5E Commuting Operators

5.71 definition: *commute*

- Two operators S and T on the same vector space *commute* if $ST = TS$.
- Two square matrices A and B of the same size *commute* if $AB = BA$.

For example, if I is the identity operator on V and $\lambda \in \mathbf{F}$, then λI commutes with every operator on V.

As another example, if T is an operator then T^2 and T^3 commute. More generally, if $p, q \in \mathcal{P}(\mathbf{F})$, then $p(T)$ and $q(T)$ commute [see 5.17(b)].

5.72 example: *partial differentiation operators commute*

Suppose m is a nonnegative integer. Let $\mathcal{P}_m(\mathbf{R}^2)$ denote the real vector space of polynomials (with real coefficients) in two real variables and of degree at most m, with the usual operations of addition and scalar multiplication of real-valued functions. Thus the elements of $\mathcal{P}_m(\mathbf{R}^2)$ are functions p on $\mathbf{R}^2$ of the form

5.73 $$p = \sum_{j+k \leq m} a_{j,k} x^j y^k,$$

where the indices j and k take on all nonnegative integer values such that $j + k \leq m$, each $a_{j,k}$ is in $\mathbf{R}$, and $x^j y^k$ denotes the function on $\mathbf{R}^2$ defined by $(x, y) \mapsto x^j y^k$.

Define operators $D_x, D_y \in \mathcal{L}(\mathcal{P}_m(\mathbf{R}^2))$ by

$$D_x p = \frac{\partial p}{\partial x} = \sum_{j+k \leq m} j a_{j,k} x^{j-1} y^k \quad \text{and} \quad D_y p = \frac{\partial p}{\partial y} = \sum_{j+k \leq m} k a_{j,k} x^j y^{k-1},$$

where p is as in 5.73. The operators D_x and D_y are called partial differentiation operators because each of these operators differentiates with respect to one of the variables while pretending that the other variable is a constant.

The operators D_x and D_y commute because if p is as in 5.73, then

$$(D_x D_y)p = \sum_{j+k \leq m} jk a_{j,k} x^{j-1} y^{k-1} = (D_y D_x)p.$$

The equation $D_x D_y = D_y D_x$ on $\mathcal{P}_m(\mathbf{R}^2)$ illustrates a more general result that the order of partial differentiation does not matter for nice functions.

Commuting matrices are unusual. For example, there are 214,358,881 ordered pairs of 2-by-2 matrices all of whose entries are integers in the interval $[-5, 5]$. Only about 0.3% of these ordered pairs of matrices commute.

All 214,358,881 (which equals 11^8) ordered pairs of the 2-by-2 matrices under consideration were checked by a computer to discover that only 674,609 of these ordered pairs of matrices commute.

The next result shows that two operators commute if and only if their matrices (with respect to the same basis) commute.

5.74 *commuting operators correspond to commuting matrices*

Suppose $S, T \in \mathcal{L}(V)$ and $v_1, \ldots, v_n$ is a basis of V. Then S and T commute if and only if $\mathcal{M}\bigl(S, (v_1, \ldots, v_n)\bigr)$ and $\mathcal{M}\bigl(T, (v_1, \ldots, v_n)\bigr)$ commute.

Proof We have

$$\begin{aligned} S \text{ and } T \text{ commute} &\iff ST = TS \\ &\iff \mathcal{M}(ST) = \mathcal{M}(TS) \\ &\iff \mathcal{M}(S)\mathcal{M}(T) = \mathcal{M}(T)\mathcal{M}(S) \\ &\iff \mathcal{M}(S) \text{ and } \mathcal{M}(T) \text{ commute,} \end{aligned}$$

as desired. ■

The next result shows that if two operators commute, then every eigenspace for one operator is invariant under the other operator. This result, which we will use several times, is one of the main reasons why a pair of commuting operators behaves better than a pair of operators that does not commute.

5.75 *eigenspace is invariant under commuting operator*

Suppose $S, T \in \mathcal{L}(V)$ commute and $\lambda \in \mathbf{F}$. Then $E(\lambda, S)$ is invariant under T.

Proof Suppose $v \in E(\lambda, S)$. Then

$$S(Tv) = (ST)v = (TS)v = T(Sv) = T(\lambda v) = \lambda Tv.$$

The equation above shows that $Tv \in E(\lambda, S)$. Thus $E(\lambda, S)$ is invariant under T. ■

Suppose we have two operators, each of which is diagonalizable. If we want to do computations involving both operators (for example, involving their sum), then we want the two operators to be diagonalizable by the same basis, which according to the next result is possible when the two operators commute.

5.76 *simultaneous diagonalizability* $\iff$ *commutativity*

Two diagonalizable operators on the same vector space have diagonal matrices with respect to the same basis if and only if the two operators commute.

Proof First suppose $S, T \in \mathcal{L}(V)$ have diagonal matrices with respect to the same basis. The product of two diagonal matrices of the same size is the diagonal matrix obtained by multiplying the corresponding elements of the two diagonals. Thus any two diagonal matrices of the same size commute. Thus S and T commute, by 5.74.

To prove the implication in the other direction, now suppose that $S, T \in \mathcal{L}(V)$ are diagonalizable operators that commute. Let $\lambda_1, \ldots, \lambda_m$ denote the distinct eigenvalues of S. Because S is diagonalizable, 5.55(c) shows that

5.77 $$V = E(\lambda_1, S) \oplus \cdots \oplus E(\lambda_m, S).$$

For each $k = 1, \ldots, m$, the subspace $E(\lambda_k, S)$ is invariant under T (by 5.75). Because T is diagonalizable, 5.65 implies that $T|_{E(\lambda_k, S)}$ is diagonalizable for each k. Hence for each $k = 1, \ldots, m$, there is a basis of $E(\lambda_k, S)$ consisting of eigenvectors of T. Putting these bases together gives a basis of V (because of 5.77), with each vector in this basis being an eigenvector of both S and T. Thus S and T both have diagonal matrices with respect to this basis, as desired. ■

See Exercise 2 for an extension of the result above to more than two operators.

Suppose V is a finite-dimensional nonzero complex vector space. Then every operator on V has an eigenvector (see 5.19). The next result shows that if two operators on V commute, then there is a vector in V that is an eigenvector for both operators (but the two commuting operators might not have a common eigenvalue). For an extension of the next result to more than two operators, see Exercise 9(a).

5.78 *common eigenvector for commuting operators*

Every pair of commuting operators on a finite-dimensional nonzero complex vector space has a common eigenvector.

Proof Suppose V is a finite-dimensional nonzero complex vector space and $S, T \in \mathcal{L}(V)$ commute. Let λ be an eigenvalue of S (5.19 tells us that S does indeed have an eigenvalue). Thus $E(\lambda, S) \neq \{0\}$. Also, $E(\lambda, S)$ is invariant under T (by 5.75).

Thus $T|_{E(\lambda, S)}$ has an eigenvector (again using 5.19), which is an eigenvector for both S and T, completing the proof. ■

5.79 example: *common eigenvector for partial differentiation operators*

Let $\mathcal{P}_m(\mathbf{R}^2)$ be as in Example 5.72 and let $D_x, D_y \in \mathcal{L}\big(\mathcal{P}_m(\mathbf{R}^2)\big)$ be the commuting partial differentiation operators in that example. As you can verify, 0 is the only eigenvalue of each of these operators. Also

$$E(0, D_x) = \left\{\sum_{k=0}^{m} a_k y^k : a_0, \ldots, a_m \in \mathbf{R}\right\},$$

$$E(0, D_y) = \left\{\sum_{j=0}^{m} c_j x^j : c_0, \ldots, c_m \in \mathbf{R}\right\}.$$

The intersection of these two eigenspaces is the set of common eigenvectors of the two operators. Because $E(0, D_x) \cap E(0, D_y)$ is the set of constant functions, we see that D_x and D_y indeed have a common eigenvector, as promised by 5.78.

The next result extends 5.47 (the existence of a basis that gives an upper-triangular matrix) to two commuting operators.

5.80 *commuting operators are simultaneously upper triangularizable*

Suppose V is a finite-dimensional complex vector space and S, T are commuting operators on V. Then there is a basis of V with respect to which both S and T have upper-triangular matrices.

Proof Let $n = \dim V$. We will use induction on n. The desired result holds if $n = 1$ because all 1-by-1 matrices are upper triangular. Now suppose $n > 1$ and the desired result holds for all complex vector spaces whose dimension is $n - 1$.

Let v_1 be any common eigenvector of S and T (using 5.78). Hence $Sv_1 \in \operatorname{span}(v_1)$ and $Tv_1 \in \operatorname{span}(v_1)$. Let W be a subspace of V such that

$$V = \operatorname{span}(v_1) \oplus W;$$

see 2.33 for the existence of W. Define a linear map $P\colon V \to W$ by

$$P(av_1 + w) = w$$

for each $a \in \mathbf{C}$ and each $w \in W$. Define $\hat{S}, \hat{T} \in \mathcal{L}(W)$ by

$$\hat{S}w = P(Sw) \quad \text{and} \quad \hat{T}w = P(Tw)$$

for each $w \in W$. To apply our induction hypothesis to $\hat{S}$ and $\hat{T}$, we must first show that these two operators on W commute. To do this, suppose $w \in W$. Then there exists $a \in \mathbf{C}$ such that

$$(\hat{S}\hat{T})w = \hat{S}\big(P(Tw)\big) = \hat{S}(Tw - av_1) = P\big(S(Tw - av_1)\big) = P\big((ST)w\big),$$

where the last equality holds because v_1 is an eigenvector of S and $Pv_1 = 0$. Similarly,

$$(\hat{T}\hat{S})w = P\big((TS)w\big).$$

Because the operators S and T commute, the last two displayed equations show that $(\hat{S}\hat{T})w = (\hat{T}\hat{S})w$. Hence $\hat{S}$ and $\hat{T}$ commute.

Thus we can use our induction hypothesis to state that there exists a basis $v_2, \dots, v_n$ of W such that $\hat{S}$ and $\hat{T}$ both have upper-triangular matrices with respect to this basis. The list $v_1, \dots, v_n$ is a basis of V.

If $k \in \{2, \dots, n\}$, then there exist $a_k, b_k \in \mathbf{C}$ such that

$$Sv_k = a_k v_1 + \hat{S}v_k \quad \text{and} \quad Tv_k = b_k v_1 + \hat{T}v_k.$$

Because $\hat{S}$ and $\hat{T}$ have upper-triangular matrices with respect to $v_2, \dots, v_n$, we know that $\hat{S}v_k \in \operatorname{span}(v_2, \dots, v_k)$ and $\hat{T}v_k \in \operatorname{span}(v_2, \dots, v_k)$. Hence the equations above imply that

$$Sv_k \in \operatorname{span}(v_1, \dots, v_k) \quad \text{and} \quad Tv_k \in \operatorname{span}(v_1, \dots, v_k).$$

Thus S and T have upper-triangular matrices with respect to $v_1, \dots, v_n$, as desired. ■

Exercise 9(b) extends the result above to more than two operators.

In general, it is not possible to determine the eigenvalues of the sum or product of two operators from the eigenvalues of the two operators. However, the next result shows that something nice happens when the two operators commute.

5.81 *eigenvalues of sum and product of commuting operators*

Suppose V is a finite-dimensional complex vector space and S, T are commuting operators on V. Then

- every eigenvalue of $S + T$ is an eigenvalue of S plus an eigenvalue of T,
- every eigenvalue of ST is an eigenvalue of S times an eigenvalue of T.

Proof There is a basis of V with respect to which both S and T have upper-triangular matrices (by 5.80). With respect to that basis,

$$\mathcal{M}(S+T) = \mathcal{M}(S) + \mathcal{M}(T) \quad \text{and} \quad \mathcal{M}(ST) = \mathcal{M}(S)\mathcal{M}(T),$$

as stated in 3.35 and 3.43.

The definition of matrix addition shows that each entry on the diagonal of $\mathcal{M}(S+T)$ equals the sum of the corresponding entries on the diagonals of $\mathcal{M}(S)$ and $\mathcal{M}(T)$. Similarly, because $\mathcal{M}(S)$ and $\mathcal{M}(T)$ are upper-triangular matrices, the definition of matrix multiplication shows that each entry on the diagonal of $\mathcal{M}(ST)$ equals the product of the corresponding entries on the diagonals of $\mathcal{M}(S)$ and $\mathcal{M}(T)$. Furthermore, $\mathcal{M}(S+T)$ and $\mathcal{M}(ST)$ are upper-triangular matrices (see Exercise 2 in Section 5C).

Every entry on the diagonal of $\mathcal{M}(S)$ is an eigenvalue of S, and every entry on the diagonal of $\mathcal{M}(T)$ is an eigenvalue of T (by 5.41). Every eigenvalue of $S + T$ is on the diagonal of $\mathcal{M}(S+T)$, and every eigenvalue of ST is on the diagonal of $\mathcal{M}(ST)$ (these assertions follow from 5.41). Putting all this together, we conclude that every eigenvalue of $S + T$ is an eigenvalue of S plus an eigenvalue of T, and every eigenvalue of ST is an eigenvalue of S times an eigenvalue of T. ■

Exercises 5E

1 Give an example of two commuting operators S, T on $\mathbf{F}^4$ such that there is a subspace of $\mathbf{F}^4$ that is invariant under S but not under T and there is a subspace of $\mathbf{F}^4$ that is invariant under T but not under S.

2 Suppose $\mathcal{E}$ is a subset of $\mathcal{L}(V)$ and every element of $\mathcal{E}$ is diagonalizable. Prove that there exists a basis of V with respect to which every element of $\mathcal{E}$ has a diagonal matrix if and only if every pair of elements of $\mathcal{E}$ commutes.

This exercise extends 5.76, which considers the case in which $\mathcal{E}$ contains only two elements. For this exercise, $\mathcal{E}$ may contain any number of elements, and $\mathcal{E}$ may even be an infinite set.

3 Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Suppose $p \in \mathcal{P}(\mathbf{F})$.

(a) Prove that $\operatorname{null} p(S)$ is invariant under T.

(b) Prove that $\operatorname{range} p(S)$ is invariant under T.

See 5.18 for the special case $S = T$.

4 Prove or give a counterexample: If A is a diagonal matrix and B is an upper-triangular matrix of the same size as A, then A and B commute.

5 Prove that a pair of operators on a finite-dimensional vector space commute if and only if their dual operators commute.

See 3.118 for the definition of the dual of an operator.

6 Suppose that V is a nonzero finite-dimensional complex vector space and $S, T \in \mathcal{L}(V)$ commute. Prove that there exist $\alpha, \lambda \in \mathbf{C}$ such that

$$\operatorname{range}(S - \alpha I) + \operatorname{range}(T - \lambda I) \neq V.$$

7 Suppose V is a complex vector space, $S \in \mathcal{L}(V)$ is diagonalizable, and $T \in \mathcal{L}(V)$ commutes with S. Prove that there is a basis of V such that S has a diagonal matrix with respect to this basis and T has an upper-triangular matrix with respect to this basis.

8 Suppose $m = 3$ in Example 5.72 and D_x, D_y are the commuting partial differentiation operators on $\mathcal{P}_3(\mathbf{R}^2)$ from that example. Find a basis of $\mathcal{P}_3(\mathbf{R}^2)$ with respect to which D_x and D_y each have an upper-triangular matrix.

9 Suppose V is a finite-dimensional nonzero complex vector space. Suppose that $\mathcal{E} \subseteq \mathcal{L}(V)$ is such that S and T commute for all $S, T \in \mathcal{E}$.

(a) Prove that there is a vector in V that is an eigenvector for every element of $\mathcal{E}$.

(b) Prove that there is a basis of V with respect to which every element of $\mathcal{E}$ has an upper-triangular matrix.

This exercise extends 5.78 and 5.80, which consider the case in which $\mathcal{E}$ contains only two elements. For this exercise, $\mathcal{E}$ may contain any number of elements, and $\mathcal{E}$ may even be an infinite set.

10 Give an example of two commuting operators S, T on a finite-dimensional real vector space such that $S + T$ has an eigenvalue that does not equal an eigenvalue of S plus an eigenvalue of T and ST has an eigenvalue that does not equal an eigenvalue of S times an eigenvalue of T.

This exercise shows that 5.81 does not hold on real vector spaces.