p(T)$ are invariant under T*

Suppose $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(\mathbf{F})$. Then $\operatorname{null} p(T)$ and $\operatorname{range} p(T)$ are invariant under T.

Proof Suppose $u \in \operatorname{null} p(T)$. Then $p(T)u = 0$. Thus

$$\big(p(T)\big)(Tu) = \big(p(T)\,T\big)(u) = \big(T\,p(T)\big)(u) = T\big(p(T)u\big) = T(0) = 0.$$

Hence $Tu \in \operatorname{null} p(T)$. Thus $\operatorname{null} p(T)$ is invariant under T, as desired.

Suppose $u \in \operatorname{range} p(T)$. Then there exists $v \in V$ such that $u = p(T)v$. Thus

$$Tu = T\big(p(T)v\big) = p(T)(Tv).$$

Hence $Tu \in \operatorname{range} p(T)$. Thus $\operatorname{range} p(T)$ is invariant under T, as desired. ■

Exercises 5A

1 Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V.

(a) Prove that if $U \subseteq \operatorname{null} T$, then U is invariant under T.

(b) Prove that if $\operatorname{range} T \subseteq U$, then U is invariant under T.

2 Suppose that $T \in \mathcal{L}(V)$ and $V_1, \ldots, V_m$ are subspaces of V invariant under T. Prove that $V_1 + \cdots + V_m$ is invariant under T.

3 Suppose $T \in \mathcal{L}(V)$. Prove that the intersection of every collection of subspaces of V invariant under T is invariant under T.

4 Prove or give a counterexample: If V is finite-dimensional and U is a subspace of V that is invariant under every operator on V, then $U = \{0\}$ or $U = V$.

5 Suppose $T \in \mathcal{L}(\mathbf{R}^2)$ is defined by $T(x, y) = (-3y, x)$. Find the eigenvalues of T.

6 Define $T \in \mathcal{L}(\mathbf{F}^2)$ by $T(w, z) = (z, w)$. Find all eigenvalues and eigenvectors of T.

7 Define $T \in \mathcal{L}(\mathbf{F}^3)$ by $T(z_1, z_2, z_3) = (2z_2, 0, 5z_3)$. Find all eigenvalues and eigenvectors of T.

8 Suppose $P \in \mathcal{L}(V)$ is such that $P^2 = P$. Prove that if λ is an eigenvalue of P, then $\lambda = 0$ or $\lambda = 1$.

9 Define $T\colon \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ by $Tp = p'$. Find all eigenvalues and eigenvectors of T.

10 Define $T \in \mathcal{L}\big(\mathcal{P}_4(\mathbf{R})\big)$ by $(Tp)(x) = xp'(x)$ for all $x \in \mathbf{R}$. Find all eigenvalues and eigenvectors of T.

11 Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $\alpha \in \mathbf{F}$. Prove that there exists $\delta > 0$ such that $T - \lambda I$ is invertible for all $\lambda \in \mathbf{F}$ such that $0 < |\alpha - \lambda| < \delta$.

12 Suppose $V = U \oplus W$, where U and W are nonzero subspaces of V. Define $P \in \mathcal{L}(V)$ by $P(u + w) = u$ for each $u \in U$ and each $w \in W$. Find all eigenvalues and eigenvectors of P.

13 Suppose $T \in \mathcal{L}(V)$. Suppose $S \in \mathcal{L}(V)$ is invertible.

(a) Prove that T and $S^{-1}TS$ have the same eigenvalues.

(b) What is the relationship between the eigenvectors of T and the eigenvectors of $S^{-1}TS$?

14 Give an example of an operator on $\mathbf{R}^4$ that has no (real) eigenvalues.

15 Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbf{F}$. Show that λ is an eigenvalue of T if and only if λ is an eigenvalue of the dual operator $T' \in \mathcal{L}(V')$.

16 Suppose $v_1, \dots, v_n$ is a basis of V and $T \in \mathcal{L}(V)$. Prove that if λ is an eigenvalue of T, then

$$|\lambda| \le n \max\{\big|\mathcal{M}(T)_{j,k}\big| : 1 \le j, k \le n\},$$

where $\mathcal{M}(T)_{j,k}$ denotes the entry in row j, column k of the matrix of T with respect to the basis $v_1, \dots, v_n$.

See Exercise 19 in Section 6A for a different bound on $|\lambda|$.

17 Suppose $\mathbf{F} = \mathbf{R}$, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbf{R}$. Prove that λ is an eigenvalue of T if and only if λ is an eigenvalue of the complexification $T_{\mathbf{C}}$.

See Exercise 33 in Section 3B for the definition of $T_{\mathbf{C}}$.

18 Suppose $\mathbf{F} = \mathbf{R}$, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbf{C}$. Prove that λ is an eigenvalue of the complexification $T_{\mathbf{C}}$ if and only if $\overline{\lambda}$ is an eigenvalue of $T_{\mathbf{C}}$.

19 Show that the forward shift operator $T \in \mathcal{L}\big(\mathbf{F}^\infty\big)$ defined by

$$T(z_1, z_2, \dots) = (0, z_1, z_2, \dots)$$

has no eigenvalues.

20 Define the backward shift operator $S \in \mathcal{L}\big(\mathbf{F}^\infty\big)$ by

$$S(z_1, z_2, z_3, \dots) = (z_2, z_3, \dots).$$

(a) Show that every element of $\mathbf{F}$ is an eigenvalue of S.

(b) Find all eigenvectors of S.

21 Suppose $T \in \mathcal{L}(V)$ is invertible.

(a) Suppose $\lambda \in \mathbf{F}$ with $\lambda \neq 0$. Prove that λ is an eigenvalue of T if and only if $\frac{1}{\lambda}$ is an eigenvalue of T^{-1}.

(b) Prove that T and T^{-1} have the same eigenvectors.

22 Suppose $T \in \mathcal{L}(V)$ and there exist nonzero vectors u and w in V such that

$$Tu = 3w \quad \text{and} \quad Tw = 3u.$$

Prove that 3 or -3 is an eigenvalue of T.

23 Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that ST and TS have the same eigenvalues.

24 Suppose A is an n-by-n matrix with entries in $\mathbf{F}$. Define $T \in \mathcal{L}(\mathbf{F}^n)$ by $Tx = Ax$, where elements of $\mathbf{F}^n$ are thought of as n-by-1 column vectors.

(a) Suppose the sum of the entries in each row of A equals 1. Prove that 1 is an eigenvalue of T.

(b) Suppose the sum of the entries in each column of A equals 1. Prove that 1 is an eigenvalue of T.

25 Suppose $T \in \mathcal{L}(V)$ and u, w are eigenvectors of T such that $u + w$ is also an eigenvector of T. Prove that u and w are eigenvectors of T corresponding to the same eigenvalue.

26 Suppose $T \in \mathcal{L}(V)$ is such that every nonzero vector in V is an eigenvector of T. Prove that T is a scalar multiple of the identity operator.

27 Suppose that V is finite-dimensional and $k \in \{1, \dots, \dim V - 1\}$. Suppose $T \in \mathcal{L}(V)$ is such that every subspace of V of dimension k is invariant under T. Prove that T is a scalar multiple of the identity operator.

28 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T has at most $1 + \dim \operatorname{range} T$ distinct eigenvalues.

29 Suppose $T \in \mathcal{L}(\mathbf{R}^3)$ and -4, 5, and $\sqrt{7}$ are eigenvalues of T. Prove that there exists $x \in \mathbf{R}^3$ such that $Tx - 9x = (-4, 5, \sqrt{7})$.

30 Suppose $T \in \mathcal{L}(V)$ and $(T - 2I)(T - 3I)(T - 4I) = 0$. Suppose λ is an eigenvalue of T. Prove that $\lambda = 2$ or $\lambda = 3$ or $\lambda = 4$.

31 Give an example of $T \in \mathcal{L}(\mathbf{R}^2)$ such that $T^4 = -I$.

32 Suppose $T \in \mathcal{L}(V)$ has no eigenvalues and $T^4 = I$. Prove that $T^2 = -I$.

33 Suppose $T \in \mathcal{L}(V)$ and m is a positive integer.

(a) Prove that T is injective if and only if T^m is injective.

(b) Prove that T is surjective if and only if T^m is surjective.

34 Suppose V is finite-dimensional and $v_1, \dots, v_m \in V$. Prove that the list $v_1, \dots, v_m$ is linearly independent if and only if there exists $T \in \mathcal{L}(V)$ such that $v_1, \dots, v_m$ are eigenvectors of T corresponding to distinct eigenvalues.

35 Suppose that $\lambda_1, \dots, \lambda_n$ is a list of distinct real numbers. Prove that the list $e^{\lambda_1 x}, \dots, e^{\lambda_n x}$ is linearly independent in the vector space of real-valued functions on $\mathbf{R}$.

Hint: Let $V = \operatorname{span}(e^{\lambda_1 x}, \dots, e^{\lambda_n x})$, and define an operator $D \in \mathcal{L}(V)$ by $Df = f'$. Find eigenvalues and eigenvectors of D.

36 Suppose that $\lambda_1, \dots, \lambda_n$ is a list of distinct positive numbers. Prove that the list $\cos(\lambda_1 x), \dots, \cos(\lambda_n x)$ is linearly independent in the vector space of real-valued functions on $\mathbf{R}$.

37 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Define $\mathcal{A} \in \mathcal{L}\big(\mathcal{L}(V)\big)$ by

$$\mathcal{A}(S) = TS$$

for each $S \in \mathcal{L}(V)$. Prove that the set of eigenvalues of T equals the set of eigenvalues of $\mathcal{A}$.

38 Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and U is a subspace of V invariant under T. The *quotient operator* $T/U \in \mathcal{L}(V/U)$ is defined by

$$(T/U)(v + U) = Tv + U$$

for each $v \in V$.

(a) Show that the definition of T/U makes sense (which requires using the condition that U is invariant under T) and show that T/U is an operator on V/U.
(b) Show that each eigenvalue of T/U is an eigenvalue of T.

39 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T has an eigenvalue if and only if there exists a subspace of V of dimension $\dim V - 1$ that is invariant under T.

40 Suppose $S, T \in \mathcal{L}(V)$ and S is invertible. Suppose $p \in \mathcal{P}(\mathbf{F})$ is a polynomial. Prove that

$$p\big(STS^{-1}\big) = Sp(T)S^{-1}.$$

41 Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V invariant under T. Prove that U is invariant under $p(T)$ for every polynomial $p \in \mathcal{P}(\mathbf{F})$.

42 Define $T \in \mathcal{L}\big(\mathbf{F}^n\big)$ by $T(x_1, x_2, x_3, \dots, x_n) = (x_1, 2x_2, 3x_3, \dots, nx_n)$.

(a) Find all eigenvalues and eigenvectors of T.
(b) Find all subspaces of $\mathbf{F}^n$ that are invariant under T.

43 Suppose that V is finite-dimensional, $\dim V > 1$, and $T \in \mathcal{L}(V)$. Prove that $\{p(T) : p \in \mathcal{P}(\mathbf{F})\} \neq \mathcal{L}(V)$.