1B Definition of Vector Space

The motivation for the definition of a vector space comes from properties of addition and scalar multiplication in $\mathbf{F}^n$: Addition is commutative, associative, and has an identity. Every element has an additive inverse. Scalar multiplication is associative. Scalar multiplication by 1 acts as expected. Addition and scalar multiplication are connected by distributive properties.

We will define a vector space to be a set V with an addition and a scalar multiplication on V that satisfy the properties in the paragraph above.

1.19 definition: *addition, scalar multiplication*

- An *addition* on a set V is a function that assigns an element $u + v \in V$ to each pair of elements $u, v \in V$.
- A *scalar multiplication* on a set V is a function that assigns an element $\lambda v \in V$ to each $\lambda \in \mathbf{F}$ and each $v \in V$.

Now we are ready to give the formal definition of a vector space.

1.20 definition: *vector space*

A *vector space* is a set V along with an addition on V and a scalar multiplication on V such that the following properties hold.

commutativity
$u + v = v + u$ for all $u, v \in V$.

associativity
$(u + v) + w = u + (v + w)$ and $(ab)v = a(bv)$ for all $u, v, w \in V$ and for all $a, b \in \mathbf{F}$.

additive identity
There exists an element $0 \in V$ such that $v + 0 = v$ for all $v \in V$.

additive inverse
For every $v \in V$, there exists $w \in V$ such that $v + w = 0$.

multiplicative identity
$1v = v$ for all $v \in V$.

distributive properties
$a(u + v) = au + av$ and $(a + b)v = av + bv$ for all $a, b \in \mathbf{F}$ and all $u, v \in V$.

The following geometric language sometimes aids our intuition.

1.21 definition: *vector, point*

Elements of a vector space are called *vectors* or *points*.

The scalar multiplication in a vector space depends on $\mathbf{F}$. Thus when we need to be precise, we will say that V is a *vector space over* $\mathbf{F}$ instead of saying simply that V is a vector space. For example, $\mathbf{R}^n$ is a vector space over $\mathbf{R}$, and $\mathbf{C}^n$ is a vector space over $\mathbf{C}$.

1.22 definition: *real vector space, complex vector space*

- A vector space over $\mathbf{R}$ is called a *real vector space*.
- A vector space over $\mathbf{C}$ is called a *complex vector space*.

Usually the choice of $\mathbf{F}$ is either clear from the context or irrelevant. Thus we often assume that $\mathbf{F}$ is lurking in the background without specifically mentioning it.

The simplest vector space is $\{0\}$, *which contains only one point.*

With the usual operations of addition and scalar multiplication, $\mathbf{F}^n$ is a vector space over $\mathbf{F}$, as you should verify. The example of $\mathbf{F}^n$ motivated our definition of vector space.

1.23 example: $\mathbf{F}^\infty$

$\mathbf{F}^\infty$ is defined to be the set of all sequences of elements of $\mathbf{F}$:

$$\mathbf{F}^\infty = \{(x_1, x_2, \ldots) : x_k \in \mathbf{F} \text{ for } k = 1, 2, \ldots\}.$$

Addition and scalar multiplication on $\mathbf{F}^\infty$ are defined as expected:

$$(x_1, x_2, \ldots) + (y_1, y_2, \ldots) = (x_1 + y_1, x_2 + y_2, \ldots),$$
$$\lambda(x_1, x_2, \ldots) = (\lambda x_1, \lambda x_2, \ldots).$$

With these definitions, $\mathbf{F}^\infty$ becomes a vector space over $\mathbf{F}$, as you should verify. The additive identity in this vector space is the sequence of all 0's.

Our next example of a vector space involves a set of functions.

1.24 notation: $\mathbf{F}^S$

- If S is a set, then $\mathbf{F}^S$ denotes the set of functions from S to $\mathbf{F}$.
- For $f, g \in \mathbf{F}^S$, the *sum* $f + g \in \mathbf{F}^S$ is the function defined by
$$(f + g)(x) = f(x) + g(x)$$
for all $x \in S$.
- For $\lambda \in \mathbf{F}$ and $f \in \mathbf{F}^S$, the *product* $\lambda f \in \mathbf{F}^S$ is the function defined by
$$(\lambda f)(x) = \lambda f(x)$$
for all $x \in S$.

As an example of the notation above, if S is the interval $[0,1]$ and $\mathbf{F} = \mathbf{R}$, then $\mathbf{R}^{[0,1]}$ is the set of real-valued functions on the interval $[0,1]$.

You should verify all three bullet points in the next example.

1.25 example: $\mathbf{F}^S$ *is a vector space*

- If S is a nonempty set, then $\mathbf{F}^S$ (with the operations of addition and scalar multiplication as defined above) is a vector space over $\mathbf{F}$.
- The additive identity of $\mathbf{F}^S$ is the function $0 : S \to \mathbf{F}$ defined by
$$0(x) = 0$$
for all $x \in S$.
- For $f \in \mathbf{F}^S$, the additive inverse of f is the function $-f : S \to \mathbf{F}$ defined by
$$(-f)(x) = -f(x)$$
for all $x \in S$.

The vector space $\mathbf{F}^n$ is a special case of the vector space $\mathbf{F}^S$ because each $(x_1, \dots, x_n) \in \mathbf{F}^n$ can be thought of as a function x from the set $\{1, 2, \dots, n\}$ to $\mathbf{F}$ by writing $x(k)$ instead of x_k for the k^{th} coordinate of $(x_1, \dots, x_n)$. In other words, we can think of $\mathbf{F}^n$ as $\mathbf{F}^{\{1,2,\dots,n\}}$. Similarly, we can think of $\mathbf{F}^\infty$ as $\mathbf{F}^{\{1,2,\dots\}}$.

The elements of the vector space $\mathbf{R}^{[0,1]}$ *are real-valued functions on* $[0,1]$, *not lists. In general, a vector space is an abstract entity whose elements might be lists, functions, or weird objects.*

Soon we will see further examples of vector spaces, but first we need to develop some of the elementary properties of vector spaces.

The definition of a vector space requires it to have an additive identity. The next result states that this identity is unique.

1.26 *unique additive identity*

A vector space has a unique additive identity.

Proof Suppose 0 and $0'$ are both additive identities for some vector space V. Then
$$0' = 0' + 0 = 0 + 0' = 0,$$
where the first equality holds because 0 is an additive identity, the second equality comes from commutativity, and the third equality holds because $0'$ is an additive identity. Thus $0' = 0$, proving that V has only one additive identity. ■

Each element v in a vector space has an additive inverse, an element w in the vector space such that $v + w = 0$. The next result shows that each element in a vector space has only one additive inverse.

1.27 *unique additive inverse*

Every element in a vector space has a unique additive inverse.

Proof Suppose V is a vector space. Let $v \in V$. Suppose w and w' are additive inverses of v. Then

$$w = w + 0 = w + (v + w') = (w + v) + w' = 0 + w' = w'.$$

Thus $w = w'$, as desired. ■

Because additive inverses are unique, the following notation now makes sense.

1.28 notation: $-v$, $w - v$

Let $v, w \in V$. Then

- $-v$ denotes the additive inverse of v;
- $w - v$ is defined to be $w + (-v)$.

Almost all results in this book involve some vector space. To avoid having to restate frequently that V is a vector space, we now make the necessary declaration once and for all.

1.29 notation: V

For the rest of this book, V denotes a vector space over $\mathbf{F}$.

In the next result, 0 denotes a scalar (the number $0 \in \mathbf{F}$) on the left side of the equation and a vector (the additive identity of V) on the right side of the equation.

1.30 *the number* 0 *times a vector*

$0v = 0$ for every $v \in V$.

Proof For $v \in V$, we have

$$0v = (0 + 0)v = 0v + 0v.$$

Adding the additive inverse of $0v$ to both sides of the equation above gives $0 = 0v$, as desired. ■

The result in 1.30 involves the additive identity of V and scalar multiplication. The only part of the definition of a vector space that connects vector addition and scalar multiplication is the distributive property. Thus the distributive property must be used in the proof of 1.30.

In the next result, 0 denotes the additive identity of V. Although their proofs are similar, 1.30 and 1.31 are not identical. More precisely, 1.30 states that the product of the scalar 0 and any vector equals the vector 0, whereas 1.31 states that the product of any scalar and the vector 0 equals the vector 0.

1.31 *a number times the vector* 0

$a0 = 0$ for every $a \in \mathbf{F}$.

Proof For $a \in \mathbf{F}$, we have

$$a0 = a(0 + 0) = a0 + a0.$$

Adding the additive inverse of $a0$ to both sides of the equation above gives $0 = a0$, as desired. ■

Now we show that if an element of V is multiplied by the scalar -1, then the result is the additive inverse of the element of V.

1.32 *the number* -1 *times a vector*

$(-1)v = -v$ for every $v \in V$.

Proof For $v \in V$, we have

$$v + (-1)v = 1v + (-1)v = \big(1 + (-1)\big)v = 0v = 0.$$

This equation says that $(-1)v$, when added to v, gives 0. Thus $(-1)v$ is the additive inverse of v, as desired. ■

Exercises 1B

1 Prove that $-(-v) = v$ for every $v \in V$.

2 Suppose $a \in \mathbf{F}$, $v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$.

3 Suppose $v, w \in V$. Explain why there exists a unique $x \in V$ such that $v + 3x = w$.

4 The empty set is not a vector space. The empty set fails to satisfy only one of the requirements listed in the definition of a vector space (1.20). Which one?

5 Show that in the definition of a vector space (1.20), the additive inverse condition can be replaced with the condition that

$$0v = 0 \text{ for all } v \in V.$$

Here the 0 on the left side is the number 0, and the 0 on the right side is the additive identity of V.

The phrase a "condition can be replaced" in a definition means that the collection of objects satisfying the definition is unchanged if the original condition is replaced with the new condition.

6 Let ∞ and $-\infty$ denote two distinct objects, neither of which is in $\mathbf{R}$. Define an addition and scalar multiplication on $\mathbf{R} \cup \{\infty, -\infty\}$ as you could guess from the notation. Specifically, the sum and product of two real numbers is as usual, and for $t \in \mathbf{R}$ define

$$t\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \qquad t(-\infty) = \begin{cases} \infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ -\infty & \text{if } t > 0, \end{cases}$$

and

$$\begin{aligned} t + \infty = \infty + t = \infty + \infty &= \infty, \\ t + (-\infty) = (-\infty) + t = (-\infty) + (-\infty) &= -\infty, \\ \infty + (-\infty) = (-\infty) + \infty &= 0. \end{aligned}$$

With these operations of addition and scalar multiplication, is $\mathbf{R} \cup \{\infty, -\infty\}$ a vector space over $\mathbf{R}$? Explain.

7 Suppose S is a nonempty set. Let V^S denote the set of functions from S to V. Define a natural addition and scalar multiplication on V^S, and show that V^S is a vector space with these definitions.

8 Suppose V is a real vector space.

- The *complexification* of V, denoted by $V_{\mathbf{C}}$, equals $V \times V$. An element of $V_{\mathbf{C}}$ is an ordered pair (u, v), where $u, v \in V$, but we write this as $u + iv$.
- Addition on $V_{\mathbf{C}}$ is defined by

  $$(u_1 + iv_1) + (u_2 + iv_2) = (u_1 + u_2) + i(v_1 + v_2)$$

  for all $u_1, v_1, u_2, v_2 \in V$.
- Complex scalar multiplication on $V_{\mathbf{C}}$ is defined by

  $$(a + bi)(u + iv) = (au - bv) + i(av + bu)$$

  for all $a, b \in \mathbf{R}$ and all $u, v \in V$.

Prove that with the definitions of addition and scalar multiplication as above, $V_{\mathbf{C}}$ is a complex vector space.

Think of V as a subset of $V_{\mathbf{C}}$ by identifying $u \in V$ with $u + i0$. The construction of $V_{\mathbf{C}}$ from V can then be thought of as generalizing the construction of $\mathbf{C}^n$ from $\mathbf{R}^n$.