3 Suppose V is a complex vector space and $\varphi \in V'$. Define $\sigma\colon V \to \mathbf{R}$ by $\sigma(v) = \operatorname{Re}\varphi(v)$ for each $v \in V$. Show that

$$\varphi(v) = \sigma(v) - i\sigma(iv)$$

for all $v \in V$.

4 Suppose m is a positive integer. Is the set

$$\{0\} \cup \{p \in \mathcal{P}(\mathbf{F}) : \deg p = m\}$$

a subspace of $\mathcal{P}(\mathbf{F})$?

5 Is the set

$$\{0\} \cup \{p \in \mathcal{P}(\mathbf{F}) : \deg p \text{ is even}\}$$

a subspace of $\mathcal{P}(\mathbf{F})$?

6 Suppose that m and n are positive integers with $m \leq n$, and suppose $\lambda_1, \dots, \lambda_m \in \mathbf{F}$. Prove that there exists a polynomial $p \in \mathcal{P}(\mathbf{F})$ with $\deg p = n$ such that $0 = p(\lambda_1) = \cdots = p(\lambda_m)$ and such that p has no other zeros.

7 Suppose that m is a nonnegative integer, $z_1, \dots, z_{m+1}$ are distinct elements of $\mathbf{F}$, and $w_1, \dots, w_{m+1} \in \mathbf{F}$. Prove that there exists a unique polynomial $p \in \mathcal{P}_m(\mathbf{F})$ such that

$$p(z_k) = w_k$$

for each $k = 1, \dots, m+1$.

This result can be proved without using linear algebra. However, try to find the clearer, shorter proof that uses some linear algebra.

8 Suppose $p \in \mathcal{P}(\mathbf{C})$ has degree m. Prove that p has m distinct zeros if and only if p and its derivative p' have no zeros in common.

9 Prove that every polynomial of odd degree with real coefficients has a real zero.

10 For $p \in \mathcal{P}(\mathbf{R})$, define $Tp\colon \mathbf{R} \to \mathbf{R}$ by

$$(Tp)(x) = \begin{cases} \dfrac{p(x) - p(3)}{x - 3} & \text{if } x \neq 3, \\ p'(3) & \text{if } x = 3 \end{cases}$$

for each $x \in \mathbf{R}$. Show that $Tp \in \mathcal{P}(\mathbf{R})$ for every polynomial $p \in \mathcal{P}(\mathbf{R})$ and also show that $T\colon \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ is a linear map.

11 Suppose $p \in \mathcal{P}(\mathbf{C})$. Define $q\colon \mathbf{C} \to \mathbf{C}$ by

$$q(z) = p(z)\,\overline{p(\overline{z})}.$$

Prove that q is a polynomial with real coefficients.

12 Suppose m is a nonnegative integer and $p \in \mathcal{P}_m(\mathbf{C})$ is such that there are distinct real numbers $x_0, x_1, \dots, x_m$ with $p(x_k) \in \mathbf{R}$ for each $k = 0, 1, \dots, m$. Prove that all coefficients of p are real.

13 Suppose $p \in \mathcal{P}(\mathbf{F})$ with $p \neq 0$. Let $U = \{pq : q \in \mathcal{P}(\mathbf{F})\}$.

(a) Show that $\dim \mathcal{P}(\mathbf{F})/U = \deg p$.

(b) Find a basis of $\mathcal{P}(\mathbf{F})/U$.

14 Suppose $p, q \in \mathcal{P}(\mathbf{C})$ are nonconstant polynomials with no zeros in common. Let $m = \deg p$ and $n = \deg q$. Use linear algebra as outlined below in (a)–(c) to prove that there exist $r \in \mathcal{P}_{n-1}(\mathbf{C})$ and $s \in \mathcal{P}_{m-1}(\mathbf{C})$ such that

$$rp + sq = 1.$$

(a) Define $T\colon \mathcal{P}_{n-1}(\mathbf{C}) \times \mathcal{P}_{m-1}(\mathbf{C}) \to \mathcal{P}_{m+n-1}(\mathbf{C})$ by

$$T(r, s) = rp + sq.$$

Show that the linear map T is injective.

(b) Show that the linear map T in (a) is surjective.

(c) Use (b) to conclude that there exist $r \in \mathcal{P}_{n-1}(\mathbf{C})$ and $s \in \mathcal{P}_{m-1}(\mathbf{C})$ such that $rp + sq = 1$.