8B Generalized Eigenspace Decomposition

Generalized Eigenspaces

8.19 definition: *generalized eigenspace, $G(\lambda, T)$*

Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbf{F}$. The *generalized eigenspace* of T corresponding to λ, denoted by $G(\lambda, T)$, is defined by

$$G(\lambda, T) = \{v \in V : (T - \lambda I)^k v = 0 \text{ for some positive integer } k\}.$$

Thus $G(\lambda, T)$ is the set of generalized eigenvectors of T corresponding to λ, along with the 0 vector.

Because every eigenvector of T is a generalized eigenvector of T (take $k = 1$ in the definition of generalized eigenvector), each eigenspace is contained in the corresponding generalized eigenspace. In other words, if $T \in \mathcal{L}(V)$ and $\lambda \in \mathbf{F}$, then $E(\lambda, T) \subseteq G(\lambda, T)$.

The next result implies that if $T \in \mathcal{L}(V)$ and $\lambda \in \mathbf{F}$, then the generalized eigenspace $G(\lambda, T)$ is a subspace of V (because the null space of each linear map on V is a subspace of V).

8.20 *description of generalized eigenspaces*

Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbf{F}$. Then $G(\lambda, T) = \operatorname{null}(T - \lambda I)^{\dim V}$.

Proof Suppose $v \in \operatorname{null}(T - \lambda I)^{\dim V}$. The definitions imply $v \in G(\lambda, T)$. Thus $G(\lambda, T) \supseteq \operatorname{null}(T - \lambda I)^{\dim V}$.

Conversely, suppose $v \in G(\lambda, T)$. Thus there is a positive integer k such that $v \in \operatorname{null}(T - \lambda I)^k$. From 8.1 and 8.3 (with $T - \lambda I$ replacing T), we get $v \in \operatorname{null}(T - \lambda I)^{\dim V}$. Thus $G(\lambda, T) \subseteq \operatorname{null}(T - \lambda I)^{\dim V}$, completing the proof. ■

8.21 example: *generalized eigenspaces of an operator on $\mathbf{C}^3$*

Define $T \in \mathcal{L}(\mathbf{C}^3)$ by

$$T(z_1, z_2, z_3) = (4z_2, 0, 5z_3).$$

In Example 8.10, we saw that the eigenvalues of T are 0 and 5, and we found the corresponding sets of generalized eigenvectors. Taking the union of those sets with $\{0\}$, we have

$$G(0, T) = \{(z_1, z_2, 0) : z_1, z_2 \in \mathbf{C}\} \quad \text{and} \quad G(5, T) = \{(0, 0, z_3) : z_3 \in \mathbf{C}\}.$$

Note that $\mathbf{C}^3 = G(0, T) \oplus G(5, T)$.

In Example 8.21, the domain space $\mathbf{C}^3$ is the direct sum of the generalized eigenspaces of the operator T in that example. Our next result shows that this behavior holds in general. Specifically, the following major result shows that if $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$, then V is the direct sum of the generalized eigenspaces of T, each of which is invariant under T and on which T is a nilpotent operator plus a scalar multiple of the identity. Thus the next result achieves our goal of decomposing V into invariant subspaces on which T has a known behavior.

As we will see, the proof follows from putting together what we have learned about generalized eigenspaces and then using our result that for each operator $T \in \mathcal{L}(V)$, there exists a basis of V consisting of generalized eigenvectors of T.

8.22 *generalized eigenspace decomposition*

Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$. Let $\lambda_1, \dots, \lambda_m$ be the distinct eigenvalues of T. Then

(a) $G(\lambda_k, T)$ is invariant under T for each $k = 1, \dots, m$;

(b) $(T - \lambda_k I)|_{G(\lambda_k, T)}$ is nilpotent for each $k = 1, \dots, m$;

(c) $V = G(\lambda_1, T) \oplus \cdots \oplus G(\lambda_m, T)$.

Proof

(a) Suppose $k \in \{1, \dots, m\}$. Then 8.20 shows that

$$G(\lambda_k, T) = \operatorname{null}(T - \lambda_k I)^{\dim V}.$$

Thus 5.18, with $p(z) = (z - \lambda_k)^{\dim V}$, implies that $G(\lambda_k, T)$ is invariant under T, proving (a).

(b) Suppose $k \in \{1, \dots, m\}$. If $v \in G(\lambda_k, T)$, then $(T - \lambda_k I)^{\dim V} v = 0$ (by 8.20). Thus $\bigl((T - \lambda_k I)|_{G(\lambda_k, T)}\bigr)^{\dim V} = 0$. Hence $(T - \lambda_k I)|_{G(\lambda_k, T)}$ is nilpotent, proving (b).

(c) To show that $G(\lambda_1, T) + \cdots + G(\lambda_m, T)$ is a direct sum, suppose

$$v_1 + \cdots + v_m = 0,$$

where each v_k is in $G(\lambda_k, T)$. Because generalized eigenvectors of T corresponding to distinct eigenvalues are linearly independent (by 8.12), this implies that each v_k equals 0. Thus $G(\lambda_1, T) + \cdots + G(\lambda_m, T)$ is a direct sum (by 1.45).

Finally, each vector in V can be written as a finite sum of generalized eigenvectors of T (by 8.9). Thus

$$V = G(\lambda_1, T) \oplus \cdots \oplus G(\lambda_m, T),$$

proving (c). ■

For the analogous result when $\mathbf{F} = \mathbf{R}$, see Exercise 8.

Multiplicity of an Eigenvalue

If V is a complex vector space and $T \in \mathcal{L}(V)$, then the decomposition of V provided by the generalized eigenspace decomposition (8.22) can be a powerful tool. The dimensions of the subspaces involved in this decomposition are sufficiently important to get a name, which is given in the next definition.

8.23 definition: ***multiplicity***

- Suppose $T \in \mathcal{L}(V)$. The *multiplicity* of an eigenvalue λ of T is defined to be the dimension of the corresponding generalized eigenspace $G(\lambda, T)$.
- In other words, the multiplicity of an eigenvalue λ of T equals

$$\dim \operatorname{null}(T - \lambda I)^{\dim V}.$$

The second bullet point above holds because $G(\lambda, T) = \operatorname{null}(T - \lambda I)^{\dim V}$ (see 8.20).

8.24 example: ***multiplicity of each eigenvalue of an operator***

Suppose $T \in \mathcal{L}(\mathbf{C}^3)$ is defined by

$$T(z_1, z_2, z_3) = (6z_1 + 3z_2 + 4z_3, 6z_2 + 2z_3, 7z_3).$$

The matrix of T (with respect to the standard basis) is

$$\begin{pmatrix} 6 & 3 & 4 \\ 0 & 6 & 2 \\ 0 & 0 & 7 \end{pmatrix}.$$

The eigenvalues of T are the diagonal entries 6 and 7, as follows from 5.41. You can verify that the generalized eigenspaces of T are as follows:

$$G(6, T) = \operatorname{span}\big((1,0,0), (0,1,0)\big) \quad \text{and} \quad G(7, T) = \operatorname{span}\big((10,2,1)\big).$$

Thus the eigenvalue 6 has multiplicity 2 and the eigenvalue 7 has multiplicity 1. The direct sum $\mathbf{C}^3 = G(6, T) \oplus G(7, T)$ is the generalized eigenspace decomposition promised by 8.22. A basis of $\mathbf{C}^3$ consisting of generalized eigenvectors of T, as promised by 8.9, is $(1,0,0), (0,1,0), (10,2,1)$. There does not exist a basis of $\mathbf{C}^3$ consisting of eigenvectors of this operator.

In this example, the multiplicity of each eigenvalue equals the number of times that eigenvalue appears on the diagonal of an upper-triangular matrix representing the operator. This behavior always happens, as we will see in 8.31.

In the example above, the sum of the multiplicities of the eigenvalues of T equals 3, which is the dimension of the domain of T. The next result shows that this holds for all operators on finite-dimensional complex vector spaces.

8.25 *sum of the multiplicities equals* $\dim V$

Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$. Then the sum of the multiplicities of all eigenvalues of T equals $\dim V$.

Proof The desired result follows from the generalized eigenspace decomposition (8.22) and the formula for the dimension of a direct sum (see 3.94). ■

The terms *algebraic multiplicity* and *geometric multiplicity* are used in some books. In case you encounter this terminology, be aware that the algebraic multiplicity is the same as the multiplicity defined here and the geometric multiplicity is the dimension of the corresponding eigenspace. In other words, if $T \in \mathcal{L}(V)$ and λ is an eigenvalue of T, then

$$\text{algebraic multiplicity of } \lambda = \dim \operatorname{null}(T - \lambda I)^{\dim V} = \dim G(\lambda, T),$$

$$\text{geometric multiplicity of } \lambda = \dim \operatorname{null}(T - \lambda I) = \dim E(\lambda, T).$$

Note that as defined above, the algebraic multiplicity also has a geometric meaning as the dimension of a certain null space. The definition of multiplicity given here is cleaner than the traditional definition that involves determinants; 9.62 implies that these definitions are equivalent.

If V is an inner product space, $T \in \mathcal{L}(V)$ is normal, and λ is an eigenvalue of T, then the algebraic multiplicity of λ equals the geometric multiplicity of λ, as can be seen from applying Exercise 27 in Section 7A to the normal operator $T - \lambda I$. As a special case, the singular values of $S \in \mathcal{L}(V, W)$ (here V and W are both finite-dimensional inner product spaces) depend on the multiplicities (either algebraic or geometric) of the eigenvalues of the self-adjoint operator S^*S.

The next definition associates a monic polynomial with each operator on a finite-dimensional complex vector space.

8.26 definition: *characteristic polynomial*

Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$. Let $\lambda_1, \dots, \lambda_m$ denote the distinct eigenvalues of T, with multiplicities $d_1, \dots, d_m$. The polynomial

$$(z - \lambda_1)^{d_1} \cdots (z - \lambda_m)^{d_m}$$

is called the *characteristic polynomial* of T.

8.27 example: *the characteristic polynomial of an operator*

Suppose $T \in \mathcal{L}(\mathbf{C}^3)$ is defined as in Example 8.24. Because the eigenvalues of T are 6, with multiplicity 2, and 7, with multiplicity 1, we see that the characteristic polynomial of T is $(z-6)^2(z-7)$.

8.28 *degree and zeros of characteristic polynomial*

Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$. Then

(a) the characteristic polynomial of T has degree $\dim V$;

(b) the zeros of the characteristic polynomial of T are the eigenvalues of T.

Proof Our result about the sum of the multiplicities (8.25) implies (a). The definition of the characteristic polynomial implies (b). ■

Most texts define the characteristic polynomial using determinants (the two definitions are equivalent by 9.62). The approach taken here, which is considerably simpler, leads to the following nice proof of the Cayley–Hamilton theorem.

8.29 *Cayley–Hamilton theorem*

Suppose $\mathbf{F} = \mathbf{C}$, $T \in \mathcal{L}(V)$, and q is the characteristic polynomial of T. Then $q(T) = 0$.

Proof Let $\lambda_1, \dots, \lambda_m$ be the distinct eigenvalues of T, and let $d_k = \dim G(\lambda_k, T)$. For each $k \in \{1, \dots, m\}$, we know that $(T - \lambda_k I)|_{G(\lambda_k, T)}$ is nilpotent. Thus we have

$$(T - \lambda_k I)^{d_k}|_{G(\lambda_k, T)} = 0$$

(by 8.16) for each $k \in \{1, \dots, m\}$.

Arthur Cayley (1821–1895) published three mathematical papers before completing his undergraduate degree.

The generalized eigenspace decomposition (8.22) states that every vector in V is a sum of vectors in $G(\lambda_1, T), \dots, G(\lambda_m, T)$. Thus to prove that $q(T) = 0$, we only need to show that $q(T)|_{G(\lambda_k, T)} = 0$ for each k.

Fix $k \in \{1, \dots, m\}$. We have

$$q(T) = (T - \lambda_1 I)^{d_1} \cdots (T - \lambda_m I)^{d_m}.$$

The operators on the right side of the equation above all commute, so we can move the factor $(T - \lambda_k I)^{d_k}$ to be the last term in the expression on the right. Because $(T - \lambda_k I)^{d_k}|_{G(\lambda_k, T)} = 0$, we have $q(T)|_{G(\lambda_k, T)} = 0$, as desired. ■

The next result implies that if the minimal polynomial of an operator $T \in \mathcal{L}(V)$ has degree $\dim V$ (as happens almost always—see the paragraphs following 5.24), then the characteristic polynomial of T equals the minimal polynomial of T.

8.30 *characteristic polynomial is a multiple of minimal polynomial*

Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$. Then the characteristic polynomial of T is a polynomial multiple of the minimal polynomial of T.

Proof The desired result follows immediately from the Cayley–Hamilton theorem (8.29) and 5.29. ■

Now we can prove that the result suggested by Example 8.24 holds for all operators on finite-dimensional complex vector spaces.

8.31 *multiplicity of an eigenvalue equals number of times on diagonal*

Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$. Suppose $v_1, \dots, v_n$ is a basis of V such that $\mathcal{M}\big(T, (v_1, \dots, v_n)\big)$ is upper triangular. Then the number of times that each eigenvalue λ of T appears on the diagonal of $\mathcal{M}\big(T, (v_1, \dots, v_n)\big)$ equals the multiplicity of λ as an eigenvalue of T.

Proof Let $A = \mathcal{M}\big(T, (v_1, \dots, v_n)\big)$. Thus A is an upper-triangular matrix. Let $\lambda_1, \dots, \lambda_n$ denote the entries on the diagonal of A. Thus for each $k \in \{1, \dots, n\}$, we have

$$Tv_k = u_k + \lambda_k v_k$$

for some $u_k \in \operatorname{span}(v_1, \dots, v_{k-1})$. Hence if $k \in \{1, \dots, n\}$ and $\lambda_k \neq 0$, then Tv_k is not a linear combination of $Tv_1, \dots, Tv_{k-1}$. The linear dependence lemma (2.19) now implies that the list of those Tv_k such that $\lambda_k \neq 0$ is linearly independent.

Let d denote the number of indices $k \in \{1, \dots, n\}$ such that $\lambda_k = 0$. The conclusion of the previous paragraph implies that

$$\dim \operatorname{range} T \geq n - d.$$

Because $n = \dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$, the inequality above implies that

8.32 $$\dim \operatorname{null} T \leq d.$$

The matrix of the operator T^n with respect to the basis $v_1, \dots, v_n$ is the upper-triangular matrix A^n, which has diagonal entries $\lambda_1^n, \dots, \lambda_n^n$ [see Exercise 2(b) in Section 5C]. Because $\lambda_k^n = 0$ if and only if $\lambda_k = 0$, the number of times that 0 appears on the diagonal of A^n equals d. Thus applying 8.32 with T replaced with T^n, we have

8.33 $$\dim \operatorname{null} T^n \leq d.$$

For λ an eigenvalue of T, let m_λ denote the multiplicity of λ as an eigenvalue of T and let d_λ denote the number of times that λ appears on the diagonal of A. Replacing T in 8.33 with $T - \lambda I$, we see that

8.34 $$m_\lambda \leq d_\lambda$$

for each eigenvalue λ of T. The sum of the multiplicities m_λ over all eigenvalues λ of T equals n, the dimension of V (by 8.25). The sum of the numbers d_λ over all eigenvalues λ of T also equals n, because the diagonal of A has length n.

Thus summing both sides of 8.34 over all eigenvalues λ of T produces an equality. Hence 8.34 must actually be an equality for each eigenvalue λ of T. Thus the multiplicity of λ as an eigenvalue of T equals the number of times that λ appears on the diagonal of A, as desired. ■

Block Diagonal Matrices

To interpret our results in matrix form, we make the following definition, generalizing the notion of a diagonal matrix. If each matrix A_k in the definition below is a 1-by-1 matrix, then we actually have a diagonal matrix.

Often we can understand a matrix better by thinking of it as composed of smaller matrices.

8.35 definition: *block diagonal matrix*

A *block diagonal matrix* is a square matrix of the form

$$\begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_m \end{pmatrix},$$

where $A_1, \ldots, A_m$ are square matrices lying along the diagonal and all other entries of the matrix equal 0.

8.36 example: *a block diagonal matrix*

The 5-by-5 matrix

$$A = \begin{pmatrix} \begin{pmatrix} 4 \end{pmatrix} & 0 \quad 0 & 0 \quad 0 \\ 0 & \begin{pmatrix} 2 & -3 \\ 0 & 2 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 \\ 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 1 & 7 \\ 0 & 1 \end{pmatrix} \end{pmatrix}$$

is a block diagonal matrix with

$$A = \begin{pmatrix} A_1 & & 0 \\ & A_2 & \\ 0 & & A_3 \end{pmatrix},$$

where

$$A_1 = \begin{pmatrix} 4 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 & -3 \\ 0 & 2 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 7 \\ 0 & 1 \end{pmatrix}.$$

Here the inner matrices in the 5-by-5 matrix above are blocked off to show how we can think of it as a block diagonal matrix.

Note that in the example above, each of A_1, A_2, A_3 is an upper-triangular matrix whose diagonal entries are all equal. The next result shows that with respect to an appropriate basis, every operator on a finite-dimensional complex vector space has a matrix of this form. Note that this result gives us many more zeros in the matrix than are needed to make it upper triangular.

8.37 *block diagonal matrix with upper-triangular blocks*

Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$. Let $\lambda_1, \dots, \lambda_m$ be the distinct eigenvalues of T, with multiplicities $d_1, \dots, d_m$. Then there is a basis of V with respect to which T has a block diagonal matrix of the form

$$\begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_m \end{pmatrix},$$

where each A_k is a d_k-by-d_k upper-triangular matrix of the form

$$A_k = \begin{pmatrix} \lambda_k & & * \\ & \ddots & \\ 0 & & \lambda_k \end{pmatrix}.$$

Proof Each $(T - \lambda_k I)|_{G(\lambda_k, T)}$ is nilpotent (see 8.22). For each k, choose a basis of $G(\lambda_k, T)$, which is a vector space of dimension d_k, such that the matrix of $(T - \lambda_k I)|_{G(\lambda_k, T)}$ with respect to this basis is as in 8.18(c). Thus with respect to this basis, the matrix of $T|_{G(\lambda_k, T)}$, which equals $(T - \lambda_k I)|_{G(\lambda_k, T)} + \lambda_k I|_{G(\lambda_k, T)}$, looks like the desired form shown above for A_k.

The generalized eigenspace decomposition (8.22) shows that putting together the bases of the $G(\lambda_k, T)$'s chosen above gives a basis of V. The matrix of T with respect to this basis has the desired form. ■

8.38 example: *block diagonal matrix via generalized eigenvectors*

Let $T \in \mathcal{L}(\mathbf{C}^3)$ be defined by $T(z_1, z_2, z_3) = (6z_1 + 3z_2 + 4z_3, 6z_2 + 2z_3, 7z_3)$. The matrix of T (with respect to the standard basis) is

$$\begin{pmatrix} 6 & 3 & 4 \\ 0 & 6 & 2 \\ 0 & 0 & 7 \end{pmatrix},$$

which is an upper-triangular matrix but is not of the form promised by 8.37.

As we saw in Example 8.24, the eigenvalues of T are 6 and 7; also,

$$G(6, T) = \operatorname{span}\big((1,0,0), (0,1,0)\big) \quad \text{and} \quad G(7, T) = \operatorname{span}\big((10,2,1)\big).$$

We also saw that a basis of $\mathbf{C}^3$ consisting of generalized eigenvectors of T is

$$(1,0,0), (0,1,0), (10,2,1).$$

The matrix of T with respect to this basis is

$$\begin{pmatrix} \begin{pmatrix} 6 & 3 \\ 0 & 6 \end{pmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \begin{pmatrix} 7 \end{pmatrix} \end{pmatrix},$$

which is a matrix of the block diagonal form promised by 8.37.

Exercises 8B

1 Define $T \in \mathcal{L}(\mathbf{C}^2)$ by $T(w, z) = (-z, w)$. Find the generalized eigenspaces corresponding to the distinct eigenvalues of T.

2 Suppose $T \in \mathcal{L}(V)$ is invertible. Prove that $G(\lambda, T) = G\bigl(\frac{1}{\lambda}, T^{-1}\bigr)$ for every $\lambda \in \mathbf{F}$ with $\lambda \neq 0$.

3 Suppose $T \in \mathcal{L}(V)$. Suppose $S \in \mathcal{L}(V)$ is invertible. Prove that T and $S^{-1}TS$ have the same eigenvalues with the same multiplicities.

4 Suppose $\dim V \geq 2$ and $T \in \mathcal{L}(V)$ is such that $\operatorname{null} T^{\dim V - 2} \neq \operatorname{null} T^{\dim V - 1}$. Prove that T has at most two distinct eigenvalues.

5 Suppose $T \in \mathcal{L}(V)$ and 3 and 8 are eigenvalues of T. Let $n = \dim V$. Prove that $V = (\operatorname{null} T^{n-2}) \oplus (\operatorname{range} T^{n-2})$.

6 Suppose $T \in \mathcal{L}(V)$ and λ is an eigenvalue of T. Explain why the exponent of $z - \lambda$ in the factorization of the minimal polynomial of T is the smallest positive integer m such that $(T - \lambda I)^m|_{G(\lambda, T)} = 0$.

7 Suppose $T \in \mathcal{L}(V)$ and λ is an eigenvalue of T with multiplicity d. Prove that $G(\lambda, T) = \operatorname{null}(T - \lambda I)^d$.

If $d < \dim V$, then this exercise improves 8.20.

8 Suppose $T \in \mathcal{L}(V)$ and $\lambda_1, \ldots, \lambda_m$ are the distinct eigenvalues of T. Prove that

$$V = G(\lambda_1, T) \oplus \cdots \oplus G(\lambda_m, T)$$

if and only if the minimal polynomial of T equals $(z - \lambda_1)^{k_1} \cdots (z - \lambda_m)^{k_m}$ for some positive integers $k_1, \ldots, k_m$.

The case $\mathbf{F} = \mathbf{C}$ follows immediately from 5.27(b) and the generalized eigenspace decomposition (8.22); thus this exercise is interesting only when $\mathbf{F} = \mathbf{R}$.

9 Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$. Prove that there exist $D, N \in \mathcal{L}(V)$ such that $T = D + N$, the operator D is diagonalizable, N is nilpotent, and $DN = ND$.

10 Suppose V is a complex inner product space, $e_1, \ldots, e_n$ is an orthonormal basis of T, and $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of T, each included as many times as its multiplicity. Prove that

$$|\lambda_1|^2 + \cdots + |\lambda_n|^2 \leq \|Te_1\|^2 + \cdots + \|Te_n\|^2.$$

See the comment after Exercise 5 in Section 7A.

11 Give an example of an operator on $\mathbf{C}^4$ whose characteristic polynomial equals $(z - 7)^2(z - 8)^2$.

12 Give an example of an operator on $\mathbf{C}^4$ whose characteristic polynomial equals $(z-1)(z-5)^3$ and whose minimal polynomial equals $(z-1)(z-5)^2$.

13 Give an example of an operator on $\mathbf{C}^4$ whose characteristic and minimal polynomials both equal $z(z-1)^2(z-3)$.

14 Give an example of an operator on $\mathbf{C}^4$ whose characteristic polynomial equals $z(z-1)^2(z-3)$ and whose minimal polynomial equals $z(z-1)(z-3)$.

15 Let T be the operator on $\mathbf{C}^4$ defined by $T(z_1, z_2, z_3, z_4) = (0, z_1, z_2, z_3)$. Find the characteristic polynomial and the minimal polynomial of T.

16 Let T be the operator on $\mathbf{C}^6$ defined by

$$T(z_1, z_2, z_3, z_4, z_5, z_6) = (0, z_1, z_2, 0, z_4, 0).$$

Find the characteristic polynomial and the minimal polynomial of T.

17 Suppose $\mathbf{F} = \mathbf{C}$ and $P \in \mathcal{L}(V)$ is such that $P^2 = P$. Prove that the characteristic polynomial of P is $z^m(z-1)^n$, where $m = \dim \operatorname{null} P$ and $n = \dim \operatorname{range} P$.

18 Suppose $T \in \mathcal{L}(V)$ and λ is an eigenvalue of T. Explain why the following four numbers equal each other.

(a) The exponent of $z - \lambda$ in the factorization of the minimal polynomial of T.

(b) The smallest positive integer m such that $(T - \lambda I)^m|_{G(\lambda, T)} = 0$.

(c) The smallest positive integer m such that

$$\operatorname{null}(T - \lambda I)^m = \operatorname{null}(T - \lambda I)^{m+1}.$$

(d) The smallest positive integer m such that

$$\operatorname{range}(T - \lambda I)^m = \operatorname{range}(T - \lambda I)^{m+1}.$$

19 Suppose $\mathbf{F} = \mathbf{C}$ and $S \in \mathcal{L}(V)$ is a unitary operator. Prove that the constant term in the characteristic polynomial of S has absolute value 1.

20 Suppose that $\mathbf{F} = \mathbf{C}$ and $V_1, \dots, V_m$ are nonzero subspaces of V such that

$$V = V_1 \oplus \cdots \oplus V_m.$$

Suppose $T \in \mathcal{L}(V)$ and each V_k is invariant under T. For each k, let p_k denote the characteristic polynomial of $T|_{V_k}$. Prove that the characteristic polynomial of T equals $p_1 \cdots p_m$.

21 Suppose $p, q \in \mathcal{P}(\mathbf{C})$ are monic polynomials with the same zeros and q is a polynomial multiple of p. Prove that there exists $T \in \mathcal{L}\big(\mathbf{C}^{\deg q}\big)$ such that the characteristic polynomial of T is q and the minimal polynomial of T is p.

This exercise implies that every monic polynomial is the characteristic polynomial of some operator.

22 Suppose A and B are block diagonal matrices of the form

$$A = \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_m \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & & 0 \\ & \ddots & \\ 0 & & B_m \end{pmatrix},$$

where A_k and B_k are square matrices of the same size for each $k = 1, \dots, m$. Show that AB is a block diagonal matrix of the form

$$AB = \begin{pmatrix} A_1 B_1 & & 0 \\ & \ddots & \\ 0 & & A_m B_m \end{pmatrix}.$$

23 Suppose $\mathbf{F} = \mathbf{R}$, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbf{C}$.

(a) Show that $u + iv \in G(\lambda, T_{\mathbf{C}})$ if and only if $u - iv \in G(\overline{\lambda}, T_{\mathbf{C}})$.

(b) Show that the multiplicity of λ as an eigenvalue of $T_{\mathbf{C}}$ equals the multiplicity of $\overline{\lambda}$ as an eigenvalue of $T_{\mathbf{C}}$.

(c) Use (b) and the result about the sum of the multiplicities (8.25) to show that if $\dim V$ is an odd number, then $T_{\mathbf{C}}$ has a real eigenvalue.

(d) Use (c) and the result about real eigenvalues of $T_{\mathbf{C}}$ (Exercise 17 in Section 5A) to show that if $\dim V$ is an odd number, then T has an eigenvalue (thus giving an alternative proof of 5.34).

See Exercise 33 in Section 3B for the definition of the complexification $T_{\mathbf{C}}$.