9C Determinants

Defining the Determinant

The next definition will lead us to a clean, beautiful, basis-free definition of the determinant of an operator.

9.40 definition: α_T

Suppose that m is a positive integer and $T \in \mathcal{L}(V)$. For $\alpha \in V_{\text{alt}}^{(m)}$, define $\alpha_T \in V_{\text{alt}}^{(m)}$ by

$$\alpha_T(v_1, \ldots, v_m) = \alpha(Tv_1, \ldots, Tv_m)$$

for each list $v_1, \ldots, v_m$ of vectors in V.

Suppose $T \in \mathcal{L}(V)$. If $\alpha \in V_{\text{alt}}^{(m)}$ and $v_1, \ldots, v_m$ is a list of vectors in V with $v_j = v_k$ for some $j \neq k$, then $Tv_j = Tv_k$, which implies that $\alpha_T(v_1, \ldots, v_m) = \alpha(Tv_1, \ldots, Tv_m) = 0$. Thus the function $\alpha \mapsto \alpha_T$ is a linear map of $V_{\text{alt}}^{(m)}$ to itself.

We know that $\dim V_{\text{alt}}^{(\dim V)} = 1$ (see 9.37). Every linear map from a one-dimensional vector space to itself is multiplication by some unique scalar. For the linear map $\alpha \mapsto \alpha_T$, we now define $\det T$ to be that scalar.

9.41 definition: *determinant of an operator,* $\det T$

Suppose $T \in \mathcal{L}(V)$. The *determinant* of T, denoted by $\det T$, is defined to be the unique number in $\mathbf{F}$ such that

$$\alpha_T = (\det T)\,\alpha$$

for all $\alpha \in V_{\text{alt}}^{(\dim V)}$.

9.42 example: *determinants of operators*

Let $n = \dim V$.

- If I is the identity operator on V, then $\alpha_I = \alpha$ for all $\alpha \in V_{\text{alt}}^{(n)}$. Thus $\det I = 1$.
- More generally, if $\lambda \in \mathbf{F}$, then $\alpha_{\lambda I} = \lambda^n \alpha$ for all $\alpha \in V_{\text{alt}}^{(n)}$. Thus $\det(\lambda I) = \lambda^n$.
- Still more generally, if $T \in \mathcal{L}(V)$ and $\lambda \in \mathbf{F}$, then $\alpha_{\lambda T} = \lambda^n \alpha_T = \lambda^n (\det T)\alpha$ for all $\alpha \in V_{\text{alt}}^{(n)}$. Thus $\det(\lambda T) = \lambda^n \det T$.
- Suppose $T \in \mathcal{L}(V)$ and there is a basis $e_1, \ldots, e_n$ of V consisting of eigenvectors of T, with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$. If $\alpha \in V_{\text{alt}}^{(n)}$, then

$$\alpha_T(e_1, \ldots, e_n) = \alpha(\lambda_1 e_1, \ldots, \lambda_n e_n) = (\lambda_1 \cdots \lambda_n)\,\alpha(e_1, \ldots, e_n).$$

  If $\alpha \neq 0$, then 9.39 implies $\alpha(e_1, \ldots, e_n) \neq 0$. Thus the equation above implies

$$\det T = \lambda_1 \cdots \lambda_n.$$

Our next task is to define and give a formula for the determinant of a square matrix. To do this, we associate with each square matrix an operator and then define the determinant of the matrix to be the determinant of the associated operator.

9.43 definition: *determinant of a matrix*, $\det A$

Suppose that n is a positive integer and A is an n-by-n square matrix with entries in $\mathbf{F}$. Let $T \in \mathcal{L}(\mathbf{F}^n)$ be the operator whose matrix with respect to the standard basis of $\mathbf{F}^n$ equals A. The *determinant* of A, denoted by $\det A$, is defined by $\det A = \det T$.

9.44 example: *determinants of matrices*

- If I is the n-by-n identity matrix, then the corresponding operator on $\mathbf{F}^n$ is the identity operator I on $\mathbf{F}^n$. Thus the first bullet point of 9.42 implies that the determinant of the identity matrix is 1.
- Suppose A is a diagonal matrix with $\lambda_1, \dots, \lambda_n$ on the diagonal. Then the corresponding operator on $\mathbf{F}^n$ has the standard basis of $\mathbf{F}^n$ as eigenvectors, with eigenvalues $\lambda_1, \dots, \lambda_n$. Thus the last bullet point of 9.42 implies that $\det A = \lambda_1 \cdots \lambda_n$.

For the next result, think of each list $v_1, \dots, v_n$ of n vectors in $\mathbf{F}^n$ as a list of n-by-1 column vectors. The notation $\begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}$ then denotes the n-by-n square matrix whose k^{th} column is v_k for each $k = 1, \dots, n$.

9.45 *determinant is an alternating multilinear form*

Suppose that n is a positive integer. The map that takes a list $v_1, \dots, v_n$ of vectors in $\mathbf{F}^n$ to $\det \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}$ is an alternating n-linear form on $\mathbf{F}^n$.

Proof Let $e_1, \dots, e_n$ be the standard basis of $\mathbf{F}^n$ and suppose $v_1, \dots, v_n$ is a list of vectors in $\mathbf{F}^n$. Let $T \in \mathcal{L}(\mathbf{F}^n)$ be the operator such that $Te_k = v_k$ for $k = 1, \dots, n$. Thus T is the operator whose matrix with respect to $e_1, \dots, e_n$ is $\begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}$. Hence $\det \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix} = \det T$, by definition of the determinant of a matrix.

Let α be an alternating n-linear form on $\mathbf{F}^n$ such that $\alpha(e_1, \dots, e_n) = 1$. Then

$$\begin{aligned} \det \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix} &= \det T \\ &= (\det T)\, \alpha(e_1, \dots, e_n) \\ &= \alpha(Te_1, \dots, Te_n) \\ &= \alpha(v_1, \dots, v_n), \end{aligned}$$

where the third line follows from the definition of the determinant of an operator. The equation above shows that the map that takes a list of vectors $v_1, \dots, v_n$ in $\mathbf{F}^n$ to $\det \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}$ is the alternating n-linear form α on $\mathbf{F}^n$. ∎

The previous result has several important consequences. For example, it immediately implies that a matrix with two identical columns has determinant 0. We will come back to other consequences later, but for now we want to give a formula for the determinant of a square matrix. Recall that if A is a matrix, then $A_{j,k}$ denotes the entry in row j, column k of A.

9.46 *formula for determinant of a matrix*

Suppose that n is a positive integer and A is an n-by-n square matrix. Then

$$\det A = \sum_{(j_1,\ldots,j_n)\in \operatorname{perm} n} \bigl(\operatorname{sign}(j_1,\ldots,j_n)\bigr) A_{j_1,1}\cdots A_{j_n,n}.$$

Proof Apply 9.36 with $V = \mathbf{F}^n$ and $e_1,\ldots,e_n$ the standard basis of $\mathbf{F}^n$ and α the alternating n-linear form on $\mathbf{F}^n$ that takes $v_1,\ldots,v_n$ to $\det\begin{pmatrix} v_1 & \cdots & v_n\end{pmatrix}$ [see 9.45]. If each v_k is the k^{th} column of A, then each $b_{j,k}$ in 9.36 equals $A_{j,k}$. Finally,

$$\alpha(e_1,\ldots,e_n) = \det\begin{pmatrix} e_1 & \cdots & e_n\end{pmatrix} = \det I = 1.$$

Thus the formula in 9.36 becomes the formula stated in this result. ■

9.47 example: *explicit formula for determinant*

- If A is a 2-by-2 matrix, then the formula in 9.46 becomes

$$\det A = A_{1,1}A_{2,2} - A_{2,1}A_{1,2}.$$

- If A is a 3-by-3 matrix, then the formula in 9.46 becomes

$$\begin{aligned}\det A = {}& A_{1,1}A_{2,2}A_{3,3} - A_{2,1}A_{1,2}A_{3,3} - A_{3,1}A_{2,2}A_{1,3}\\ & - A_{1,1}A_{3,2}A_{2,3} + A_{3,1}A_{1,2}A_{2,3} + A_{2,1}A_{3,2}A_{1,3}.\end{aligned}$$

The sum in the formula in 9.46 contains $n!$ terms. Because $n!$ grows rapidly as n increases, the formula in 9.46 is not a viable method to evaluate determinants even for moderately sized n. For example, $10!$ is over three million, and $100!$ is approximately 10^{158}, leading to a sum that the fastest computer cannot evaluate. We will soon see some results that lead to faster evaluations of determinants than direct use of the sum in 9.46.

9.48 *determinant of upper-triangular matrix*

Suppose that A is an upper-triangular matrix with $\lambda_1,\ldots,\lambda_n$ on the diagonal. Then $\det A = \lambda_1\cdots\lambda_n$.

Proof If $(j_1,\ldots,j_n)\in \operatorname{perm} n$ with $(j_1,\ldots,j_n)\neq(1,\ldots,n)$, then $j_k > k$ for some $k\in\{1,\ldots,n\}$, which implies that $A_{j_k,k} = 0$. Thus the only permutation that can make a nonzero contribution to the sum in 9.46 is the permutation $(1,\ldots,n)$. Because $A_{k,k} = \lambda_k$ for each $k = 1,\ldots,n$, this implies that $\det A = \lambda_1\cdots\lambda_n$. ■

Properties of Determinants

Our definition of the determinant leads to the following magical proof that the determinant is multiplicative.

9.49 *determinant is multiplicative*

(a) Suppose $S, T \in \mathcal{L}(V)$. Then $\det(ST) = (\det S)(\det T)$.

(b) Suppose A and B are square matrices of the same size. Then

$$\det(AB) = (\det A)(\det B)$$

Proof

(a) Let $n = \dim V$. Suppose $\alpha \in V_{\text{alt}}^{(n)}$ and $v_1, \ldots, v_n \in V$. Then

$$\begin{aligned}\alpha_{ST}(v_1, \ldots, v_n) &= \alpha(STv_1, \ldots, STv_n) \\ &= (\det S)\,\alpha(Tv_1, \ldots, Tv_n) \\ &= (\det S)(\det T)\,\alpha(v_1, \ldots, v_n),\end{aligned}$$

where the first equation follows from the definition of α_{ST}, the second equation follows from the definition of $\det S$, and the third equation follows from the definition of $\det T$. The equation above implies that $\det(ST) = (\det S)(\det T)$.

(b) Let $S, T \in \mathcal{L}(\mathbf{F}^n)$ be such that $\mathcal{M}(S) = A$ and $\mathcal{M}(T) = B$, where all matrices of operators in this proof are with respect to the standard basis of $\mathbf{F}^n$. Then $\mathcal{M}(ST) = \mathcal{M}(S)\mathcal{M}(T) = AB$ (see 3.43). Thus

$$\det(AB) = \det(ST) = (\det S)(\det T) = (\det A)(\det B),$$

where the second equality comes from the result in (a). ∎

The determinant of an operator determines whether the operator is invertible.

9.50 *invertible $\Longleftrightarrow$ nonzero determinant*

An operator $T \in \mathcal{L}(V)$ is invertible if and only if $\det T \neq 0$. Furthermore, if T is invertible, then $\det(T^{-1}) = \frac{1}{\det T}$.

Proof First suppose T is invertible. Thus $TT^{-1} = I$. Now 9.49 implies that

$$1 = \det I = \det(TT^{-1}) = (\det T)\bigl(\det(T^{-1})\bigr).$$

Hence $\det T \neq 0$ and $\det(T^{-1})$ is the multiplicative inverse of $\det T$.

To prove the other direction, now suppose $\det T \neq 0$. Suppose $v \in V$ and $v \neq 0$. Let $v, e_2, \ldots, e_n$ be a basis of V and let $\alpha \in V_{\text{alt}}^{(n)}$ be such that $\alpha \neq 0$. Then $\alpha(v, e_2, \ldots, e_n) \neq 0$ (by 9.39). Now

$$\alpha(Tv, Te_2, \ldots, Te_n) = (\det T)\,\alpha(v, e_2, \ldots, e_n) \neq 0.$$

Thus $Tv \neq 0$. Hence T is invertible. ∎

An n-by-n matrix A is invertible (see 3.80 for the definition of an invertible matrix) if and only if the operator on $\mathbf{F}^n$ associated with A (via the standard basis of $\mathbf{F}^n$) is invertible. Thus the previous result shows that a square matrix A is invertible if and only if $\det A \neq 0$.

9.51 *eigenvalues and determinants*

Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbf{F}$. Then λ is an eigenvalue of T if and only if $\det(\lambda I - T) = 0$.

Proof The number λ is an eigenvalue of T if and only if $T - \lambda I$ is not invertible (see 5.7), which happens if and only if $\lambda I - T$ is not invertible, which happens if and only if $\det(\lambda I - T) = 0$ (by 9.50). ■

Suppose $T \in \mathcal{L}(V)$ and $S \colon W \to V$ is an invertible linear map. To prove that $\det(S^{-1}TS) = \det T$, we could try to use 9.49 and 9.50, writing

$$\begin{aligned}\det(S^{-1}TS) &= (\det S^{-1})(\det T)(\det S)\\ &= \det T.\end{aligned}$$

That proof works if $W = V$, but if $W \neq V$ then it makes no sense because the determinant is defined only for linear maps from a vector space to itself, and S maps W to V, making $\det S$ undefined. The proof given below works around this issue and is valid when $W \neq V$.

9.52 *determinant is a similarity invariant*

Suppose $T \in \mathcal{L}(V)$ and $S \colon W \to V$ is an invertible linear map. Then

$$\det(S^{-1}TS) = \det T.$$

Proof Let $n = \dim W = \dim V$. Suppose $\tau \in W^{(n)}_{\text{alt}}$. Define $\alpha \in V^{(n)}_{\text{alt}}$ by

$$\alpha(v_1, \ldots, v_n) = \tau(S^{-1}v_1, \ldots, S^{-1}v_n)$$

for $v_1, \ldots, v_n \in V$. Suppose $w_1, \ldots, w_n \in W$. Then

$$\begin{aligned}\tau_{S^{-1}TS}(w_1, \ldots, w_n) &= \tau(S^{-1}TSw_1, \ldots, S^{-1}TSw_n)\\ &= \alpha(TSw_1, \ldots, TSw_n)\\ &= \alpha_T(Sw_1, \ldots, Sw_n)\\ &= (\det T)\,\alpha(Sw_1, \ldots, Sw_n)\\ &= (\det T)\,\tau(w_1, \ldots, w_n).\end{aligned}$$

The equation above and the definition of the determinant of the operator $S^{-1}TS$ imply that $\det(S^{-1}TS) = \det T$. ■

For the special case in which $V = \mathbf{F}^n$ and $e_1, \dots, e_n$ is the standard basis of $\mathbf{F}^n$, the next result is true by the definition of the determinant of a matrix. The left side of the equation in the next result does not depend on a choice of basis, which means that the right side is independent of the choice of basis.

9.53 *determinant of operator equals determinant of its matrix*

Suppose $T \in \mathcal{L}(V)$ and $e_1, \dots, e_n$ is a basis of V. Then

$$\det T = \det \mathcal{M}\big(T, (e_1, \dots, e_n)\big).$$

Proof Let $f_1, \dots, f_n$ be the standard basis of $\mathbf{F}^n$. Let $S \colon \mathbf{F}^n \to V$ be the linear map such that $Sf_k = e_k$ for each $k = 1, \dots, n$. Thus $\mathcal{M}\big(S, (f_1, \dots, f_n), (e_1, \dots, e_n)\big)$ and $\mathcal{M}\big(S^{-1}, (e_1, \dots, e_n), (f_1, \dots, f_n)\big)$ both equal the n-by-n identity matrix. Hence

9.54 $$\mathcal{M}\big(S^{-1}TS, (f_1, \dots, f_n)\big) = \mathcal{M}\big(T, (e_1, \dots, e_n)\big),$$

as follows from two applications of 3.43. Thus

$$\begin{aligned} \det T &= \det\big(S^{-1}TS\big) \\ &= \det \mathcal{M}\big(S^{-1}TS, (f_1, \dots, f_n)\big) \\ &= \det \mathcal{M}\big(T, (e_1, \dots, e_n)\big), \end{aligned}$$

where the first line comes from 9.52, the second line comes from the definition of the determinant of a matrix, and the third line follows from 9.54. ■

The next result gives a more intuitive way to think about determinants than the definition or the formula in 9.46. We could make the characterization in the result below the definition of the determinant of an operator on a finite-dimensional complex vector space, with the current definition then becoming a consequence of that definition.

9.55 *if* $\mathbf{F} = \mathbf{C}$, *then determinant equals product of eigenvalues*

Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$. Then $\det T$ equals the product of the eigenvalues of T, with each eigenvalue included as many times as its multiplicity.

Proof There is a basis of V with respect to which T has an upper-triangular matrix with the diagonal entries of the matrix consisting of the eigenvalues of T, with each eigenvalue included as many times as its multiplicity—see 8.37. Thus 9.53 and 9.48 imply that $\det T$ equals the product of the eigenvalues of T, with each eigenvalue included as many times as its multiplicity. ■

As the next result shows, the determinant interacts nicely with the transpose of a square matrix, with the dual of an operator, and with the adjoint of an operator on an inner product space.

9.56 *determinant of transpose, dual, or adjoint*

(a) Suppose A is a square matrix. Then $\det A^{\mathrm{t}} = \det A$.

(b) Suppose $T \in \mathcal{L}(V)$. Then $\det T' = \det T$.

(c) Suppose V is an inner product space and $T \in \mathcal{L}(V)$. Then

$$\det(T^*) = \overline{\det T}.$$

Proof

(a) Let n be a positive integer. Define $\alpha\colon (\mathbf{F}^n)^n \to \mathbf{F}$ by

$$\alpha\bigl(\begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}\bigr) = \det\bigl(\begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}^{\mathrm{t}}\bigr)$$

for all $v_1, \ldots, v_n \in \mathbf{F}^n$. The formula in 9.46 for the determinant of a matrix shows that α is an n-linear form on $\mathbf{F}^n$.

Suppose $v_1, \ldots, v_n \in \mathbf{F}^n$ and $v_j = v_k$ for some $j \neq k$. If B is an n-by-n matrix, then $\begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}^{\mathrm{t}} B$ cannot equal the identity matrix because row j and row k of $\begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}^{\mathrm{t}} B$ are equal. Thus $\begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}^{\mathrm{t}}$ is not invertible, which implies that $\alpha\bigl(\begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}\bigr) = 0$. Hence α is an alternating n-linear form on $\mathbf{F}^n$.

Note that α applied to the standard basis of $\mathbf{F}^n$ equals 1. Because the vector space of alternating n-linear forms on $\mathbf{F}^n$ has dimension one (by 9.37), this implies that α is the determinant function. Thus (a) holds.

(b) The equation $\det T' = \det T$ follows from (a) and 9.53 and 3.132.

(c) Pick an orthonormal basis of V. The matrix of T^* with respect to that basis is the conjugate transpose of the matrix of T with respect to that basis (by 7.9). Thus 9.53, 9.46, and (a) imply that $\det(T^*) = \overline{\det T}$. ■

9.57 *helpful results in evaluating determinants*

(a) If either two columns or two rows of a square matrix are equal, then the determinant of the matrix equals 0.

(b) Suppose A is a square matrix and B is the matrix obtained from A by swapping either two columns or two rows. Then $\det A = -\det B$.

(c) If one column or one row of a square matrix is multiplied by a scalar, then the value of the determinant is multiplied by the same scalar.

(d) If a scalar multiple of one column of a square matrix is added to another column, then the value of the determinant is unchanged.

(e) If a scalar multiple of one row of a square matrix is added to another row, then the value of the determinant is unchanged.

Proof All the assertions in this result follow from the result that the maps $v_1, \dots, v_n \mapsto \det\begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}$ and $v_1, \dots, v_n \mapsto \det\begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}^{\mathrm{t}}$ are both alternating n-linear forms on $\mathbf{F}^n$ [see 9.45 and 9.56(a)].

For example, to prove (d) suppose $v_1, \dots, v_n \in \mathbf{F}^n$ and $c \in \mathbf{F}$. Then

$$\begin{aligned}
&\det\begin{pmatrix} v_1 + cv_2 & v_2 & \cdots & v_n \end{pmatrix} \\
&\quad = \det\begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix} + c \det\begin{pmatrix} v_2 & v_2 & v_3 & \cdots & v_n \end{pmatrix} \\
&\quad = \det\begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix},
\end{aligned}$$

where the first equation follows from the multilinearity property and the second equation follows from the alternating property. The equation above shows that adding a multiple of the second column to the first column does not change the value of the determinant. The same conclusion holds for any two columns. Thus (d) holds.

The proof of (e) follows from (d) and from 9.56(a). The proofs of (a), (b), and (c) use similar tools and are left to the reader. ■

For matrices whose entries are concrete numbers, the result above leads to a much faster way to evaluate the determinant than direct application of the formula in 9.46. Specifically, apply the Gaussian elimination procedure of swapping rows [by 9.57(b), this changes the determinant by a factor of -1], multiplying a row by a nonzero constant [by 9.57(c), this changes the determinant by the same constant], and adding a multiple of one row to another row [by 9.57(e), this does not change the determinant] to produce an upper-triangular matrix, whose determinant is the product of the diagonal entries (by 9.48). If your software keeps track of the number of row swaps and of the constants used when multiplying a row by a constant, then the determinant of the original matrix can be computed.

Because a number $\lambda \in \mathbf{F}$ is an eigenvalue of an operator $T \in \mathcal{L}(V)$ if and only if $\det(\lambda I - T) = 0$ (by 9.51), you may be tempted to think that one way to find eigenvalues quickly is to choose a basis of V, let $A = \mathcal{M}(T)$, evaluate $\det(\lambda I - A)$, and then solve the equation $\det(\lambda I - A) = 0$ for λ. However, that procedure is rarely efficient, except when $\dim V = 2$ (or when $\dim V$ equals 3 or 4 if you are willing to use the cubic or quartic formulas). One problem is that the procedure described in the paragraph above for evaluating a determinant does not work when the matrix includes a symbol (such as the λ in $\lambda I - A$). This problem arises because decisions need to be made in the Gaussian elimination procedure about whether certain quantities equal 0, and those decisions become complicated in expressions involving a symbol λ.

Recall that an operator on a finite-dimensional inner product space is unitary if it preserves norms (see 7.51 and the paragraph following it). Every eigenvalue of a unitary operator has absolute value 1 (by 7.54). Thus the product of the eigenvalues of a unitary operator has absolute value 1. Hence (at least in the case $\mathbf{F} = \mathbf{C}$) the determinant of a unitary operator has absolute value 1 (by 9.55). The next result gives a proof that works without the assumption that $\mathbf{F} = \mathbf{C}$.

9.58 *every unitary operator has determinant with absolute value* 1

Suppose V is an inner product space and $S \in \mathcal{L}(V)$ is a unitary operator. Then $|\det S| = 1$.

Proof Because S is unitary, $I = S^*S$ (see 7.53). Thus

$$1 = \det(S^*S) = (\det S^*)(\det S) = \overline{(\det S)}(\det S) = |\det S|^2,$$

where the second equality comes from 9.49(a) and the third equality comes from 9.56(c). The equation above implies that $|\det S| = 1$. ■

The determinant of a positive operator on an inner product space meshes well with the analogy that such operators correspond to the nonnegative real numbers.

9.59 *every positive operator has nonnegative determinant*

Suppose V is an inner product space and $T \in \mathcal{L}(V)$ is a positive operator. Then $\det T \geq 0$.

Proof By the spectral theorem (7.29 or 7.31), V has an orthonormal basis consisting of eigenvectors of T. Thus by the last bullet point of 9.42, $\det T$ equals a product of the eigenvalues of T, possibly with repetitions. Each eigenvalue of T is a nonnegative number (by 7.38). Thus we conclude that $\det T \geq 0$. ■

Suppose V is an inner product space and $T \in \mathcal{L}(V)$. Recall that the list of nonnegative square roots of the eigenvalues of T^*T (each included as many times as its multiplicity) is called the list of singular values of T (see Section 7E).

9.60 $|\det T|$ = *product of singular values of* T

Suppose V is an inner product space and $T \in \mathcal{L}(V)$. Then

$$|\det T| = \sqrt{\det\bigl(T^*T\bigr)} = \text{product of singular values of } T.$$

Proof We have

$$|\det T|^2 = \overline{(\det T)}(\det T) = \Bigl(\det\bigl(T^*\bigr)\Bigr)(\det T) = \det\bigl(T^*T\bigr),$$

where the middle equality comes from 9.56(c) and the last equality comes from 9.49(a). Taking square roots of both sides of the equation above shows that $|\det T| = \sqrt{\det\bigl(T^*T\bigr)}$.

Let $s_1, \dots, s_n$ denote the list of singular values of T. Thus $s_1^2, \dots, s_n^2$ is the list of eigenvalues of T^*T (with appropriate repetitions), corresponding to an orthonormal basis of V consisting of eigenvectors of T^*T. Hence the last bullet point of 9.42 implies that

$$\det\bigl(T^*T\bigr) = s_1^2 \cdots s_n^2.$$

Thus $|\det T| = s_1 \cdots s_n$, as desired. ■

An operator T on a real inner product space changes volume by a factor of the product of the singular values (by 7.111). Thus the next result follows immediately from 7.111 and 9.60. This result explains why the absolute value of a determinant appears in the change of variables formula in multivariable calculus.

9.61 *T changes volume by factor of* $|\det T|$

Suppose $T \in \mathcal{L}(\mathbf{R}^n)$ and $\Omega \subseteq \mathbf{R}^n$. Then

$$\text{volume}\, T(\Omega) = |\det T|(\text{volume}\, \Omega).$$

For operators on finite-dimensional complex vector spaces, we now connect the determinant to a polynomial that we have previously seen.

9.62 *if $\mathbf{F} = \mathbf{C}$, then characteristic polynomial of T equals* $\det(zI - T)$

Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_m$ denote the distinct eigenvalues of T, and let $d_1, \ldots, d_m$ denote their multiplicities. Then

$$\det(zI - T) = (z - \lambda_1)^{d_1} \cdots (z - \lambda_m)^{d_m}.$$

Proof There exists a basis of V with respect to which T has an upper-triangular matrix with each λ_k appearing on the diagonal exactly d_k times (by 8.37). With respect to this basis, $zI - T$ has an upper-triangular matrix with $z - \lambda_k$ appearing on the diagonal exactly d_k times for each k. Thus 9.48 gives the desired equation. ■

Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$. The characteristic polynomial of T was defined in 8.26 as the polynomial on the right side of the equation in 9.62. We did not previously define the characteristic polynomial of an operator on a finite-dimensional real vector space because such operators may have no eigenvalues, making a definition using the right side of the equation in 9.62 inappropriate.

We now present a new definition of the characteristic polynomial, motivated by 9.62. This new definition is valid for both real and complex vector spaces. The equation in 9.62 shows that this new definition is equivalent to our previous definition when $\mathbf{F} = \mathbf{C}$ (8.26).

9.63 definition: *characteristic polynomial*

Suppose $T \in \mathcal{L}(V)$. The polynomial defined by

$$z \mapsto \det(zI - T)$$

is called the *characteristic polynomial* of T.

The formula in 9.46 shows that the characteristic polynomial of an operator $T \in \mathcal{L}(V)$ is a monic polynomial of degree $\dim V$. The zeros in $\mathbf{F}$ of the characteristic polynomial of T are exactly the eigenvalues of T (by 9.51).

Previously we proved the Cayley–Hamilton theorem (8.29) in the complex case. Now we can extend that result to operators on real vector spaces.

9.64 *Cayley–Hamilton theorem*

Suppose $T \in \mathcal{L}(V)$ and q is the characteristic polynomial of T. Then $q(T) = 0$.

Proof If $\mathbf{F} = \mathbf{C}$, then the equation $q(T) = 0$ follows from 9.62 and 8.29.

Now suppose $\mathbf{F} = \mathbf{R}$. Fix a basis of V, and let A be the matrix of T with respect to this basis. Let S be the operator on $\mathbf{C}^{\dim V}$ such that the matrix of S (with respect to the standard basis of $\mathbf{C}^{\dim V}$) is A. For all $z \in \mathbf{R}$ we have

$$q(z) = \det(zI - T) = \det(zI - A) = \det(zI - S).$$

Thus q is the characteristic polynomial of S. The case $\mathbf{F} = \mathbf{C}$ (first sentence of this proof) now implies that $0 = q(S) = q(A) = q(T)$. ■

The Cayley–Hamilton theorem (9.64) implies that the characteristic polynomial of an operator $T \in \mathcal{L}(V)$ is a polynomial multiple of the minimal polynomial of T (by 5.29). Thus if the degree of the minimal polynomial of T equals $\dim V$, then the characteristic polynomial of T equals the minimal polynomial of T. This happens for a very large percentage of operators, including over 99.999% of 4-by-4 matrices with integer entries in $[-100, 100]$ (see the paragraph following 5.25).

The last sentence in our next result was previously proved in the complex case (see 8.54). Now we can give a proof that works on both real and complex vector spaces.

9.65 *characteristic polynomial, trace, and determinant*

Suppose $T \in \mathcal{L}(V)$. Let $n = \dim V$. Then the characteristic polynomial of T can be written as

$$z^n - (\operatorname{tr} T)z^{n-1} + \cdots + (-1)^n(\det T).$$

Proof The constant term of a polynomial function of z is the value of the polynomial when $z = 0$. Thus the constant term of the characteristic polynomial of T equals $\det(-T)$, which equals $(-1)^n \det T$ (by the third bullet point of 9.42).

Fix a basis of V, and let A be the matrix of T with respect to this basis. The matrix of $zI - T$ with respect to this basis is $zI - A$. The term coming from the identity permutation $\{1, \dots, n\}$ in the formula 9.46 for $\det(zI - A)$ is

$$(z - A_{1,1}) \cdots (z - A_{n,n}).$$

The coefficient of z^{n-1} in the expression above is $-(A_{1,1} + \cdots + A_{n,n})$, which equals $-\operatorname{tr} T$. The terms in the formula for $\det(zI - A)$ coming from other elements of $\operatorname{perm} n$ contain at most $n-2$ factors of the form $z - A_{k,k}$ and thus do not contribute to the coefficient of z^{n-1} in the characteristic polynomial of T. ■

In the result below, think of the columns of the n-by-n matrix A as elements of $\mathbf{F}^n$. The norms appearing below then arise from the standard inner product on $\mathbf{F}^n$. Recall that the notation $R_{\cdot,k}$ in the proof below means the k^{th} column of the matrix R (as was defined in 3.44).

The next result was proved by Jacques Hadamard (1865–1963) in 1893.

9.66 *Hadamard's inequality*

Suppose A is an n-by-n matrix. Let $v_1, \dots, v_n$ denote the columns of A. Then

$$|\det A| \leq \prod_{k=1}^{n} \|v_k\|.$$

Proof If A is not invertible, then $\det A = 0$ and hence the desired inequality holds in this case.

Thus assume that A is invertible. The QR factorization (7.58) tells us that there exist a unitary matrix Q and an upper-triangular matrix R whose diagonal contains only positive numbers such that $A = QR$. We have

$$\begin{aligned} |\det A| &= |\det Q|\,|\det R| \\ &= |\det R| \\ &= \prod_{k=1}^{n} R_{k,k} \\ &\leq \prod_{k=1}^{n} \|R_{\cdot,k}\| \\ &= \prod_{k=1}^{n} \|QR_{\cdot,k}\| \\ &= \prod_{k=1}^{n} \|v_k\|, \end{aligned}$$

where the first line comes from 9.49(b), the second line comes from 9.58, the third line comes from 9.48, and the fifth line holds because Q is an isometry. ■

To give a geometric interpretation to Hadamard's inequality, suppose $\mathbf{F} = \mathbf{R}$. Let $T \in \mathcal{L}(\mathbf{R}^n)$ be the operator such that $Te_k = v_k$ for each $k = 1, \dots, n$, where $e_1, \dots, e_n$ is the standard basis of $\mathbf{R}^n$. Then T maps the box $P(e_1, \dots, e_n)$ onto the parallelepiped $P(v_1, \dots, v_n)$ [see 7.102 and 7.105 for a review of this notation and terminology]. Because the box $P(e_1, \dots, e_n)$ has volume 1, this implies (by 9.61) that the parallelepiped $P(v_1, \dots, v_n)$ has volume $|\det T|$, which equals $|\det A|$. Thus Hadamard's inequality above can be interpreted to say that among all parallelepipeds whose edges have lengths $\|v_1\|, \dots, \|v_n\|$, the ones with largest volume have orthogonal edges $\left(\text{and thus have volume } \prod_{k=1}^{n} \|v_k\|\right)$.

For a necessary and sufficient condition for Hadamard's inequality to be an equality, see Exercise 18.

The matrix in the next result is called the *Vandermonde matrix*. Vandermonde matrices have important applications in polynomial interpolation, the discrete Fourier transform, and other areas of mathematics. The proof of the next result is a nice illustration of the power of switching between matrices and linear maps.

9.67 *determinant of Vandermonde matrix*

Suppose $n > 1$ and $\beta_1, \dots, \beta_n \in \mathbf{F}$. Then

$$\det \begin{pmatrix} 1 & \beta_1 & \beta_1^2 & \cdots & \beta_1^{n-1} \\ 1 & \beta_2 & \beta_2^2 & \cdots & \beta_2^{n-1} \\ & & \ddots & & \\ 1 & \beta_n & \beta_n^2 & \cdots & \beta_n^{n-1} \end{pmatrix} = \prod_{1 \le j < k \le n} (\beta_k - \beta_j).$$

Proof Let $1, z, \dots, z^{n-1}$ be the standard basis of $\mathcal{P}_{n-1}(\mathbf{F})$ and let $e_1, \dots, e_n$ denote the standard basis of $\mathbf{F}^n$. Define a linear map $S \colon \mathcal{P}_{n-1}(\mathbf{F}) \to \mathbf{F}^n$ by

$$Sp = \big(p(\beta_1), \dots, p(\beta_n)\big).$$

Let A denote the Vandermonde matrix shown in the statement of this result. Note that

$$A = \mathcal{M}\big(S, (1, z, \dots, z^{n-1}), (e_1, \dots, e_n)\big).$$

Let $T \colon \mathcal{P}_{n-1}(\mathbf{F}) \to \mathcal{P}_{n-1}(\mathbf{F})$ be the operator on $\mathcal{P}_{n-1}(\mathbf{F})$ such that $T1 = 1$ and

$$Tz^k = (z - \beta_1)(z - \beta_2)\cdots(z - \beta_k)$$

for $k = 1, \dots, n-1$. Let $B = \mathcal{M}\big(T, (1, z, \dots, z^{n-1}), (1, z, \dots, z^{n-1})\big)$. Then B is an upper-triangular matrix all of whose diagonal entries equal 1. Thus $\det B = 1$ (by 9.48).

Let $C = \mathcal{M}\big(ST, (1, z, \dots, z^{n-1}), (e_1, \dots, e_n)\big)$. Thus $C = AB$ (by 3.81), which implies that

$$\det A = (\det A)(\det B) = \det C.$$

The definitions of C, S, and T show that C equals

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & \beta_2 - \beta_1 & 0 & \cdots & 0 \\ 1 & \beta_3 - \beta_1 & (\beta_3 - \beta_1)(\beta_3 - \beta_2) & \cdots & 0 \\ & & & \ddots & \\ 1 & \beta_n - \beta_1 & (\beta_n - \beta_1)(\beta_n - \beta_2) & \cdots & (\beta_n - \beta_1)(\beta_n - \beta_2)\cdots(\beta_n - \beta_{n-1}) \end{pmatrix}.$$

Now $\det A = \det C = \displaystyle\prod_{1 \le j < k \le n} (\beta_k - \beta_j)$, where we have used 9.56(a) and 9.48. ■

Exercises 9C

1 Prove or give a counterexample: $S, T \in \mathcal{L}(V) \implies \det(S+T) = \det S + \det T$.

2 Suppose the first column of a square matrix A consists of all zeros except possibly the first entry $A_{1,1}$. Let B be the matrix obtained from A by deleting the first row and the first column of A. Show that $\det A = A_{1,1} \det B$.

3 Suppose $T \in \mathcal{L}(V)$ is nilpotent. Prove that $\det(I + T) = 1$.

4 Suppose $S \in \mathcal{L}(V)$. Prove that S is unitary if and only if $|\det S| = \|S\| = 1$.

5 Suppose A is a block upper-triangular matrix

$$A = \begin{pmatrix} A_1 & & * \\ & \ddots & \\ 0 & & A_m \end{pmatrix},$$

where each A_k along the diagonal is a square matrix. Prove that

$$\det A = (\det A_1) \cdots (\det A_m).$$

6 Suppose $A = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}$ is an n-by-n matrix, with v_k denoting the k^{th} column of A. Show that if $(m_1, \ldots, m_n) \in \operatorname{perm} n$, then

$$\det \begin{pmatrix} v_{m_1} & \cdots & v_{m_n} \end{pmatrix} = \bigl(\operatorname{sign}(m_1, \ldots, m_n)\bigr) \det A.$$

7 Suppose $T \in \mathcal{L}(V)$ is invertible. Let p denote the characteristic polynomial of T and let q denote the characteristic polynomial of T^{-1}. Prove that

$$q(z) = \frac{1}{p(0)}\, z^{\dim V} p\Bigl(\frac{1}{z}\Bigr)$$

for all nonzero $z \in \mathbf{F}$.

8 Suppose $T \in \mathcal{L}(V)$ is an operator with no eigenvalues (which implies that $\mathbf{F} = \mathbf{R}$). Prove that $\det T > 0$.

9 Suppose that V is a real vector space of even dimension, $T \in \mathcal{L}(V)$, and $\det T < 0$. Prove that T has at least two distinct eigenvalues.

10 Suppose V is a real vector space of odd dimension and $T \in \mathcal{L}(V)$. Without using the minimal polynomial, prove that T has an eigenvalue.

This result was previously proved without using determinants or the characteristic polynomial—see 5.34.

11 Prove or give a counterexample: If $\mathbf{F} = \mathbf{R}$, $T \in \mathcal{L}(V)$, and $\det T > 0$, then T has a square root.

If $\mathbf{F} = \mathbf{C}$, $T \in \mathcal{L}(V)$, *and* $\det T \neq 0$, *then* T *has a square root (see 8.41).*

12 Suppose $S, T \in \mathcal{L}(V)$ and S is invertible. Define $p \colon \mathbf{F} \to \mathbf{F}$ by

$$p(z) = \det(zS - T).$$

Prove that p is a polynomial of degree $\dim V$ and that the coefficient of $z^{\dim V}$ in this polynomial is $\det S$.

13 Suppose $\mathbf{F} = \mathbf{C}$, $T \in \mathcal{L}(V)$, and $n = \dim V > 2$. Let $\lambda_1, \dots, \lambda_n$ denote the eigenvalues of T, with each eigenvalue included as many times as its multiplicity.

(a) Find a formula for the coefficient of z^{n-2} in the characteristic polynomial of T in terms of $\lambda_1, \dots, \lambda_n$.

(b) Find a formula for the coefficient of z in the characteristic polynomial of T in terms of $\lambda_1, \dots, \lambda_n$.

14 Suppose V is an inner product space and T is a positive operator on V. Prove that

$$\det \sqrt{T} = \sqrt{\det T}.$$

15 Suppose V is an inner product space and $T \in \mathcal{L}(V)$. Use the polar decomposition to give a proof that

$$|\det T| = \sqrt{\det(T^*T)}$$

that is different from the proof given earlier (see 9.60).

16 Suppose $T \in \mathcal{L}(V)$. Define $g \colon \mathbf{F} \to \mathbf{F}$ by $g(x) = \det(I + xT)$. Show that $g'(0) = \operatorname{tr} T$.

Look for a clean solution to this exercise, without using the explicit but complicated formula for the determinant of a matrix.

17 Suppose a, b, c are positive numbers. Find the volume of the ellipsoid

$$\left\{(x, y, z) \in \mathbf{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} < 1\right\}$$

by finding a set $\Omega \subseteq \mathbf{R}^3$ whose volume you know and an operator T on $\mathbf{R}^3$ such that $T(\Omega)$ equals the ellipsoid above.

18 Suppose that A is an invertible square matrix. Prove that Hadamard's inequality (9.66) is an equality if and only if each column of A is orthogonal to the other columns.

19 Suppose V is an inner product space, $e_1, \dots, e_n$ is an orthonormal basis of V, and $T \in \mathcal{L}(V)$ is a positive operator.

(a) Prove that $\det T \le \prod_{k=1}^{n} \langle Te_k, e_k \rangle$.

(b) Prove that if T is invertible, then the inequality in (a) is an equality if and only if e_k is an eigenvector of T for each $k = 1, \dots, n$.

20 Suppose A is an n-by-n matrix, and suppose c is such that $|A_{j,k}| \leq c$ for all $j, k \in \{1, \dots, n\}$. Prove that

$$|\det A| \leq c^n n^{n/2}.$$

The formula for the determinant of a matrix (9.46) shows that $|\det A| \leq c^n n!$. However, the estimate given by this exercise is much better. For example, if $c = 1$ and $n = 100$, then $c^n n! \approx 10^{158}$, but the estimate given by this exercise is the much smaller number 10^{100}. If n is an integer power of 2, then the inequality above is sharp and cannot be improved.

21 Suppose n is a positive integer and $\delta \colon \mathbf{C}^{n,n} \to \mathbf{C}$ is a function such that

$$\delta(AB) = \delta(A) \cdot \delta(B)$$

for all $A, B \in \mathbf{C}^{n,n}$ and $\delta(A)$ equals the product of the diagonal entries of A for each diagonal matrix $A \in \mathbf{C}^{n,n}$. Prove that

$$\delta(A) = \det A$$

for all $A \in \mathbf{C}^{n,n}$.

Recall that $\mathbf{C}^{n,n}$ denotes the set of n-by-n matrices with entries in $\mathbf{C}$. This exercise shows that the determinant is the unique function defined on square matrices that is multiplicative and has the desired behavior on diagonal matrices. This result is analogous to Exercise 10 in Section 8D, which shows that the trace is uniquely determined by its algebraic properties.

I find that in my own elementary lectures, I have, for pedagogical reasons, pushed determinants more and more into the background. Too often I have had the experience that, while the students acquired facility with the formulas, which are so useful in abbreviating long expressions, they often failed to gain familiarity with their *meaning*, and skill in manipulation prevented the student from going into all the details of the subject and so gaining a mastery.

—*Elementary Mathematics from an Advanced Standpoint*: *Geometry*, Felix Klein