9B Alternating Multilinear Forms

Multilinear Forms

9.24 definition: V^m

For m a positive integer, define V^m by

$$V^m = \underbrace{V \times \cdots \times V}_{m \text{ times}}.$$

Now we can define m-linear forms as a generalization of the bilinear forms that we discussed in the previous section.

9.25 definition: *m-linear form, $V^{(m)}$, multilinear form*

- For m a positive integer, an *m-linear form* on V is a function $\beta\colon V^m \to \mathbf{F}$ that is linear in each slot when the other slots are held fixed. This means that for each $k \in \{1, \ldots, m\}$ and all $u_1, \ldots, u_m \in V$, the function

$$v \mapsto \beta(u_1, \ldots, u_{k-1}, v, u_{k+1}, \ldots, u_m)$$

  is a linear map from V to $\mathbf{F}$.
- The set of m-linear forms on V is denoted by $V^{(m)}$.
- A function β is called a *multilinear form* on V if it is an m-linear form on V for some positive integer m.

In the definition above, the expression $\beta(u_1, \ldots, u_{k-1}, v, u_{k+1}, \ldots, u_m)$ means $\beta(v, u_2, \ldots, u_m)$ if $k = 1$ and means $\beta(u_1, \ldots, u_{m-1}, v)$ if $k = m$.

A 1-linear form on V is a linear functional on V. A 2-linear form on V is a bilinear form on V. You can verify that with the usual addition and scalar multiplication of functions, $V^{(m)}$ is a vector space.

9.26 example: *m-linear forms*

- Suppose $\alpha, \rho \in V^{(2)}$. Define a function $\beta\colon V^4 \to \mathbf{F}$ by

$$\beta(v_1, v_2, v_3, v_4) = \alpha(v_1, v_2)\, \rho(v_3, v_4).$$

  Then $\beta \in V^{(4)}$.
- Define $\beta\colon \big(\mathcal{L}(V)\big)^m \to \mathbf{F}$ by

$$\beta(T_1, \ldots, T_m) = \operatorname{tr}(T_1 \cdots T_m).$$

  Then β is an m-linear form on $\mathcal{L}(V)$.

Alternating multilinear forms, which we now define, play an important role as we head toward defining determinants.

9.27 definition: *alternating forms,* $V_{\text{alt}}^{(m)}$

Suppose m is a positive integer.

- An m-linear form α on V is called *alternating* if $\alpha(v_1, \dots, v_m) = 0$ whenever $v_1, \dots, v_m$ is a list of vectors in V with $v_j = v_k$ for some two distinct values of j and k in $\{1, \dots, m\}$.
- $V_{\text{alt}}^{(m)} = \{\alpha \in V^{(m)} : \alpha \text{ is an alternating } m\text{-linear form on } V\}$.

You should verify that $V_{\text{alt}}^{(m)}$ is a subspace of $V^{(m)}$. See Example 9.15 for examples of alternating 2-linear forms. See Exercise 2 for an example of an alternating 3-linear form.

The next result tells us that if a linearly dependent list is input to an alternating multilinear form, then the output equals 0.

9.28 *alternating multilinear forms and linear dependence*

Suppose m is a positive integer and α is an alternating m-linear form on V. If $v_1, \dots, v_m$ is a linearly dependent list in V, then

$$\alpha(v_1, \dots, v_m) = 0.$$

Proof Suppose $v_1, \dots, v_m$ is a linearly dependent list in V. By the linear dependence lemma (2.19), some v_k is a linear combination of $v_1, \dots, v_{k-1}$. Thus there exist $b_1, \dots, b_{k-1}$ such that $v_k = b_1 v_1 + \cdots + b_{k-1} v_{k-1}$. Now

$$\begin{aligned} \alpha(v_1, \dots, v_m) &= \alpha\Bigl(v_1, \dots, v_{k-1}, \sum_{j=1}^{k-1} b_j v_j, v_{k+1}, \dots, v_m\Bigr) \\ &= \sum_{j=1}^{k-1} b_j\, \alpha(v_1, \dots, v_{k-1}, v_j, v_{k+1}, \dots, v_m) \\ &= 0. \end{aligned}$$

■

The next result states that if $m > \dim V$, then there are no alternating m-linear forms on V other than the function on V^m that is identically 0.

9.29 *no nonzero alternating m-linear forms for $m > \dim V$*

Suppose $m > \dim V$. Then 0 is the only alternating m-linear form on V.

Proof Suppose that α is an alternating m-linear form on V and $v_1, \dots, v_m \in V$. Because $m > \dim V$, this list is not linearly independent (by 2.22). Thus 9.28 implies that $\alpha(v_1, \dots, v_m) = 0$. Hence α is the zero function from V^m to $\mathbf{F}$. ■

Alternating Multilinear Forms and Permutations

9.30 *swapping input vectors in an alternating multilinear form*

Suppose m is a positive integer, α is an alternating m-linear form on V, and $v_1, \ldots, v_m$ is a list of vectors in V. Then swapping the vectors in any two slots of $\alpha(v_1, \ldots, v_m)$ changes the value of α by a factor of -1.

Proof Put $v_1 + v_2$ in both the first two slots, getting

$$0 = \alpha(v_1 + v_2, v_1 + v_2, v_3, \ldots, v_m).$$

Use the multilinear properties of α to expand the right side of the equation above (as in the proof of 9.16) to get

$$\alpha(v_2, v_1, v_3, \ldots, v_m) = -\alpha(v_1, v_2, v_3, \ldots, v_m).$$

Similarly, swapping the vectors in any two slots of $\alpha(v_1, \ldots, v_m)$ changes the value of α by a factor of -1. ∎

To see what can happen with multiple swaps, suppose α is an alternating 3-linear form on V and $v_1, v_2, v_3 \in V$. To evaluate $\alpha(v_3, v_1, v_2)$ in terms of $\alpha(v_1, v_2, v_3)$, start with $\alpha(v_3, v_1, v_2)$ and swap the entries in the first and third slots, getting $\alpha(v_3, v_1, v_2) = -\alpha(v_2, v_1, v_3)$. Now in the last expression, swap the entries in the first and second slots, getting

$$\alpha(v_3, v_1, v_2) = -\alpha(v_2, v_1, v_3) = \alpha(v_1, v_2, v_3).$$

More generally, we see that if we do an odd number of swaps, then the value of α changes by a factor of -1, and if we do an even number of swaps, then the value of α does not change.

To deal with arbitrary multiple swaps, we need a bit of information about permutations.

9.31 definition: *permutation*, $\operatorname{perm} m$

Suppose m is a positive integer.

- A *permutation* of $(1, \ldots, m)$ is a list $(j_1, \ldots, j_m)$ that contains each of the numbers $1, \ldots, m$ exactly once.
- The set of all permutations of $(1, \ldots, m)$ is denoted by $\operatorname{perm} m$.

For example, $(2, 3, 4, 5, 1) \in \operatorname{perm} 5$. You should think of an element of $\operatorname{perm} m$ as a rearrangement of the first m positive integers.

The number of swaps used to change a permutation $(j_1, \ldots, j_m)$ to the standard order $(1, \ldots, m)$ can depend on the specific swaps selected. The following definition has the advantage of assigning a well-defined sign to every permutation.

9.32 definition: *sign of a permutation*

The *sign* of a permutation $(j_1, \dots, j_m)$ is defined by

$$\operatorname{sign}(j_1, \dots, j_m) = (-1)^N,$$

where N is the number of pairs of integers (k, ℓ) with $1 \le k < \ell \le m$ such that k appears after ℓ in the list $(j_1, \dots, j_m)$.

Hence the sign of a permutation equals 1 if the natural order has been changed an even number of times and equals -1 if the natural order has been changed an odd number of times.

9.33 example: *signs*

- The permutation $(1, \dots, m)$ [no changes in the natural order] has sign 1.
- The only pair of integers (k, ℓ) with $k < \ell$ such that k appears after ℓ in the list $(2, 1, 3, 4)$ is $(1, 2)$. Thus the permutation $(2, 1, 3, 4)$ has sign -1.
- In the permutation $(2, 3, \dots, m, 1)$, the only pairs (k, ℓ) with $k < \ell$ that appear with changed order are $(1, 2), (1, 3), \dots, (1, m)$. Because we have $m - 1$ such pairs, the sign of this permutation equals $(-1)^{m-1}$.

9.34 *swapping two entries in a permutation*

Swapping two entries in a permutation multiplies the sign of the permutation by -1.

Proof Suppose we have two permutations, where the second permutation is obtained from the first by swapping two entries. The two swapped entries were in their natural order in the first permutation if and only if they are not in their natural order in the second permutation. Thus we have a net change (so far) of 1 or -1 (both odd numbers) in the number of pairs not in their natural order.

Consider each entry between the two swapped entries. If an intermediate entry was originally in the natural order with respect to both swapped entries, then it is now in the natural order with respect to neither swapped entry. Similarly, if an intermediate entry was originally in the natural order with respect to neither of the swapped entries, then it is now in the natural order with respect to both swapped entries. If an intermediate entry was originally in the natural order with respect to exactly one of the swapped entries, then that is still true. Thus the net change (for each pair containing an entry between the two swapped entries) in the number of pairs not in their natural order is 2, -2, or 0 (all even numbers).

For all other pairs of entries, there is no change in whether or not they are in their natural order. Thus the total net change in the number of pairs not in their natural order is an odd number. Hence the sign of the second permutation equals -1 times the sign of the first permutation. ■

9.35 *permutations and alternating multilinear forms*

Suppose m is a positive integer and $\alpha \in V_{\mathrm{alt}}^{(m)}$. Then

$$\alpha(v_{j_1}, \dots, v_{j_m}) = \bigl(\operatorname{sign}(j_1, \dots, j_m)\bigr)\alpha(v_1, \dots, v_m)$$

for every list $v_1, \dots, v_m$ of vectors in V and all $(j_1, \dots, j_m) \in \operatorname{perm} m$.

Proof Suppose $v_1, \dots, v_m \in V$ and $(j_1, \dots, j_m) \in \operatorname{perm} m$. We can get from $(j_1, \dots, j_m)$ to $(1, \dots, m)$ by a series of swaps of entries in different slots. Each such swap changes the value of α by a factor of -1 (by 9.30) and also changes the sign of the remaining permutation by a factor of -1 (by 9.34). After an appropriate number of swaps, we reach the permutation $1, \dots, m$, which has sign 1. Thus the value of α changed signs an even number of times if $\operatorname{sign}(j_1, \dots, j_m) = 1$ and an odd number of times if $\operatorname{sign}(j_1, \dots, j_m) = -1$, which gives the desired result. ∎

Our use of permutations now leads in a natural way to the following beautiful formula for alternating n-linear forms on an n-dimensional vector space.

9.36 *formula for* $(\dim V)$*-linear alternating forms on* V

Let $n = \dim V$. Suppose $e_1, \dots, e_n$ is a basis of V and $v_1, \dots, v_n \in V$. For each $k \in \{1, \dots, n\}$, let $b_{1,k}, \dots, b_{n,k} \in \mathbf{F}$ be such that

$$v_k = \sum_{j=1}^{n} b_{j,k} e_j.$$

Then

$$\alpha(v_1, \dots, v_n) = \alpha(e_1, \dots, e_n) \sum_{(j_1, \dots, j_n) \in \operatorname{perm} n} \bigl(\operatorname{sign}(j_1, \dots, j_n)\bigr) b_{j_1,1} \cdots b_{j_n,n}$$

for every alternating n-linear form α on V.

Proof Suppose α is an alternating n-linear form α on V. Then

$$\begin{aligned}
\alpha(v_1, \dots, v_n) &= \alpha\Bigl(\sum_{j_1=1}^{n} b_{j_1,1} e_{j_1}, \dots, \sum_{j_n=1}^{n} b_{j_n,n} e_{j_n}\Bigr)\\
&= \sum_{j_1=1}^{n} \cdots \sum_{j_n=1}^{n} b_{j_1,1} \cdots b_{j_n,n}\, \alpha(e_{j_1}, \dots, e_{j_n})\\
&= \sum_{(j_1, \dots, j_n) \in \operatorname{perm} n} b_{j_1,1} \cdots b_{j_n,n}\, \alpha(e_{j_1}, \dots, e_{j_n})\\
&= \alpha(e_1, \dots, e_n) \sum_{(j_1, \dots, j_n) \in \operatorname{perm} n} \bigl(\operatorname{sign}(j_1, \dots, j_n)\bigr) b_{j_1,1} \cdots b_{j_n,n},
\end{aligned}$$

where the third line holds because $\alpha(e_{j_1}, \dots, e_{j_n}) = 0$ if $j_1, \dots, j_n$ are not distinct integers, and the last line holds by 9.35. ∎

The following result will be the key to our definition of the determinant in the next section.

9.37 $\dim V_{\text{alt}}^{(\dim V)} = 1$

The vector space $V_{\text{alt}}^{(\dim V)}$ has dimension one.

Proof Let $n = \dim V$. Suppose α and α' are alternating n-linear forms on V with $\alpha \neq 0$. Let $e_1, \ldots, e_n$ be such that $\alpha(e_1, \ldots, e_n) \neq 0$. There exists $c \in \mathbf{F}$ such that

$$\alpha'(e_1, \ldots, e_n) = c\alpha(e_1, \ldots, e_n).$$

Furthermore, 9.28 implies that $e_1, \ldots, e_n$ is linearly independent and thus is a basis of V.

Suppose $v_1, \ldots, v_n \in V$. Let $b_{j,k}$ be as in 9.36 for $j, k = 1, \ldots, n$. Then

$$\begin{aligned}
\alpha'(v_1, \ldots, v_n) &= \alpha'(e_1, \ldots, e_n) \sum_{(j_1, \ldots, j_n) \in \operatorname{perm} n} \big(\operatorname{sign}(j_1, \ldots, j_n)\big) b_{j_1,1} \cdots b_{j_n,n} \\
&= c\alpha(e_1, \ldots, e_n) \sum_{(j_1, \ldots, j_n) \in \operatorname{perm} n} \big(\operatorname{sign}(j_1, \ldots, j_n)\big) b_{j_1,1} \cdots b_{j_n,n} \\
&= c\alpha(v_1, \ldots, v_n),
\end{aligned}$$

where the first and last lines above come from 9.36. The equation above implies that $\alpha' = c\alpha$. Thus α', α is not a linearly independent list, which implies that $\dim V_{\text{alt}}^{(n)} \leq 1$.

To complete the proof, we only need to show that there exists a nonzero alternating n-linear form α on V $\big($thus eliminating the possibility that $\dim V_{\text{alt}}^{(n)}$ equals $0\big)$. To do this, let $e_1, \ldots, e_n$ be any basis of V, and let $\varphi_1, \ldots, \varphi_n \in V'$ be the linear functionals on V that allow us to express each element of V as a linear combination of $e_1, \ldots, e_n$. In other words,

$$v = \sum_{j=1}^{n} \varphi_j(v) e_j$$

for every $v \in V$ (see 3.114). Now for $v_1, \ldots, v_n \in V$, define

9.38 $$\alpha(v_1, \ldots, v_n) = \sum_{(j_1, \ldots, j_n) \in \operatorname{perm} n} \big(\operatorname{sign}(j_1, \ldots, j_n)\big) \varphi_{j_1}(v_1) \cdots \varphi_{j_n}(v_n).$$

The verification that α is an n-linear form on V is straightforward.

To see that α is alternating, suppose $v_1, \ldots, v_n \in V$ with $v_1 = v_2$. For each $(j_1, \ldots, j_n) \in \operatorname{perm} n$, the permutation $(j_2, j_1, j_3, \ldots, j_n)$ has the opposite sign. Because $v_1 = v_2$, the contributions from these two permutations to the sum in 9.38 cancel either other. Hence $\alpha(v_1, v_1, v_3, \ldots, v_n) = 0$. Similarly, $\alpha(v_1, \ldots, v_n) = 0$ if any two vectors in the list $v_1, \ldots, v_n$ are equal. Thus α is alternating.

Finally, consider 9.38 with each $v_k = e_k$. Because $\varphi_j(e_k)$ equals 0 if $j \neq k$ and equals 1 if $j = k$, only the permutation $(1, \ldots, n)$ makes a nonzero contribution to the right side of 9.38 in this case, giving the equation $\alpha(e_1, \ldots, e_n) = 1$. Thus we have produced a nonzero alternating n-linear form α on V, as desired. ■

Earlier we showed that the value of an alternating multilinear form applied to a linearly dependent list is 0; see 9.28. The next result provides a converse of 9.28 for n-linear multilinear forms when $n = \dim V$. In the following result, the statement that α is nonzero means (as usual for a function) that α is not the function on V^n that is identically 0.

The formula 9.38 used in the last proof to construct a nonzero alternating n-linear form came from the formula in 9.36, and that formula arose naturally from the properties of an alternating multilinear form.

9.39 *alternating* $(\dim V)$*-linear forms and linear independence*

Let $n = \dim V$. Suppose α is a nonzero alternating n-linear form on V and $e_1, \ldots, e_n$ is a list of vectors in V. Then

$$\alpha(e_1, \ldots, e_n) \neq 0$$

if and only if $e_1, \ldots, e_n$ is linearly independent.

Proof First suppose $\alpha(e_1, \ldots, e_n) \neq 0$. Then 9.28 implies that $e_1, \ldots, e_n$ is linearly independent.

To prove the implication in the other direction, now suppose $e_1, \ldots, e_n$ is linearly independent. Because $n = \dim V$, this implies that $e_1, \ldots, e_n$ is a basis of V (see 2.38).

Because α is not the zero n-linear form, there exist $v_1, \ldots, v_n \in V$ such that $\alpha(v_1, \ldots, v_n) \neq 0$. Now 9.36 implies that $\alpha(e_1, \ldots, e_n) \neq 0$. ■

Exercises 9B

1 Suppose m is a positive integer. Show that $\dim V^{(m)} = (\dim V)^m$.

2 Suppose $n \geq 3$ and $\alpha : \mathbf{F}^n \times \mathbf{F}^n \times \mathbf{F}^n \to \mathbf{F}$ is defined by

$$\begin{aligned}\alpha\big((x_1, \ldots, x_n), (y_1, \ldots, y_n), (z_1, \ldots, z_n)\big)\\ = x_1y_2z_3 - x_2y_1z_3 - x_3y_2z_1 - x_1y_3z_2 + x_3y_1z_2 + x_2y_3z_1.\end{aligned}$$

Show that α is an alternating 3-linear form on $\mathbf{F}^n$.

3 Suppose m is a positive integer and α is an m-linear form on V such that $\alpha(v_1, \ldots, v_m) = 0$ whenever $v_1, \ldots, v_m$ is a list of vectors in V with $v_j = v_{j+1}$ for some $j \in \{1, \ldots, m-1\}$. Prove that α is an alternating m-linear form on V.

4 Prove or give a counterexample: If $\alpha \in V^{(4)}_{\text{alt}}$, then

$$\{(v_1, v_2, v_3, v_4) \in V^4 : \alpha(v_1, v_2, v_3, v_4) = 0\}$$

is a subspace of V^4.

5 Suppose m is a positive integer and β is an m-linear form on V. Define an m-linear form α on V by

$$\alpha(v_1, \ldots, v_m) = \sum_{(j_1,\ldots,j_m) \in \operatorname{perm} m} \bigl(\operatorname{sign}(j_1, \ldots, j_m)\bigr) \beta(v_{j_1}, \ldots, v_{j_m})$$

for $v_1, \ldots, v_m \in V$. Explain why $\alpha \in V_{\text{alt}}^{(m)}$.

6 Suppose m is a positive integer and β is an m-linear form on V. Define an m-linear form α on V by

$$\alpha(v_1, \ldots, v_m) = \sum_{(j_1,\ldots,j_m) \in \operatorname{perm} m} \beta(v_{j_1}, \ldots, v_{j_m})$$

for $v_1, \ldots, v_m \in V$. Explain why

$$\alpha(v_{k_1}, \ldots, v_{k_m}) = \alpha(v_1, \ldots, v_m)$$

for all $v_1, \ldots, v_m \in V$ and all $(k_1, \ldots, k_m) \in \operatorname{perm} m$.

7 Give an example of a nonzero alternating 2-linear form α on $\mathbf{R}^3$ and a linearly independent list v_1, v_2 in $\mathbf{R}^3$ such that $\alpha(v_1, v_2) = 0$.

This exercise shows that 9.39 can fail if the hypothesis that $n = \dim V$ is deleted.