9D Tensor Products

Tensor Product of Two Vector Spaces

The motivation for our next topic comes from wanting to form the product of a vector $v \in V$ and a vector $w \in W$. This product will be denoted by $v \otimes w$, pronounced "v tensor w", and will be an element of some new vector space called $V \otimes W$ (also pronounced "V tensor W").

We already have a vector space $V \times W$ (see Section 3E), called the product of V and W. However, $V \times W$ will not serve our purposes here because it does not provide a natural way to multiply an element of V by an element of W. We would like our tensor product to satisfy some of the usual properties of multiplication. For example, we would like the distributive property to be satisfied, meaning that if $v_1, v_2, v \in V$ and $w_1, w_2, w \in W$, then

$$(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w \quad \text{and} \quad v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2.$$

To produce $\otimes$ in TEX, type `\otimes`.

We would also like scalar multiplication to interact well with this new multiplication, meaning that

$$\lambda(v \otimes w) = (\lambda v) \otimes w = v \otimes (\lambda w)$$

for all $\lambda \in \mathbf{F}$, $v \in V$, and $w \in W$.

Furthermore, it would be nice if each basis of V when combined with each basis of W produced a basis of $V \otimes W$. Specifically, if $e_1, \ldots, e_m$ is a basis of V and $f_1, \ldots, f_n$ is a basis of W, then we would like a list (in any order) consisting of $e_j \otimes f_k$, as j ranges from 1 to m and k ranges from 1 to n, to be a basis of $V \otimes W$. This implies that $\dim(V \otimes W)$ should equal $(\dim V)(\dim W)$. Recall that $\dim(V \times W) = \dim V + \dim W$ (see 3.92), which shows that the product $V \times W$ will not serve our purposes here.

To produce a vector space whose dimension is $(\dim V)(\dim W)$ in a natural fashion from V and W, we look at the vector space of bilinear functionals, as defined below.

9.68 definition: *bilinear functional on $V \times W$, the vector space $\mathcal{B}(V, W)$*

- A *bilinear functional* on $V \times W$ is a function $\beta\colon V \times W \to \mathbf{F}$ such that $v \mapsto \beta(v, w)$ is a linear functional on V for each $w \in W$ and $w \mapsto \beta(v, w)$ is a linear functional on W for each $v \in V$.
- The vector space of bilinear functionals on $V \times W$ is denoted by $\mathcal{B}(V, W)$.

If $W = V$, then a bilinear functional on $V \times W$ is a bilinear form; see 9.1.

The operations of addition and scalar multiplication on $\mathcal{B}(V, W)$ are defined to be the usual operations of addition and scalar multiplication of functions. As you can verify, these operations make $\mathcal{B}(V, W)$ into a vector space whose additive identity is the zero function from $V \times W$ to $\mathbf{F}$.

9.69 example: *bilinear functionals*

- Suppose $\varphi \in V'$ and $\tau \in W'$. Define $\beta \colon V \times W \to \mathbf{F}$ by $\beta(v, w) = \varphi(v)\tau(w)$. Then β is a bilinear functional on $V \times W$.
- Suppose $v \in V$ and $w \in W$. Define $\beta \colon V' \times W' \to \mathbf{F}$ by $\beta(\varphi, \tau) = \varphi(v)\tau(w)$. Then β is a bilinear functional on $V' \times W'$.
- Define $\beta \colon V \times V' \to \mathbf{F}$ by $\beta(v, \varphi) = \varphi(v)$. Then β is a bilinear functional on $V \times V'$.
- Suppose $\varphi \in V'$. Define $\beta \colon V \times \mathcal{L}(V) \to \mathbf{F}$ by $\beta(v, T) = \varphi(Tv)$. Then β is a bilinear functional on $V \times \mathcal{L}(V)$.
- Suppose m and n are positive integers. Define $\beta \colon \mathbf{F}^{m,n} \times \mathbf{F}^{n,m} \to \mathbf{F}$ by $\beta(A, B) = \operatorname{tr}(AB)$. Then β is a bilinear functional on $\mathbf{F}^{m,n} \times \mathbf{F}^{n,m}$.

9.70 *dimension of the vector space of bilinear functionals*

$\dim \mathcal{B}(V, W) = (\dim V)(\dim W)$.

Proof Let $e_1, \ldots, e_m$ be a basis of V and $f_1, \ldots, f_n$ be a basis of W. For a bilinear functional $\beta \in \mathcal{B}(V, W)$, let $\mathcal{M}(\beta)$ be the m-by-n matrix whose entry in row j, column k is $\beta(e_j, f_k)$. The map $\beta \mapsto \mathcal{M}(\beta)$ is a linear map of $\mathcal{B}(V, W)$ into $\mathbf{F}^{m,n}$.

For a matrix $C \in \mathbf{F}^{m,n}$, define a bilinear functional β_C on $V \times W$ by

$$\beta_C(a_1 e_1 + \cdots + a_m e_m, b_1 f_1 + \cdots + b_n f_n) = \sum_{k=1}^{n} \sum_{j=1}^{m} C_{j,k} a_j b_k$$

for $a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathbf{F}$.

The linear map $\beta \mapsto \mathcal{M}(\beta)$ from $\mathcal{B}(V, W)$ to $\mathbf{F}^{m,n}$ and the linear map $C \mapsto \beta_C$ from $\mathbf{F}^{m,n}$ to $\mathcal{B}(V, W)$ are inverses of each other because $\beta_{\mathcal{M}(\beta)} = \beta$ for all $\beta \in \mathcal{B}(V, W)$ and $\mathcal{M}(\beta_C) = C$ for all $C \in \mathbf{F}^{m,n}$, as you should verify.

Thus both maps are isomorphisms and the two spaces that they connect have the same dimension. Hence $\dim \mathcal{B}(V, W) = \dim \mathbf{F}^{m,n} = mn = (\dim V)(\dim W)$. ∎

Several different definitions of $V \otimes W$ appear in the mathematical literature. These definitions are equivalent to each other, at least in the finite-dimensional context, because any two vector spaces of the same dimension are isomorphic.

The result above states that $\mathcal{B}(V, W)$ has the dimension that we seek, as do $\mathcal{L}(V, W)$ and $\mathbf{F}^{\dim V, \dim W}$. Thus it may be tempting to define $V \otimes W$ to be $\mathcal{B}(V, W)$ or $\mathcal{L}(V, W)$ or $\mathbf{F}^{\dim V, \dim W}$. However, none of those definitions would lead to a basis-free definition of $v \otimes w$ for $v \in V$ and $w \in W$.

The following definition, while it may seem a bit strange and abstract at first, has the huge advantage that it defines $v \otimes w$ in a basis-free fashion. We define $V \otimes W$ to be the vector space of bilinear functionals on $V' \times W'$ instead of the more tempting choice of the vector space of bilinear functionals on $V \times W$.

9.71 definition: *tensor product, $V \otimes W$, $v \otimes w$*

- The *tensor product* $V \otimes W$ is defined to be $\mathcal{B}(V', W')$.
- For $v \in V$ and $w \in W$, the *tensor product* $v \otimes w$ is the element of $V \otimes W$ defined by
$$(v \otimes w)(\varphi, \tau) = \varphi(v)\tau(w)$$
for all $(\varphi, \tau) \in V' \times W'$.

We can quickly prove that the definition of $V \otimes W$ gives it the desired dimension.

9.72 *dimension of the tensor product of two vector spaces*

$\dim(V \otimes W) = (\dim V)(\dim W)$.

Proof Because a vector space and its dual have the same dimension (by 3.111), we have $\dim V' = \dim V$ and $\dim W' = \dim W$. Thus 9.70 tells us that the dimension of $\mathcal{B}(V', W')$ equals $(\dim V)(\dim W)$. ∎

To understand the definition of the tensor product $v \otimes w$ of two vectors $v \in V$ and $w \in W$, focus on the kind of object it is. An element of $V \otimes W$ is a bilinear functional on $V' \times W'$, and in particular it is a function from $V' \times W'$ to $\mathbf{F}$. Thus for each element of $V' \times W'$, it should produce an element of $\mathbf{F}$. The definition above has this behavior, because $v \otimes w$ applied to a typical element (φ, τ) of $V' \times W'$ produces the number $\varphi(v)\tau(w)$.

The somewhat abstract nature of $v \otimes w$ should not matter. The important point is the behavior of these objects. The next result shows that tensor products of vectors have the desired bilinearity properties.

9.73 *bilinearity of tensor product*

Suppose $v, v_1, v_2 \in V$ and $w, w_1, w_2 \in W$ and $\lambda \in \mathbf{F}$. Then

$$(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w \quad \text{and} \quad v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$$

and

$$\lambda(v \otimes w) = (\lambda v) \otimes w = v \otimes (\lambda w).$$

Proof Suppose $(\varphi, \tau) \in V' \times W'$. Then

$$\begin{aligned} \big((v_1 + v_2) \otimes w\big)(\varphi, \tau) &= \varphi(v_1 + v_2)\tau(w) \\ &= \varphi(v_1)\tau(w) + \varphi(v_2)\tau(w) \\ &= (v_1 \otimes w)(\varphi, \tau) + (v_2 \otimes w)(\varphi, \tau) \\ &= (v_1 \otimes w + v_2 \otimes w)(\varphi, \tau). \end{aligned}$$

Thus $(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$.

The other two equalities are proved similarly. ∎

Lists are, by definition, ordered. The order matters when, for example, we form the matrix of an operator with respect to a basis. For lists in this section with two indices, such as $\{e_j \otimes f_k\}_{j=1,\dots,m;k=1,\dots,n}$ in the next result, the ordering does not matter and we do not specify it—just choose any convenient ordering.

The linear independence of elements of $V \otimes W$ in (a) of the result below captures the idea that there are no relationships among vectors in $V \otimes W$ other than the relationships that come from bilinearity of the tensor product (see 9.73) and the relationships that may be present due to linear dependence of a list of vectors in V or a list of vectors in W.

9.74 *basis of $V \otimes W$*

Suppose $e_1, \dots, e_m$ is a list of vectors in V and $f_1, \dots, f_n$ is a list of vectors in W.

(a) If $e_1, \dots, e_m$ and $f_1, \dots, f_n$ are both linearly independent lists, then

$$\{e_j \otimes f_k\}_{j=1,\dots,m;k=1,\dots,n}$$

is a linearly independent list in $V \otimes W$.

(b) If $e_1, \dots, e_m$ is a basis of V and $f_1, \dots, f_n$ is a basis of W, then the list $\{e_j \otimes f_k\}_{j=1,\dots,m;k=1,\dots,n}$ is a basis of $V \otimes W$.

Proof To prove (a), suppose $e_1, \dots, e_m$ and $f_1, \dots, f_n$ are both linearly independent lists. This linear independence and the linear map lemma (3.4) imply that there exist $\varphi_1, \dots, \varphi_m \in V'$ and $\tau_1, \dots, \tau_n \in W'$ such that

$$\varphi_j(e_k) = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k \end{cases} \quad \text{and} \quad \tau_j(f_k) = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k, \end{cases}$$

where $j, k \in \{1, \dots, m\}$ in the first equation and $j, k \in \{1, \dots, n\}$ in the second equation.

Suppose $\{a_{j,k}\}_{j=1,\dots,m;k=1,\dots,n}$ is a list of scalars such that

9.75
$$\sum_{k=1}^{n} \sum_{j=1}^{m} a_{j,k}(e_j \otimes f_k) = 0.$$

Note that $(e_j \otimes f_k)(\varphi_M, \tau_N)$ equals 1 if $j = M$ and $k = N$, and equals 0 otherwise. Thus applying both sides of 9.75 to (φ_M, τ_N) shows that $a_{M,N} = 0$, proving that $\{e_j \otimes f_k\}_{j=1,\dots,m;k=1,\dots,n}$ is linearly independent.

Now (b) follows from (a), the equation $\dim V \otimes W = (\dim V)(\dim W)$ [see 9.72], and the result that a linearly independent list of the right length is a basis (see 2.38). ■

Every element of $V \otimes W$ is a finite sum of elements of the form $v \otimes w$, where $v \in V$ and $w \in W$, as implied by (b) in the result above. However, if $\dim V > 1$ and $\dim W > 1$, then Exercise 4 shows that

$$\{v \otimes w : (v, w) \in V \times W\} \neq V \otimes W.$$

9.76 example: *tensor product of element of* $\mathbf{F}^m$ *with element of* $\mathbf{F}^n$

Suppose m and n are positive integers. Let $e_1, \dots, e_m$ denote the standard basis of $\mathbf{F}^m$ and let $f_1, \dots, f_n$ denote the standard basis of $\mathbf{F}^n$. Suppose

$$v = (v_1, \dots, v_m) \in \mathbf{F}^m \quad \text{and} \quad w = (w_1, \dots, w_n) \in \mathbf{F}^n.$$

Then

$$\begin{aligned} v \otimes w &= \Bigl(\sum_{j=1}^{m} v_j e_j\Bigr) \otimes \Bigl(\sum_{k=1}^{n} w_k f_k\Bigr) \\ &= \sum_{k=1}^{n} \sum_{j=1}^{m} (v_j w_k)(e_j \otimes f_k). \end{aligned}$$

Thus with respect to the basis $\{e_j \otimes f_k\}_{j=1,\dots,m;\,k=1,\dots,n}$ of $\mathbf{F}^m \otimes \mathbf{F}^n$ provided by 9.74(b), the coefficients of $v \otimes w$ are the numbers $\{v_j w_k\}_{j=1,\dots,m;\,k=1,\dots,n}$. If instead of writing these numbers in a list, we write them in an m-by-n matrix with $v_j w_k$ in row j, column k, then we can identify $v \otimes w$ with the m-by-n matrix

$$\begin{pmatrix} v_1 w_1 & \cdots & v_1 w_n \\ & \ddots & \\ v_m w_1 & \cdots & v_m w_n \end{pmatrix}.$$

See Exercises 5 and 6 for practice in using the identification from the example above.

We now define bilinear maps, which differ from bilinear functionals in that the target space can be an arbitrary vector space rather than just the scalar field.

9.77 definition: *bilinear map*

A *bilinear map* from $V \times W$ to a vector space U is a function $\Gamma\colon V \times W \to U$ such that $v \mapsto \Gamma(v, w)$ is a linear map from V to U for each $w \in W$ and $w \mapsto \Gamma(v, w)$ is a linear map from W to U for each $v \in V$.

9.78 example: *bilinear maps*

- Every bilinear functional on $V \times W$ is a bilinear map from $V \times W$ to $\mathbf{F}$.
- The function $\Gamma\colon V \times W \to V \otimes W$ defined by $\Gamma(v, w) = v \otimes w$ is a bilinear map from $V \times W$ to $V \otimes W$ (by 9.73).
- The function $\Gamma\colon \mathcal{L}(V) \times \mathcal{L}(V) \to \mathcal{L}(V)$ defined by $\Gamma(S, T) = ST$ is a bilinear map from $\mathcal{L}(V) \times \mathcal{L}(V)$ to $\mathcal{L}(V)$.
- The function $\Gamma\colon V \times \mathcal{L}(V, W) \to W$ defined by $\Gamma(v, T) = Tv$ is a bilinear map from $V \times \mathcal{L}(V, W)$ to W.

Tensor products allow us to convert bilinear maps on $V \times W$ into linear maps on $V \otimes W$ (and vice versa), as shown by the next result. In the mathematical literature, (a) of the result below is called the "universal property" of tensor products.

9.79 *converting bilinear maps to linear maps*

Suppose U is a vector space.

(a) Suppose $\Gamma\colon V \times W \to U$ is a bilinear map. Then there exists a unique linear map $\hat{\Gamma}\colon V \otimes W \to U$ such that

$$\hat{\Gamma}(v \otimes w) = \Gamma(v, w)$$

for all $(v, w) \in V \times W$.

(b) Conversely, suppose $T\colon V \otimes W \to U$ is a linear map. There there exists a unique bilinear map $T^{\#}\colon V \times W \to U$ such that

$$T^{\#}(v, w) = T(v \otimes w)$$

for all $(v, w) \in V \times W$.

Proof Let $e_1, \dots, e_m$ be a basis of V and let $f_1, \dots, f_n$ be a basis of W. By the linear map lemma (3.4) and 9.74(b), there exists a unique linear map $\hat{\Gamma}\colon V \otimes W \to U$ such that

$$\hat{\Gamma}(e_j \otimes f_k) = \Gamma(e_j, f_k)$$

for all $j \in \{1, \dots, m\}$ and $k \in \{1, \dots, n\}$.

Now suppose $(v, w) \in V \times W$. There exist $a_1, \dots, a_m, b_1, \dots, b_n \in \mathbf{F}$ such that $v = a_1 e_1 + \cdots + a_m e_m$ and $w = b_1 f_1 + \cdots + b_n f_n$. Thus

$$\begin{aligned}
\hat{\Gamma}(v \otimes w) &= \hat{\Gamma}\Bigl(\sum_{k=1}^{n}\sum_{j=1}^{m}(a_j b_k)(e_j \otimes f_k)\Bigr)\\
&= \sum_{k=1}^{n}\sum_{j=1}^{m} a_j b_k \hat{\Gamma}(e_j \otimes f_k)\\
&= \sum_{k=1}^{n}\sum_{j=1}^{m} a_j b_k \Gamma(e_j, f_k)\\
&= \Gamma(v, w),
\end{aligned}$$

as desired, where the second line holds because $\hat{\Gamma}$ is linear, the third line holds by the definition of $\hat{\Gamma}$, and the fourth line holds because Γ is bilinear.

The uniqueness of the linear map $\hat{\Gamma}$ satisfying $\hat{\Gamma}(v \otimes w) = \Gamma(v, w)$ follows from 9.74(b), completing the proof of (a).

To prove (b), define a function $T^{\#}\colon V \times W \to U$ by $T^{\#}(v, w) = T(v \otimes w)$ for all $(v, w) \in V \times W$. The bilinearity of the tensor product (see 9.73) and the linearity of T imply that $T^{\#}$ is bilinear.

Clearly the choice of $T^{\#}$ that satisfies the conditions is unique. ■

To prove 9.79(a), we could not just define $\hat{\Gamma}(v \otimes w) = \Gamma(v, w)$ for all $v \in V$ and $w \in W$ (and then extend $\hat{\Gamma}$ linearly to all of $V \otimes W$) because elements of $V \otimes W$ do not have unique representations as finite sums of elements of the form $v \otimes w$. Our proof used a basis of V and a basis of W to get around this problem.

Although our construction of $\hat{\Gamma}$ in the proof of 9.79(a) depended on a basis of V and a basis of W, the equation $\hat{\Gamma}(v \otimes w) = \Gamma(v, w)$ that holds for all $v \in V$ and $w \in W$ shows that $\hat{\Gamma}$ does not depend on the choice of bases for V and W.

Tensor Product of Inner Product Spaces

The result below features three inner products—one on $V \otimes W$, one on V, and one on W, although we use the same symbol $\langle \cdot, \cdot \rangle$ for all three inner products.

9.80 *inner product on tensor product of two inner product spaces*

Suppose V and W are inner product spaces. Then there is a unique inner product on $V \otimes W$ such that

$$\langle v \otimes w, u \otimes x \rangle = \langle v, u \rangle \langle w, x \rangle$$

for all $v, u \in V$ and $w, x \in W$.

Proof Suppose $e_1, \dots, e_m$ is an orthonormal basis of V and $f_1, \dots, f_n$ is an orthonormal basis of W. Define an inner product on $V \otimes W$ by

9.81
$$\left\langle \sum_{k=1}^{n} \sum_{j=1}^{m} b_{j,k}\, e_j \otimes f_k, \sum_{k=1}^{n} \sum_{j=1}^{m} c_{j,k}\, e_j \otimes f_k \right\rangle = \sum_{k=1}^{n} \sum_{j=1}^{m} b_{j,k} \overline{c_{j,k}}.$$

The straightforward verification that 9.81 defines an inner product on $V \otimes W$ is left to the reader [use 9.74(b)].

Suppose that $v, u \in V$ and $w, x \in W$. Let $v_1, \dots, v_m \in \mathbf{F}$ be such that $v = v_1 e_1 + \cdots + v_m e_m$, with similar expressions for u, w, and x. Then

$$\begin{aligned}
\langle v \otimes w, u \otimes x \rangle &= \left\langle \sum_{j=1}^{m} v_j e_j \otimes \sum_{k=1}^{n} w_k f_k, \sum_{j=1}^{m} u_j e_j \otimes \sum_{k=1}^{n} x_k f_k \right\rangle \\
&= \left\langle \sum_{k=1}^{n} \sum_{j=1}^{m} v_j w_k\, e_j \otimes f_k, \sum_{k=1}^{n} \sum_{j=1}^{m} u_j x_k\, e_j \otimes f_k \right\rangle \\
&= \sum_{k=1}^{n} \sum_{j=1}^{m} v_j \overline{u_j} w_k \overline{x_k} \\
&= \left(\sum_{j=1}^{m} v_j \overline{u_j} \right) \left(\sum_{k=1}^{n} w_k \overline{x_k} \right) \\
&= \langle v, u \rangle \langle w, x \rangle .
\end{aligned}$$

There is only one inner product on $V \otimes W$ such that $\langle v \otimes w, u \otimes x \rangle = \langle v, u \rangle \langle w, x \rangle$ for all $v, u \in V$ and $w, x \in W$ because every element of $V \otimes W$ can be written as a linear combination of elements of the form $v \otimes w$ [by 9.74(b)]. ■

The definition below of a natural inner product on $V \otimes W$ is now justified by 9.80. We could not have simply defined $\langle v \otimes w, u \otimes x \rangle$ to be $\langle v, u \rangle \langle w, x \rangle$ (and then used additivity in each slot separately to extend the definition to $V \otimes W$) without some proof because elements of $V \otimes W$ do not have unique representations as finite sums of elements of the form $v \otimes w$.

9.82 definition: *inner product on tensor product of two inner product spaces*

Suppose V and W are inner product spaces. The inner product on $V \otimes W$ is the unique function $\langle \cdot, \cdot \rangle$ from $(V \otimes W) \times (V \otimes W)$ to $\mathbf{F}$ such that

$$\langle v \otimes w, u \otimes x \rangle = \langle v, u \rangle \langle w, x \rangle$$

for all $v, u \in V$ and $w, x \in W$.

Take $u = v$ and $x = w$ in the equation above and then take square roots to show that

$$\|v \otimes w\| = \|v\| \, \|w\|$$

for all $v \in V$ and all $w \in W$.

The construction of the inner product in the proof of 9.80 depended on an orthonormal basis $e_1, \ldots, e_m$ of V and an orthonormal basis $f_1, \ldots, f_n$ of W. Formula 9.81 implies that $\{e_j \otimes f_k\}_{j=1,\ldots,m;\, k=1,\ldots,n}$ is a doubly indexed orthonormal list in $V \otimes W$ and hence is an orthonormal basis of $V \otimes W$ [by 9.74(b)]. The importance of the next result arises because the orthonormal bases used there can be different from the orthonormal bases used to define the inner product in 9.80. Although the notation for the bases is the same in the proof of 9.80 and in the result below, think of them as two different sets of orthonormal bases.

9.83 *orthonormal basis of $V \otimes W$*

Suppose V and W are inner product spaces, and $e_1, \ldots, e_m$ is an orthonormal basis of V and $f_1, \ldots, f_n$ is an orthonormal basis of W. Then

$$\{e_j \otimes f_k\}_{j=1,\ldots,m;\, k=1,\ldots,n}$$

is an orthonormal basis of $V \otimes W$.

Proof We know that $\{e_j \otimes f_k\}_{j=1,\ldots,m;\, k=1,\ldots,n}$ is a basis of $V \otimes W$ [by 9.74(b)]. Thus we only need to verify orthonormality. To do this, suppose $j, M \in \{1, \ldots, m\}$ and $k, N \in \{1, \ldots, n\}$. Then

$$\langle e_j \otimes f_k, e_M \otimes f_N \rangle = \langle e_j, e_M \rangle \langle f_k, f_N \rangle = \begin{cases} 1 & \text{if } j = M \text{ and } k = N, \\ 0 & \text{otherwise.} \end{cases}$$

Hence the doubly indexed list $\{e_j \otimes f_k\}_{j=1,\ldots,m;\, k=1,\ldots,n}$ is indeed an orthonormal basis of $V \otimes W$. ■

See Exercise 11 for an example of how the inner product structure on $V \otimes W$ interacts with operators on V and W.

Tensor Product of Multiple Vector Spaces

We have been discussing properties of the tensor product of two finite-dimensional vector spaces. Now we turn our attention to the tensor product of multiple finite-dimensional vector spaces. This generalization requires no new ideas, only some slightly more complicated notation. Readers with a good understanding of the tensor product of two vector spaces should be able to make the extension to the tensor product of more than two vector spaces.

Thus in this subsection, no proofs will be provided. The definitions and the statements of results that will be provided should be enough information to enable readers to fill in the details, using what has already been learned about the tensor product of two vector spaces.

We begin with the following notational assumption.

9.84 notation: $V_1, \dots, V_m$

For the rest of this subsection, m denotes an integer greater than 1 and $V_1, \dots, V_m$ denote finite-dimensional vector spaces.

The notion of an m-linear functional, which we are about to define, generalizes the notion of a bilinear functional (see 9.68). Recall that the use of the word "functional" indicates that we are mapping into the scalar field $\mathbf{F}$. Recall also that the terminology "m-linear form" is used in the special case $V_1 = \cdots = V_m$ (see 9.25). The notation $\mathcal{B}(V_1, \dots, V_m)$ generalizes our previous notation $\mathcal{B}(V, W)$.

9.85 definition: *m-linear functional, the vector space $\mathcal{B}(V_1, \dots, V_m)$*

- An *m-linear functional* on $V_1 \times \cdots \times V_m$ is a function $\beta\colon V_1 \times \cdots \times V_m \to \mathbf{F}$ that is a linear functional in each slot when the other slots are held fixed.
- The vector space of m-linear functionals on $V_1 \times \cdots \times V_m$ is denoted by $\mathcal{B}(V_1, \dots, V_m)$.

9.86 example: *m-linear functional*

Suppose $\varphi_k \in V_k{}'$ for each $k \in \{1, \dots, m\}$. Define $\beta\colon V_1 \times \cdots \times V_m \to \mathbf{F}$ by

$$\beta(v_1, \dots, v_m) = \varphi_1(v_1) \times \cdots \times \varphi_m(v_m).$$

Then β is an m-linear functional on $V_1 \times \cdots \times V_m$.

The next result can be proved by imitating the proof of 9.70.

9.87 *dimension of the vector space of m-linear functionals*

$\dim \mathcal{B}(V_1, \dots, V_m) = (\dim V_1) \times \cdots \times (\dim V_m)$.

Now we can define the tensor product of multiple vector spaces and the tensor product of elements of those vector spaces. The following definition is completely analogous to our previous definition (9.71) in the case $m = 2$.

9.88 definition: *tensor product,* $V_1 \otimes \cdots \otimes V_m$, $v_1 \otimes \cdots \otimes v_m$

- The *tensor product* $V_1 \otimes \cdots \otimes V_m$ is defined to be $\mathcal{B}(V_1', \dots, V_m')$.
- For $v_1 \in V_1, \dots, v_m \in V_m$, the *tensor product* $v_1 \otimes \cdots \otimes v_m$ is the element of $V_1 \otimes \cdots \otimes V_m$ defined by
$$(v_1 \otimes \cdots \otimes v_m)(\varphi_1, \dots, \varphi_m) = \varphi_1(v_1) \cdots \varphi_m(v_m)$$
for all $(\varphi_1, \dots, \varphi_m) \in V_1' \times \cdots \times V_m'$.

The next result can be proved by following the pattern of the proof of the analogous result when $m = 2$ (see 9.72).

9.89 *dimension of the tensor product*

$\dim(V_1 \otimes \cdots \otimes V_m) = (\dim V_1) \cdots (\dim V_m)$.

Our next result generalizes 9.74.

9.90 *basis of* $V_1 \otimes \cdots \otimes V_m$

Suppose $\dim V_k = n_k$ and $e_1^k, \dots, e_{n_k}^k$ is a basis of V_k for $k = 1, \dots, m$. Then
$$\{e_{j_1}^1 \otimes \cdots \otimes e_{j_m}^m\}_{j_1 = 1, \dots, n_1; \cdots; j_m = 1, \dots, n_m}$$
is a basis of $V_1 \otimes \cdots \otimes V_m$.

Suppose $m = 2$ and $e_1^1, \dots, e_{n_1}^1$ is a basis of V_1 and $e_1^2, \dots, e_{n_2}^2$ is a basis of V_2. Then with respect to the basis $\{e_{j_1}^1 \otimes e_{j_2}^2\}_{j_1 = 1, \dots, n_1; j_2 = 1, \dots, n_2}$ in the result above, the coefficients of an element of $V_1 \otimes V_2$ can be represented by an n_1-by-n_2 matrix that contains the coefficient of $e_{j_1}^1 \otimes e_{j_2}^2$ in row j_1, column j_2. Thus we need a matrix, which is an array specified by two indices, to represent an element of $V_1 \otimes V_2$.

If $m > 2$, then the result above shows that we need an array specified by m indices to represent an arbitrary element of $V_1 \otimes \cdots \otimes V_m$. Thus tensor products may appear when we deal with objects specified by arrays with multiple indices.

The next definition generalizes the notion of a bilinear map (see 9.77). As with bilinear maps, the target space can be an arbitrary vector space.

9.91 definition: *m-linear map*

An *m-linear map* from $V_1 \times \cdots \times V_m$ to a vector space U is a function $\Gamma \colon V_1 \times \cdots \times V_m \to U$ that is a linear map in each slot when the other slots are held fixed.

The next result can be proved by following the pattern of the proof of 9.79.

9.92 *converting m-linear maps to linear maps*

Suppose U is a vector space.

(a) Suppose that $\Gamma\colon V_1 \times \cdots \times V_m \to U$ is an m-linear map. Then there exists a unique linear map $\hat{\Gamma}\colon V_1 \otimes \cdots \otimes V_m \to U$ such that

$$\hat{\Gamma}(v_1 \otimes \cdots \otimes v_m) = \Gamma(v_1, \ldots, v_m)$$

for all $(v_1, \ldots, v_m) \in V_1 \times \cdots \times V_m$.

(b) Conversely, suppose $T\colon V_1 \otimes \cdots \otimes V_m \to U$ is a linear map. There there exists a unique m-linear map $T^{\#}\colon V_1 \times \cdots \times V_m \to U$ such that

$$T^{\#}(v_1, \ldots, v_m) = T(v_1 \otimes \cdots \otimes v_m)$$

for all $(v_1, \ldots, v_m) \in V_1 \times \cdots \times V_m$.

See Exercises 12 and 13 for tensor products of multiple inner product spaces.

Exercises 9D

1 Suppose $v \in V$ and $w \in W$. Prove that $v \otimes w = 0$ if and only if $v = 0$ or $w = 0$.

2 Give an example of six distinct vectors $v_1, v_2, v_3, w_1, w_2, w_3$ in $\mathbf{R}^3$ such that

$$v_1 \otimes w_1 + v_2 \otimes w_2 + v_3 \otimes w_3 = 0$$

but none of $v_1 \otimes w_1, v_2 \otimes w_2, v_3 \otimes w_3$ is a scalar multiple of another element of this list.

3 Suppose that $v_1, \ldots, v_m$ is a linearly independent list in V. Suppose also that $w_1, \ldots, w_m$ is a list in W such that

$$v_1 \otimes w_1 + \cdots + v_m \otimes w_m = 0.$$

Prove that $w_1 = \cdots = w_m = 0$.

4 Suppose $\dim V > 1$ and $\dim W > 1$. Prove that

$$\{v \otimes w : (v, w) \in V \times W\}$$

is not a subspace of $V \otimes W$.

This exercise implies that if $\dim V > 1$ *and* $\dim W > 1$, *then*

$$\{v \otimes w : (v, w) \in V \times W\} \neq V \otimes W.$$

5 Suppose m and n are positive integers. For $v \in \mathbf{F}^m$ and $w \in \mathbf{F}^n$, identify $v \otimes w$ with an m-by-n matrix as in Example 9.76. With that identification, show that the set

$$\{v \otimes w : v \in \mathbf{F}^m \text{ and } w \in \mathbf{F}^n\}$$

is the set of m-by-n matrices (with entries in $\mathbf{F}$) that have rank at most one.

6 Suppose m and n are positive integers. Give a description, analogous to Exercise 5, of the set of m-by-n matrices (with entries in $\mathbf{F}$) that have rank at most two.

7 Suppose $\dim V > 2$ and $\dim W > 2$. Prove that

$$\{v_1 \otimes w_1 + v_2 \otimes w_2 : v_1, v_2 \in V \text{ and } w_1, w_2 \in W\} \neq V \otimes W.$$

8 Suppose $v_1, \dots, v_m \in V$ and $w_1, \dots, w_m \in W$ are such that

$$v_1 \otimes w_1 + \cdots + v_m \otimes w_m = 0.$$

Suppose that U is a vector space and $\Gamma \colon V \times W \to U$ is a bilinear map. Show that

$$\Gamma(v_1, w_1) + \cdots + \Gamma(v_m, w_m) = 0.$$

9 Suppose $S \in \mathcal{L}(V)$ and $T \in \mathcal{L}(W)$. Prove that there exists a unique operator on $V \otimes W$ that takes $v \otimes w$ to $Sv \otimes Tw$ for all $v \in V$ and $w \in W$.

In an abuse of notation, the operator on $V \otimes W$ given by this exercise is often called $S \otimes T$.

10 Suppose $S \in \mathcal{L}(V)$ and $T \in \mathcal{L}(W)$. Prove that $S \otimes T$ is an invertible operator on $V \otimes W$ if and only if both S and T are invertible operators. Also, prove that if both S and T are invertible operators, then $(S \otimes T)^{-1} = S^{-1} \otimes T^{-1}$, where we are using the notation from the comment after Exercise 9.

11 Suppose V and W are inner product spaces. Prove that if $S \in \mathcal{L}(V)$ and $T \in \mathcal{L}(W)$, then $(S \otimes T)^* = S^* \otimes T^*$, where we are using the notation from the comment after Exercise 9.

12 Suppose that $V_1, \dots, V_m$ are finite-dimensional inner product spaces. Prove that there is a unique inner product on $V_1 \otimes \cdots \otimes V_m$ such that

$$\langle v_1 \otimes \cdots \otimes v_m, u_1 \otimes \cdots \otimes u_m \rangle = \langle v_1, u_1 \rangle \cdots \langle v_m, u_m \rangle$$

for all $(v_1, \dots, v_m)$ and $(u_1, \dots, u_m)$ in $V_1 \times \cdots \times V_m$.

Note that the equation above implies that

$$\|v_1 \otimes \cdots \otimes v_m\| = \|v_1\| \times \cdots \times \|v_m\|$$

for all $(v_1, \dots, v_m) \in V_1 \times \cdots \times V_m$.

13 Suppose that $V_1, \dots, V_m$ are finite-dimensional inner product spaces and $V_1 \otimes \cdots \otimes V_m$ is made into an inner product space using the inner product from Exercise 12. Suppose $e_1^k, \dots, e_{n_k}^k$ is an orthonormal basis of V_k for each $k = 1, \dots, m$. Show that the list

$$\{e_{j_1}^1 \otimes \cdots \otimes e_{j_m}^m\}_{j_1 = 1, \dots, n_1; \cdots; j_m = 1, \dots, n_m}$$

is an orthonormal basis of $V_1 \otimes \cdots \otimes V_m$.