2C Dimension

Although we have been discussing finite-dimensional vector spaces, we have not yet defined the dimension of such an object. How should dimension be defined? A reasonable definition should force the dimension of $\mathbf{F}^n$ to equal n. Notice that the standard basis

$$(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$$

of $\mathbf{F}^n$ has length n. Thus we are tempted to define the dimension as the length of a basis. However, a finite-dimensional vector space in general has many different bases, and our attempted definition makes sense only if all bases in a given vector space have the same length. Fortunately that turns out to be the case, as we now show.

2.34 *basis length does not depend on basis*

Any two bases of a finite-dimensional vector space have the same length.

Proof Suppose V is finite-dimensional. Let B_1 and B_2 be two bases of V. Then B_1 is linearly independent in V and B_2 spans V, so the length of B_1 is at most the length of B_2 (by 2.22). Interchanging the roles of B_1 and B_2, we also see that the length of B_2 is at most the length of B_1. Thus the length of B_1 equals the length of B_2, as desired. ■

Now that we know that any two bases of a finite-dimensional vector space have the same length, we can formally define the dimension of such spaces.

2.35 definition: *dimension*, $\dim V$

- The *dimension* of a finite-dimensional vector space is the length of any basis of the vector space.
- The dimension of a finite-dimensional vector space V is denoted by $\dim V$.

2.36 example: *dimensions*

- $\dim \mathbf{F}^n = n$ because the standard basis of $\mathbf{F}^n$ has length n.
- $\dim \mathcal{P}_m(\mathbf{F}) = m + 1$ because the standard basis $1, z, \ldots, z^m$ of $\mathcal{P}_m(\mathbf{F})$ has length $m + 1$.
- If $U = \{(x, x, y) \in \mathbf{F}^3 : x, y \in \mathbf{F}\}$, then $\dim U = 2$ because $(1, 1, 0), (0, 0, 1)$ is a basis of U.
- If $U = \{(x, y, z) \in \mathbf{F}^3 : x + y + z = 0\}$, then $\dim U = 2$ because the list $(1, -1, 0), (1, 0, -1)$ is a basis of U.

Every subspace of a finite-dimensional vector space is finite-dimensional (by 2.25) and so has a dimension. The next result gives the expected inequality about the dimension of a subspace.

2.37 *dimension of a subspace*

If V is finite-dimensional and U is a subspace of V, then $\dim U \leq \dim V$.

Proof Suppose V is finite-dimensional and U is a subspace of V. Think of a basis of U as a linearly independent list in V, and think of a basis of V as a spanning list in V. Now use 2.22 to conclude that $\dim U \leq \dim V$. ■

To check that a list of vectors in V is a basis of V, we must, according to the definition, show that the list in question satisfies two properties: it must be linearly independent and it must span V. The next two results show that if the list in question has the right length, then we only need to check that it satisfies one of the two required properties. First we prove that every linearly independent list of the right length is a basis.

The real vector space $\mathbf{R}^2$ has dimension two; the complex vector space $\mathbf{C}$ has dimension one. As sets, $\mathbf{R}^2$ can be identified with $\mathbf{C}$ (and addition is the same on both spaces, as is scalar multiplication by real numbers). Thus when we talk about the dimension of a vector space, the role played by the choice of $\mathbf{F}$ cannot be neglected.

2.38 *linearly independent list of the right length is a basis*

Suppose V is finite-dimensional. Then every linearly independent list of vectors in V of length $\dim V$ is a basis of V.

Proof Suppose $\dim V = n$ and $v_1, \ldots, v_n$ is linearly independent in V. The list $v_1, \ldots, v_n$ can be extended to a basis of V (by 2.32). However, every basis of V has length n, so in this case the extension is the trivial one, meaning that no elements are adjoined to $v_1, \ldots, v_n$. Thus $v_1, \ldots, v_n$ is a basis of V, as desired. ■

The next result is a useful consequence of the previous result.

2.39 *subspace of full dimension equals the whole space*

Suppose that V is finite-dimensional and U is a subspace of V such that $\dim U = \dim V$. Then $U = V$.

Proof Let $u_1, \ldots, u_n$ be a basis of U. Thus $n = \dim U$, and by hypothesis we also have $n = \dim V$. Thus $u_1, \ldots, u_n$ is a linearly independent list of vectors in V (because it is a basis of U) of length $\dim V$. From 2.38, we see that $u_1, \ldots, u_n$ is a basis of V. In particular every vector in V is a linear combination of $u_1, \ldots, u_n$. Thus $U = V$. ■

2.40 example: *a basis of* $\mathbf{F}^2$

Consider the list $(5,7),(4,3)$ of vectors in $\mathbf{F}^2$. This list of length two is linearly independent in $\mathbf{F}^2$ (because neither vector is a scalar multiple of the other). Note that $\mathbf{F}^2$ has dimension two. Thus 2.38 implies that the linearly independent list $(5,7),(4,3)$ of length two is a basis of $\mathbf{F}^2$ (we do not need to bother checking that it spans $\mathbf{F}^2$).

2.41 example: *a basis of a subspace of* $\mathcal{P}_3(\mathbf{R})$

Let U be the subspace of $\mathcal{P}_3(\mathbf{R})$ defined by

$$U = \{p \in \mathcal{P}_3(\mathbf{R}) : p'(5) = 0\}.$$

To find a basis of U, first note that each of the polynomials 1, $(x-5)^2$, and $(x-5)^3$ is in U.

Suppose $a, b, c \in \mathbf{R}$ and

$$a + b(x-5)^2 + c(x-5)^3 = 0$$

for every $x \in \mathbf{R}$. Without explicitly expanding the left side of the equation above, we can see that the left side has a cx^3 term. Because the right side has no x^3 term, this implies that $c = 0$. Because $c = 0$, we see that the left side has a bx^2 term, which implies that $b = 0$. Because $b = c = 0$, we can also conclude that $a = 0$. Thus the equation above implies that $a = b = c = 0$. Hence the list $1, (x-5)^2, (x-5)^3$ is linearly independent in U. Thus $3 \leq \dim U$. Hence

$$3 \leq \dim U \leq \dim \mathcal{P}_3(\mathbf{R}) = 4,$$

where we have used 2.37.

The polynomial x is not in U because its derivative is the constant function 1. Thus $U \neq \mathcal{P}_3(\mathbf{R})$. Hence $\dim U \neq 4$ (by 2.39). The inequality above now implies that $\dim U = 3$. Thus the linearly independent list $1, (x-5)^2, (x-5)^3$ in U has length $\dim U$ and hence is a basis of U (by 2.38).

Now we prove that a spanning list of the right length is a basis.

2.42 *spanning list of the right length is a basis*

Suppose V is finite-dimensional. Then every spanning list of vectors in V of length $\dim V$ is a basis of V.

Proof Suppose $\dim V = n$ and $v_1, \ldots, v_n$ spans V. The list $v_1, \ldots, v_n$ can be reduced to a basis of V (by 2.30). However, every basis of V has length n, so in this case the reduction is the trivial one, meaning that no elements are deleted from $v_1, \ldots, v_n$. Thus $v_1, \ldots, v_n$ is a basis of V, as desired. ■

The next result gives a formula for the dimension of the sum of two subspaces of a finite-dimensional vector space. This formula is analogous to a familiar counting formula: the number of elements in the union of two finite sets equals the number of elements in the first set, plus the number of elements in the second set, minus the number of elements in the intersection of the two sets.

2.43 *dimension of a sum*

If V_1 and V_2 are subspaces of a finite-dimensional vector space, then

$$\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2).$$

Proof Let $v_1, \dots, v_m$ be a basis of $V_1 \cap V_2$; thus $\dim(V_1 \cap V_2) = m$. Because $v_1, \dots, v_m$ is a basis of $V_1 \cap V_2$, it is linearly independent in V_1. Hence this list can be extended to a basis $v_1, \dots, v_m, u_1, \dots, u_j$ of V_1 (by 2.32). Thus $\dim V_1 = m + j$. Also extend $v_1, \dots, v_m$ to a basis $v_1, \dots, v_m, w_1, \dots, w_k$ of V_2; thus $\dim V_2 = m + k$.

We will show that

2.44 $$v_1, \dots, v_m, u_1, \dots, u_j, w_1, \dots, w_k$$

is a basis of $V_1 + V_2$. This will complete the proof, because then we will have

$$\begin{aligned}\dim(V_1 + V_2) &= m + j + k \\ &= (m + j) + (m + k) - m \\ &= \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2).\end{aligned}$$

The list 2.44 is contained in $V_1 \cup V_2$ and thus is contained in $V_1 + V_2$. The span of this list contains V_1 and contains V_2 and hence is equal to $V_1 + V_2$. Thus to show that 2.44 is a basis of $V_1 + V_2$ we only need to show that it is linearly independent.

To prove that 2.44 is linearly independent, suppose

$$a_1 v_1 + \dots + a_m v_m + b_1 u_1 + \dots + b_j u_j + c_1 w_1 + \dots + c_k w_k = 0,$$

where all the a's, b's, and c's are scalars. We need to prove that all the a's, b's, and c's equal 0. The equation above can be rewritten as

2.45 $$c_1 w_1 + \dots + c_k w_k = -a_1 v_1 - \dots - a_m v_m - b_1 u_1 - \dots - b_j u_j,$$

which shows that $c_1 w_1 + \dots + c_k w_k \in V_1$. All the w's are in V_2, so this implies that $c_1 w_1 + \dots + c_k w_k \in V_1 \cap V_2$. Because $v_1, \dots, v_m$ is a basis of $V_1 \cap V_2$, we have

$$c_1 w_1 + \dots + c_k w_k = d_1 v_1 + \dots + d_m v_m$$

for some scalars $d_1, \dots, d_m$. But $v_1, \dots, v_m, w_1, \dots, w_k$ is linearly independent, so the last equation implies that all the c's (and d's) equal 0. Thus 2.45 becomes the equation

$$a_1 v_1 + \dots + a_m v_m + b_1 u_1 + \dots + b_j u_j = 0.$$

Because the list $v_1, \dots, v_m, u_1, \dots, u_j$ is linearly independent, this equation implies that all the a's and b's are 0, completing the proof. ■

For S a finite set, let $\#S$ denote the number of elements of S. The table below compares finite sets with finite-dimensional vector spaces, showing the analogy between $\#S$ (for sets) and $\dim V$ (for vector spaces), as well as the analogy between unions of subsets (in the context of sets) and sums of subspaces (in the context of vector spaces).

sets	**vector spaces**
S is a finite set	V is a finite-dimensional vector space
$\#S$	$\dim V$
for subsets S_1, S_2 of S, the union $S_1 \cup S_2$ is the smallest subset of S containing S_1 and S_2	for subspaces V_1, V_2 of V, the sum $V_1 + V_2$ is the smallest subspace of V containing V_1 and V_2
$\#(S_1 \cup S_2)$ $= \#S_1 + \#S_2 - \#(S_1 \cap S_2)$	$\dim(V_1 + V_2)$ $= \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$
$\#(S_1 \cup S_2) = \#S_1 + \#S_2$ $\iff S_1 \cap S_2 = \emptyset$	$\dim(V_1 + V_2) = \dim V_1 + \dim V_2$ $\iff V_1 \cap V_2 = \{0\}$
$S_1 \cup \cdots \cup S_m$ is a disjoint union $\iff$ $\#(S_1 \cup \cdots \cup S_m) = \#S_1 + \cdots + \#S_m$	$V_1 + \cdots + V_m$ is a direct sum $\iff$ $\dim(V_1 + \cdots + V_m)$ $= \dim V_1 + \cdots + \dim V_m$

The last row above focuses on the analogy between disjoint unions (for sets) and direct sums (for vector spaces). The proof of the result in the last box above will be given in 3.94.

You should be able to find results about sets that correspond, via analogy, to the results about vector spaces in Exercises 12 through 18.

Exercises 2C

1 Show that the subspaces of $\mathbf{R}^2$ are precisely $\{0\}$, all lines in $\mathbf{R}^2$ containing the origin, and $\mathbf{R}^2$.

2 Show that the subspaces of $\mathbf{R}^3$ are precisely $\{0\}$, all lines in $\mathbf{R}^3$ containing the origin, all planes in $\mathbf{R}^3$ containing the origin, and $\mathbf{R}^3$.

3 (a) Let $U = \{p \in \mathcal{P}_4(\mathbf{F}) : p(6) = 0\}$. Find a basis of U.
(b) Extend the basis in (a) to a basis of $\mathcal{P}_4(\mathbf{F})$.
(c) Find a subspace W of $\mathcal{P}_4(\mathbf{F})$ such that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.

4 (a) Let $U = \{p \in \mathcal{P}_4(\mathbf{R}) : p''(6) = 0\}$. Find a basis of U.
(b) Extend the basis in (a) to a basis of $\mathcal{P}_4(\mathbf{R})$.
(c) Find a subspace W of $\mathcal{P}_4(\mathbf{R})$ such that $\mathcal{P}_4(\mathbf{R}) = U \oplus W$.

5 (a) Let $U = \{p \in \mathcal{P}_4(\mathbf{F}) : p(2) = p(5)\}$. Find a basis of U.
(b) Extend the basis in (a) to a basis of $\mathcal{P}_4(\mathbf{F})$.
(c) Find a subspace W of $\mathcal{P}_4(\mathbf{F})$ such that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.

6 (a) Let $U = \{p \in \mathcal{P}_4(\mathbf{F}) : p(2) = p(5) = p(6)\}$. Find a basis of U.
(b) Extend the basis in (a) to a basis of $\mathcal{P}_4(\mathbf{F})$.
(c) Find a subspace W of $\mathcal{P}_4(\mathbf{F})$ such that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.

7 (a) Let $U = \left\{p \in \mathcal{P}_4(\mathbf{R}) : \int_{-1}^{1} p = 0\right\}$. Find a basis of U.
(b) Extend the basis in (a) to a basis of $\mathcal{P}_4(\mathbf{R})$.
(c) Find a subspace W of $\mathcal{P}_4(\mathbf{R})$ such that $\mathcal{P}_4(\mathbf{R}) = U \oplus W$.

8 Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Prove that

$$\dim \operatorname{span}(v_1 + w, \ldots, v_m + w) \geq m - 1.$$

9 Suppose m is a positive integer and $p_0, p_1, \ldots, p_m \in \mathcal{P}(\mathbf{F})$ are such that each p_k has degree k. Prove that $p_0, p_1, \ldots, p_m$ is a basis of $\mathcal{P}_m(\mathbf{F})$.

10 Suppose m is a positive integer. For $0 \leq k \leq m$, let

$$p_k(x) = x^k (1 - x)^{m-k}.$$

Show that $p_0, \ldots, p_m$ is a basis of $\mathcal{P}_m(\mathbf{F})$.

The basis in this exercise leads to what are called ***Bernstein polynomials****. You can do a web search to learn how Bernstein polynomials are used to approximate continuous functions on* $[0, 1]$.

11 Suppose U and W are both four-dimensional subspaces of $\mathbf{C}^6$. Prove that there exist two vectors in $U \cap W$ such that neither of these vectors is a scalar multiple of the other.

12 Suppose that U and W are subspaces of $\mathbf{R}^8$ such that $\dim U = 3$, $\dim W = 5$, and $U + W = \mathbf{R}^8$. Prove that $\mathbf{R}^8 = U \oplus W$.

13 Suppose U and W are both five-dimensional subspaces of $\mathbf{R}^9$. Prove that $U \cap W \neq \{0\}$.

14 Suppose V is a ten-dimensional vector space and V_1, V_2, V_3 are subspaces of V with $\dim V_1 = \dim V_2 = \dim V_3 = 7$. Prove that $V_1 \cap V_2 \cap V_3 \neq \{0\}$.

15 Suppose V is finite-dimensional and V_1, V_2, V_3 are subspaces of V with $\dim V_1 + \dim V_2 + \dim V_3 > 2 \dim V$. Prove that $V_1 \cap V_2 \cap V_3 \neq \{0\}$.

16 Suppose V is finite-dimensional and U is a subspace of V with $U \neq V$. Let $n = \dim V$ and $m = \dim U$. Prove that there exist $n - m$ subspaces of V, each of dimension $n - 1$, whose intersection equals U.

17 Suppose that $V_1, \ldots, V_m$ are finite-dimensional subspaces of V. Prove that $V_1 + \cdots + V_m$ is finite-dimensional and

$$\dim(V_1 + \cdots + V_m) \leq \dim V_1 + \cdots + \dim V_m.$$

The inequality above is an equality if and only if $V_1 + \cdots + V_m$ *is a direct sum, as will be shown in* *3.94*.

18 Suppose V is finite-dimensional, with $\dim V = n \geq 1$. Prove that there exist one-dimensional subspaces $V_1, \ldots, V_n$ of V such that

$$V = V_1 \oplus \cdots \oplus V_n.$$

19 Explain why you might guess, motivated by analogy with the formula for the number of elements in the union of three finite sets, that if V_1, V_2, V_3 are subspaces of a finite-dimensional vector space, then

$$\begin{aligned}\dim(V_1 + V_2 + V_3) \\ &= \dim V_1 + \dim V_2 + \dim V_3 \\ &\quad - \dim(V_1 \cap V_2) - \dim(V_1 \cap V_3) - \dim(V_2 \cap V_3) \\ &\quad + \dim(V_1 \cap V_2 \cap V_3).\end{aligned}$$

Then either prove the formula above or give a counterexample.

20 Prove that if V_1, V_2, and V_3 are subspaces of a finite-dimensional vector space, then

$$\begin{aligned}\dim(V_1 + V_2 + V_3) \\ &= \dim V_1 + \dim V_2 + \dim V_3 \\ &\quad - \frac{\dim(V_1 \cap V_2) + \dim(V_1 \cap V_3) + \dim(V_2 \cap V_3)}{3} \\ &\quad - \frac{\dim\bigl((V_1 + V_2) \cap V_3\bigr) + \dim\bigl((V_1 + V_3) \cap V_2\bigr) + \dim\bigl((V_2 + V_3) \cap V_1\bigr)}{3}.\end{aligned}$$

The formula above may seem strange because the right side does not look like an integer.

I at once gave up my former occupations, set down natural history and all its progeny as a deformed and abortive creation, and entertained the greatest disdain for a would-be science which could never even step within the threshold of real knowledge. In this mood I betook myself to the mathematics and the branches of study appertaining to that science as being built upon secure foundations, and so worthy of my consideration.

—*Frankenstein*, Mary Wollstonecraft Shelley