Chapter 3

Linear Maps

So far our attention has focused on vector spaces. No one gets excited about vector spaces. The interesting part of linear algebra is the subject to which we now turn—linear maps.

We will frequently use the powerful fundamental theorem of linear maps, which states that the dimension of the domain of a linear map equals the dimension of the subspace that gets sent to 0 plus the dimension of the range. This will imply the striking result that a linear map from a finite-dimensional vector space to itself is one-to-one if and only if its range is the whole space.

A major concept that we will introduce in this chapter is the matrix associated with a linear map and with a basis of the domain space and a basis of the target space. This correspondence between linear maps and matrices provides much insight into key aspects of linear algebra.

This chapter concludes by introducing product, quotient, and dual spaces.

In this chapter we will need additional vector spaces, which we call U and W, in addition to V. Thus our standing assumptions are now as follows.

standing assumptions for this chapter

- $\mathbf{F}$ denotes $\mathbf{R}$ or $\mathbf{C}$.
- U, V, and W denote vector spaces over $\mathbf{F}$.

The twelfth-century Dankwarderode Castle in Brunswick (Braunschweig), where Carl Friedrich Gauss (1777–1855) was born and grew up. In 1809 Gauss published a method for solving systems of linear equations. This method, now called Gaussian elimination, was used in a Chinese book written over 1600 years earlier.

3A Vector Space of Linear Maps

Definition and Examples of Linear Maps

Now we are ready for one of the key definitions in linear algebra.

3.1 definition: *linear map*

A *linear map* from V to W is a function $T \colon V \to W$ with the following properties.

additivity

$T(u + v) = Tu + Tv$ for all $u, v \in V$.

homogeneity

$T(\lambda v) = \lambda(Tv)$ for all $\lambda \in \mathbf{F}$ and all $v \in V$.

Note that for linear maps we often use the notation Tv as well as the usual function notation $T(v)$.

Some mathematicians use the phrase ***linear transformation****, which means the same as linear map.*

3.2 notation: $\mathcal{L}(V, W)$, $\mathcal{L}(V)$

- The set of linear maps from V to W is denoted by $\mathcal{L}(V, W)$.
- The set of linear maps from V to V is denoted by $\mathcal{L}(V)$. In other words, $\mathcal{L}(V) = \mathcal{L}(V, V)$.

Let's look at some examples of linear maps. Make sure you verify that each of the functions defined in the next example is indeed a linear map:

3.3 example: *linear maps*

zero

In addition to its other uses, we let the symbol 0 denote the linear map that takes every element of some vector space to the additive identity of another (or possibly the same) vector space. To be specific, $0 \in \mathcal{L}(V, W)$ is defined by

$$0v = 0.$$

The 0 on the left side of the equation above is a function from V to W, whereas the 0 on the right side is the additive identity in W. As usual, the context should allow you to distinguish between the many uses of the symbol 0.

identity operator

The *identity operator*, denoted by I, is the linear map on some vector space that takes each element to itself. To be specific, $I \in \mathcal{L}(V)$ is defined by

$$Iv = v.$$

differentiation
Define $D \in \mathcal{L}\big(\mathcal{P}(\mathbf{R})\big)$ by

$$Dp = p'.$$

The assertion that this function is a linear map is another way of stating a basic result about differentiation: $(f+g)' = f' + g'$ and $(\lambda f)' = \lambda f'$ whenever f, g are differentiable and λ is a constant.

integration
Define $T \in \mathcal{L}\big(\mathcal{P}(\mathbf{R}), \mathbf{R}\big)$ by

$$Tp = \int_0^1 p.$$

The assertion that this function is linear is another way of stating a basic result about integration: the integral of the sum of two functions equals the sum of the integrals, and the integral of a constant times a function equals the constant times the integral of the function.

multiplication by x^2
Define a linear map $T \in \mathcal{L}\big(\mathcal{P}(\mathbf{R})\big)$ by

$$(Tp)(x) = x^2 p(x)$$

for each $x \in \mathbf{R}$.

backward shift
Recall that $\mathbf{F}^\infty$ denotes the vector space of all sequences of elements of $\mathbf{F}$. Define a linear map $T \in \mathcal{L}(\mathbf{F}^\infty)$ by

$$T(x_1, x_2, x_3, \dots) = (x_2, x_3, \dots).$$

from $\mathbf{R}^3$ to $\mathbf{R}^2$
Define a linear map $T \in \mathcal{L}(\mathbf{R}^3, \mathbf{R}^2)$ by

$$T(x, y, z) = (2x - y + 3z, 7x + 5y - 6z).$$

from $\mathbf{F}^n$ to $\mathbf{F}^m$
To generalize the previous example, let m and n be positive integers, let $A_{j,k} \in \mathbf{F}$ for each $j = 1, \dots, m$ and each $k = 1, \dots, n$, and define a linear map $T \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$ by

$$T(x_1, \dots, x_n) = (A_{1,1}x_1 + \cdots + A_{1,n}x_n, \dots, A_{m,1}x_1 + \cdots + A_{m,n}x_n).$$

Actually every linear map from $\mathbf{F}^n$ to $\mathbf{F}^m$ is of this form.

composition
Fix a polynomial $q \in \mathcal{P}(\mathbf{R})$. Define a linear map $T \in \mathcal{L}\big(\mathcal{P}(\mathbf{R})\big)$ by

$$(Tp)(x) = p\big(q(x)\big).$$

The existence part of the next result means that we can find a linear map that takes on whatever values we wish on the vectors in a basis. The uniqueness part of the next result means that a linear map is completely determined by its values on a basis.

3.4 *linear map lemma*

Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_n \in W$. Then there exists a unique linear map $T\colon V \to W$ such that

$$Tv_k = w_k$$

for each $k = 1, \ldots, n$.

Proof First we show the existence of a linear map T with the desired property. Define $T\colon V \to W$ by

$$T(c_1 v_1 + \cdots + c_n v_n) = c_1 w_1 + \cdots + c_n w_n,$$

where $c_1, \ldots, c_n$ are arbitrary elements of $\mathbf{F}$. The list $v_1, \ldots, v_n$ is a basis of V. Thus the equation above does indeed define a function T from V to W (because each element of V can be uniquely written in the form $c_1 v_1 + \cdots + c_n v_n$).

For each k, taking $c_k = 1$ and the other c's equal to 0 in the equation above shows that $Tv_k = w_k$.

If $u, v \in V$ with $u = a_1 v_1 + \cdots + a_n v_n$ and $v = c_1 v_1 + \cdots + c_n v_n$, then

$$\begin{aligned} T(u + v) &= T\big((a_1 + c_1)v_1 + \cdots + (a_n + c_n)v_n\big) \\ &= (a_1 + c_1)w_1 + \cdots + (a_n + c_n)w_n \\ &= (a_1 w_1 + \cdots + a_n w_n) + (c_1 w_1 + \cdots + c_n w_n) \\ &= Tu + Tv. \end{aligned}$$

Similarly, if $\lambda \in \mathbf{F}$ and $v = c_1 v_1 + \cdots + c_n v_n$, then

$$\begin{aligned} T(\lambda v) &= T(\lambda c_1 v_1 + \cdots + \lambda c_n v_n) \\ &= \lambda c_1 w_1 + \cdots + \lambda c_n w_n \\ &= \lambda(c_1 w_1 + \cdots + c_n w_n) \\ &= \lambda Tv. \end{aligned}$$

Thus T is a linear map from V to W.

To prove uniqueness, now suppose that $T \in \mathcal{L}(V, W)$ and that $Tv_k = w_k$ for each $k = 1, \ldots, n$. Let $c_1, \ldots, c_n \in \mathbf{F}$. Then the homogeneity of T implies that $T(c_k v_k) = c_k w_k$ for each $k = 1, \ldots, n$. The additivity of T now implies that

$$T(c_1 v_1 + \cdots + c_n v_n) = c_1 w_1 + \cdots + c_n w_n.$$

Thus T is uniquely determined on $\operatorname{span}(v_1, \ldots, v_n)$ by the equation above. Because $v_1, \ldots, v_n$ is a basis of V, this implies that T is uniquely determined on V, as desired. ■

Algebraic Operations on $\mathcal{L}(V, W)$

We begin by defining addition and scalar multiplication on $\mathcal{L}(V, W)$.

3.5 definition: *addition and scalar multiplication on $\mathcal{L}(V, W)$*

Suppose $S, T \in \mathcal{L}(V, W)$ and $\lambda \in \mathbf{F}$. The *sum* $S + T$ and the *product* λT are the linear maps from V to W defined by

$$(S + T)(v) = Sv + Tv \quad \text{and} \quad (\lambda T)(v) = \lambda(Tv)$$

for all $v \in V$.

You should verify that $S + T$ and λT as defined above are indeed linear maps. In other words, if $S, T \in \mathcal{L}(V, W)$ and $\lambda \in \mathbf{F}$, then $S + T \in \mathcal{L}(V, W)$ and $\lambda T \in \mathcal{L}(V, W)$.

Because we took the trouble to define addition and scalar multiplication on $\mathcal{L}(V, W)$, the next result should not be a surprise.

Linear maps are pervasive throughout mathematics. However, they are not as ubiquitous as imagined by people who seem to think cos *is a linear map from* **R** *to* **R** *when they incorrectly write that* $\cos(x + y)$ *equals* $\cos x + \cos y$ *and that* $\cos 2x$ *equals* $2\cos x$.

3.6 *$\mathcal{L}(V, W)$ is a vector space*

With the operations of addition and scalar multiplication as defined above, $\mathcal{L}(V, W)$ is a vector space.

The routine proof of the result above is left to the reader. Note that the additive identity of $\mathcal{L}(V, W)$ is the zero linear map defined in Example 3.3.

Usually it makes no sense to multiply together two elements of a vector space, but for some pairs of linear maps a useful product exists, as in the next definition.

3.7 definition: *product of linear maps*

If $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, then the *product* $ST \in \mathcal{L}(U, W)$ is defined by

$$(ST)(u) = S(Tu)$$

for all $u \in U$.

Thus ST is just the usual composition $S \circ T$ of two functions, but when both functions are linear, we usually write ST instead of $S \circ T$. The product notation ST helps make the distributive properties (see next result) seem natural.

Note that ST is defined only when T maps into the domain of S. You should verify that ST is indeed a linear map from U to W whenever $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$.

3.8 *algebraic properties of products of linear maps*

associativity

$(T_1T_2)T_3 = T_1(T_2T_3)$ whenever T_1, T_2, and T_3 are linear maps such that the products make sense (meaning T_3 maps into the domain of T_2, and T_2 maps into the domain of T_1).

identity

$TI = IT = T$ whenever $T \in \mathcal{L}(V, W)$; here the first I is the identity operator on V, and the second I is the identity operator on W.

distributive properties

$(S_1 + S_2)T = S_1T + S_2T$ and $S(T_1 + T_2) = ST_1 + ST_2$ whenever $T, T_1, T_2 \in \mathcal{L}(U, V)$ and $S, S_1, S_2 \in \mathcal{L}(V, W)$.

The routine proof of the result above is left to the reader.

Multiplication of linear maps is not commutative. In other words, it is not necessarily true that $ST = TS$, even if both sides of the equation make sense.

3.9 example: *two noncommuting linear maps from $\mathcal{P}(\mathbf{R})$ to $\mathcal{P}(\mathbf{R})$*

Suppose $D \in \mathcal{L}\big(\mathcal{P}(\mathbf{R})\big)$ is the differentiation map defined in Example 3.3 and $T \in \mathcal{L}\big(\mathcal{P}(\mathbf{R})\big)$ is the multiplication by x^2 map defined earlier in this section. Then

$$\big((TD)p\big)(x) = x^2p'(x) \quad \text{but} \quad \big((DT)p\big)(x) = x^2p'(x) + 2xp(x).$$

Thus $TD \neq DT$—differentiating and then multiplying by x^2 is not the same as multiplying by x^2 and then differentiating.

3.10 *linear maps take 0 to 0*

Suppose T is a linear map from V to W. Then $T(0) = 0$.

Proof By additivity, we have

$$T(0) = T(0 + 0) = T(0) + T(0).$$

Add the additive inverse of $T(0)$ to each side of the equation above to conclude that $T(0) = 0$. ■

Suppose $m, b \in \mathbf{R}$. The function $f\colon \mathbf{R} \to \mathbf{R}$ defined by

$$f(x) = mx + b$$

is a linear map if and only if $b = 0$ (use 3.10). Thus the linear functions of high school algebra are not the same as linear maps in the context of linear algebra.

Exercises 3A

1 Suppose $b, c \in \mathbf{R}$. Define $T\colon \mathbf{R}^3 \to \mathbf{R}^2$ by

$$T(x, y, z) = (2x - 4y + 3z + b, 6x + cxyz).$$

Show that T is linear if and only if $b = c = 0$.

2 Suppose $b, c \in \mathbf{R}$. Define $T\colon \mathcal{P}(\mathbf{R}) \to \mathbf{R}^2$ by

$$Tp = \left(3p(4) + 5p'(6) + bp(1)p(2), \int_{-1}^{2} x^3 p(x)\,dx + c \sin p(0)\right).$$

Show that T is linear if and only if $b = c = 0$.

3 Suppose that $T \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$. Show that there exist scalars $A_{j,k} \in \mathbf{F}$ for $j = 1, \ldots, m$ and $k = 1, \ldots, n$ such that

$$T(x_1, \ldots, x_n) = (A_{1,1}x_1 + \cdots + A_{1,n}x_n, \ldots, A_{m,1}x_1 + \cdots + A_{m,n}x_n)$$

for every $(x_1, \ldots, x_n) \in \mathbf{F}^n$.

This exercise shows that the linear map T has the form promised in the second to last item of Example 3.3.

4 Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \ldots, v_m$ is a list of vectors in V such that $Tv_1, \ldots, Tv_m$ is a linearly independent list in W. Prove that $v_1, \ldots, v_m$ is linearly independent.

5 Prove that $\mathcal{L}(V, W)$ is a vector space, as was asserted in 3.6.

6 Prove that multiplication of linear maps has the associative, identity, and distributive properties asserted in 3.8.

7 Show that every linear map from a one-dimensional vector space to itself is multiplication by some scalar. More precisely, prove that if $\dim V = 1$ and $T \in \mathcal{L}(V)$, then there exists $\lambda \in \mathbf{F}$ such that $Tv = \lambda v$ for all $v \in V$.

8 Give an example of a function $\varphi\colon \mathbf{R}^2 \to \mathbf{R}$ such that

$$\varphi(av) = a\varphi(v)$$

for all $a \in \mathbf{R}$ and all $v \in \mathbf{R}^2$ but φ is not linear.

This exercise and the next exercise show that neither homogeneity nor additivity alone is enough to imply that a function is a linear map.

9 Give an example of a function $\varphi\colon \mathbf{C} \to \mathbf{C}$ such that

$$\varphi(w + z) = \varphi(w) + \varphi(z)$$

for all $w, z \in \mathbf{C}$ but φ is not linear. (Here $\mathbf{C}$ is thought of as a complex vector space.)

There also exists a function $\varphi\colon \mathbf{R} \to \mathbf{R}$ such that φ satisfies the additivity condition above but φ is not linear. However, showing the existence of such a function involves considerably more advanced tools.

10 Prove or give a counterexample: If $q \in \mathcal{P}(\mathbf{R})$ and $T\colon \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ is defined by $Tp = q \circ p$, then T is a linear map.

The function T defined here differs from the function T defined in the last bullet point of 3.3 by the order of the functions in the compositions.

11 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T is a scalar multiple of the identity if and only if $ST = TS$ for every $S \in \mathcal{L}(V)$.

12 Suppose U is a subspace of V with $U \neq V$. Suppose $S \in \mathcal{L}(U, W)$ and $S \neq 0$ (which means that $Su \neq 0$ for some $u \in U$). Define $T\colon V \to W$ by

$$Tv = \begin{cases} Sv & \text{if } v \in U, \\ 0 & \text{if } v \in V \text{ and } v \notin U. \end{cases}$$

Prove that T is not a linear map on V.

13 Suppose V is finite-dimensional. Prove that every linear map on a subspace of V can be extended to a linear map on V. In other words, show that if U is a subspace of V and $S \in \mathcal{L}(U, W)$, then there exists $T \in \mathcal{L}(V, W)$ such that $Tu = Su$ for all $u \in U$.

The result in this exercise is used in the proof of 3.125.

14 Suppose V is finite-dimensional with $\dim V > 0$, and suppose W is infinite-dimensional. Prove that $\mathcal{L}(V, W)$ is infinite-dimensional.

15 Suppose $v_1, \ldots, v_m$ is a linearly dependent list of vectors in V. Suppose also that $W \neq \{0\}$. Prove that there exist $w_1, \ldots, w_m \in W$ such that no $T \in \mathcal{L}(V, W)$ satisfies $Tv_k = w_k$ for each $k = 1, \ldots, m$.

16 Suppose V is finite-dimensional with $\dim V > 1$. Prove that there exist $S, T \in \mathcal{L}(V)$ such that $ST \neq TS$.

17 Suppose V is finite-dimensional. Show that the only two-sided ideals of $\mathcal{L}(V)$ are $\{0\}$ and $\mathcal{L}(V)$.

A subspace $\mathcal{E}$ of $\mathcal{L}(V)$ is called a ***two-sided ideal*** *of $\mathcal{L}(V)$ if $TE \in \mathcal{E}$ and $ET \in \mathcal{E}$ for all $E \in \mathcal{E}$ and all $T \in \mathcal{L}(V)$.*