Superdense Coding

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Superdesnse coding is about transformations

1 setup

Task: Send 2 classical bits by sending 1 qubit[1] across

To carry out this task we use an entangled pair and transform it to encode information;

There are four entangled pairs:

$\frac{|00\rangle+|11\rangle}{\sqrt{2}}$, $\frac{|00\rangle-|11\rangle}{\sqrt{2}}$, $\frac{|01\rangle+|10\rangle}{\sqrt{2}}$, $\frac{|01\rangle-|10\rangle}{\sqrt{2}}$,

and the possibilities of four values of 2 classical bits:

00, 11, 01, 10;

00: $\frac{|00\rangle+|11\rangle}{\sqrt{2}}$, 11: $\frac{|00\rangle-|11\rangle}{\sqrt{2}}$, 01: $\frac{|01\rangle+|10\rangle}{\sqrt{2}}$, 10: $\frac{|01\rangle-|10\rangle}{\sqrt{2}}$

The four entangles pairs are I, X, Z and XZ transformations apart [2] [3]

Using these transformations Alice can send her two classical bits to Bob by sending over a tranformed qubit. Upon receipt of the qubit, Bob can measure it in the Bell basis[4] since it forms an orthonormal basis [5].

2 solution

steps:

1. Alice and Bob share the mapping from classical information of 2 bits to quantum information of entangled pairs
2. Alice performs the necessary transformation to reflect her choice
3. Transfer the entangled pair through a quantum channel

[1] is an entangled pair considered one qubit or two? If it is considered one, why is it considered one and how do you refer to the information in the subsystems? If it is considered two qubits, why does the text say that superdense coding is about sending 2 classical bits by sending *1* qubit?

[2] XZ = -2iY, but -2 is global amplitude scalar, and this does not affect our bell state to classical doublt-bit encoding; the phase is global and so can be ignored and the magnitude disappears upon renormalization

[3] The transformations are wrt to the qubit of the second subsystem

[5] Refer to appendix B, to see how bell basis forms an orthonormal basis.

[4] You have to measure it in the bell basis and not any other basis like the computational basis, since measuring it in other bases would not return any bell state with probability 1, but when measured int he bell basis, one of the four bell states is returned with probability 1 and the other three with probability 0, accomodating determinstic communication.

In a thought experiment, if you are Bob and trying to decode the information Alice sent (assuming you have the mapping from qubit state to classical information), you might be tempted to measure it in the computational basis, but you will quickly (or taking your time) realize that you will lose information about relative phase and will only be able to the information down to one of two camps of the bell states, where members of the camp only differ by a relative phase. Note then that you want to extract phase information. If you want to extract any information you will have to measure, or do it before measurement. All we can done before the measuremen, is unitary transformations and so to recover/extract the relative phase, perform a unitary transformation. It turns out the unitary transformation (for there could be any number of unitary transformations) that does the job is basis change transformation. Or you could just shorten the process and directly measure it in the bell basis. But if you want to measure it in the computational basis, first perform the necessary unitary transformation.

4. Bob measures the qubit in bell basis and uses the map to get the classical information.

Note that it cannot be a shared system with each person carrying one subsystem since the receiver needs both subsystems to ascertaint both the classical bits

3 Appendix A

When there are 32 superpositions, why are the 4 states so special that they get their own two names (EPR pairs and bell states)?

There are 32 equally weighted superpositions of $|10\rangle, |01\rangle, |10\rangle, |11\rangle$. Square the cardinality of the set of equally weighted superpositions. Taking into account negative relative phase, we double that number. $\Rightarrow 2 \times 4^2 = 32$.

Of these 8 are not actually superpositions[5]:

$\frac{|00\rangle+|00\rangle}{2}, \frac{|00\rangle-|00\rangle}{2}, \frac{|01\rangle+|01\rangle}{2}, \frac{|01\rangle-|01\rangle}{2},$

$\frac{|10\rangle+|10\rangle}{2}, \frac{|10\rangle-|10\rangle}{2}, \frac{|11\rangle+|11\rangle}{2}, \frac{|11\rangle-|11\rangle}{2},$

Of the remaining 12 are duplicates:

$\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$ and $\frac{1}{\sqrt{2}}(|10\rangle + |00\rangle)$

and so on; they may at most differ by global phase like in

$\frac{1}{\sqrt{2}}(|100\rangle - |110\rangle)$ and $\frac{1}{\sqrt{2}}(|110\rangle - |100\rangle)$

Of the remaining, 8 are not entangled.

This leaves us with $2 \times 4^2 - 12 - 8 = 32 - 20 = 4$ which are

$\frac{|100\rangle+|110\rangle}{\sqrt{2}}, \frac{|100\rangle-|111\rangle}{\sqrt{2}}, \frac{|101\rangle+|111\rangle}{\sqrt{2}}, \frac{|101\rangle-|110\rangle}{\sqrt{2}}$

4 Appendix B

Bell states are pair wise orthogonal[6]a $\frac{|00\rangle+|11\rangle}{\sqrt{2}}$ and $\frac{|00\rangle-|11\rangle}{\sqrt{2}}$ are orthogonal since they differ by relative phase

Same with $\frac{|01\rangle+|10\rangle}{\sqrt{2}}$ and $\frac{|01\rangle-|10\rangle}{\sqrt{2}}$

Now consider $\frac{|00\rangle+|11\rangle}{\sqrt{2}}$ and $\frac{|01\rangle-|10\rangle}{\sqrt{2}}$

These two are also orthogonal since, $(\frac{\langle 00|+\langle 11|}{\sqrt{2}})(\frac{|01\rangle-|10\rangle}{\sqrt{2}})$

$= (\frac{\langle 00|01\rangle-\langle 00|10\rangle+\langle 11|01\rangle-\langle 11|10\rangle}{\sqrt{2}})$

$= (\frac{\langle 0|0\rangle\cdot\langle 0|1\rangle-\langle 0|1\rangle\cdot\langle 0|0\rangle+\langle 1|0\rangle\cdot\langle 1|1\rangle-\langle 1|1\rangle\cdot\langle 1|0\rangle}{2})$

$= \frac{1\cdot 0-0\cdot 1+0\cdot 1-1\cdot 0}{2}$

$= \frac{0-0+0-0}{2}$

$= 0$

By transitivity all the bell states are orthogonal to each other

[5]consider $\frac{|00\rangle-|00\rangle}{2}$, what would phase mean here? especially since differing by relative phase does not yield orthogonality in this case. That is, when $\frac{|00\rangle-|00\rangle}{2} = \frac{|00\rangle+|00\rangle}{2}$ what does relative phase signify then?

[6]Where inner product is defined as product of inner product of qubits in the respective subsystems ie constituent Hilbert spaces of the tensor product hilbert space