2B *Bases*

In the previous section, we discussed linearly independent lists and we also discussed spanning lists. Now we bring these concepts together by considering lists that have both properties.

2.26 definition: *basis*

A *basis* of V is a list of vectors in V that is linearly independent and spans V.

2.27 example: *bases*

(a) The list $(1,0,\dots,0),(0,1,0,\dots,0),\dots,(0,\dots,0,1)$ is a basis of $\mathbf{F}^n$, called the *standard basis* of $\mathbf{F}^n$.

(b) The list $(1,2),(3,5)$ is a basis of $\mathbf{F}^2$. Note that this list has length two, which is the same as the length of the standard basis of $\mathbf{F}^2$. In the next section, we will see that this is not a coincidence.

(c) The list $(1,2,-4),(7,-5,6)$ is linearly independent in $\mathbf{F}^3$ but is not a basis of $\mathbf{F}^3$ because it does not span $\mathbf{F}^3$.

(d) The list $(1,2),(3,5),(4,13)$ spans $\mathbf{F}^2$ but is not a basis of $\mathbf{F}^2$ because it is not linearly independent.

(e) The list $(1,1,0),(0,0,1)$ is a basis of $\{(x,x,y)\in\mathbf{F}^3 : x,y\in\mathbf{F}\}$.

(f) The list $(1,-1,0),(1,0,-1)$ is a basis of

$$\{(x,y,z)\in\mathbf{F}^3 : x+y+z=0\}.$$

(g) The list $1,z,\dots,z^m$ is a basis of $\mathcal{P}_m(\mathbf{F})$, called the *standard basis* of $\mathcal{P}_m(\mathbf{F})$.

In addition to the standard basis, $\mathbf{F}^n$ has many other bases. For example,

$$(7,5),(-4,9) \quad \text{and} \quad (1,2),(3,5)$$

are both bases of $\mathbf{F}^2$.

The next result helps explain why bases are useful. Recall that "uniquely" means "in only one way".

2.28 *criterion for basis*

A list $v_1,\dots,v_n$ of vectors in V is a basis of V if and only if every $v\in V$ can be written uniquely in the form

$$v = a_1v_1+\cdots+a_nv_n, \tag{2.29}$$

where $a_1,\dots,a_n\in\mathbf{F}$.

This proof is essentially a repetition of the ideas that led us to the definition of linear independence.

Proof First suppose that $v_1, \dots, v_n$ is a basis of V. Let $v \in V$. Because $v_1, \dots, v_n$ spans V, there exist $a_1, \dots, a_n \in \mathbf{F}$ such that 2.29 holds. To show that the representation in 2.29 is unique, suppose $c_1, \dots, c_n$ are scalars such that we also have

$$v = c_1 v_1 + \cdots + c_n v_n.$$

Subtracting the last equation from 2.29, we get

$$0 = (a_1 - c_1)v_1 + \cdots + (a_n - c_n)v_n.$$

This implies that each $a_k - c_k$ equals 0 (because $v_1, \dots, v_n$ is linearly independent). Hence $a_1 = c_1, \dots, a_n = c_n$. We have the desired uniqueness, completing the proof in one direction.

For the other direction, suppose every $v \in V$ can be written uniquely in the form given by 2.29. This implies that the list $v_1, \dots, v_n$ spans V. To show that $v_1, \dots, v_n$ is linearly independent, suppose $a_1, \dots, a_n \in \mathbf{F}$ are such that

$$0 = a_1 v_1 + \cdots + a_n v_n.$$

The uniqueness of the representation 2.29 (taking $v = 0$) now implies that $a_1 = \cdots = a_n = 0$. Thus $v_1, \dots, v_n$ is linearly independent and hence is a basis of V. ■

A spanning list in a vector space may not be a basis because it is not linearly independent. Our next result says that given any spanning list, some (possibly none) of the vectors in it can be discarded so that the remaining list is linearly independent and still spans the vector space.

As an example in the vector space $\mathbf{F}^2$, if the procedure in the proof below is applied to the list $(1,2), (3,6), (4,7), (5,9)$, then the second and fourth vectors will be removed. This leaves $(1,2), (4,7)$, which is a basis of $\mathbf{F}^2$.

2.30 *every spanning list contains a basis*

Every spanning list in a vector space can be reduced to a basis of the vector space.

Proof Suppose $v_1, \dots, v_n$ spans V. We want to remove some of the vectors from $v_1, \dots, v_n$ so that the remaining vectors form a basis of V. We do this through the multistep process described below.

Start with B equal to the list $v_1, \dots, v_n$.

Step 1

If $v_1 = 0$, then delete v_1 from B. If $v_1 \neq 0$, then leave B unchanged.

Step *k*

If v_k is in $\operatorname{span}(v_1, \dots, v_{k-1})$, then delete v_k from the list B. If v_k is not in $\operatorname{span}(v_1, \dots, v_{k-1})$, then leave B unchanged.

Stop the process after step n, getting a list B. This list B spans V because our original list spanned V and we have discarded only vectors that were already in the span of the previous vectors. The process ensures that no vector in B is in the span of the previous ones. Thus B is linearly independent, by the linear dependence lemma (2.19). Hence B is a basis of V. ■

We now come to an important corollary of the previous result.

2.31 *basis of finite-dimensional vector space*

Every finite-dimensional vector space has a basis.

Proof By definition, a finite-dimensional vector space has a spanning list. The previous result tells us that each spanning list can be reduced to a basis. ■

Our next result is in some sense a dual of 2.30, which said that every spanning list can be reduced to a basis. Now we show that given any linearly independent list, we can adjoin some additional vectors (this includes the possibility of adjoining no additional vectors) so that the extended list is still linearly independent but also spans the space.

2.32 *every linearly independent list extends to a basis*

Every linearly independent list of vectors in a finite-dimensional vector space can be extended to a basis of the vector space.

Proof Suppose $u_1, \ldots, u_m$ is linearly independent in a finite-dimensional vector space V. Let $w_1, \ldots, w_n$ be a list of vectors in V that spans V. Thus the list

$$u_1, \ldots, u_m, w_1, \ldots, w_n$$

spans V. Applying the procedure of the proof of 2.30 to reduce this list to a basis of V produces a basis consisting of the vectors $u_1, \ldots, u_m$ and some of the w's (none of the u's get deleted in this procedure because $u_1, \ldots, u_m$ is linearly independent). ■

As an example in $\mathbf{F}^3$, suppose we start with the linearly independent list $(2, 3, 4), (9, 6, 8)$. If we take w_1, w_2, w_3 to be the standard basis of $\mathbf{F}^3$, then applying the procedure in the proof above produces the list

$$(2, 3, 4), (9, 6, 8), (0, 1, 0),$$

which is a basis of $\mathbf{F}^3$.

As an application of the result above, we now show that every subspace of a finite-dimensional vector space can be paired with another subspace to form a direct sum of the whole space.

Using the same ideas but more advanced tools, the next result can be proved without the hypothesis that V is finite-dimensional.

2.33 *every subspace of V is part of a direct sum equal to V*

Suppose V is finite-dimensional and U is a subspace of V. Then there is a subspace W of V such that $V = U \oplus W$.

Proof Because V is finite-dimensional, so is U (see 2.25). Thus there is a basis $u_1, \dots, u_m$ of U (by 2.31). Of course $u_1, \dots, u_m$ is a linearly independent list of vectors in V. Hence this list can be extended to a basis $u_1, \dots, u_m, w_1, \dots, w_n$ of V (by 2.32). Let $W = \operatorname{span}(w_1, \dots, w_n)$.

To prove that $V = U \oplus W$, by 1.46 we only need to show that

$$V = U + W \quad \text{and} \quad U \cap W = \{0\}.$$

To prove the first equation above, suppose $v \in V$. Then, because the list $u_1, \dots, u_m, w_1, \dots, w_n$ spans V, there exist $a_1, \dots, a_m, b_1, \dots, b_n \in \mathbf{F}$ such that

$$v = \underbrace{a_1 u_1 + \cdots + a_m u_m}_{u} + \underbrace{b_1 w_1 + \cdots + b_n w_n}_{w}.$$

We have $v = u + w$, where $u \in U$ and $w \in W$ are defined as above. Thus $v \in U + W$, completing the proof that $V = U + W$.

To show that $U \cap W = \{0\}$, suppose $v \in U \cap W$. Then there exist scalars $a_1, \dots, a_m, b_1, \dots, b_n \in \mathbf{F}$ such that

$$v = a_1 u_1 + \cdots + a_m u_m = b_1 w_1 + \cdots + b_n w_n.$$

Thus

$$a_1 u_1 + \cdots + a_m u_m - b_1 w_1 - \cdots - b_n w_n = 0.$$

Because $u_1, \dots, u_m, w_1, \dots, w_n$ is linearly independent, this implies that

$$a_1 = \cdots = a_m = b_1 = \cdots = b_n = 0.$$

Thus $v = 0$, completing the proof that $U \cap W = \{0\}$. ■

Exercises 2B

1 Find all vector spaces that have exactly one basis.

2 Verify all assertions in Example 2.27.

3 (a) Let U be the subspace of $\mathbf{R}^5$ defined by

$$U = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{R}^5 : x_1 = 3x_2 \text{ and } x_3 = 7x_4\}.$$

Find a basis of U.

(b) Extend the basis in (a) to a basis of $\mathbf{R}^5$.

(c) Find a subspace W of $\mathbf{R}^5$ such that $\mathbf{R}^5 = U \oplus W$.

4 (a) Let U be the subspace of $\mathbf{C}^5$ defined by

$$U = \{(z_1, z_2, z_3, z_4, z_5) \in \mathbf{C}^5 : 6z_1 = z_2 \text{ and } z_3 + 2z_4 + 3z_5 = 0\}.$$

Find a basis of U.

(b) Extend the basis in (a) to a basis of $\mathbf{C}^5$.

(c) Find a subspace W of $\mathbf{C}^5$ such that $\mathbf{C}^5 = U \oplus W$.

5 Suppose V is finite-dimensional and U, W are subspaces of V such that $V = U + W$. Prove that there exists a basis of V consisting of vectors in $U \cup W$.

6 Prove or give a counterexample: If p_0, p_1, p_2, p_3 is a list in $\mathcal{P}_3(\mathbf{F})$ such that none of the polynomials p_0, p_1, p_2, p_3 has degree 2, then p_0, p_1, p_2, p_3 is not a basis of $\mathcal{P}_3(\mathbf{F})$.

7 Suppose v_1, v_2, v_3, v_4 is a basis of V. Prove that

$$v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$$

is also a basis of V.

8 Prove or give a counterexample: If v_1, v_2, v_3, v_4 is a basis of V and U is a subspace of V such that $v_1, v_2 \in U$ and $v_3 \notin U$ and $v_4 \notin U$, then v_1, v_2 is a basis of U.

9 Suppose $v_1, \dots, v_m$ is a list of vectors in V. For $k \in \{1, \dots, m\}$, let

$$w_k = v_1 + \cdots + v_k.$$

Show that $v_1, \dots, v_m$ is a basis of V if and only if $w_1, \dots, w_m$ is a basis of V.

10 Suppose U and W are subspaces of V such that $V = U \oplus W$. Suppose also that $u_1, \dots, u_m$ is a basis of U and $w_1, \dots, w_n$ is a basis of W. Prove that

$$u_1, \dots, u_m, w_1, \dots, w_n$$

is a basis of V.

11 Suppose V is a real vector space. Show that if $v_1, \dots, v_n$ is a basis of V (as a real vector space), then $v_1, \dots, v_n$ is also a basis of the complexification $V_{\mathbf{C}}$ (as a complex vector space).

See Exercise 8 in Section 1B for the definition of the complexification $V_{\mathbf{C}}$.