Exercise 56

Use the spectral decomposition to show that K $\equiv$ -i log(U) is Hermitian for any Unitary U, and thus U = exp(iK) for some Hermitian K.

Suffices to show that -i log(U) = (-i log(U))†

Suffices to show that -i log(U) = i(log(U))†

Suffices to show that -i log(U) = i(log(U†))

Suffices to show that i log(U^{-1}) = i log(U†)

Suffices to show that U^{-1} = U†, but unitary operation is defined so. $\blacksquare$

Exercise 57

To show that cascaded measurements are single measurements, we would have to show two things, that the cascaded measurement takes any arbitrary state to the state state as the single measurement and tha the the probability distribution of eigenvalues of the cascaded measurement is the same as that of the single measurement:

i) same eigenvector: Let $|\psi\rangle$ be the initial arbitrary state

post first measurement: $\frac{L_l|\psi\rangle}{\sqrt{\langle\psi|L^\dagger L|\psi\rangle}}$

post second measurement:

numerator:

$$M_m(\frac{M_m(\frac{L_l|psik}{\sqrt{\langle\psi|L_l^\dagger L_l\langle\psi|}})}{\sqrt{\frac{\langle\psi|L_l^\dagger}{\langle\psi|L_l^\dagger L_l|\psi\rangle}M_m^\dagger M_m\frac{L_l|\psi\rangle}{\sqrt{\langle\psi|L_l^\dagger L_l|\psi\rangle}}}})$$

denominator:

$$= \sqrt{\frac{\langle\psi|L_l^\dagger M_m^\dagger M_m L_l|\psi\rangle}{\langle\psi|L_l^\dagger L_l|\psi\rangle}} = \sqrt{\frac{\langle\psi|L_l^\dagger M_m^\dagger M_m L_l|\psi\rangle}{\langle\psi|L_l^\dagger L_l|\psi\rangle}}$$

the state after second measurement:

$$\frac{M_m L_l|\psi\rangle}{\sqrt{\langle\psi|L_l^\dagger L|\psi\rangle}\frac{\sqrt{\langle\psi|L_l^\dagger M_m^\dagger M_m L_l|\psi\rangle}}{\sqrt{\langle\psi|L-l^\dagger L|\psi\rangle}}}$$

$$= \frac{M_m L_l|\psi\rangle}{\cancel{\sqrt{\langle\psi|L_l^\dagger L|\psi\rangle}}\frac{\sqrt{\langle\psi|L_l^\dagger M_m^\dagger M_m L_l|\psi\rangle}}{\cancel{\sqrt{\langle\psi|L-l^\dagger L|\psi\rangle}}}}$$

$$= \frac{M_m L_l|\psi\rangle}{\sqrt{\langle\psi|L_l^\dagger M_m^\dagger M_m L_l|\psi\rangle}} \quad \blacksquare$$

Exercise 58

Suppose we prepare a quantum system in an eigenstate $|\psi\rangle$ of some observable M,w with corresponding eigenvalue m.What is the average observed value of M, and the standard deviation?

Average value $= \langle\phi|M|\phi\rangle$

$$\Rightarrow \frac{\langle\psi|M_m^\dagger}{\sqrt{\langle\psi|M_m^\dagger M_m|\psi\rangle}} M \frac{M_m|\psi\rangle}{\sqrt{\langle\psi|M_m^\dagger M_m|\psi\rangle}}$$

$$= \frac{\langle\psi|M_m^\dagger M M_m|\psi\rangle}{\langle\psi|M_m^\dagger M_m|\psi\rangle}$$

$$= \frac{\langle\psi|M_m^\dagger M_m|\psi\rangle}{\langle\psi|M_m^\dagger M_m|\psi\rangle} = 1?$$

Note that since the effect of the M on M_m was leave it unchanged, the same will happen with M^2, meaning $\sqrt{\langle M^2\rangle - \langle M\rangle^2} = \sqrt{1-1} = 0$ ■

Bonus

Show that $\sqrt{\langle (M - \langle M\rangle)^2\rangle} = \sqrt{\langle M^2\rangle - \langle M\rangle^2}$

Ex 2.59

Suppose we have a qubit in the state $|0\rangle$, and we measure the observable X. What is the average value of X? What is the standard deviation of X?

$\langle X\rangle = \langle 0|X|0\rangle = \langle 0|1\rangle = 0$

$\sqrt{\langle X^2\rangle = \langle X\rangle^2} = \sqrt{\langle 0|X|0\rangle - (\langle 0|X|0\rangle)^2} = \sqrt{1-0} = \pm 1$ ■

Ex 2.60

i) Show that $\vec{v} \cdot \vec{\sigma}$ has eigenvalues $\pm$ 1.

Solution:

Let $\vec{v}$ be $\begin{pmatrix} a \\ b \\ \sqrt{1-a^2-b^2} \end{pmatrix}$.

$\Rightarrow \vec{v} \cdot \vec{\sigma} = a\sigma_1 + b\sigma_2 + c\sigma_3$

$= \begin{bmatrix} \sqrt{1-a^2-b^2} & a-ib \\ a+ib & -\sqrt{1-a^2-b^2} \end{bmatrix}$

$\Rightarrow$ the characteristic equation is $(\sqrt{1-a^2-b^2} - \lambda)(-\sqrt{1-a^2-b^2} - \lambda) - (a-ib)(a+ib) = 0$

$\Rightarrow \lambda^2 - (1-a^2-b^2) - a^2 - b^2 = 0$

$\Rightarrow \lambda^2 = 1$

$\Rightarrow \lambda = \pm 1$ ■

ii) Show that the projectors onto the corresponding eigenspaces are given by $P_{\pm} = (I \pm \vec{v} \cdot \vec{\sigma})/2$

So long as $\vec{v} \cdot \vec{\sigma}$ defines a measurement,

$\vec{v} \cdot \vec{\sigma} = \Sigma_i \lambda_i P_i$ (where P_i are the projectors.)

$= (+1P_+) + (-1P_-)$

$\Rightarrow \vec{v} \cdot \vec{\sigma} = P_+ - P_-$ (Let this be eq 1)

$I = \Sigma_i P_i$ (Let this be eq 2)

Using eq2 in eq $1 we have \vec{v} \cdot \vec{\sigma} = P_+ - (I + P_+)$

$\Rightarrow \vec{v} \cdot \vec{\sigma} = 2P_+ - I \Rightarrow P_+ = (I + \vec{v} \cdot \vec{\sigma})/2$

Similarly,

Using eq2 in eq $1 we have \vec{v} \cdot \vec{\sigma} = P_+ - (I + P_+)$

$\Rightarrow \vec{v} \cdot \vec{\sigma} = I - 2P_- \Rightarrow P_- = (I - \vec{v} \cdot \vec{\sigma})/2$

Thus, $P_{\pm} = (I \pm \vec{v} \cdot \vec{\sigma})/2$ ■

Ex 2.61

Calculate the probability of obtaining the result $+1$ for a measurement of $\vec{v}\cdot\vec{\sigma}$, given that the state prior to measurement is $|0\rangle$. What is the state of the system after the measurement if $+1$ is obtained?

Probability of measuring $+1$ when the system is in the state $|0\rangle$ is given by

p$(+1)$

$= \langle 0|(\vec{v}\cdot\vec{\sigma})|0\rangle$

$= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{bmatrix} \sqrt{1-a^2-b^2} & a-ib \\ a+ib & -\sqrt{1-a^2-b^2} \end{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{bmatrix} \sqrt{1-a^2-b^2} \\ a+ib \end{bmatrix}$

$= \sqrt{1-a^2-b^2}$

The state of the system after the measurement is obatined is P_+ ie, $(I+\vec{v}\cdot\vec{\sigma})/2$ ■

Ex 2.62

Show that any measurement where the Measurement opertors and the POVM elements coincide is a projective measurement.

POVM elements: P_i such that $\Sigma_i P_i = I$

Measurement Operators: M_m describe a measurement such that $\langle\psi|M_m|\psi\rangle = pr(m)$

Show that if these two coincide such that we have $\Sigma_m M_m = I$, then

$M_m M_n = 0$, where m $\neq$ n

Solution:

Consider two projectors M_m, M_n such that they are together complete

$\Rightarrow M_m = I - M_n$

Now consider $M_m M_n|\psi\rangle$:

$= M_m(M_n|\psi\rangle)$

$= (I - M_n)(M_n|\psi\rangle)$

$= (M_n|\psi\rangle) - (M_n^2|\psi\rangle)$

$= M_n|\psi\rangle - M_n|\psi\rangle$

$= 0$

Thus, $M_m M_n = 0$

This solves it for the case where there are two projectors

What about the case where there are 3 or n projectors? Is it that as long as a projector is orthogonal with its complement with Identity, every pair of projectors conceivable for that observable is orthogonal?

This solution is not right

Ex 2.63

Before we dive into the problem, let's first understand what measurement is, what measurement operators are, what the POVM formalism is. This is where we are ultimately headed in this section.

Hilbert space describes a quantum system: Lets start at the beginning, a very good place to start. You have an isolated quantum system. This is described by a Hilbert space. The Hilbert space could be a tensor product space or an ordinary Hilbert space depending on whether the system is considered composite or not.

Now the system could be represented by a state vector in the Hilbert space at a given instant of time. (Time and energy have an uncertainity relationship, but let's set that aside for now). But we dont know the state of the system. It could be in any state in the Hilbert space. Now, since we dont know the state of the system, it is in a mixed state of infinite possibilities. This is best described [for convenience] by a density matrix.

Now, as we move along the time dimension, the system changes. The isolated system has unitary evolution. In that, the system itself changes. If it is in a Pure state, it remains a pure state, but the amplitudes change.

Can we say that the density matrix has a unitary evolution?

The state certainly has unitary freedom. The density matrix has unitary freedom. These two freedoms are not the same. But are they related?