3F Duality

Dual Space and Dual Map

Linear maps into the scalar field $\mathbf{F}$ play a special role in linear algebra, and thus they get a special name.

3.108 definition: *linear functional*

A *linear functional* on V is a linear map from V to $\mathbf{F}$. In other words, a linear functional is an element of $\mathcal{L}(V, \mathbf{F})$.

3.109 example: *linear functionals*

- Define $\varphi: \mathbf{R}^3 \to \mathbf{R}$ by $\varphi(x, y, z) = 4x - 5y + 2z$. Then φ is a linear functional on $\mathbf{R}^3$.
- Fix $(c_1, \dots, c_n) \in \mathbf{F}^n$. Define $\varphi: \mathbf{F}^n \to \mathbf{F}$ by $\varphi(x_1, \dots, x_n) = c_1 x_1 + \cdots + c_n x_n$. Then φ is a linear functional on $\mathbf{F}^n$.
- Define $\varphi: \mathcal{P}(\mathbf{R}) \to \mathbf{R}$ by
$$\varphi(p) = 3p''(5) + 7p(4).$$
Then φ is a linear functional on $\mathcal{P}(\mathbf{R})$.
- Define $\varphi: \mathcal{P}(\mathbf{R}) \to \mathbf{R}$ by
$$\varphi(p) = \int_0^1 p$$
for each $p \in \mathcal{P}(\mathbf{R})$. Then φ is a linear functional on $\mathcal{P}(\mathbf{R})$.

The vector space $\mathcal{L}(V, \mathbf{F})$ also gets a special name and special notation.

3.110 definition: *dual space, V'*

The *dual space* of V, denoted by V', is the vector space of all linear functionals on V. In other words, $V' = \mathcal{L}(V, \mathbf{F})$.

3.111 $\dim V' = \dim V$

Suppose V is finite-dimensional. Then V' is also finite-dimensional and
$$\dim V' = \dim V.$$

Proof By 3.72 we have
$$\dim V' = \dim \mathcal{L}(V, \mathbf{F}) = (\dim V)(\dim \mathbf{F}) = \dim V,$$
as desired. ■

In the following definition, the linear map lemma (3.4) implies that each φ_j is well defined.

3.112 definition: *dual basis*

If $v_1, \ldots, v_n$ is a basis of V, then the *dual basis* of $v_1, \ldots, v_n$ is the list $\varphi_1, \ldots, \varphi_n$ of elements of V', where each φ_j is the linear functional on V such that

$$\varphi_j(v_k) = \begin{cases} 1 & \text{if } k = j, \\ 0 & \text{if } k \neq j. \end{cases}$$

3.113 example: *the dual basis of the standard basis of* $\mathbf{F}^n$

Suppose n is a positive integer. For $1 \leq j \leq n$, define φ_j to be the linear functional on $\mathbf{F}^n$ that selects the j^{th} coordinate of a vector in $\mathbf{F}^n$. Thus

$$\varphi_j(x_1, \ldots, x_n) = x_j$$

for each $(x_1, \ldots, x_n) \in \mathbf{F}^n$.

Let $e_1, \ldots, e_n$ be the standard basis of $\mathbf{F}^n$. Then

$$\varphi_j(e_k) = \begin{cases} 1 & \text{if } k = j, \\ 0 & \text{if } k \neq j. \end{cases}$$

Thus $\varphi_1, \ldots, \varphi_n$ is the dual basis of the standard basis $e_1, \ldots, e_n$ of $\mathbf{F}^n$.

The next result shows that the dual basis of a basis of V consists of the linear functionals on V that give the coefficients for expressing a vector in V as a linear combination of the basis vectors.

3.114 *dual basis gives coefficients for linear combination*

Suppose $v_1, \ldots, v_n$ is a basis of V and $\varphi_1, \ldots, \varphi_n$ is the dual basis. Then

$$v = \varphi_1(v)v_1 + \cdots + \varphi_n(v)v_n$$

for each $v \in V$.

Proof Suppose $v \in V$. Then there exist $c_1, \ldots, c_n \in \mathbf{F}$ such that

3.115 $$v = c_1v_1 + \cdots + c_nv_n.$$

If $j \in \{1, \ldots, n\}$, then applying φ_j to both sides of the equation above gives

$$\varphi_j(v) = c_j.$$

Substituting the values for $c_1, \ldots, c_n$ given by the equation above into 3.115 shows that $v = \varphi_1(v)v_1 + \cdots + \varphi_n(v)v_n$. ■

The next result shows that the dual basis is indeed a basis of the dual space. Thus the terminology "dual basis" is justified.

3.116 *dual basis is a basis of the dual space*

Suppose V is finite-dimensional. Then the dual basis of a basis of V is a basis of V'.

Proof Suppose $v_1, \ldots, v_n$ is a basis of V. Let $\varphi_1, \ldots, \varphi_n$ denote the dual basis.

To show that $\varphi_1, \ldots, \varphi_n$ is a linearly independent list of elements of V', suppose $a_1, \ldots, a_n \in \mathbf{F}$ are such that

$$a_1\varphi_1 + \cdots + a_n\varphi_n = 0. \tag{3.117}$$

Now

$$(a_1\varphi_1 + \cdots + a_n\varphi_n)(v_k) = a_k$$

for each $k = 1, \ldots, n$. Thus 3.117 shows that $a_1 = \cdots = a_n = 0$. Hence $\varphi_1, \ldots, \varphi_n$ is linearly independent.

Because $\varphi_1, \ldots, \varphi_n$ is a linearly independent list in V' whose length equals $\dim V'$ (by 3.111), we can conclude that $\varphi_1, \ldots, \varphi_n$ is a basis of V' (see 2.38). ■

In the definition below, note that if T is a linear map from V to W then T' is a linear map from W' to V'.

3.118 definition: *dual map, T'*

Suppose $T \in \mathcal{L}(V, W)$. The *dual map* of T is the linear map $T' \in \mathcal{L}(W', V')$ defined for each $\varphi \in W'$ by

$$T'(\varphi) = \varphi \circ T.$$

If $T \in \mathcal{L}(V, W)$ and $\varphi \in W'$, then $T'(\varphi)$ is defined above to be the composition of the linear maps φ and T. Thus $T'(\varphi)$ is indeed a linear map from V to $\mathbf{F}$; in other words, $T'(\varphi) \in V'$.

The following two bullet points show that T' is a linear map from W' to V'.

- If $\varphi, \psi \in W'$, then
$$T'(\varphi + \psi) = (\varphi + \psi) \circ T = \varphi \circ T + \psi \circ T = T'(\varphi) + T'(\psi).$$
- If $\lambda \in \mathbf{F}$ and $\varphi \in W'$, then
$$T'(\lambda\varphi) = (\lambda\varphi) \circ T = \lambda(\varphi \circ T) = \lambda T'(\varphi).$$

The prime notation appears with two unrelated meanings in the next example: D' denotes the dual of the linear map D, and p' denotes the derivative of a polynomial p.

3.119 example: *dual map of the differentiation linear map*

Define $D \colon \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ by $Dp = p'$.

- Suppose φ is the linear functional on $\mathcal{P}(\mathbf{R})$ defined by $\varphi(p) = p(3)$. Then $D'(\varphi)$ is the linear functional on $\mathcal{P}(\mathbf{R})$ given by
$$\bigl(D'(\varphi)\bigr)(p) = (\varphi \circ D)(p) = \varphi(Dp) = \varphi\bigl(p'\bigr) = p'(3).$$
Thus $D'(\varphi)$ is the linear functional on $\mathcal{P}(\mathbf{R})$ taking p to $p'(3)$.

- Suppose φ is the linear functional on $\mathcal{P}(\mathbf{R})$ defined by $\varphi(p) = \int_0^1 p$. Then $D'(\varphi)$ is the linear functional on $\mathcal{P}(\mathbf{R})$ given by
$$\begin{aligned}\bigl(D'(\varphi)\bigr)(p) &= (\varphi \circ D)(p)\\ &= \varphi(Dp)\\ &= \varphi\bigl(p'\bigr)\\ &= \int_0^1 p'\\ &= p(1) - p(0).\end{aligned}$$
Thus $D'(\varphi)$ is the linear functional on $\mathcal{P}(\mathbf{R})$ taking p to $p(1) - p(0)$.

In the next result, (a) and (b) imply that the function that takes T to T' is a linear map from $\mathcal{L}(V, W)$ to $\mathcal{L}\bigl(W', V'\bigr)$.

In (c) below, note the reversal of order from ST on the left to $T'S'$ on the right.

3.120 *algebraic properties of dual maps*

Suppose $T \in \mathcal{L}(V, W)$. Then

(a) $(S + T)' = S' + T'$ for all $S \in \mathcal{L}(V, W)$;

(b) $(\lambda T)' = \lambda T'$ for all $\lambda \in \mathbf{F}$;

(c) $(ST)' = T'S'$ for all $S \in \mathcal{L}(W, U)$.

Proof The proofs of (a) and (b) are left to the reader.

To prove (c), suppose $\varphi \in U'$. Then
$$(ST)'(\varphi) = \varphi \circ (ST) = (\varphi \circ S) \circ T = T'(\varphi \circ S) = T'\bigl(S'(\varphi)\bigr) = \bigl(T'S'\bigr)(\varphi),$$
where the first, third, and fourth equalities above hold because of the definition of the dual map, the second equality holds because composition of functions is associative, and the last equality follows from the definition of composition.

Some books use the notation V^* and T^* for duality instead of V' and T'. However, here we reserve the notation T^* for the adjoint, which will be introduced when we study linear maps on inner product spaces in Chapter 7.

The equation above shows that $(ST)'(\varphi) = \bigl(T'S'\bigr)(\varphi)$ for all $\varphi \in U'$. Thus $(ST)' = T'S'$. ■

Null Space and Range of Dual of Linear Map

Our goal in this subsection is to describe null T' and range T' in terms of range T and null T. To do this, we will need the next definition.

3.121 definition: *annihilator, U^0*

For $U \subseteq V$, the *annihilator* of U, denoted by U^0, is defined by

$$U^0 = \{\varphi \in V' : \varphi(u) = 0 \text{ for all } u \in U\}.$$

3.122 example: *element of an annihilator*

Suppose U is the subspace of $\mathcal{P}(\mathbf{R})$ consisting of polynomial multiples of x^2. If φ is the linear functional on $\mathcal{P}(\mathbf{R})$ defined by $\varphi(p) = p'(0)$, then $\varphi \in U^0$.

For $U \subseteq V$, the annihilator U^0 is a subset of the dual space V'. Thus U^0 depends on the vector space containing U, so a notation such as U_V^0 would be more precise. However, the containing vector space will always be clear from the context, so we will use the simpler notation U^0.

3.123 example: *the annihilator of a two-dimensional subspace of $\mathbf{R}^5$*

Let e_1, e_2, e_3, e_4, e_5 denote the standard basis of $\mathbf{R}^5$; let $\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5 \in (\mathbf{R}^5)'$ denote the dual basis of e_1, e_2, e_3, e_4, e_5. Suppose

$$U = \operatorname{span}(e_1, e_2) = \{(x_1, x_2, 0, 0, 0) \in \mathbf{R}^5 : x_1, x_2 \in \mathbf{R}\}.$$

We want to show that $U^0 = \operatorname{span}(\varphi_3, \varphi_4, \varphi_5)$.

Recall (see 3.113) that φ_j is the linear functional on $\mathbf{R}^5$ that selects the j^{th} coordinate: $\varphi_j(x_1, x_2, x_3, x_4, x_5) = x_j$.

First suppose $\varphi \in \operatorname{span}(\varphi_3, \varphi_4, \varphi_5)$. Then there exist $c_3, c_4, c_5 \in \mathbf{R}$ such that $\varphi = c_3\varphi_3 + c_4\varphi_4 + c_5\varphi_5$. If $(x_1, x_2, 0, 0, 0) \in U$, then

$$\varphi(x_1, x_2, 0, 0, 0) = (c_3\varphi_3 + c_4\varphi_4 + c_5\varphi_5)(x_1, x_2, 0, 0, 0) = 0.$$

Thus $\varphi \in U^0$. Hence we have shown that $\operatorname{span}(\varphi_3, \varphi_4, \varphi_5) \subseteq U^0$.

To show the inclusion in the other direction, suppose that $\varphi \in U^0$. Because the dual basis is a basis of $(\mathbf{R}^5)'$, there exist $c_1, c_2, c_3, c_4, c_5 \in \mathbf{R}$ such that $\varphi = c_1\varphi_1 + c_2\varphi_2 + c_3\varphi_3 + c_4\varphi_4 + c_5\varphi_5$. Because $e_1 \in U$ and $\varphi \in U^0$, we have

$$0 = \varphi(e_1) = (c_1\varphi_1 + c_2\varphi_2 + c_3\varphi_3 + c_4\varphi_4 + c_5\varphi_5)(e_1) = c_1.$$

Similarly, $e_2 \in U$ and thus $c_2 = 0$. Hence $\varphi = c_3\varphi_3 + c_4\varphi_4 + c_5\varphi_5$. Thus $\varphi \in \operatorname{span}(\varphi_3, \varphi_4, \varphi_5)$, which shows that $U^0 \subseteq \operatorname{span}(\varphi_3, \varphi_4, \varphi_5)$.

Thus $U^0 = \operatorname{span}(\varphi_3, \varphi_4, \varphi_5)$.

3.124 *the annihilator is a subspace*

Suppose $U \subseteq V$. Then U^0 is a subspace of V'.

Proof Note that $0 \in U^0$ (here 0 is the zero linear functional on V) because the zero linear functional applied to every vector in U equals $0 \in \mathbf{F}$.

Suppose $\varphi, \psi \in U^0$. Thus $\varphi, \psi \in V'$ and $\varphi(u) = \psi(u) = 0$ for every $u \in U$. If $u \in U$, then

$$(\varphi + \psi)(u) = \varphi(u) + \psi(u) = 0 + 0 = 0.$$

Thus $\varphi + \psi \in U^0$.

Similarly, U^0 is closed under scalar multiplication. Thus 1.34 implies that U^0 is a subspace of V'. ■

The next result shows that $\dim U^0$ is the difference of $\dim V$ and $\dim U$. For example, this shows that if U is a two-dimensional subspace of $\mathbf{R}^5$, then U^0 is a three-dimensional subspace of $(\mathbf{R}^5)'$, as in Example 3.123.

The next result can be proved following the pattern of Example 3.123: choose a basis $u_1, \dots, u_m$ of U, extend to a basis $u_1, \dots, u_m, \dots, u_n$ of V, let $\varphi_1, \dots, \varphi_m, \dots, \varphi_n$ be the dual basis of V', and then show that $\varphi_{m+1}, \dots, \varphi_n$ is a basis of U^0, which implies the desired result. You should construct the proof just outlined, even though a slicker proof is presented here.

3.125 *dimension of the annihilator*

Suppose V is finite-dimensional and U is a subspace of V. Then

$$\dim U^0 = \dim V - \dim U.$$

Proof Let $i \in \mathcal{L}(U, V)$ be the inclusion map defined by $i(u) = u$ for each $u \in U$. Thus i' is a linear map from V' to U'. The fundamental theorem of linear maps (3.21) applied to i' shows that

$$\dim \operatorname{range} i' + \dim \operatorname{null} i' = \dim V'.$$

However, $\operatorname{null} i' = U^0$ (as can be seen by thinking about the definitions) and $\dim V' = \dim V$ (by 3.111), so we can rewrite the equation above as

3.126 $$\dim \operatorname{range} i' + \dim U^0 = \dim V.$$

If $\varphi \in U'$, then φ can be extended to a linear functional ψ on V (see, for example, Exercise 13 in Section 3A). The definition of i' shows that $i'(\psi) = \varphi$. Thus $\varphi \in \operatorname{range} i'$, which implies that $\operatorname{range} i' = U'$. Hence

$$\dim \operatorname{range} i' = \dim U' = \dim U,$$

and then 3.126 becomes the equation $\dim U + \dim U^0 = \dim V$, as desired. ■

The next result can be a useful tool to show that a subspace is as big as possible—see (a)—or to show that a subspace is as small as possible—see (b).

3.127 *condition for the annihilator to equal $\{0\}$ or the whole space*

Suppose V is finite-dimensional and U is a subspace of V. Then

(a) $U^0 = \{0\} \iff U = V$;

(b) $U^0 = V' \iff U = \{0\}$.

Proof To prove (a), we have

$$\begin{aligned} U^0 = \{0\} &\iff \dim U^0 = 0 \\ &\iff \dim U = \dim V \\ &\iff U = V, \end{aligned}$$

where the second equivalence follows from 3.125 and the third equivalence follows from 2.39.

Similarly, to prove (b) we have

$$\begin{aligned} U^0 = V' &\iff \dim U^0 = \dim V' \\ &\iff \dim U^0 = \dim V \\ &\iff \dim U = 0 \\ &\iff U = \{0\}, \end{aligned}$$

where one direction of the first equivalence follows from 2.39, the second equivalence follows from 3.111, and the third equivalence follows from 3.125. ■

The proof of (a) in the next result does not use the hypothesis that V and W are finite-dimensional.

3.128 *the null space of T'*

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then

(a) $\operatorname{null} T' = (\operatorname{range} T)^0$;

(b) $\dim \operatorname{null} T' = \dim \operatorname{null} T + \dim W - \dim V$.

Proof

(a) First suppose $\varphi \in \operatorname{null} T'$. Thus $0 = T'(\varphi) = \varphi \circ T$. Hence

$$0 = (\varphi \circ T)(v) = \varphi(Tv) \quad \text{for every } v \in V.$$

Thus $\varphi \in (\operatorname{range} T)^0$. This implies that $\operatorname{null} T' \subseteq (\operatorname{range} T)^0$.

To prove the inclusion in the opposite direction, now suppose $\varphi \in (\operatorname{range} T)^0$. Thus $\varphi(Tv) = 0$ for every vector $v \in V$. Hence $0 = \varphi \circ T = T'(\varphi)$. In other words, $\varphi \in \operatorname{null} T'$, which shows that $(\operatorname{range} T)^0 \subseteq \operatorname{null} T'$, completing the proof of (a).

(b) We have

$$\begin{aligned}\dim \operatorname{null} T' &= \dim(\operatorname{range} T)^0\\ &= \dim W - \dim \operatorname{range} T\\ &= \dim W - (\dim V - \dim \operatorname{null} T)\\ &= \dim \operatorname{null} T + \dim W - \dim V,\end{aligned}$$

where the first equality comes from (a), the second equality comes from 3.125, and the third equality comes from the fundamental theorem of linear maps (3.21). ■

The next result can be useful because sometimes it is easier to verify that T' is injective than to show directly that T is surjective.

3.129 *T surjective is equivalent to T' injective*

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then

$$T \text{ is surjective} \iff T' \text{ is injective.}$$

Proof We have

$$\begin{aligned} T \in \mathcal{L}(V, W) \text{ is surjective} &\iff \operatorname{range} T = W\\ &\iff (\operatorname{range} T)^0 = \{0\}\\ &\iff \operatorname{null} T' = \{0\}\\ &\iff T' \text{ is injective,}\end{aligned}$$

where the second equivalence comes from 3.127(a) and the third equivalence comes from 3.128(a). ■

3.130 *the range of T'*

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then

(a) $\dim \operatorname{range} T' = \dim \operatorname{range} T$;

(b) $\operatorname{range} T' = (\operatorname{null} T)^0$.

Proof

(a) We have

$$\begin{aligned}\dim \operatorname{range} T' &= \dim W' - \dim \operatorname{null} T'\\ &= \dim W - \dim(\operatorname{range} T)^0\\ &= \dim \operatorname{range} T,\end{aligned}$$

where the first equality comes from 3.21, the second equality comes from 3.111 and 3.128(a), and the third equality comes from 3.125.

(b) First suppose $\varphi \in \operatorname{range} T'$. Thus there exists $\psi \in W'$ such that $\varphi = T'(\psi)$. If $v \in \operatorname{null} T$, then

$$\varphi(v) = \big(T'(\psi)\big)v = (\psi \circ T)(v) = \psi(Tv) = \psi(0) = 0.$$

Hence $\varphi \in (\operatorname{null} T)^0$. This implies that $\operatorname{range} T' \subseteq (\operatorname{null} T)^0$.

We will complete the proof by showing that $\operatorname{range} T'$ and $(\operatorname{null} T)^0$ have the same dimension. To do this, note that

$$\begin{aligned} \dim \operatorname{range} T' &= \dim \operatorname{range} T \\ &= \dim V - \dim \operatorname{null} T \\ &= \dim (\operatorname{null} T)^0, \end{aligned}$$

where the first equality comes from (a), the second equality comes from 3.21, and the third equality comes from 3.125. ■

The next result should be compared to 3.129.

3.131 *T injective is equivalent to T' surjective*

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then

$$T \text{ is injective} \iff T' \text{ is surjective.}$$

Proof We have

$$\begin{aligned} T \text{ is injective} &\iff \operatorname{null} T = \{0\} \\ &\iff (\operatorname{null} T)^0 = V' \\ &\iff \operatorname{range} T' = V', \end{aligned}$$

where the second equivalence follows from 3.127(b) and the third equivalence follows from 3.130(b). ■

Matrix of Dual of Linear Map

The setting for the next result is the assumption that we have a basis $v_1, \ldots, v_n$ of V, along with its dual basis $\varphi_1, \ldots, \varphi_n$ of V'. We also have a basis $w_1, \ldots, w_m$ of W, along with its dual basis $\psi_1, \ldots, \psi_m$ of W'. Thus $\mathcal{M}(T)$ is computed with respect to the bases just mentioned of V and W, and $\mathcal{M}(T')$ is computed with respect to the dual bases just mentioned of W' and V'. Using these bases gives the following pretty result.

3.132 *matrix of T' is transpose of matrix of T*

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then

$$\mathcal{M}(T') = \big(\mathcal{M}(T)\big)^{\mathrm{t}}.$$

Proof Let $A = \mathcal{M}(T)$ and $C = \mathcal{M}(T')$. Suppose $1 \leq j \leq m$ and $1 \leq k \leq n$. From the definition of $\mathcal{M}(T')$ we have

$$T'(\psi_j) = \sum_{r=1}^{n} C_{r,j}\varphi_r.$$

The left side of the equation above equals $\psi_j \circ T$. Thus applying both sides of the equation above to v_k gives

$$\begin{aligned}(\psi_j \circ T)(v_k) &= \sum_{r=1}^{n} C_{r,j}\varphi_r(v_k)\\ &= C_{k,j}.\end{aligned}$$

We also have

$$\begin{aligned}(\psi_j \circ T)(v_k) &= \psi_j(Tv_k)\\ &= \psi_j\Bigl(\sum_{r=1}^{m} A_{r,k}w_r\Bigr)\\ &= \sum_{r=1}^{m} A_{r,k}\psi_j(w_r)\\ &= A_{j,k}.\end{aligned}$$

Comparing the last line of the last two sets of equations, we have $C_{k,j} = A_{j,k}$. Thus $C = A^{\mathrm{t}}$. In other words, $\mathcal{M}(T') = \bigl(\mathcal{M}(T)\bigr)^{\mathrm{t}}$, as desired. ∎

Now we use duality to give an alternative proof that the column rank of a matrix equals the row rank of the matrix. This result was previously proved using different tools—see 3.57.

3.133 *column rank equals row rank*

Suppose $A \in \mathbf{F}^{m,n}$. Then the column rank of A equals the row rank of A.

Proof Define $T\colon \mathbf{F}^{n,1} \to \mathbf{F}^{m,1}$ by $Tx = Ax$. Thus $\mathcal{M}(T) = A$, where $\mathcal{M}(T)$ is computed with respect to the standard bases of $\mathbf{F}^{n,1}$ and $\mathbf{F}^{m,1}$. Now

$$\begin{aligned}\text{column rank of } A &= \text{column rank of } \mathcal{M}(T)\\ &= \dim \operatorname{range} T\\ &= \dim \operatorname{range} T'\\ &= \text{column rank of } \mathcal{M}(T')\\ &= \text{column rank of } A^{\mathrm{t}}\\ &= \text{row rank of } A,\end{aligned}$$

where the second equality comes from 3.78, the third equality comes from 3.130(a), the fourth equality comes from 3.78, the fifth equality comes from 3.132, and the last equality follows from the definitions of row and column rank. ∎

See Exercise 8 in Section 7A for another alternative proof of the result above.

Exercises 3F

1 Explain why each linear functional is surjective or is the zero map.

2 Give three distinct examples of linear functionals on $\mathbf{R}^{[0,1]}$.

3 Suppose V is finite-dimensional and $v \in V$ with $v \neq 0$. Prove that there exists $\varphi \in V'$ such that $\varphi(v) = 1$.

4 Suppose V is finite-dimensional and U is a subspace of V such that $U \neq V$. Prove that there exists $\varphi \in V'$ such that $\varphi(u) = 0$ for every $u \in U$ but $\varphi \neq 0$.

5 Suppose $T \in \mathcal{L}(V, W)$ and $w_1, \dots, w_m$ is a basis of range T. Hence for each $v \in V$, there exist unique numbers $\varphi_1(v), \dots, \varphi_m(v)$ such that

$$Tv = \varphi_1(v)w_1 + \cdots + \varphi_m(v)w_m,$$

thus defining functions $\varphi_1, \dots, \varphi_m$ from V to $\mathbf{F}$. Show that each of the functions $\varphi_1, \dots, \varphi_m$ is a linear functional on V.

6 Suppose $\varphi, \beta \in V'$. Prove that $\operatorname{null}\varphi \subseteq \operatorname{null}\beta$ if and only if there exists $c \in \mathbf{F}$ such that $\beta = c\varphi$.

7 Suppose that $V_1, \dots, V_m$ are vector spaces. Prove that $(V_1 \times \cdots \times V_m)'$ and $V_1{}' \times \cdots \times V_m{}'$ are isomorphic vector spaces.

8 Suppose $v_1, \dots, v_n$ is a basis of V and $\varphi_1, \dots, \varphi_n$ is the dual basis of V'. Define $\Gamma\colon V \to \mathbf{F}^n$ and $\Lambda\colon \mathbf{F}^n \to V$ by

$$\Gamma(v) = \bigl(\varphi_1(v), \dots, \varphi_n(v)\bigr) \quad \text{and} \quad \Lambda(a_1, \dots, a_n) = a_1v_1 + \cdots + a_nv_n.$$

Explain why Γ and Λ are inverses of each other.

9 Suppose m is a positive integer. Show that the dual basis of the basis $1, x, \dots, x^m$ of $\mathcal{P}_m(\mathbf{R})$ is $\varphi_0, \varphi_1, \dots, \varphi_m$, where

$$\varphi_k(p) = \frac{p^{(k)}(0)}{k!}.$$

Here $p^{(k)}$ denotes the k^{th} derivative of p, with the understanding that the 0^{th} derivative of p is p.

10 Suppose m is a positive integer.

(a) Show that $1, x-5, \dots, (x-5)^m$ is a basis of $\mathcal{P}_m(\mathbf{R})$.
(b) What is the dual basis of the basis in (a)?

11 Suppose $v_1, \dots, v_n$ is a basis of V and $\varphi_1, \dots, \varphi_n$ is the corresponding dual basis of V'. Suppose $\psi \in V'$. Prove that

$$\psi = \psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n.$$

12 Suppose $S, T \in \mathcal{L}(V, W)$.

(a) Prove that $(S+T)' = S' + T'$.

(b) Prove that $(\lambda T)' = \lambda T'$ for all $\lambda \in \mathbf{F}$.

This exercise asks you to verify (*a*) *and* (*b*) *in 3.120.*

13 Show that the dual map of the identity operator on V is the identity operator on V'.

14 Define $T \colon \mathbf{R}^3 \to \mathbf{R}^2$ by

$$T(x, y, z) = (4x + 5y + 6z, 7x + 8y + 9z).$$

Suppose φ_1, φ_2 denotes the dual basis of the standard basis of $\mathbf{R}^2$ and ψ_1, ψ_2, ψ_3 denotes the dual basis of the standard basis of $\mathbf{R}^3$.

(a) Describe the linear functionals $T'(\varphi_1)$ and $T'(\varphi_2)$.

(b) Write $T'(\varphi_1)$ and $T'(\varphi_2)$ as linear combinations of ψ_1, ψ_2, ψ_3.

15 Define $T \colon \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ by

$$(Tp)(x) = x^2 p(x) + p''(x)$$

for each $x \in \mathbf{R}$.

(a) Suppose $\varphi \in \mathcal{P}(\mathbf{R})'$ is defined by $\varphi(p) = p'(4)$. Describe the linear functional $T'(\varphi)$ on $\mathcal{P}(\mathbf{R})$.

(b) Suppose $\varphi \in \mathcal{P}(\mathbf{R})'$ is defined by $\varphi(p) = \int_0^1 p$. Evaluate $\bigl(T'(\varphi)\bigr)(x^3)$.

16 Suppose W is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that

$$T' = 0 \iff T = 0.$$

17 Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that T is invertible if and only if $T' \in \mathcal{L}\bigl(W', V'\bigr)$ is invertible.

18 Suppose V and W are finite-dimensional. Prove that the map that takes $T \in \mathcal{L}(V, W)$ to $T' \in \mathcal{L}\bigl(W', V'\bigr)$ is an isomorphism of $\mathcal{L}(V, W)$ onto $\mathcal{L}\bigl(W', V'\bigr)$.

19 Suppose $U \subseteq V$. Explain why

$$U^0 = \{\varphi \in V' : U \subseteq \operatorname{null} \varphi\}.$$

20 Suppose V is finite-dimensional and U is a subspace of V. Show that

$$U = \{v \in V : \varphi(v) = 0 \text{ for every } \varphi \in U^0\}.$$

21 Suppose V is finite-dimensional and U and W are subspaces of V.

(a) Prove that $W^0 \subseteq U^0$ if and only if $U \subseteq W$.

(b) Prove that $W^0 = U^0$ if and only if $U = W$.

22 Suppose V is finite-dimensional and U and W are subspaces of V.

(a) Show that $(U + W)^0 = U^0 \cap W^0$.
(b) Show that $(U \cap W)^0 = U^0 + W^0$.

23 Suppose V is finite-dimensional and $\varphi_1, \dots, \varphi_m \in V'$. Prove that the following three sets are equal to each other.
(a) $\operatorname{span}(\varphi_1, \dots, \varphi_m)$
(b) $\big((\operatorname{null} \varphi_1) \cap \cdots \cap (\operatorname{null} \varphi_m)\big)^0$
(c) $\{\varphi \in V' : (\operatorname{null} \varphi_1) \cap \cdots \cap (\operatorname{null} \varphi_m) \subseteq \operatorname{null} \varphi\}$

24 Suppose V is finite-dimensional and $v_1, \dots, v_m \in V$. Define a linear map $\Gamma\colon V' \to \mathbf{F}^m$ by $\Gamma(\varphi) = \big(\varphi(v_1), \dots, \varphi(v_m)\big)$.

(a) Prove that $v_1, \dots, v_m$ spans V if and only if Γ is injective.
(b) Prove that $v_1, \dots, v_m$ is linearly independent if and only if Γ is surjective.

25 Suppose V is finite-dimensional and $\varphi_1, \dots, \varphi_m \in V'$. Define a linear map $\Gamma\colon V \to \mathbf{F}^m$ by $\Gamma(v) = \big(\varphi_1(v), \dots, \varphi_m(v)\big)$.

(a) Prove that $\varphi_1, \dots, \varphi_m$ spans V' if and only if Γ is injective.
(b) Prove that $\varphi_1, \dots, \varphi_m$ is linearly independent if and only if Γ is surjective.

26 Suppose V is finite-dimensional and Ω is a subspace of V'. Prove that

$$\Omega = \{v \in V : \varphi(v) = 0 \text{ for every } \varphi \in \Omega\}^0.$$

27 Suppose $T \in \mathcal{L}\big(\mathcal{P}_5(\mathbf{R})\big)$ and $\operatorname{null} T' = \operatorname{span}(\varphi)$, where φ is the linear functional on $\mathcal{P}_5(\mathbf{R})$ defined by $\varphi(p) = p(8)$. Prove that

$$\operatorname{range} T = \{p \in \mathcal{P}_5(\mathbf{R}) : p(8) = 0\}.$$

28 Suppose V is finite-dimensional and $\varphi_1, \dots, \varphi_m$ is a linearly independent list in V'. Prove that

$$\dim\big((\operatorname{null} \varphi_1) \cap \cdots \cap (\operatorname{null} \varphi_m)\big) = (\dim V) - m.$$

29 Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$.

(a) Prove that if $\varphi \in W'$ and $\operatorname{null} T' = \operatorname{span}(\varphi)$, then $\operatorname{range} T = \operatorname{null} \varphi$.
(b) Prove that if $\psi \in V'$ and $\operatorname{range} T' = \operatorname{span}(\psi)$, then $\operatorname{null} T = \operatorname{null} \psi$.

30 Suppose V is finite-dimensional and $\varphi_1, \dots, \varphi_n$ is a basis of V'. Show that there exists a basis of V whose dual basis is $\varphi_1, \dots, \varphi_n$.

31 Suppose U is a subspace of V. Let $i\colon U \to V$ be the inclusion map defined by $i(u) = u$. Thus $i' \in \mathcal{L}\big(V', U'\big)$.

(a) Show that $\operatorname{null} i' = U^0$.
(b) Prove that if V is finite-dimensional, then $\operatorname{range} i' = U'$.
(c) Prove that if V is finite-dimensional, then $\tilde{i'}$ is an isomorphism from V'/U^0 onto U'.

The isomorphism in (c) is natural in that it does not depend on a choice of basis in either vector space.

32 The *double dual space* of V, denoted by V'', is defined to be the dual space of V'. In other words, $V'' = \bigl(V'\bigr)'$. Define $\Lambda\colon V \to V''$ by

$$(\Lambda v)(\varphi) = \varphi(v)$$

for each $v \in V$ and each $\varphi \in V'$.

(a) Show that Λ is a linear map from V to V''.

(b) Show that if $T \in \mathcal{L}(V)$, then $T'' \circ \Lambda = \Lambda \circ T$, where $T'' = \bigl(T'\bigr)'$.

(c) Show that if V is finite-dimensional, then Λ is an isomorphism from V onto V''.

Suppose V is finite-dimensional. Then V and V' are isomorphic, but finding an isomorphism from V onto V' generally requires choosing a basis of V. In contrast, the isomorphism Λ from V onto V'' does not require a choice of basis and thus is considered more natural.

33 Suppose U is a subspace of V. Let $\pi\colon V \to V/U$ be the usual quotient map. Thus $\pi' \in \mathcal{L}\bigl((V/U)', V'\bigr)$.

(a) Show that π' is injective.

(b) Show that range $\pi' = U^0$.

(c) Conclude that π' is an isomorphism from $(V/U)'$ onto U^0.

The isomorphism in (c) is natural in that it does not depend on a choice of basis in either vector space. In fact, there is no assumption here that any of these vector spaces are finite-dimensional.