7.38 *characterizations of positive operators*

Let $T \in \mathcal{L}(V)$. Then the following are equivalent.

(a) T is a positive operator.

(b) T is self-adjoint and all eigenvalues of T are nonnegative.

(c) With respect to some orthonormal basis of V, the matrix of T is a diagonal matrix with only nonnegative numbers on the diagonal.

(d) T has a positive square root.

(e) T has a self-adjoint square root.

(f) $T = R^*R$ for some $R \in \mathcal{L}(V)$.

Proof We will prove that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (f) $\Rightarrow$ (a).

First suppose (a) holds, so that T is positive, which implies that T is self-adjoint (by definition of positive operator). To prove the other condition in (b), suppose λ is an eigenvalue of T. Let v be an eigenvector of T corresponding to λ. Then

$$0 \le \langle Tv, v\rangle = \langle \lambda v, v\rangle = \lambda\langle v, v\rangle.$$

Thus λ is a nonnegative number. Hence (b) holds, showing that (a) implies (b).

Now suppose (b) holds, so that T is self-adjoint and all eigenvalues of T are nonnegative. By the spectral theorem (7.29 and 7.31), there is an orthonormal basis $e_1, \dots, e_n$ of V consisting of eigenvectors of T. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of T corresponding to $e_1, \dots, e_n$; thus each λ_k is a nonnegative number. The matrix of T with respect to $e_1, \dots, e_n$ is the diagonal matrix with $\lambda_1, \dots, \lambda_n$ on the diagonal, which shows that (b) implies (c).

Now suppose (c) holds. Suppose $e_1, \dots, e_n$ is an orthonormal basis of V such that the matrix of T with respect to this basis is a diagonal matrix with nonnegative numbers $\lambda_1, \dots, \lambda_n$ on the diagonal. The linear map lemma (3.4) implies that there exists $R \in \mathcal{L}(V)$ such that

$$Re_k = \sqrt{\lambda_k}e_k$$

for each $k = 1, \dots, n$. As you should verify, R is a positive operator. Furthermore, $R^2e_k = \lambda_k e_k = Te_k$ for each k, which implies that $R^2 = T$. Thus R is a positive square root of T. Hence (d) holds, which shows that (c) implies (d).

Every positive operator is self-adjoint (by definition of positive operator). Thus (d) implies (e).

Now suppose (e) holds, meaning that there exists a self-adjoint operator R on V such that $T = R^2$. Then $T = R^*R$ $\bigl($because $R^* = R\bigr)$. Hence (e) implies (f).

Finally, suppose (f) holds. Let $R \in \mathcal{L}(V)$ be such that $T = R^*R$. Then $T^* = \bigl(R^*R\bigr)^* = R^*\bigl(R^*\bigr)^* = R^*R = T$. Hence T is self-adjoint. To complete the proof that (a) holds, note that

$$\langle Tv, v\rangle = \bigl\langle R^*Rv, v\bigr\rangle = \langle Rv, Rv\rangle \ge 0$$

for every $v \in V$. Thus T is positive, showing that (f) implies (a). ■

Every nonnegative number has a unique nonnegative square root. The next result shows that positive operators enjoy a similar property.

7.39 *each positive operator has only one positive square root*

Every positive operator on V has a unique positive square root.

A positive operator can have infinitely many square roots (although only one of them can be positive). For example, the identity operator on V has infinitely many square roots if $\dim V > 1$.

Proof Suppose $T \in \mathcal{L}(V)$ is positive. Suppose $v \in V$ is an eigenvector of T. Hence there exists a real number $\lambda \geq 0$ such that $Tv = \lambda v$.

Let R be a positive square root of T. We will prove that $Rv = \sqrt{\lambda}v$. This will imply that the behavior of R on the eigenvectors of T is uniquely determined. Because there is a basis of V consisting of eigenvectors of T (by the spectral theorem), this will imply that R is uniquely determined.

To prove that $Rv = \sqrt{\lambda}v$, note that the spectral theorem asserts that there is an orthonormal basis $e_1, \ldots, e_n$ of V consisting of eigenvectors of R. Because R is a positive operator, all its eigenvalues are nonnegative. Thus there exist nonnegative numbers $\lambda_1, \ldots, \lambda_n$ such that $Re_k = \sqrt{\lambda_k}e_k$ for each $k = 1, \ldots, n$.

Because $e_1, \ldots, e_n$ is a basis of V, we can write

$$v = a_1e_1 + \cdots + a_ne_n$$

for some numbers $a_1, \ldots, a_n \in \mathbf{F}$. Thus

$$Rv = a_1\sqrt{\lambda_1}e_1 + \cdots + a_n\sqrt{\lambda_n}e_n.$$

Hence

$$\lambda v = Tv = R^2v = a_1\lambda_1e_1 + \cdots + a_n\lambda_ne_n.$$

The equation above implies that

$$a_1\lambda e_1 + \cdots + a_n\lambda e_n = a_1\lambda_1e_1 + \cdots + a_n\lambda_ne_n.$$

Thus $a_k(\lambda - \lambda_k) = 0$ for each $k = 1, \ldots, n$. Hence

$$v = \sum_{\{k\,:\,\lambda_k = \lambda\}} a_ke_k.$$

Thus

$$Rv = \sum_{\{k\,:\,\lambda_k = \lambda\}} a_k\sqrt{\lambda}e_k = \sqrt{\lambda}v,$$

as desired. ■

The notation defined below makes sense thanks to the result above.

7.40 notation: $\sqrt{T}$

For T a positive operator, $\sqrt{T}$ denotes the unique positive square root of T.

7.41 example: *square root of positive operators*

Define operators S, T on $\mathbf{R}^2$ (with the usual Euclidean inner product) by

$$S(x,y) = (x, 2y) \quad \text{and} \quad T(x,y) = (x+y, x+y).$$

Then with respect to the standard basis of $\mathbf{R}^2$ we have

7.42
$$\mathcal{M}(S) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad \mathcal{M}(T) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Each of these matrices equals its transpose; thus S and T are self-adjoint.

If $(x,y) \in \mathbf{R}^2$, then

$$\langle S(x,y), (x,y) \rangle = x^2 + 2y^2 \geq 0$$

and

$$\langle T(x,y), (x,y) \rangle = x^2 + 2xy + y^2 = (x+y)^2 \geq 0.$$

Thus S and T are positive operators.

The standard basis of $\mathbf{R}^2$ is an orthonormal basis consisting of eigenvectors of S. Note that

$$\left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right), \left(\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}\right)$$

is an orthonormal basis of eigenvectors of T, with eigenvalue 2 for the first eigenvector and eigenvalue 0 for the second eigenvector. Thus $\sqrt{T}$ has the same eigenvectors, with eigenvalues $\sqrt{2}$ and 0.

You can verify that

$$\mathcal{M}\left(\sqrt{S}\right) = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{pmatrix} \quad \text{and} \quad \mathcal{M}\left(\sqrt{T}\right) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

with respect to the standard basis by showing that the squares of the matrices above are the matrices in 7.42 and that each matrix above is the matrix of a positive operator.

The statement of the next result does not involve a square root, but the clean proof makes nice use of the square root of a positive operator.

7.43 *T positive and $\langle Tv, v \rangle = 0 \implies Tv = 0$*

Suppose T is a positive operator on V and $v \in V$ is such that $\langle Tv, v \rangle = 0$. Then $Tv = 0$.

Proof We have

$$0 = \langle Tv, v \rangle = \left\langle \sqrt{T}\sqrt{T}v, v \right\rangle = \left\langle \sqrt{T}v, \sqrt{T}v \right\rangle = \left\| \sqrt{T}v \right\|^2.$$

Hence $\sqrt{T}v = 0$. Thus $Tv = \sqrt{T}\left(\sqrt{T}v\right) = 0$, as desired. ∎

Exercises 7C

1 Suppose $T \in \mathcal{L}(V)$. Prove that if both T and $-T$ are positive operators, then $T = 0$.

2 Suppose $T \in \mathcal{L}(\mathbf{F}^4)$ is the operator whose matrix (with respect to the standard basis) is

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

Show that T is an invertible positive operator.

3 Suppose n is a positive integer and $T \in \mathcal{L}(\mathbf{F}^n)$ is the operator whose matrix (with respect to the standard basis) consists of all 1's. Show that T is a positive operator.

4 Suppose n is an integer with $n > 1$. Show that there exists an n-by-n matrix A such that all of the entries of A are positive numbers and $A = A^*$, but the operator on $\mathbf{F}^n$ whose matrix (with respect to the standard basis) equals A is not a positive operator.

5 Suppose $T \in \mathcal{L}(V)$ is self-adjoint. Prove that T is a positive operator if and only if for every orthonormal basis $e_1, \dots, e_n$ of V, all entries on the diagonal of $\mathcal{M}\big(T, (e_1, \dots, e_n)\big)$ are nonnegative numbers.

6 Prove that the sum of two positive operators on V is a positive operator.

7 Suppose $S \in \mathcal{L}(V)$ is an invertible positive operator and $T \in \mathcal{L}(V)$ is a positive operator. Prove that $S + T$ is invertible.

8 Suppose $T \in \mathcal{L}(V)$. Prove that T is a positive operator if and only if the pseudoinverse $T^\dagger$ is a positive operator.

9 Suppose $T \in \mathcal{L}(V)$ is a positive operator and $S \in \mathcal{L}(W, V)$. Prove that S^*TS is a positive operator on W.

10 Suppose T is a positive operator on V. Suppose $v, w \in V$ are such that

$$Tv = w \quad \text{and} \quad Tw = v.$$

Prove that $v = w$.

11 Suppose T is a positive operator on V and U is a subspace of V invariant under T. Prove that $T|_U \in \mathcal{L}(U)$ is a positive operator on U.

12 Suppose $T \in \mathcal{L}(V)$ is a positive operator. Prove that T^k is a positive operator for every positive integer k.

13 Suppose $T \in \mathcal{L}(V)$ is self-adjoint and $\alpha \in \mathbf{R}$.

(a) Prove that $T - \alpha I$ is a positive operator if and only if α is less than or equal to every eigenvalue of T.

(b) Prove that $\alpha I - T$ is a positive operator if and only if α is greater than or equal to every eigenvalue of T.

14 Suppose T is a positive operator on V and $v_1, \dots, v_m \in V$. Prove that

$$\sum_{j=1}^{m} \sum_{k=1}^{m} \langle T v_k, v_j \rangle \geq 0.$$

15 Suppose $T \in \mathcal{L}(V)$ is self-adjoint. Prove that there exist positive operators $A, B \in \mathcal{L}(V)$ such that

$$T = A - B \quad \text{and} \quad \sqrt{T^* T} = A + B \quad \text{and} \quad AB = BA = 0.$$

16 Suppose T is a positive operator on V. Prove that

$$\operatorname{null} \sqrt{T} = \operatorname{null} T \quad \text{and} \quad \operatorname{range} \sqrt{T} = \operatorname{range} T.$$

17 Suppose that $T \in \mathcal{L}(V)$ is a positive operator. Prove that there exists a polynomial p with real coefficients such that $\sqrt{T} = p(T)$.

18 Suppose S and T are positive operators on V. Prove that ST is a positive operator if and only if S and T commute.

19 Show that the identity operator on $\mathbf{F}^2$ has infinitely many self-adjoint square roots.

20 Suppose $T \in \mathcal{L}(V)$ and $e_1, \dots, e_n$ is an orthonormal basis of V. Prove that T is a positive operator if and only if there exist $v_1, \dots, v_n \in V$ such that

$$\langle T e_k, e_j \rangle = \langle v_k, v_j \rangle$$

for all $j, k = 1, \dots, n$.

The numbers $\{\langle T e_k, e_j \rangle\}_{j,k=1,\dots,n}$ are the entries in the matrix of T with respect to the orthonormal basis $e_1, \dots, e_n$.

21 Suppose n is a positive integer. The n-by-n *Hilbert matrix* is the n-by-n matrix whose entry in row j, column k is $\frac{1}{j+k-1}$. Suppose $T \in \mathcal{L}(V)$ is an operator whose matrix with respect to some orthonormal basis of V is the n-by-n Hilbert matrix. Prove that T is a positive invertible operator.

Example: The 4-by-4 Hilbert matrix is

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{pmatrix}.$$

22 Suppose $T \in \mathcal{L}(V)$ is a positive operator and $u \in V$ is such that $\|u\| = 1$ and $\|Tu\| \geq \|Tv\|$ for all $v \in V$ with $\|v\| = 1$. Show that u is an eigenvector of T corresponding to the largest eigenvalue of T.

23 For $T \in \mathcal{L}(V)$ and $u, v \in V$, define $\langle u, v \rangle_T$ by $\langle u, v \rangle_T = \langle Tu, v \rangle$.

(a) Suppose $T \in \mathcal{L}(V)$. Prove that $\langle \cdot, \cdot \rangle_T$ is an inner product on V if and only if T is an invertible positive operator (with respect to the original inner product $\langle \cdot, \cdot \rangle$).

(b) Prove that every inner product on V is of the form $\langle \cdot, \cdot \rangle_T$ for some positive invertible operator $T \in \mathcal{L}(V)$.

24 Suppose S and T are positive operators on V. Prove that

$$\operatorname{null}(S + T) = \operatorname{null} S \cap \operatorname{null} T.$$

25 Let T be the second derivative operator in Exercise 31(b) in Section 7A. Show that $-T$ is a positive operator.