Solved Math Problems

1 solved problems

Exercise 2.1: Linear dependence

We are asked to consider the linear independence of the vectors

$$\begin{pmatrix}-1\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}$$

In a 2-dimensional space ($\mathbb{R}^2$), any set of three or more vectors is linearly dependent. Therefore, this set of four vectors is linearly dependent.

As a proof by example, we can show one vector can be written as a linear combination of others. For instance:

$$\begin{pmatrix}-1\\1\end{pmatrix} = \alpha \begin{pmatrix}1\\2\end{pmatrix} + \beta \begin{pmatrix}2\\1\end{pmatrix}$$

This gives us a system of two equations:

$$\begin{aligned} -1 &= \alpha + 2\beta \\ 1 &= 2\alpha + \beta \end{aligned}$$

Solving this system, we find $\alpha = 1$ and $\beta = -1$.

$$\begin{pmatrix}-1\\1\end{pmatrix} = (1) \begin{pmatrix}1\\2\end{pmatrix} + (-1) \begin{pmatrix}2\\1\end{pmatrix} \blacksquare$$

Exercise 2.2: Matrix Representation of an Operator

Let $A : V_0 \to V_0$ be a linear operator on the vector space spanned by the basis vectors $|0\rangle$ and $|1\rangle$. The action of the operator is given by:

$$A|0\rangle = |1\rangle \quad \text{and} \quad A|1\rangle = |0\rangle$$

The matrix representation of A in the basis $\{|0\rangle, |1\rangle\}$ is found by determining how A acts on each basis vector. Let the matrix be $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

- $A|0\rangle = A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}a\\c\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} \implies a = 0, c = 1.$
- $A|1\rangle = A\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}b\\d\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \implies b = 1, d = 0.$

Thus, the matrix representation of A is:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

This matrix is known as the Pauli-X matrix, often denoted by σ_x.

Exercise 2.2, rabbithole: Change of Basis

We have the same operator A (the Pauli-X matrix), but we want to find a new output basis such that its matrix representation is diagonal, specifically:

$$A' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

This new basis, the eigenbasis of A, is composed of the eigenvectors of A. We can find them by solving the eigenvalue equation $A\,|v\rangle = \lambda\,|v\rangle$.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}$$

The characteristic equation is $\det(A - \lambda I) = 0$:

$$\det \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - 1 = 0 \implies \lambda = \pm 1$$

The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -1$.

- For $\lambda_1 = 1$:

  $$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies -a + b = 0 \implies b = a$$

  The normalized eigenvector is $|v_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

- For $\lambda_2 = -1$:

  $$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies a + b = 0 \implies b = -a$$

  The normalized eigenvector is $|v_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

The new output basis is the set of eigenvectors $\{|v_1\rangle, |v_2\rangle\}$. The action of A on these basis vectors is:

$$A\,|v_1\rangle = 1\,|v_1\rangle \quad \text{and} \quad A\,|v_2\rangle = -1\,|v_2\rangle$$

In this new basis, the matrix representation of A is indeed $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.