Chapter 9

Multilinear Algebra and Determinants

We begin this chapter by investigating bilinear forms and quadratic forms on a vector space. Then we will move on to multilinear forms. We will show that the vector space of alternating n-linear forms has dimension one on a vector space of dimension n. This result will allow us to give a clean basis-free definition of the determinant of an operator.

This approach to the determinant via alternating multilinear forms leads to straightforward proofs of key properties of determinants. For example, we will see that the determinant is multiplicative, meaning that $\det(ST) = (\det S)(\det T)$ for all operators S and T on the same vector space. We will also see that T is invertible if and only if $\det T \neq 0$. Another important result states that the determinant of an operator on a complex vector space equals the product of the eigenvalues of the operator, with each eigenvalue included as many times as its multiplicity.

The chapter concludes with an introduction to tensor products.

standing assumptions for this chapter

- **F** denotes **R** or **C**.
- V and W denote finite-dimensional nonzero vector spaces over **F**.

The Mathematical Institute at the University of Göttingen. This building opened in 1930, when Emmy Noether (1882–1935) had already been a research mathematician and faculty member at the university for 15 years (the first eight years without salary). Noether was fired by the Nazi government in 1933. By then Noether and her collaborators had created many of the foundations of modern algebra, including an abstract algebra viewpoint that contributed to the development of linear algebra.

9A Bilinear Forms and Quadratic Forms

Bilinear Forms

A bilinear form on V is a function from $V \times V$ to $\mathbf{F}$ that is linear in each slot separately, meaning that if we hold either slot fixed then we have a linear function in the other slot. Here is the formal definition.

9.1 definition: *bilinear form*

A *bilinear form* on V is a function $\beta \colon V \times V \to \mathbf{F}$ such that

$$v \mapsto \beta(v, u) \quad \text{and} \quad v \mapsto \beta(u, v)$$

are both linear functionals on V for every $u \in V$.

*Recall that the term **linear functional**, used in the definition above, means a linear function that maps into the scalar field* $\mathbf{F}$. *Thus the term **bilinear functional** would be more consistent terminology than **bilinear form**, which unfortunately has become standard.*

For example, if V is a real inner product space, then the function that takes an ordered pair $(u, v) \in V \times V$ to $\langle u, v \rangle$ is a bilinear form on V. If V is a nonzero complex inner product space, then this function is not a bilinear form because the inner product is not linear in the second slot (complex scalars come out of the second slot as their complex conjugates).

If $\mathbf{F} = \mathbf{R}$, then a bilinear form differs from an inner product in that an inner product requires symmetry [meaning that $\beta(v, w) = \beta(w, v)$ for all $v, w \in V$] and positive definiteness [meaning that $\beta(v, v) > 0$ for all $v \in V \backslash \{0\}$], but these properties are not required for a bilinear form.

9.2 example: *bilinear forms*

- The function $\beta \colon \mathbf{F}^3 \times \mathbf{F}^3 \to \mathbf{F}$ defined by

$$\beta\big((x_1, x_2, x_3), (y_1, y_2, y_3)\big) = x_1 y_2 - 5x_2 y_3 + 2x_3 y_1$$

  is a bilinear form on $\mathbf{F}^3$.

- Suppose A is an n-by-n matrix with $A_{j,k} \in \mathbf{F}$ in row j, column k. Define a bilinear form β_A on $\mathbf{F}^n$ by

$$\beta_A\big((x_1, \dots, x_n), (y_1, \dots, y_n)\big) = \sum_{k=1}^{n} \sum_{j=1}^{n} A_{j,k} x_j y_k.$$

  The first bullet point is a special case of this bullet point with $n = 3$ and

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -5 \\ 2 & 0 & 0 \end{pmatrix}.$$

- Suppose V is a real inner product space and $T \in \mathcal{L}(V)$. Then the function $\beta\colon V \times V \to \mathbf{R}$ defined by
$$\beta(u, v) = \langle u, Tv \rangle$$
is a bilinear form on V.
- If n is a positive integer, then the function $\beta\colon \mathcal{P}_n(\mathbf{R}) \times \mathcal{P}_n(\mathbf{R}) \to \mathbf{R}$ defined by
$$\beta(p, q) = p(2) \cdot q'(3)$$
is a bilinear form on $\mathcal{P}_n(\mathbf{R})$.
- Suppose $\varphi, \tau \in V'$. Then the function $\beta\colon V \times V \to \mathbf{F}$ defined by
$$\beta(u, v) = \varphi(u) \cdot \tau(v)$$
is a bilinear form on V.
- More generally, suppose that $\varphi_1, \dots, \varphi_n, \tau_1, \dots, \tau_n \in V'$. Then the function $\beta\colon V \times V \to \mathbf{F}$ defined by
$$\beta(u, v) = \varphi_1(u) \cdot \tau_1(v) + \cdots + \varphi_n(u) \cdot \tau_n(v)$$
is a bilinear form on V.

A bilinear form on V is a function from $V \times V$ to $\mathbf{F}$. Because $V \times V$ is a vector space, this raises the question of whether a bilinear form can also be a linear map from $V \times V$ to $\mathbf{F}$. Note that none of the bilinear forms in 9.2 are linear maps except in some special cases in which the bilinear form is the zero map. Exercise 3 shows that a bilinear form β on V is a linear map on $V \times V$ only if $\beta = 0$.

9.3 definition: $V^{(2)}$

The set of bilinear forms on V is denoted by $V^{(2)}$.

With the usual operations of addition and scalar multiplication of functions, $V^{(2)}$ is a vector space.

For T an operator on an n-dimensional vector space V and a basis $e_1, \dots, e_n$ of V, we used an n-by-n matrix to provide information about T. We now do the same thing for bilinear forms on V.

9.4 definition: *matrix of a bilinear form, $\mathcal{M}(\beta)$*

Suppose β is a bilinear form on V and $e_1, \dots, e_n$ is a basis of V. The *matrix* of β with respect to this basis is the n-by-n matrix $\mathcal{M}(\beta)$ whose entry $\mathcal{M}(\beta)_{j,k}$ in row j, column k is given by
$$\mathcal{M}(\beta)_{j,k} = \beta(e_j, e_k).$$
If the basis $e_1, \dots, e_n$ is not clear from the context, then the notation $\mathcal{M}\bigl(\beta, (e_1, \dots, e_n)\bigr)$ is used.

Recall that $\mathbf{F}^{n,n}$ denotes the vector space of n-by-n matrices with entries in $\mathbf{F}$ and that $\dim \mathbf{F}^{n,n} = n^2$ (see 3.39 and 3.40).

9.5 $\dim V^{(2)} = (\dim V)^2$

Suppose $e_1, \dots, e_n$ is a basis of V. Then the map $\beta \mapsto \mathcal{M}(\beta)$ is an isomorphism of $V^{(2)}$ onto $\mathbf{F}^{n,n}$. Furthermore, $\dim V^{(2)} = (\dim V)^2$.

Proof The map $\beta \mapsto \mathcal{M}(\beta)$ is clearly a linear map of $V^{(2)}$ into $\mathbf{F}^{n,n}$. For $A \in \mathbf{F}^{n,n}$, define a bilinear form β_A on V by

$$\beta_A(x_1 e_1 + \cdots + x_n e_n, y_1 e_1 + \cdots + y_n e_n) = \sum_{k=1}^{n} \sum_{j=1}^{n} A_{j,k} x_j y_k$$

for $x_1, \dots, x_n, y_1, \dots, y_n \in \mathbf{F}$ $\big($if $V = \mathbf{F}^n$ and $e_1, \dots, e_n$ is the standard basis of $\mathbf{F}^n$, this β_A is the same as the bilinear form β_A in the second bullet point of Example 9.2$\big)$.

The linear map $\beta \mapsto \mathcal{M}(\beta)$ from $V^{(2)}$ to $\mathbf{F}^{n,n}$ and the linear map $A \mapsto \beta_A$ from $\mathbf{F}^{n,n}$ to $V^{(2)}$ are inverses of each other because $\beta_{\mathcal{M}(\beta)} = \beta$ for all $\beta \in V^{(2)}$ and $\mathcal{M}(\beta_A) = A$ for all $A \in \mathbf{F}^{n,n}$, as you should verify.

Thus both maps are isomorphisms and the two spaces that they connect have the same dimension. Hence $\dim V^{(2)} = \dim \mathbf{F}^{n,n} = n^2 = (\dim V)^2$. ∎

Recall that C^{t} denotes the transpose of a matrix C. The matrix C^{t} is obtained by interchanging the rows and the columns of C.

9.6 *composition of a bilinear form and an operator*

Suppose β is a bilinear form on V and $T \in \mathcal{L}(V)$. Define bilinear forms α and ρ on V by

$$\alpha(u, v) = \beta(u, Tv) \quad \text{and} \quad \rho(u, v) = \beta(Tu, v).$$

Let $e_1, \dots, e_n$ be a basis of V. Then

$$\mathcal{M}(\alpha) = \mathcal{M}(\beta)\mathcal{M}(T) \quad \text{and} \quad \mathcal{M}(\rho) = \mathcal{M}(T)^{\mathrm{t}} \mathcal{M}(\beta).$$

Proof If $j, k \in \{1, \dots, n\}$, then

$$\begin{aligned}
\mathcal{M}(\alpha)_{j,k} &= \alpha(e_j, e_k) \\
&= \beta(e_j, Te_k) \\
&= \beta\Big(e_j, \sum_{m=1}^{n} \mathcal{M}(T)_{m,k}\, e_m\Big) \\
&= \sum_{m=1}^{n} \beta(e_j, e_m)\, \mathcal{M}(T)_{m,k} \\
&= \big(\mathcal{M}(\beta)\mathcal{M}(T)\big)_{j,k}.
\end{aligned}$$

Thus $\mathcal{M}(\alpha) = \mathcal{M}(\beta)\mathcal{M}(T)$. The proof that $\mathcal{M}(\rho) = \mathcal{M}(T)^{\mathrm{t}} \mathcal{M}(\beta)$ is similar. ∎

The result below shows how the matrix of a bilinear form changes if we change the basis. The formula in the result below should be compared to the change-of-basis formula for the matrix of an operator (see 3.84). The two formulas are similar, except that the transpose C^{t} appears in the formula below and the inverse C^{-1} appears in the change-of-basis formula for the matrix of an operator.

9.7 *change-of-basis formula*

Suppose $\beta \in V^{(2)}$. Suppose $e_1, \dots, e_n$ and $f_1, \dots, f_n$ are bases of V. Let

$$A = \mathcal{M}\big(\beta, (e_1, \dots, e_n)\big) \quad \text{and} \quad B = \mathcal{M}\big(\beta, (f_1, \dots, f_n)\big)$$

and $C = \mathcal{M}\big(I, (e_1, \dots, e_n), (f_1, \dots, f_n)\big)$. Then

$$A = C^{\mathrm{t}}BC.$$

Proof The linear map lemma (3.4) tells us that there exists an operator $T \in \mathcal{L}(V)$ such that $Tf_k = e_k$ for each $k = 1, \dots, n$. The definition of the matrix of an operator with respect to a basis implies that

$$\mathcal{M}\big(T, (f_1, \dots, f_n)\big) = C.$$

Define bilinear forms α, ρ on V by

$$\alpha(u, v) = \beta(u, Tv) \quad \text{and} \quad \rho(u, v) = \alpha(Tu, v) = \beta(Tu, Tv).$$

Then $\beta(e_j, e_k) = \beta(Tf_j, Tf_k) = \rho(f_j, f_k)$ for all $j, k \in \{1, \dots, n\}$. Thus

$$\begin{aligned} A &= \mathcal{M}\big(\rho, (f_1, \dots, f_n)\big) \\ &= C^{\mathrm{t}}\mathcal{M}\big(\alpha, (f_1, \dots, f_n)\big) \\ &= C^{\mathrm{t}}BC, \end{aligned}$$

where the second and third lines each follow from 9.6. ■

9.8 example: *the matrix of a bilinear form on $\mathcal{P}_2(\mathbf{R})$*

Define a bilinear form β on $\mathcal{P}_2(\mathbf{R})$ by $\beta(p, q) = p(2) \cdot q'(3)$. Let

$$A = \mathcal{M}\big(\beta, (1, x - 2, (x - 3)^2)\big) \quad \text{and} \quad B = \mathcal{M}\big(\beta, (1, x, x^2)\big)$$

and $C = \mathcal{M}\big(I, (1, x - 2, (x - 3)^2), (1, x, x^2)\big)$. Then

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 & 6 \\ 0 & 2 & 12 \\ 0 & 4 & 24 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 & -2 & 9 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now the change-of-basis formula 9.7 asserts that $A = C^{\mathrm{t}}BC$, which you can verify with matrix multiplication using the matrices above.

Symmetric Bilinear Forms

9.9 definition: *symmetric bilinear form,* $V^{(2)}_{\mathrm{sym}}$

A bilinear form $\rho \in V^{(2)}$ is called *symmetric* if

$$\rho(u, w) = \rho(w, u)$$

for all $u, w \in V$. The set of symmetric bilinear forms on V is denoted by $V^{(2)}_{\mathrm{sym}}$.

9.10 example: *symmetric bilinear forms*

- If V is a real inner product space and $\rho \in V^{(2)}$ is defined by

$$\rho(u, w) = \langle u, w \rangle,$$

then ρ is a symmetric bilinear form on V.

- Suppose V is a real inner product space and $T \in \mathcal{L}(V)$. Define $\rho \in V^{(2)}$ by

$$\rho(u, w) = \langle u, Tw \rangle.$$

Then ρ is a symmetric bilinear form on V if and only if T is a self-adjoint operator (the previous bullet point is the special case $T = I$).

- Suppose $\rho \colon \mathcal{L}(V) \times \mathcal{L}(V) \to \mathbf{F}$ is defined by

$$\rho(S, T) = \operatorname{tr}(ST).$$

Then ρ is a symmetric bilinear form on $\mathcal{L}(V)$ because trace is a linear functional on $\mathcal{L}(V)$ and $\operatorname{tr}(ST) = \operatorname{tr}(TS)$ for all $S, T \in \mathcal{L}(V)$; see 8.56.

9.11 definition: *symmetric matrix*

A square matrix A is called *symmetric* if it equals its transpose.

An operator on V may have a symmetric matrix with respect to some but not all bases of V. In contrast, the next result shows that a bilinear form on V has a symmetric matrix with respect to either all bases of V or with respect to no bases of V.

9.12 *symmetric bilinear forms are diagonalizable*

Suppose $\rho \in V^{(2)}$. Then the following are equivalent.

(a) ρ is a symmetric bilinear form on V.

(b) $\mathcal{M}\big(\rho, (e_1, \dots, e_n)\big)$ is a symmetric matrix for every basis $e_1, \dots, e_n$ of V.

(c) $\mathcal{M}\big(\rho, (e_1, \dots, e_n)\big)$ is a symmetric matrix for some basis $e_1, \dots, e_n$ of V.

(d) $\mathcal{M}\big(\rho, (e_1, \dots, e_n)\big)$ is a diagonal matrix for some basis $e_1, \dots, e_n$ of V.

Proof First suppose (a) holds, so ρ is a symmetric bilinear form. Suppose $e_1, \dots, e_n$ is a basis of V and $j, k \in \{1, \dots, n\}$. Then $\rho(e_j, e_k) = \rho(e_k, e_j)$ because ρ is symmetric. Thus $\mathcal{M}\big(\rho, (e_1, \dots, e_n)\big)$ is a symmetric matrix, showing that (a) implies (b).

Clearly (b) implies (c).

Now suppose (c) holds and $e_1, \dots, e_n$ is a basis of V such that $\mathcal{M}\big(\rho, (e_1, \dots, e_n)\big)$ is a symmetric matrix. Suppose $u, w \in V$. There exist $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbf{F}$ such that $u = a_1 e_1 + \cdots + a_n e_n$ and $w = b_1 e_1 + \cdots + b_n e_n$. Now

$$\begin{aligned}
\rho(u, w) &= \rho\Big(\sum_{j=1}^{n} a_j e_j, \sum_{k=1}^{n} b_k e_k\Big) \\
&= \sum_{j=1}^{n} \sum_{k=1}^{n} a_j b_k \rho(e_j, e_k) \\
&= \sum_{j=1}^{n} \sum_{k=1}^{n} a_j b_k \rho(e_k, e_j) \\
&= \rho\Big(\sum_{k=1}^{n} b_k e_k, \sum_{j=1}^{n} a_j e_j\Big) \\
&= \rho(w, u),
\end{aligned}$$

where the third line holds because $\mathcal{M}(\rho)$ is a symmetric matrix. The equation above shows that ρ is a symmetric bilinear form, proving that (c) implies (a).

At this point, we have proved that (a), (b), (c) are equivalent. Because every diagonal matrix is symmetric, (d) implies (c). To complete the proof, we will show that (a) implies (d) by induction on $n = \dim V$.

If $n = 1$, then (a) implies (d) because every 1-by-1 matrix is diagonal. Now suppose $n > 1$ and the implication (a) $\Longrightarrow$ (d) holds for one less dimension. Suppose (a) holds, so ρ is a symmetric bilinear form. If $\rho = 0$, then the matrix of ρ with respect to every basis of V is the zero matrix, which is a diagonal matrix. Hence we can assume that $\rho \neq 0$, which means there exist $u, w \in V$ such that $\rho(u, w) \neq 0$. Now

$$2\rho(u, w) = \rho(u + w, u + w) - \rho(u, u) - \rho(w, w).$$

Because the left side of the equation above is nonzero, the three terms on the right cannot all equal 0. Hence there exists $v \in V$ such that $\rho(v, v) \neq 0$.

Let $U = \{u \in V : \rho(u, v) = 0\}$. Thus U is the null space of the linear functional $u \mapsto \rho(u, v)$ on V. This linear functional is not the zero linear functional because $v \notin U$. Thus $\dim U = n - 1$. By our induction hypothesis, there is a basis $e_1, \dots, e_{n-1}$ of U such that the symmetric bilinear form $\rho|_{U \times U}$ has a diagonal matrix with respect to this basis.

Because $v \notin U$, the list $e_1, \dots, e_{n-1}, v$ is a basis of V. Suppose $k \in \{1, \dots, n-1\}$. Then $\rho(e_k, v) = 0$ by the construction of U. Because ρ is symmetric, we also have $\rho(v, e_k) = 0$. Thus the matrix of ρ with respect to $e_1, \dots, e_{n-1}, v$ is a diagonal matrix, completing the proof that (a) implies (d). ■

The previous result states that every symmetric bilinear form has a diagonal matrix with respect to some basis. If our vector space happens to be a real inner product space, then the next result shows that every symmetric bilinear form has a diagonal matrix with respect to some *orthonormal* basis. Note that the inner product here is unrelated to the bilinear form.

9.13 *diagonalization of a symmetric bilinear form by an orthonormal basis*

Suppose V is a real inner product space and ρ is a symmetric bilinear form on V. Then ρ has a diagonal matrix with respect to some orthonormal basis of V.

Proof Let $f_1, \dots, f_n$ be an orthonormal basis of V. Let $B = \mathcal{M}\big(\rho, (f_1, \dots, f_n)\big)$. Then B is a symmetric matrix (by 9.12). Let $T \in \mathcal{L}(V)$ be the operator such that $\mathcal{M}\big(T, (f_1, \dots, f_n)\big) = B$. Thus T is self-adjoint.

The real spectral theorem (7.29) states that T has a diagonal matrix with respect to some orthonormal basis $e_1, \dots, e_n$ of V. Let $C = \mathcal{M}\big(I, (e_1, \dots, e_n), (f_1, \dots, f_n)\big)$. Thus $C^{-1}BC$ is the matrix of T with respect to the basis $e_1, \dots, e_n$ (by 3.84). Hence $C^{-1}BC$ is a diagonal matrix. Now

$$M\big(\rho, (e_1, \dots, e_n)\big) = C^{\mathrm{t}}BC = C^{-1}BC,$$

where the first equality holds by 9.7 and the second equality holds because C is a unitary matrix with real entries $\big($which implies that $C^{-1} = C^{\mathrm{t}}$; see 7.57$\big)$. ■

Now we turn our attention to alternating bilinear forms. Alternating multilinear forms will play a major role in our approach to determinants later in this chapter.

9.14 definition: *alternating bilinear form, $V^{(2)}_{\mathrm{alt}}$*

A bilinear form $\alpha \in V^{(2)}$ is called *alternating* if

$$\alpha(v, v) = 0$$

for all $v \in V$. The set of alternating bilinear forms on V is denoted by $V^{(2)}_{\mathrm{alt}}$.

9.15 example: *alternating bilinear forms*

- Suppose $n \geq 3$ and $\alpha \colon \mathbf{F}^n \times \mathbf{F}^n \to \mathbf{F}$ is defined by

  $$\alpha\big((x_1, \dots, x_n), (y_1, \dots, y_n)\big) = x_1y_2 - x_2y_1 + x_1y_3 - x_3y_1.$$

  Then α is an alternating bilinear form on $\mathbf{F}^n$.
- Suppose $\varphi, \tau \in V'$. Then the bilinear form α on V defined by

  $$\alpha(u, w) = \varphi(u)\tau(w) - \varphi(w)\tau(u)$$

  is alternating.

The next result shows that a bilinear form is alternating if and only if switching the order of the two inputs multiplies the output by -1.

9.16 *characterization of alternating bilinear forms*

A bilinear form α on V is alternating if and only if

$$\alpha(u, w) = -\alpha(w, u)$$

for all $u, w \in V$.

Proof First suppose that α is alternating. If $u, w \in V$, then

$$\begin{aligned} 0 &= \alpha(u + w, u + w) \\ &= \alpha(u, u) + \alpha(u, w) + \alpha(w, u) + \alpha(w, w) \\ &= \alpha(u, w) + \alpha(w, u). \end{aligned}$$

Thus $\alpha(u, w) = -\alpha(w, u)$, as desired.

To prove the implication in the other direction, suppose $\alpha(u, w) = -\alpha(w, u)$ for all $u, w \in V$. Then $\alpha(v, v) = -\alpha(v, v)$ for all $v \in V$, which implies that $\alpha(v, v) = 0$ for all $v \in V$. Thus α is alternating. ∎

Now we show that the vector space of bilinear forms on V is the direct sum of the symmetric bilinear forms on V and the alternating bilinear forms on V.

9.17 $V^{(2)} = V^{(2)}_{\mathrm{sym}} \oplus V^{(2)}_{\mathrm{alt}}$

The sets $V^{(2)}_{\mathrm{sym}}$ and $V^{(2)}_{\mathrm{alt}}$ are subspaces of $V^{(2)}$. Furthermore,

$$V^{(2)} = V^{(2)}_{\mathrm{sym}} \oplus V^{(2)}_{\mathrm{alt}}.$$

Proof The definition of symmetric bilinear form implies that the sum of any two symmetric bilinear forms on V is a symmetric bilinear form on V, and every scalar multiple of any symmetric bilinear form on V is a symmetric bilinear form on V. Also, the zero bilinear form is in $V^{(2)}_{\mathrm{sym}}$. Thus $V^{(2)}_{\mathrm{sym}}$ is a subspace of $V^{(2)}$. Similarly, the verification that $V^{(2)}_{\mathrm{alt}}$ is a subspace of $V^{(2)}$ is straightforward.

Next, we want to show that $V^{(2)} = V^{(2)}_{\mathrm{sym}} + V^{(2)}_{\mathrm{alt}}$. To do this, suppose $\beta \in V^{(2)}$. Define $\rho, \alpha \in V^{(2)}$ by

$$\rho(u, w) = \frac{\beta(u, w) + \beta(w, u)}{2} \quad \text{and} \quad \alpha(u, w) = \frac{\beta(u, w) - \beta(w, u)}{2}.$$

Then $\rho \in V^{(2)}_{\mathrm{sym}}$ and $\alpha \in V^{(2)}_{\mathrm{alt}}$, and $\beta = \rho + \alpha$. Thus $V^{(2)} = V^{(2)}_{\mathrm{sym}} + V^{(2)}_{\mathrm{alt}}$.

Finally, to show that the intersection of the two subspaces under consideration equals $\{0\}$, suppose $\beta \in V^{(2)}_{\mathrm{sym}} \cap V^{(2)}_{\mathrm{alt}}$. If $u, w \in V$, then 9.16 implies that

$$\beta(u, w) = -\beta(w, u) = -\beta(u, w)$$

and hence $\beta(u, w) = 0$. Thus $\beta = 0$. Hence $V^{(2)} = V^{(2)}_{\mathrm{sym}} \oplus V^{(2)}_{\mathrm{alt}}$ (by 1.46). ∎

Quadratic Forms

9.18 definition: *quadratic form associated with a bilinear form,* q_β

For β a bilinear form on V, define a function $q_\beta \colon V \to \mathbf{F}$ by $q_\beta(v) = \beta(v, v)$. A function $q \colon V \to \mathbf{F}$ is called a *quadratic form* on V if there exists a bilinear form β on V such that $q = q_\beta$.

Note that if β is a bilinear form, then $q_\beta = 0$ if and only if β is alternating.

9.19 example: *quadratic form*

Suppose β is the bilinear form on $\mathbf{R}^3$ defined by

$$\beta\big((x_1, x_2, x_3), (y_1, y_2, y_3)\big) = x_1y_1 - 4x_1y_2 + 8x_1y_3 - 3x_3y_3.$$

Then q_β is the quadratic form on $\mathbf{R}^3$ given by the formula

$$q_\beta(x_1, x_2, x_3) = x_1^2 - 4x_1x_2 + 8x_1x_3 - 3x_3^2.$$

The quadratic form in the example above is typical of quadratic forms on $\mathbf{F}^n$, as shown in the next result.

9.20 *quadratic forms on* $\mathbf{F}^n$

Suppose n is a positive integer and q is a function from $\mathbf{F}^n$ to $\mathbf{F}$. Then q is a quadratic form on $\mathbf{F}^n$ if and only if there exist numbers $A_{j,k} \in \mathbf{F}$ for $j, k \in \{1, \dots, n\}$ such that

$$q(x_1, \dots, x_n) = \sum_{k=1}^{n} \sum_{j=1}^{n} A_{j,k} x_j x_k$$

for all $(x_1, \dots, x_n) \in \mathbf{F}^n$.

Proof First suppose q is a quadratic form on $\mathbf{F}^n$. Thus there exists a bilinear form β on $\mathbf{F}^n$ such that $q = q_\beta$. Let A be the matrix of β with respect to the standard basis of $\mathbf{F}^n$. Then for all $(x_1, \dots, x_n) \in \mathbf{F}^n$, we have the desired equation

$$q(x_1, \dots, x_n) = \beta\big((x_1, \dots, x_n), (x_1, \dots, x_n)\big) = \sum_{k=1}^{n} \sum_{j=1}^{n} A_{j,k} x_j x_k.$$

Conversely, suppose there exist numbers $A_{j,k} \in \mathbf{F}$ for $j, k \in \{1, \dots, n\}$ such that

$$q(x_1, \dots, x_n) = \sum_{k=1}^{n} \sum_{j=1}^{n} A_{j,k} x_j x_k$$

for all $(x_1, \dots, x_n) \in \mathbf{F}^n$. Define a bilinear form β on $\mathbf{F}^n$ by

$$\beta\big((x_1, \dots, x_n), (y_1, \dots, y_n)\big) = \sum_{k=1}^{n} \sum_{j=1}^{n} A_{j,k} x_j y_k.$$

Then $q = q_\beta$, as desired. ■

Although quadratic forms are defined in terms of an arbitrary bilinear form, the equivalence of (a) and (b) in the result below shows that a *symmetric* bilinear form can always be used.

9.21 *characterizations of quadratic forms*

Suppose $q\colon V \to \mathbf{F}$ is a function. The following are equivalent.

(a) q is a quadratic form.

(b) There exists a unique symmetric bilinear form ρ on V such that $q = q_\rho$.

(c) $q(\lambda v) = \lambda^2 q(v)$ for all $\lambda \in \mathbf{F}$ and all $v \in V$, and the function

$$(u, w) \mapsto q(u+w) - q(u) - q(w)$$

is a symmetric bilinear form on V.

(d) $q(2v) = 4q(v)$ for all $v \in V$, and the function

$$(u, w) \mapsto q(u+w) - q(u) - q(w)$$

is a symmetric bilinear form on V.

Proof First suppose (a) holds, so q is a quadratic form. Hence there exists a bilinear form β such that $q = q_\beta$. By 9.17, there exist a symmetric bilinear form ρ on V and an alternating bilinear form α on V such that $\beta = \rho + \alpha$. Now

$$q = q_\beta = q_\rho + q_\alpha = q_\rho.$$

If $\rho' \in V^{(2)}_{\mathrm{sym}}$ also satisfies $q_{\rho'} = q$, then $q_{\rho'-\rho} = 0$; thus $\rho' - \rho \in V^{(2)}_{\mathrm{sym}} \cap V^{(2)}_{\mathrm{alt}}$, which implies that $\rho' = \rho$ (by 9.17). This completes the proof that (a) implies (b).

Now suppose (b) holds, so there exists a symmetric bilinear form ρ on V such that $q = q_\rho$. If $\lambda \in \mathbf{F}$ and $v \in V$ then

$$q(\lambda v) = \rho(\lambda v, \lambda v) = \lambda\rho(v, \lambda v) = \lambda^2\rho(v, v) = \lambda^2 q(v),$$

showing that the first part of (c) holds.

If $u, w \in V$, then

$$q(u+w) - q(u) - q(w) = \rho(u+w, u+w) - \rho(u,u) - \rho(w,w) = 2\rho(u,w).$$

Thus the function $(u, w) \mapsto q(u+w) - q(u) - q(w)$ equals 2ρ, which is a symmetric bilinear form on V, completing the proof that (b) implies (c).

Clearly (c) implies (d).

Now suppose (d) holds. Let ρ be the symmetric bilinear form on V defined by

$$\rho(u, w) = \frac{q(u+w) - q(u) - q(w)}{2}.$$

If $v \in V$, then

$$\rho(v, v) = \frac{q(2v) - q(v) - q(v)}{2} = \frac{4q(v) - 2q(v)}{2} = q(v).$$

Thus $q = q_\rho$, completing the proof that (d) implies (a). ∎

9.22 example: *symmetric bilinear form associated with a quadratic form*

Suppose q is the quadratic form on $\mathbf{R}^3$ given by the formula

$$q(x_1, x_2, x_3) = x_1^2 - 4x_1x_2 + 8x_1x_3 - 3x_3^2.$$

A bilinear form β on $\mathbf{R}^3$ such that $q = q_\beta$ is given by Example 9.19, but this bilinear form is not symmetric, as promised by 9.21(b). However, the bilinear form ρ on $\mathbf{R}^3$ defined by

$$\rho\big((x_1, x_2, x_3), (y_1, y_2, y_3)\big) = x_1y_1 - 2x_1y_2 - 2x_2y_1 + 4x_1y_3 + 4x_3y_1 - 3x_3y_3$$

is symmetric and satisfies $q = q_\rho$.

The next result states that for each quadratic form we can choose a basis such that the quadratic form looks like a weighted sum of squares of the coordinates, meaning that there are no cross terms of the form x_jx_k with $j \neq k$.

9.23 *diagonalization of quadratic form*

Suppose q is a quadratic form on V.

(a) There exist a basis $e_1, \dots, e_n$ of V and $\lambda_1, \dots, \lambda_n \in \mathbf{F}$ such that

$$q(x_1e_1 + \cdots + x_ne_n) = \lambda_1x_1^2 + \cdots + \lambda_nx_n^2$$

for all $x_1, \dots, x_n \in \mathbf{F}$.

(b) If $\mathbf{F} = \mathbf{R}$ and V is an inner product space, then the basis in (a) can be chosen to be an orthonormal basis of V.

Proof

(a) There exists a symmetric bilinear form ρ on V such that $q = q_\rho$ (by 9.21). Now there exists a basis $e_1, \dots, e_n$ of V such that $\mathcal{M}\big(\rho, (e_1, \dots, e_n)\big)$ is a diagonal matrix (by 9.12). Let $\lambda_1, \dots, \lambda_n$ denote the entries on the diagonal of this matrix. Thus

$$\rho(e_j, e_k) = \begin{cases} \lambda_j & \text{if } j = k, \\ 0 & \text{if } j \neq k \end{cases}$$

for all $j, k \in \{1, \dots, n\}$. If $x_1, \dots, x_n \in \mathbf{F}$, then

$$\begin{aligned} q(x_1e_1 + \cdots + x_ne_n) &= \rho(x_1e_1 + \cdots + x_ne_n, x_1e_1 + \cdots + x_ne_n) \\ &= \sum_{k=1}^{n} \sum_{j=1}^{n} x_jx_k\rho(e_j, e_k) \\ &= \lambda_1x_1^2 + \cdots + \lambda_nx_n^2, \end{aligned}$$

as desired.

(b) Suppose $\mathbf{F} = \mathbf{R}$ and V is an inner product space. Then 9.13 tells us that the basis in (a) can be chosen to be an orthonormal basis of V. ∎

Exercises 9A

1 Prove that if β is a bilinear form on $\mathbf{F}$, then there exists $c \in \mathbf{F}$ such that

$$\beta(x, y) = cxy$$

for all $x, y \in \mathbf{F}$.

2 Let $n = \dim V$. Suppose β is a bilinear form on V. Prove that there exist $\varphi_1, \ldots, \varphi_n, \tau_1, \ldots, \tau_n \in V'$ such that

$$\beta(u, v) = \varphi_1(u) \cdot \tau_1(v) + \cdots + \varphi_n(u) \cdot \tau_n(v)$$

for all $u, v \in V$.

This exercise shows that if $n = \dim V$, then every bilinear form on V is of the form given by the last bullet point of Example 9.2.

3 Suppose $\beta \colon V \times V \to \mathbf{F}$ is a bilinear form on V and also is a linear functional on $V \times V$. Prove that $\beta = 0$.

4 Suppose V is a real inner product space and β is a bilinear form on V. Show that there exists a unique operator $T \in \mathcal{L}(V)$ such that

$$\beta(u, v) = \langle u, Tv \rangle$$

for all $u, v \in V$.

This exercise states that if V is a real inner product space, then every bilinear form on V is of the form given by the third bullet point in 9.2.

5 Suppose β is a bilinear form on a real inner product space V and T is the unique operator on V such that $\beta(u, v) = \langle u, Tv \rangle$ for all $u, v \in V$ (see Exercise 4). Show that β is an inner product on V if and only if T is an invertible positive operator on V.

6 Prove or give a counterexample: If ρ is a symmetric bilinear form on V, then

$$\{v \in V : \rho(v, v) = 0\}$$

is a subspace of V.

7 Explain why the proof of 9.13 (diagonalization of a symmetric bilinear form by an orthonormal basis on a real inner product space) fails if the hypothesis that $\mathbf{F} = \mathbf{R}$ is dropped.

8 Find formulas for $\dim V^{(2)}_{\text{sym}}$ and $\dim V^{(2)}_{\text{alt}}$ in terms of $\dim V$.

9 Suppose that n is a positive integer and $V = \{p \in \mathcal{P}_n(\mathbf{R}) : p(0) = p(1)\}$. Define $\alpha \colon V \times V \to \mathbf{R}$ by

$$\alpha(p, q) = \int_0^1 pq'.$$

Show that α is an alternating bilinear form on V.

10 Suppose that n is a positive integer and

$$V = \{p \in \mathcal{P}_n(\mathbf{R}) : p(0) = p(1) \text{ and } p'(0) = p'(1)\}.$$

Define $\rho\colon V \times V \to \mathbf{R}$ by

$$\rho(p, q) = \int_0^1 pq''.$$

Show that ρ is a symmetric bilinear form on V.