5D *Diagonalizable Operators*

Diagonal Matrices

5.48 definition: *diagonal matrix*

A *diagonal matrix* is a square matrix that is 0 everywhere except possibly on the diagonal.

5.49 example: *diagonal matrix*

$$\begin{pmatrix} 8 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

is a diagonal matrix.

If an operator has a diagonal matrix with respect to some basis, then the entries on the diagonal are precisely the eigenvalues of the operator; this follows from 5.41 (or find an easier direct proof for diagonal matrices).

Every diagonal matrix is upper triangular. Diagonal matrices typically have many more 0's than most upper-triangular matrices of the same size.

5.50 definition: *diagonalizable*

An operator on V is called *diagonalizable* if the operator has a diagonal matrix with respect to some basis of V.

5.51 example: *diagonalization may require a different basis*

Define $T \in \mathcal{L}(\mathbf{R}^2)$ by

$$T(x, y) = (41x + 7y, -20x + 74y).$$

The matrix of T with respect to the standard basis of $\mathbf{R}^2$ is

$$\begin{pmatrix} 41 & 7 \\ -20 & 74 \end{pmatrix},$$

which is not a diagonal matrix. However, T is diagonalizable. Specifically, the matrix of T with respect to the basis $(1, 4), (7, 5)$ is

$$\begin{pmatrix} 69 & 0 \\ 0 & 46 \end{pmatrix}$$

because $T(1, 4) = (69, 276) = 69(1, 4)$ and $T(7, 5) = (322, 230) = 46(7, 5)$.

For $\lambda \in \mathbf{F}$, we will find it convenient to have a name and a notation for the set of vectors that an operator T maps to λ times the vector.

5.52 definition: *eigenspace, $E(\lambda, T)$*

Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbf{F}$. The *eigenspace* of T corresponding to λ is the subspace $E(\lambda, T)$ of V defined by

$$E(\lambda, T) = \operatorname{null}(T - \lambda I) = \{v \in V : Tv = \lambda v\}.$$

Hence $E(\lambda, T)$ is the set of all eigenvectors of T corresponding to λ, along with the 0 vector.

For $T \in \mathcal{L}(V)$ and $\lambda \in \mathbf{F}$, the set $E(\lambda, T)$ is a subspace of V because the null space of each linear map on V is a subspace of V. The definitions imply that λ is an eigenvalue of T if and only if $E(\lambda, T) \neq \{0\}$.

5.53 example: *eigenspaces of an operator*

Suppose the matrix of an operator $T \in \mathcal{L}(V)$ with respect to a basis v_1, v_2, v_3 of V is the matrix in Example 5.49. Then

$$E(8, T) = \operatorname{span}(v_1), \quad E(5, T) = \operatorname{span}(v_2, v_3).$$

If λ is an eigenvalue of an operator $T \in \mathcal{L}(V)$, then T restricted to $E(\lambda, T)$ is just the operator of multiplication by λ.

5.54 *sum of eigenspaces is a direct sum*

Suppose $T \in \mathcal{L}(V)$ and $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of T. Then

$$E(\lambda_1, T) + \cdots + E(\lambda_m, T)$$

is a direct sum. Furthermore, if V is finite-dimensional, then

$$\dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T) \leq \dim V.$$

Proof To show that $E(\lambda_1, T) + \cdots + E(\lambda_m, T)$ is a direct sum, suppose

$$v_1 + \cdots + v_m = 0,$$

where each v_k is in $E(\lambda_k, T)$. Because eigenvectors corresponding to distinct eigenvalues are linearly independent (by 5.11), this implies that each v_k equals 0. Thus $E(\lambda_1, T) + \cdots + E(\lambda_m, T)$ is a direct sum (by 1.45), as desired.

Now suppose V is finite-dimensional. Then

$$\begin{aligned} \dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T) &= \dim\bigl(E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)\bigr) \\ &\leq \dim V, \end{aligned}$$

where the first line follows from 3.94 and the second line follows from 2.37. ∎

Conditions for Diagonalizability

The following characterizations of diagonalizable operators will be useful.

5.55 *conditions equivalent to diagonalizability*

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_m$ denote the distinct eigenvalues of T. Then the following are equivalent.

(a) T is diagonalizable.

(b) V has a basis consisting of eigenvectors of T.

(c) $V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$.

(d) $\dim V = \dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T)$.

Proof An operator $T \in \mathcal{L}(V)$ has a diagonal matrix

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

with respect to a basis $v_1, \ldots, v_n$ of V if and only if $Tv_k = \lambda_k v_k$ for each k. Thus (a) and (b) are equivalent.

Suppose (b) holds; thus V has a basis consisting of eigenvectors of T. Hence every vector in V is a linear combination of eigenvectors of T, which implies that

$$V = E(\lambda_1, T) + \cdots + E(\lambda_m, T).$$

Now 5.54 shows that (c) holds, proving that (b) implies (c).

That (c) implies (d) follows immediately from 3.94.

Finally, suppose (d) holds; thus

5.56
$$\dim V = \dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T).$$

Choose a basis of each $E(\lambda_k, T)$; put all these bases together to form a list $v_1, \ldots, v_n$ of eigenvectors of T, where $n = \dim V$ (by 5.56). To show that this list is linearly independent, suppose

$$a_1 v_1 + \cdots + a_n v_n = 0,$$

where $a_1, \ldots, a_n \in \mathbf{F}$. For each $k = 1, \ldots, m$, let u_k denote the sum of all the terms $a_j v_j$ such that $v_j \in E(\lambda_k, T)$. Thus each u_k is in $E(\lambda_k, T)$, and

$$u_1 + \cdots + u_m = 0.$$

Because eigenvectors corresponding to distinct eigenvalues are linearly independent (see 5.11), this implies that each u_k equals 0. Because each u_k is a sum of terms $a_j v_j$, where the v_j's were chosen to be a basis of $E(\lambda_k, T)$, this implies that all a_j's equal 0. Thus $v_1, \ldots, v_n$ is linearly independent and hence is a basis of V (by 2.38). Thus (d) implies (b), completing the proof. ■

For additional conditions equivalent to diagonalizability, see 5.62, Exercises 5 and 15 in this section, Exercise 24 in Section 7B, and Exercise 15 in Section 8A.

As we know, every operator on a nonzero finite-dimensional complex vector space has an eigenvalue. However, not every operator on a nonzero finite-dimensional complex vector space has enough eigenvectors to be diagonalizable, as shown by the next example.

5.57 example: *an operator that is not diagonalizable*

Define an operator $T \in \mathcal{L}(\mathbf{F}^3)$ by $T(a,b,c) = (b,c,0)$. The matrix of T with respect to the standard basis of $\mathbf{F}^3$ is

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

which is an upper-triangular matrix but is not a diagonal matrix.

As you should verify, 0 is the only eigenvalue of T and furthermore

$$E(0,T) = \{(a,0,0) \in \mathbf{F}^3 : a \in \mathbf{F}\}.$$

Hence conditions (b), (c), and (d) of 5.55 fail (of course, because these conditions are equivalent, it is sufficient to check that only one of them fails). Thus condition (a) of 5.55 also fails. Hence T is not diagonalizable, regardless of whether $\mathbf{F} = \mathbf{R}$ or $\mathbf{F} = \mathbf{C}$.

The next result shows that if an operator has as many distinct eigenvalues as the dimension of its domain, then the operator is diagonalizable.

5.58 *enough eigenvalues implies diagonalizability*

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$ has $\dim V$ distinct eigenvalues. Then T is diagonalizable.

Proof Suppose T has distinct eigenvalues $\lambda_1, \dots, \lambda_{\dim V}$. For each k, let $v_k \in V$ be an eigenvector corresponding to the eigenvalue λ_k. Because eigenvectors corresponding to distinct eigenvalues are linearly independent (see 5.11), $v_1, \dots, v_{\dim V}$ is linearly independent.

A linearly independent list of $\dim V$ vectors in V is a basis of V (see 2.38); thus $v_1, \dots, v_{\dim V}$ is a basis of V. With respect to this basis consisting of eigenvectors, T has a diagonal matrix. ∎

In later chapters we will find additional conditions that imply that certain operators are diagonalizable. For example, see the real spectral theorem (7.29) and the complex spectral theorem (7.31).

The result above gives a sufficient condition for an operator to be diagonalizable. However, this condition is not necessary. For example, the operator T on $\mathbf{F}^3$ defined by $T(x,y,z) = (6x,6y,7z)$ has only two eigenvalues (6 and 7) and $\dim \mathbf{F}^3 = 3$, but T is diagonalizable (by the standard basis of $\mathbf{F}^3$).

The next example illustrates the importance of diagonalization, which can be used to compute high powers of an operator, taking advantage of the equation $T^k v = \lambda^k v$ if v is an eigenvector of T with eigenvalue λ.

For a spectacular application of these techniques, see Exercise 21, which shows how to use diagonalization to find an exact formula for the n^{th} term of the Fibonacci sequence.

5.59 example: *using diagonalization to compute T^{100}*

Define $T \in \mathcal{L}(\mathbf{F}^3)$ by $T(x, y, z) = (2x + y, 5y + 3z, 8z)$. With respect to the standard basis, the matrix of T is

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 5 & 3 \\ 0 & 0 & 8 \end{pmatrix}.$$

The matrix above is an upper-triangular matrix but it is not a diagonal matrix. By 5.41, the eigenvalues of T are 2, 5, and 8. Because T is an operator on a vector space of dimension three and T has three distinct eigenvalues, 5.58 assures us that there exists a basis of $\mathbf{F}^3$ with respect to which T has a diagonal matrix.

To find this basis, we only have to find an eigenvector for each eigenvalue. In other words, we have to find a nonzero solution to the equation

$$T(x, y, z) = \lambda(x, y, z)$$

for $\lambda = 2$, then for $\lambda = 5$, and then for $\lambda = 8$. Solving these simple equations shows that for $\lambda = 2$ we have an eigenvector $(1, 0, 0)$, for $\lambda = 5$ we have an eigenvector $(1, 3, 0)$, and for $\lambda = 8$ we have an eigenvector $(1, 6, 6)$.

Thus $(1, 0, 0), (1, 3, 0), (1, 6, 6)$ is a basis of $\mathbf{F}^3$ consisting of eigenvectors of T, and with respect to this basis the matrix of T is the diagonal matrix

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 8 \end{pmatrix}.$$

To compute $T^{100}(0, 0, 1)$, for example, write $(0, 0, 1)$ as a linear combination of our basis of eigenvectors:

$$(0, 0, 1) = \tfrac{1}{6}(1, 0, 0) - \tfrac{1}{3}(1, 3, 0) + \tfrac{1}{6}(1, 6, 6).$$

Now apply T^{100} to both sides of the equation above, getting

$$\begin{aligned} T^{100}(0, 0, 1) &= \tfrac{1}{6}\bigl(T^{100}(1, 0, 0)\bigr) - \tfrac{1}{3}\bigl(T^{100}(1, 3, 0)\bigr) + \tfrac{1}{6}\bigl(T^{100}(1, 6, 6)\bigr) \\ &= \tfrac{1}{6}\bigl(2^{100}(1, 0, 0) - 2 \cdot 5^{100}(1, 3, 0) + 8^{100}(1, 6, 6)\bigr) \\ &= \tfrac{1}{6}\bigl(2^{100} - 2 \cdot 5^{100} + 8^{100}, 6 \cdot 8^{100} - 6 \cdot 5^{100}, 6 \cdot 8^{100}\bigr). \end{aligned}$$

We saw earlier that an operator T on a finite-dimensional vector space V has an upper-triangular matrix with respect to some basis of V if and only if the minimal polynomial of T equals $(z - \lambda_1) \cdots (z - \lambda_m)$ for some $\lambda_1, \ldots, \lambda_m \in \mathbf{F}$ (see 5.44). As we previously noted (see 5.47), this condition is always satisfied if $\mathbf{F} = \mathbf{C}$.

Our next result 5.62 states that an operator $T \in \mathcal{L}(V)$ has a diagonal matrix with respect to some basis of V if and only if the minimal polynomial of T equals $(z - \lambda_1) \cdots (z - \lambda_m)$ for some ***distinct*** $\lambda_1, \ldots, \lambda_m \in \mathbf{F}$. Before formally stating this result, we give two examples of using it.

5.60 example: *diagonalizable, but with no known exact eigenvalues*

Define $T \in \mathcal{L}(\mathbf{C}^5)$ by

$$T(z_1, z_2, z_3, z_4, z_5) = (-3z_5, z_1 + 6z_5, z_2, z_3, z_4).$$

The matrix of T is shown in Example 5.26, where we showed that the minimal polynomial of T is $3 - 6z + z^5$.

As mentioned in Example 5.28, no exact expression is known for any of the zeros of this polynomial, but numeric techniques show that the zeros of this polynomial are approximately -1.67, 0.51, 1.40, $-0.12 + 1.59i$, $-0.12 - 1.59i$.

The software that produces these approximations is accurate to more than three digits. Thus these approximations are good enough to show that the five numbers above are distinct. The minimal polynomial of T equals the fifth degree monic polynomial with these zeros. Now 5.62 shows that T is diagonalizable.

5.61 example: *showing that an operator is not diagonalizable*

Define $T \in \mathcal{L}(\mathbf{F}^3)$ by

$$T(z_1, z_2, z_3) = (6z_1 + 3z_2 + 4z_3, 6z_2 + 2z_3, 7z_3).$$

The matrix of T with respect to the standard basis of $\mathbf{F}^3$ is

$$\begin{pmatrix} 6 & 3 & 4 \\ 0 & 6 & 2 \\ 0 & 0 & 7 \end{pmatrix}.$$

The matrix above is an upper-triangular matrix but is not a diagonal matrix. Might T have a diagonal matrix with respect to some other basis of $\mathbf{F}^3$?

To answer this question, we will find the minimal polynomial of T. First note that the eigenvalues of T are the diagonal entries of the matrix above (by 5.41). Thus the zeros of the minimal polynomial of T are $6, 7$ [by 5.27(a)]. The diagonal of the matrix above tells us that $(T - 6I)^2(T - 7I) = 0$ (by 5.40). The minimal polynomial of T has degree at most 3 (by 5.22). Putting all this together, we see that the minimal polynomial of T is either $(z - 6)(z - 7)$ or $(z - 6)^2(z - 7)$.

A simple computation shows that $(T - 6I)(T - 7I) \neq 0$. Thus the minimal polynomial of T is $(z - 6)^2(z - 7)$.

Now 5.62 shows that T is not diagonalizable.

5.62 *necessary and sufficient condition for diagonalizability*

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Then T is diagonalizable if and only if the minimal polynomial of T equals $(z - \lambda_1) \cdots (z - \lambda_m)$ for some list of distinct numbers $\lambda_1, \ldots, \lambda_m \in \mathbf{F}$.

Proof First suppose T is diagonalizable. Thus there is a basis $v_1, \ldots, v_n$ of V consisting of eigenvectors of T. Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of T. Then for each v_j, there exists λ_k with $(T - \lambda_k I)v_j = 0$. Thus

$$(T - \lambda_1 I) \cdots (T - \lambda_m I)v_j = 0,$$

which implies that the minimal polynomial of T equals $(z - \lambda_1) \cdots (z - \lambda_m)$.

To prove the implication in the other direction, now suppose the minimal polynomial of T equals $(z - \lambda_1) \cdots (z - \lambda_m)$ for some list of distinct numbers $\lambda_1, \ldots, \lambda_m \in \mathbf{F}$. Thus

5.63
$$(T - \lambda_1 I) \cdots (T - \lambda_m I) = 0.$$

We will prove that T is diagonalizable by induction on m. To get started, suppose $m = 1$. Then $T - \lambda_1 I = 0$, which means that T is a scalar multiple of the identity operator, which implies that T is diagonalizable.

Now suppose that $m > 1$ and the desired result holds for all smaller values of m. The subspace $\operatorname{range}(T - \lambda_m I)$ is invariant under T [this is a special case of 5.18 with $p(z) = z - \lambda_m$]. Thus T restricted to $\operatorname{range}(T - \lambda_m I)$ is an operator on $\operatorname{range}(T - \lambda_m I)$.

If $u \in \operatorname{range}(T - \lambda_m I)$, then $u = (T - \lambda_m I)v$ for some $v \in V$, and 5.63 implies

5.64
$$(T - \lambda_1 I) \cdots (T - \lambda_{m-1} I)u = (T - \lambda_1 I) \cdots (T - \lambda_m I)v = 0.$$

Hence $(z - \lambda_1) \cdots (z - \lambda_{m-1})$ is a polynomial multiple of the minimal polynomial of T restricted to $\operatorname{range}(T - \lambda_m I)$ [by 5.29]. Thus by our induction hypothesis, there is a basis of $\operatorname{range}(T - \lambda_m I)$ consisting of eigenvectors of T.

Suppose that $u \in \operatorname{range}(T - \lambda_m I) \cap \operatorname{null}(T - \lambda_m I)$. Then $Tu = \lambda_m u$. Now 5.64 implies that

$$\begin{aligned} 0 &= (T - \lambda_1 I) \cdots (T - \lambda_{m-1} I)u \\ &= (\lambda_m - \lambda_1) \cdots (\lambda_m - \lambda_{m-1})u. \end{aligned}$$

Because $\lambda_1, \ldots, \lambda_m$ are distinct, the equation above implies that $u = 0$. Hence $\operatorname{range}(T - \lambda_m I) \cap \operatorname{null}(T - \lambda_m I) = \{0\}$.

Thus $\operatorname{range}(T - \lambda_m I) + \operatorname{null}(T - \lambda_m I)$ is a direct sum (by 1.46) whose dimension is $\dim V$ (by 3.94 and 3.21). Hence $\operatorname{range}(T - \lambda_m I) \oplus \operatorname{null}(T - \lambda_m I) = V$. Every nonzero vector in $\operatorname{null}(T - \lambda_m I)$ is an eigenvector of T with eigenvalue λ_m. Earlier in this proof we saw that there is a basis of $\operatorname{range}(T - \lambda_m I)$ consisting of eigenvectors of T. Adjoining to that basis a basis of $\operatorname{null}(T - \lambda_m I)$ gives a basis of V consisting of eigenvectors of T. The matrix of T with respect to this basis is a diagonal matrix, as desired. ■

No formula exists for the zeros of polynomials of degree 5 or greater. However, the previous result can be used to determine whether an operator on a complex vector space is diagonalizable without even finding approximations of the zeros of the minimal polynomial—see Exercise 15.

The next result will be a key tool when we prove a result about the simultaneous diagonalization of two operators; see 5.76. Note how the use of a characterization of diagonalizable operators in terms of the minimal polynomial (see 5.62) leads to a short proof of the next result.

5.65 *restriction of diagonalizable operator to invariant subspace*

Suppose $T \in \mathcal{L}(V)$ is diagonalizable and U is a subspace of V that is invariant under T. Then $T|_U$ is a diagonalizable operator on U.

Proof Because the operator T is diagonalizable, the minimal polynomial of T equals $(z - \lambda_1) \cdots (z - \lambda_m)$ for some list of distinct numbers $\lambda_1, \dots, \lambda_m \in \mathbf{F}$ (by 5.62). The minimal polynomial of T is a polynomial multiple of the minimal polynomial of $T|_U$ (by 5.31). Hence the minimal polynomial of $T|_U$ has the form required by 5.62, which shows that $T|_U$ is diagonalizable. ■

Gershgorin Disk Theorem

5.66 definition: *Gershgorin disks*

Suppose $T \in \mathcal{L}(V)$ and $v_1, \dots, v_n$ is a basis of V. Let A denote the matrix of T with respect to this basis. A *Gershgorin disk* of T with respect to the basis $v_1, \dots, v_n$ is a set of the form

$$\Big\{ z \in \mathbf{F} : |z - A_{j,j}| \leq \sum_{\substack{k=1 \\ k \neq j}}^{n} |A_{j,k}| \Big\},$$

where $j \in \{1, \dots, n\}$.

Because there are n choices for j in the definition above, T has n Gershgorin disks. If $\mathbf{F} = \mathbf{C}$, then for each $j \in \{1, \dots, n\}$, the corresponding Gershgorin disk is a closed disk in $\mathbf{C}$ centered at $A_{j,j}$, which is the j^{th} entry on the diagonal of A. The radius of this closed disk is the sum of the absolute values of the entries in row j of A, excluding the diagonal entry. If $\mathbf{F} = \mathbf{R}$, then the Gershgorin disks are closed intervals in $\mathbf{R}$.

In the special case that the square matrix A above is a diagonal matrix, each Gershgorin disk consists of a single point that is a diagonal entry of A (and each eigenvalue of T is one of those points, as required by the next result). One consequence of our next result is that if the nondiagonal entries of A are small, then each eigenvalue of T is near a diagonal entry of A.

5.67 *Gershgorin disk theorem*

Suppose $T \in \mathcal{L}(V)$ and $v_1, \ldots, v_n$ is a basis of V. Then each eigenvalue of T is contained in some Gershgorin disk of T with respect to the basis $v_1, \ldots, v_n$.

Proof Suppose $\lambda \in \mathbf{F}$ is an eigenvalue of T. Let $w \in V$ be a corresponding eigenvector. There exist $c_1, \ldots, c_n \in \mathbf{F}$ such that

5.68
$$w = c_1 v_1 + \cdots + c_n v_n.$$

Let A denote the matrix of T with respect to the basis $v_1, \ldots, v_n$. Applying T to both sides of the equation above gives

5.69
$$\lambda w = \sum_{k=1}^{n} c_k T v_k$$

$$= \sum_{k=1}^{n} c_k \sum_{j=1}^{n} A_{j,k} v_j$$

5.70
$$= \sum_{j=1}^{n} \Bigl(\sum_{k=1}^{n} A_{j,k} c_k \Bigr) v_j.$$

Let $j \in \{1, \ldots, n\}$ be such that

$$|c_j| = \max\{|c_1|, \ldots, |c_n|\}.$$

Using 5.68, we see that the coefficient of v_j on the left side of 5.69 equals λc_j, which must equal the coefficient of v_j on the right side of 5.70. In other words,

$$\lambda c_j = \sum_{k=1}^{n} A_{j,k}\, c_k.$$

Subtract $A_{j,j}\, c_j$ from each side of the equation above and then divide both sides by c_j to get

$$|\lambda - A_{j,j}| = \Biggl| \sum_{\substack{k=1\\ k \neq j}}^{n} A_{j,k} \frac{c_k}{c_j} \Biggr|$$

$$\leq \sum_{\substack{k=1\\ k \neq j}}^{n} |A_{j,k}|.$$

Thus λ is in the j^{th} Gershgorin disk with respect to the basis $v_1, \ldots, v_n$. ■

The Gershgorin disk theorem is named for Semyon Aronovich Gershgorin, who published this result in 1931.

Exercise 22 gives a nice application of the Gershgorin disk theorem.

Exercise 23 states that the radius of each Gershgorin disk could be changed to the sum of the absolute values of corresponding column entries (instead of row entries), excluding the diagonal entry, and the theorem above would still hold.

Exercises 5D

1 Suppose V is a finite-dimensional complex vector space and $T \in \mathcal{L}(V)$.

(a) Prove that if $T^4 = I$, then T is diagonalizable.

(b) Prove that if $T^4 = T$, then T is diagonalizable.

(c) Give an example of an operator $T \in \mathcal{L}(\mathbf{C}^2)$ such that $T^4 = T^2$ and T is not diagonalizable.

2 Suppose $T \in \mathcal{L}(V)$ has a diagonal matrix A with respect to some basis of V. Prove that if $\lambda \in \mathbf{F}$, then λ appears on the diagonal of A precisely $\dim E(\lambda, T)$ times.

3 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that if the operator T is diagonalizable, then $V = \operatorname{null} T \oplus \operatorname{range} T$.

4 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that the following are equivalent.

(a) $V = \operatorname{null} T \oplus \operatorname{range} T$.

(b) $V = \operatorname{null} T + \operatorname{range} T$.

(c) $\operatorname{null} T \cap \operatorname{range} T = \{0\}$.

5 Suppose V is a finite-dimensional complex vector space and $T \in \mathcal{L}(V)$. Prove that T is diagonalizable if and only if

$$V = \operatorname{null}(T - \lambda I) \oplus \operatorname{range}(T - \lambda I)$$

for every $\lambda \in \mathbf{C}$.

6 Suppose $T \in \mathcal{L}(\mathbf{F}^5)$ and $\dim E(8, T) = 4$. Prove that $T - 2I$ or $T - 6I$ is invertible.

7 Suppose $T \in \mathcal{L}(V)$ is invertible. Prove that

$$E(\lambda, T) = E\left(\tfrac{1}{\lambda}, T^{-1}\right)$$

for every $\lambda \in \mathbf{F}$ with $\lambda \neq 0$.

8 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_m$ denote the distinct nonzero eigenvalues of T. Prove that

$$\dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T) \leq \dim \operatorname{range} T.$$

9 Suppose $R, T \in \mathcal{L}(\mathbf{F}^3)$ each have 2, 6, 7 as eigenvalues. Prove that there exists an invertible operator $S \in \mathcal{L}(\mathbf{F}^3)$ such that $R = S^{-1}TS$.

10 Find $R, T \in \mathcal{L}(\mathbf{F}^4)$ such that R and T each have 2, 6, 7 as eigenvalues, R and T have no other eigenvalues, and there does not exist an invertible operator $S \in \mathcal{L}(\mathbf{F}^4)$ such that $R = S^{-1}TS$.

11 Find $T \in \mathcal{L}(\mathbf{C}^3)$ such that 6 and 7 are eigenvalues of T and such that T does not have a diagonal matrix with respect to any basis of $\mathbf{C}^3$.

12 Suppose $T \in \mathcal{L}(\mathbf{C}^3)$ is such that 6 and 7 are eigenvalues of T. Furthermore, suppose T does not have a diagonal matrix with respect to any basis of $\mathbf{C}^3$. Prove that there exists $(z_1, z_2, z_3) \in \mathbf{C}^3$ such that

$$T(z_1, z_2, z_3) = (6 + 8z_1, 7 + 8z_2, 13 + 8z_3).$$

13 Suppose A is a diagonal matrix with distinct entries on the diagonal and B is a matrix of the same size as A. Show that $AB = BA$ if and only if B is a diagonal matrix.

14 (a) Give an example of a finite-dimensional complex vector space and an operator T on that vector space such that T^2 is diagonalizable but T is not diagonalizable.

(b) Suppose $\mathbf{F} = \mathbf{C}$, k is a positive integer, and $T \in \mathcal{L}(V)$ is invertible. Prove that T is diagonalizable if and only if T^k is diagonalizable.

15 Suppose V is a finite-dimensional complex vector space, $T \in \mathcal{L}(V)$, and p is the minimal polynomial of T. Prove that the following are equivalent.

(a) T is diagonalizable.

(b) There does not exist $\lambda \in \mathbf{C}$ such that p is a polynomial multiple of $(z - \lambda)^2$.

(c) p and its derivative p' have no zeros in common.

(d) The greatest common divisor of p and p' is the constant polynomial 1.

*The **greatest common divisor** of p and p' is the monic polynomial q of largest degree such that p and p' are both polynomial multiples of q. The Euclidean algorithm for polynomials (look it up) can quickly determine the greatest common divisor of two polynomials, without requiring any information about the zeros of the polynomials. Thus the equivalence of (a) and (d) above shows that we can determine whether T is diagonalizable without knowing anything about the zeros of p.*

16 Suppose that $T \in \mathcal{L}(V)$ is diagonalizable. Let $\lambda_1, \ldots, \lambda_m$ denote the distinct eigenvalues of T. Prove that a subspace U of V is invariant under T if and only if there exist subspaces $U_1, \ldots, U_m$ of V such that $U_k \subseteq E(\lambda_k, T)$ for each k and $U = U_1 \oplus \cdots \oplus U_m$.

17 Suppose V is finite-dimensional. Prove that $\mathcal{L}(V)$ has a basis consisting of diagonalizable operators.

18 Suppose that $T \in \mathcal{L}(V)$ is diagonalizable and U is a subspace of V that is invariant under T. Prove that the quotient operator T/U is a diagonalizable operator on V/U.

The quotient operator T/U was defined in Exercise 38 in Section 5A.

19 Prove or give a counterexample: If $T \in \mathcal{L}(V)$ and there exists a subspace U of V that is invariant under T such that $T|_U$ and T/U are both diagonalizable, then T is diagonalizable.

See Exercise 13 in Section 5C for an analogous statement about upper-triangular matrices.

20 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T is diagonalizable if and only if the dual operator T' is diagonalizable.

21 The *Fibonacci sequence* $F_0, F_1, F_2, \ldots$ is defined by

$$F_0 = 0,\ F_1 = 1,\ \text{and } F_n = F_{n-2} + F_{n-1} \text{ for } n \geq 2.$$

Define $T \in \mathcal{L}(\mathbf{R}^2)$ by $T(x, y) = (y, x + y)$.

(a) Show that $T^n(0,1) = (F_n, F_{n+1})$ for each nonnegative integer n.
(b) Find the eigenvalues of T.
(c) Find a basis of $\mathbf{R}^2$ consisting of eigenvectors of T.
(d) Use the solution to (c) to compute $T^n(0,1)$. Conclude that

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

for each nonnegative integer n.
(e) Use (d) to conclude that if n is a nonnegative integer, then the Fibonacci number F_n is the integer that is closest to

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n.$$

Each F_n is a nonnegative integer, even though the right side of the formula in (d) does not look like an integer. The number

$$\frac{1+\sqrt{5}}{2}$$

is called the ***golden ratio****.*

22 Suppose $T \in \mathcal{L}(V)$ and A is an n-by-n matrix that is the matrix of T with respect to some basis of V. Prove that if

$$|A_{j,j}| > \sum_{\substack{k=1 \\ k \neq j}}^{n} |A_{j,k}|$$

for each $j \in \{1, \ldots, n\}$, then T is invertible.

This exercise states that if the diagonal entries of the matrix of T are large compared to the nondiagonal entries, then T is invertible.

23 Suppose the definition of the Gershgorin disks is changed so that the radius of the k^{th} disk is the sum of the absolute values of the entries in column (instead of row) k of A, excluding the diagonal entry. Show that the Gershgorin disk theorem (5.67) still holds with this changed definition.