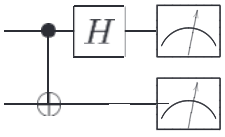
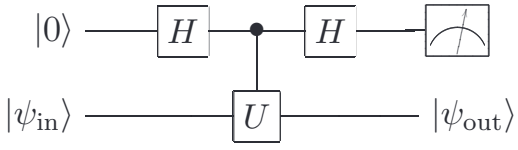
in the computational basis. However, often we want to perform a measurement in some other basis, defined by a complete set of orthonormal states. To perform this measurement, simply unitarily transform from the basis we wish to perform the measurement in to the computational basis, then measure. For example, show that the circuit

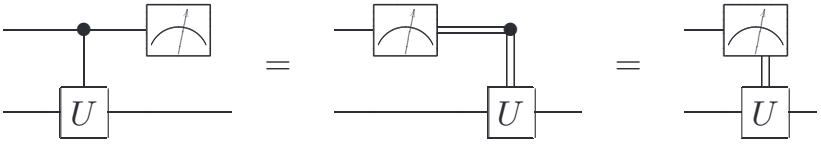


performs a measurement in the basis of the Bell states. More precisely, show that this circuit results in a measurement being performed with corresponding POVM elements the four projectors onto the Bell states. What are the corresponding measurement operators?

Exercise 4.34: (Measuring an operator) Suppose we have a single qubit operator U with eigenvalues ± 1, so that U is both Hermitian and unitary, so it can be regarded both as an observable and a quantum gate. Suppose we wish to measure the observable U. That is, we desire to obtain a measurement result indicating one of the two eigenvalues, and leaving a post-measurement state which is the corresponding eigenvector. How can this be implemented by a quantum circuit? Show that the following circuit implements a measurement of U:



Exercise 4.35: (Measurement commutes with controls) A consequence of the principle of deferred measurement is that measurements commute with quantum gates when the qubit being measured is a control qubit, that is:



(Recall that the double lines represent classical bits in this diagram.) Prove the first equality. The rightmost circuit is simply a convenient notation to depict the use of a measurement result to classically control a quantum gate.

4.5 Universal quantum gates

A small set of gates (e.g. AND, OR, NOT) can be used to compute an arbitrary classical function, as we saw in Section 3.1.2. We say that such a set of gates is *universal* for classical computation. In fact, since the Toffoli gate is universal for classical computation, quantum circuits subsume classical circuits. A similar universality result is true for quantum computation, where a set of gates is said to be *universal for quantum computation* if any unitary operation may be approximated to arbitrary accuracy by a quantum circuit

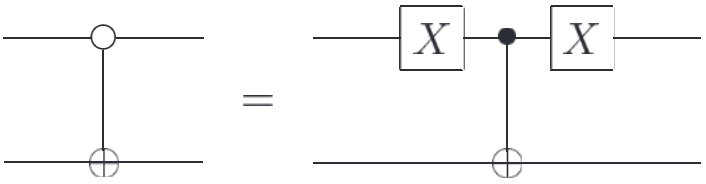


Figure 4.11. Controlled operation with a NOT gate being performed on the second qubit, conditional on the first qubit being set to zero.

to one and circuits which condition on qubits being set to zero, by insertion of X gates in appropriate locations, as illustrated in Figure 4.12.

Another convention which is sometimes useful is to allow controlled-NOT gates to have multiple targets, as shown in Figure 4.13. This natural notation means that when the control qubit is 1, then all the qubits marked with a $\oplus$ are flipped, and otherwise nothing happens. It is convenient to use, for example, in constructing classical functions such as permutations, or in encoders and decoders for quantum error-correction circuits, as we shall see in Chapter 10.

Exercise 4.31: (More circuit identities) Let subscripts denote which qubit an operator acts on, and let C be a CNOT with qubit 1 the control qubit and qubit 2 the target qubit. Prove the following identities:

$$CX_1C = X_1X_2 \tag{4.32}$$
$$CY_1C = Y_1X_2 \tag{4.33}$$
$$CZ_1C = Z_1 \tag{4.34}$$
$$CX_2C = X_2 \tag{4.35}$$
$$CY_2C = Z_1Y_2 \tag{4.36}$$
$$CZ_2C = Z_1Z_2 \tag{4.37}$$
$$R_{z,1}(\theta)C = CR_{z,1}(\theta) \tag{4.38}$$
$$R_{x,2}(\theta)C = CR_{x,2}(\theta). \tag{4.39}$$

4.4 Measurement

A final element used in quantum circuits, almost implicitly sometimes, is measurement. In our circuits, we shall denote a projective measurement in the computational basis (Section 2.2.5) using a 'meter' symbol, illustrated in Figure 4.14. In the theory of quantum circuits it is conventional to not use any special symbols to denote more general measurements, because, as explained in Chapter 2, they can always be represented by unitary transforms with ancilla qubits followed by projective measurements.

There are two important principles that it is worth bearing in mind about quantum circuits. Both principles are rather obvious; however, they are of such great utility that they are worth emphasizing early. The first principle is that classically conditioned operations can be replaced by quantum conditioned operations:

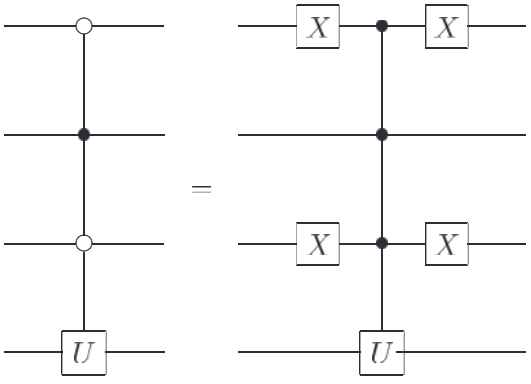


Figure 4.12. Controlled-U operation and its equivalent in terms of circuit elements we already know how to implement. The fourth qubit has U applied if the first and third qubits are set to zero, and the second qubit is set to one.

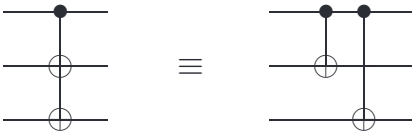


Figure 4.13. Controlled-NOT gate with multiple targets.

> **Principle of deferred measurement:** Measurements can always be moved from an intermediate stage of a quantum circuit to the end of the circuit; if the measurement results are used at any stage of the circuit then the classically controlled operations can be replaced by conditional quantum operations.

Often, quantum measurements are performed as an intermediate step in a quantum circuit, and the measurement results are used to conditionally control subsequent quantum gates. This is the case, for example, in the teleportation circuit of Figure 1.13 on page 27. However, such measurements can *always* be moved to the end of the circuit. Figure 4.15 illustrates how this may be done by replacing all the classical conditional operations by corresponding quantum conditional operations. (Of course, some of the interpretation of this circuit as performing 'teleportation' is lost, because no classical information is transmitted from Alice to Bob, but it is clear that the overall action of the two quantum circuits is the same, which is the key point.)

The second principle is even more obvious – and surprisingly useful!

Figure 4.14. Symbol for projective measurement on a single qubit. In this circuit nothing further is done with the measurement result, but in more general quantum circuits it is possible to change later parts of the quantum circuit, *conditional* on measurement outcomes in earlier parts of the circuit. Such a usage of classical information is depicted using wires drawn with double lines (not shown here).

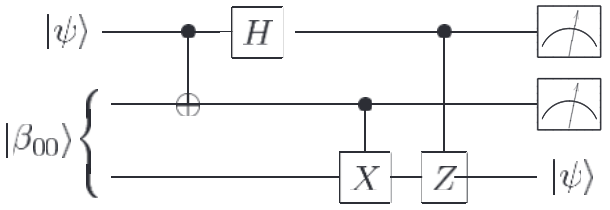


Figure 4.15. Quantum teleportation circuit in which measurements are done at the end, instead of in the middle of the circuit. As in Figure 1.13, the top two qubits belong to Alice, and the bottom one to Bob.

Principle of implicit measurement: Without loss of generality, any unterminated quantum wires (qubits which are not measured) at the end of a quantum circuit may be assumed to be measured.

To understand why this is true, imagine you have a quantum circuit containing just two qubits, and only the first qubit is measured at the end of the circuit. Then the measurement statistics observed at this time are completely determined by the reduced density matrix of the first qubit. However, if a measurement had also been performed on the second qubit, then it would be highly surprising if that measurement could change the statistics of measurement on the first qubit. You'll prove this in Exercise 4.32 by showing that the reduced density matrix of the first qubit is not affected by performing a measurement on the second.

As you consider the role of measurements in quantum circuits, it is important to keep in mind that in its role as an interface between the quantum and classical worlds, measurement is generally considered to be an irreversible operation, destroying quantum information and replacing it with classical information. In certain carefully designed cases, however, this need not be true, as is vividly illustrated by teleportation and quantum error-correction (Chapter 10). What teleportation and quantum error-correction have in common is that in neither instance does the measurement result reveal any information about the identity of the quantum state being measured. Indeed, we will see in Chapter 10 that this is a more general feature of measurement – in order for a measurement to be reversible, it must reveal no information about the quantum state being measured!

Exercise 4.32: Suppose ρ is the density matrix describing a two qubit system. Suppose we perform a projective measurement in the computational basis of the second qubit. Let $P_0 = |0\rangle\langle 0|$ and $P_1 = |1\rangle\langle 1|$ be the projectors onto the $|0\rangle$ and $|1\rangle$ states of the second qubit, respectively. Let ρ' be the density matrix which would be assigned to the system after the measurement by an observer who did not learn the measurement result. Show that

$$\rho' = P_0 \rho P_0 + P_1 \rho P_1 \,. \tag{4.40}$$

Also show that the reduced density matrix for the first qubit is not affected by the measurement, that is, $\mathrm{tr}_2(\rho) = \mathrm{tr}_2(\rho')$.

Exercise 4.33: (Measurement in the Bell basis) The measurement model we have specified for the quantum circuit model is that measurements are performed only