3B Null Spaces and Ranges

Null Space and Injectivity

In this section we will learn about two subspaces that are intimately connected with each linear map. We begin with the set of vectors that get mapped to 0.

3.11 definition: *null space*, null T

For $T \in \mathcal{L}(V, W)$, the *null space* of T, denoted by null T, is the subset of V consisting of those vectors that T maps to 0:

$$\operatorname{null} T = \{v \in V : Tv = 0\}.$$

3.12 example: *null space*

- If T is the zero map from V to W, meaning that $Tv = 0$ for every $v \in V$, then null $T = V$.
- Suppose $\varphi \in \mathcal{L}(\mathbf{C}^3, \mathbf{C})$ is defined by $\varphi(z_1, z_2, z_3) = z_1 + 2z_2 + 3z_3$. Then null φ equals $\{(z_1, z_2, z_3) \in \mathbf{C}^3 : z_1 + 2z_2 + 3z_3 = 0\}$, which is a subspace of the domain of φ. We will soon see that the null space of each linear map is a subspace of its domain.
- Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ is the differentiation map defined by $Dp = p'$. The only functions whose derivative equals the zero function are the constant functions. Thus the null space of D equals the set of constant functions.

*The word "null" means zero. Thus the term "null space" should remind you of the connection to 0. Some mathematicians use the term **kernel** instead of null space.*

- Suppose that $T \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ is the multiplication by x^2 map defined by $(Tp)(x) = x^2 p(x)$. The only polynomial p such that $x^2 p(x) = 0$ for all $x \in \mathbf{R}$ is the 0 polynomial. Thus null $T = \{0\}$.
- Suppose $T \in \mathcal{L}(\mathbf{F}^\infty)$ is the backward shift defined by

$$T(x_1, x_2, x_3, \dots) = (x_2, x_3, \dots).$$

Then $T(x_1, x_2, x_3, \dots)$ equals 0 if and only if the numbers $x_2, x_3, \dots$ are all 0. Thus null $T = \{(a, 0, 0, \dots) : a \in \mathbf{F}\}$.

The next result shows that the null space of each linear map is a subspace of the domain. In particular, 0 is in the null space of every linear map.

3.13 *the null space is a subspace*

Suppose $T \in \mathcal{L}(V, W)$. Then null T is a subspace of V.

Proof Because T is a linear map, $T(0) = 0$ (by 3.10). Thus $0 \in \operatorname{null} T$.

Suppose $u, v \in \operatorname{null} T$. Then

$$T(u + v) = Tu + Tv = 0 + 0 = 0.$$

Hence $u + v \in \operatorname{null} T$. Thus $\operatorname{null} T$ is closed under addition.

Suppose $u \in \operatorname{null} T$ and $\lambda \in \mathbf{F}$. Then

$$T(\lambda u) = \lambda Tu = \lambda 0 = 0.$$

Hence $\lambda u \in \operatorname{null} T$. Thus $\operatorname{null} T$ is closed under scalar multiplication.

We have shown that $\operatorname{null} T$ contains 0 and is closed under addition and scalar multiplication. Thus $\operatorname{null} T$ is a subspace of V (by 1.34). ■

As we will soon see, for a linear map the next definition is closely connected to the null space.

3.14 definition: ***injective***

A function $T\colon V \to W$ is called *injective* if $Tu = Tv$ implies $u = v$.

We could rephrase the definition above to say that T is injective if $u \neq v$ implies that $Tu \neq Tv$. Thus T is injective if and only if it maps distinct inputs to distinct outputs.

The term ***one-to-one*** *means the same as injective.*

The next result says that we can check whether a linear map is injective by checking whether 0 is the only vector that gets mapped to 0. As a simple application of this result, we see that of the linear maps whose null spaces we computed in 3.12, only multiplication by x^2 is injective (except that the zero map is injective in the special case $V = \{0\}$).

3.15 ***injectivity*** $\Longleftrightarrow$ ***null space equals*** $\{0\}$

Let $T \in \mathcal{L}(V, W)$. Then T is injective if and only if $\operatorname{null} T = \{0\}$.

Proof First suppose T is injective. We want to prove that $\operatorname{null} T = \{0\}$. We already know that $\{0\} \subseteq \operatorname{null} T$ (by 3.10). To prove the inclusion in the other direction, suppose $v \in \operatorname{null} T$. Then

$$T(v) = 0 = T(0).$$

Because T is injective, the equation above implies that $v = 0$. Thus we can conclude that $\operatorname{null} T = \{0\}$, as desired.

To prove the implication in the other direction, now suppose $\operatorname{null} T = \{0\}$. We want to prove that T is injective. To do this, suppose $u, v \in V$ and $Tu = Tv$. Then

$$0 = Tu - Tv = T(u - v).$$

Thus $u - v$ is in $\operatorname{null} T$, which equals $\{0\}$. Hence $u - v = 0$, which implies that $u = v$. Hence T is injective, as desired. ■

Range and Surjectivity

Now we give a name to the set of outputs of a linear map.

3.16 definition: *range*

For $T \in \mathcal{L}(V, W)$, the *range* of T is the subset of W consisting of those vectors that are equal to Tv for some $v \in V$:

$$\operatorname{range} T = \{Tv : v \in V\}.$$

3.17 example: *range*

- If T is the zero map from V to W, meaning that $Tv = 0$ for every $v \in V$, then $\operatorname{range} T = \{0\}$.
- Suppose $T \in \mathcal{L}(\mathbf{R}^2, \mathbf{R}^3)$ is defined by $T(x, y) = (2x, 5y, x + y)$. Then

$$\operatorname{range} T = \{(2x, 5y, x + y) : x, y \in \mathbf{R}\}.$$

  Note that $\operatorname{range} T$ is a subspace of $\mathbf{R}^3$. We will soon see that the range of each element of $\mathcal{L}(V, W)$ is a subspace of W.
- Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ is the differentiation map defined by $Dp = p'$. Because for every polynomial $q \in \mathcal{P}(\mathbf{R})$ there exists a polynomial $p \in \mathcal{P}(\mathbf{R})$ such that $p' = q$, the range of D is $\mathcal{P}(\mathbf{R})$.

The next result shows that the range of each linear map is a subspace of the vector space into which it is being mapped.

3.18 *the range is a subspace*

If $T \in \mathcal{L}(V, W)$, then $\operatorname{range} T$ is a subspace of W.

Proof Suppose $T \in \mathcal{L}(V, W)$. Then $T(0) = 0$ (by 3.10), which implies that $0 \in \operatorname{range} T$.

If $w_1, w_2 \in \operatorname{range} T$, then there exist $v_1, v_2 \in V$ such that $Tv_1 = w_1$ and $Tv_2 = w_2$. Thus

$$T(v_1 + v_2) = Tv_1 + Tv_2 = w_1 + w_2.$$

Hence $w_1 + w_2 \in \operatorname{range} T$. Thus $\operatorname{range} T$ is closed under addition.

If $w \in \operatorname{range} T$ and $\lambda \in \mathbf{F}$, then there exists $v \in V$ such that $Tv = w$. Thus

$$T(\lambda v) = \lambda Tv = \lambda w.$$

Hence $\lambda w \in \operatorname{range} T$. Thus $\operatorname{range} T$ is closed under scalar multiplication.

We have shown that $\operatorname{range} T$ contains 0 and is closed under addition and scalar multiplication. Thus $\operatorname{range} T$ is a subspace of W (by 1.34). ■

3.19 definition: *surjective*

A function $T: V \to W$ is called *surjective* if its range equals W.

To illustrate the definition above, note that of the ranges we computed in 3.17, only the differentiation map is surjective (except that the zero map is surjective in the special case $W = \{0\}$).

Whether a linear map is surjective depends on what we are thinking of as the vector space into which it maps.

Some people use the term ***onto****, which means the same as surjective.*

3.20 example: *surjectivity depends on the target space*

The differentiation map $D \in \mathcal{L}(\mathcal{P}_5(\mathbf{R}))$ defined by $Dp = p'$ is not surjective, because the polynomial x^5 is not in the range of D. However, the differentiation map $S \in \mathcal{L}(\mathcal{P}_5(\mathbf{R}), \mathcal{P}_4(\mathbf{R}))$ defined by $Sp = p'$ is surjective, because its range equals $\mathcal{P}_4(\mathbf{R})$, which is the vector space into which S maps.

Fundamental Theorem of Linear Maps

The next result is so important that it gets a dramatic name.

3.21 *fundamental theorem of linear maps*

Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$. Then range T is finite-dimensional and

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T.$$

Proof Let $u_1, \ldots, u_m$ be a basis of null T; thus $\dim \operatorname{null} T = m$. The linearly independent list $u_1, \ldots, u_m$ can be extended to a basis

$$u_1, \ldots, u_m, v_1, \ldots, v_n$$

of V (by 2.32). Thus $\dim V = m + n$. To complete the proof, we need to show that range T is finite-dimensional and $\dim \operatorname{range} T = n$. We will do this by proving that $Tv_1, \ldots, Tv_n$ is a basis of range T.

Let $v \in V$. Because $u_1, \ldots, u_m, v_1, \ldots, v_n$ spans V, we can write

$$v = a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_n v_n,$$

where the a's and b's are in $\mathbf{F}$. Applying T to both sides of this equation, we get

$$Tv = b_1 Tv_1 + \cdots + b_n Tv_n,$$

where the terms of the form Tu_k disappeared because each u_k is in null T. The last equation implies that the list $Tv_1, \ldots, Tv_n$ spans range T. In particular, range T is finite-dimensional.

To show $Tv_1, \dots, Tv_n$ is linearly independent, suppose $c_1, \dots, c_n \in \mathbf{F}$ and

$$c_1 Tv_1 + \cdots + c_n Tv_n = 0.$$

Then

$$T(c_1 v_1 + \cdots + c_n v_n) = 0.$$

Hence

$$c_1 v_1 + \cdots + c_n v_n \in \operatorname{null} T.$$

Because $u_1, \dots, u_m$ spans $\operatorname{null} T$, we can write

$$c_1 v_1 + \cdots + c_n v_n = d_1 u_1 + \cdots + d_m u_m,$$

where the d's are in $\mathbf{F}$. This equation implies that all the c's (and d's) are 0 (because $u_1, \dots, u_m, v_1, \dots, v_n$ is linearly independent). Thus $Tv_1, \dots, Tv_n$ is linearly independent and hence is a basis of range T, as desired. ■

Now we can show that no linear map from a finite-dimensional vector space to a "smaller" vector space can be injective, where "smaller" is measured by dimension.

3.22 *linear map to a lower-dimensional space is not injective*

Suppose V and W are finite-dimensional vector spaces such that $\dim V > \dim W$. Then no linear map from V to W is injective.

Proof Let $T \in \mathcal{L}(V, W)$. Then

$$\begin{aligned} \dim \operatorname{null} T &= \dim V - \dim \operatorname{range} T \\ &\geq \dim V - \dim W \\ &> 0, \end{aligned}$$

where the first line above comes from the fundamental theorem of linear maps (3.21) and the second line follows from 2.37. The inequality above states that $\dim \operatorname{null} T > 0$. This means that $\operatorname{null} T$ contains vectors other than 0. Thus T is not injective (by 3.15). ■

3.23 example: *linear map from $\mathbf{F}^4$ to $\mathbf{F}^3$ is not injective*

Define a linear map $T \colon \mathbf{F}^4 \to \mathbf{F}^3$ by

$$T(z_1, z_2, z_3, z_4) = \left(\sqrt{7} z_1 + \pi z_2 + z_4, 97 z_1 + 3 z_2 + 2 z_3, z_2 + 6 z_3 + 7 z_4\right).$$

Because $\dim \mathbf{F}^4 > \dim \mathbf{F}^3$, we can use 3.22 to assert that T is not injective, without doing any calculations.

The next result shows that no linear map from a finite-dimensional vector space to a "bigger" vector space can be surjective, where "bigger" is measured by dimension.

3.24 *linear map to a higher-dimensional space is not surjective*

Suppose V and W are finite-dimensional vector spaces such that $\dim V < \dim W$. Then no linear map from V to W is surjective.

Proof Let $T \in \mathcal{L}(V, W)$. Then

$$\begin{aligned} \dim \operatorname{range} T &= \dim V - \dim \operatorname{null} T \\ &\leq \dim V \\ &< \dim W, \end{aligned}$$

where the equality above comes from the fundamental theorem of linear maps (3.21). The inequality above states that $\dim \operatorname{range} T < \dim W$. This means that $\operatorname{range} T$ cannot equal W. Thus T is not surjective. ∎

As we will soon see, 3.22 and 3.24 have important consequences in the theory of linear equations. The idea is to express questions about systems of linear equations in terms of linear maps. Let's begin by rephrasing in terms of linear maps the question of whether a homogeneous system of linear equations has a nonzero solution.

__Homogeneous__, in this context, means that the constant term on the right side of each equation below is 0.

Fix positive integers m and n, and let $A_{j,k} \in \mathbf{F}$ for $j = 1, \ldots, m$ and $k = 1, \ldots, n$. Consider the homogeneous system of linear equations

$$\begin{gathered} \sum_{k=1}^{n} A_{1,k} x_k = 0 \\ \vdots \\ \sum_{k=1}^{n} A_{m,k} x_k = 0. \end{gathered}$$

Clearly $x_1 = \cdots = x_n = 0$ is a solution of the system of equations above; the question here is whether any other solutions exist.

Define $T \colon \mathbf{F}^n \to \mathbf{F}^m$ by

3.25 $$T(x_1, \ldots, x_n) = \Bigl(\sum_{k=1}^{n} A_{1,k} x_k, \ldots, \sum_{k=1}^{n} A_{m,k} x_k\Bigr).$$

The equation $T(x_1, \ldots, x_n) = 0$ (the 0 here is the additive identity in $\mathbf{F}^m$, namely, the list of length m of all 0's) is the same as the homogeneous system of linear equations above.

Thus we want to know if $\operatorname{null} T$ is strictly bigger than $\{0\}$, which is equivalent to T not being injective (by 3.15). The next result gives an important condition for ensuring that T is not injective.

3.26 *homogeneous system of linear equations*

A homogeneous system of linear equations with more variables than equations has nonzero solutions.

Proof Use the notation and result from the discussion above. Thus T is a linear map from $\mathbf{F}^n$ to $\mathbf{F}^m$, and we have a homogeneous system of m linear equations with n variables $x_1, \dots, x_n$. From 3.22 we see that T is not injective if $n > m$. ■

Example of the result above: a homogeneous system of four linear equations with five variables has nonzero solutions.

Now we consider the question of whether a system of linear equations has no solutions for some choice of the constant terms. To rephrase this question in terms of a linear map, fix positive integers m and n, and let $A_{j,k} \in \mathbf{F}$ for all $j = 1, \dots, m$ and all $k = 1, \dots, n$. For $c_1, \dots, c_m \in \mathbf{F}$, consider the system of linear equations

3.27
$$\begin{aligned} \sum_{k=1}^{n} A_{1,k} x_k &= c_1 \\ &\vdots \\ \sum_{k=1}^{n} A_{m,k} x_k &= c_m. \end{aligned}$$

With this notation, the question here is whether there is some choice of the constant terms $c_1, \dots, c_m \in \mathbf{F}$ such that no solution exists to the system above.

Define $T: \mathbf{F}^n \to \mathbf{F}^m$ as in 3.25. The equation $T(x_1, \dots, x_n) = (c_1, \dots, c_m)$ is the same as the system of equations 3.27. Thus we want to know if range $T \neq \mathbf{F}^m$. Hence we can rephrase our question about not having a solution for some choice of $c_1, \dots, c_m \in \mathbf{F}$ as follows: What condition ensures that T is not surjective? The next result gives one such condition.

The results 3.26 and 3.28, which compare the number of variables and the number of equations, can also be proved using Gaussian elimination. The abstract approach taken here seems to provide cleaner proofs.

3.28 *system of linear equations with more equations than variables*

A system of linear equations with more equations than variables has no solution for some choice of the constant terms.

Proof Use the notation from the discussion above. Thus T is a linear map from $\mathbf{F}^n$ to $\mathbf{F}^m$, and we have a system of m equations with n variables $x_1, \dots, x_n$; see 3.27. If $n < m$, then 3.24 implies that T is not surjective. As discussed above, this shows that if we have more equations than variables in a system of linear equations, then there is no solution for some choice of the constant terms. ■

Example of the result above: a system of five linear equations with four variables has no solution for some choice of the constant terms.

Exercises 3B

1 Give an example of a linear map T with $\dim \operatorname{null} T = 3$ and $\dim \operatorname{range} T = 2$.

2 Suppose $S, T \in \mathcal{L}(V)$ are such that $\operatorname{range} S \subseteq \operatorname{null} T$. Prove that $(ST)^2 = 0$.

3 Suppose $v_1, \ldots, v_m$ is a list of vectors in V. Define $T \in \mathcal{L}(\mathbf{F}^m, V)$ by

$$T(z_1, \ldots, z_m) = z_1 v_1 + \cdots + z_m v_m.$$

(a) What property of T corresponds to $v_1, \ldots, v_m$ spanning V?
(b) What property of T corresponds to the list $v_1, \ldots, v_m$ being linearly independent?

4 Show that $\{T \in \mathcal{L}(\mathbf{R}^5, \mathbf{R}^4) : \dim \operatorname{null} T > 2\}$ is not a subspace of $\mathcal{L}(\mathbf{R}^5, \mathbf{R}^4)$.

5 Give an example of $T \in \mathcal{L}(\mathbf{R}^4)$ such that $\operatorname{range} T = \operatorname{null} T$.

6 Prove that there does not exist $T \in \mathcal{L}(\mathbf{R}^5)$ such that $\operatorname{range} T = \operatorname{null} T$.

7 Suppose V and W are finite-dimensional with $2 \leq \dim V \leq \dim W$. Show that $\{T \in \mathcal{L}(V, W) : T \text{ is not injective}\}$ is not a subspace of $\mathcal{L}(V, W)$.

8 Suppose V and W are finite-dimensional with $\dim V \geq \dim W \geq 2$. Show that $\{T \in \mathcal{L}(V, W) : T \text{ is not surjective}\}$ is not a subspace of $\mathcal{L}(V, W)$.

9 Suppose $T \in \mathcal{L}(V, W)$ is injective and $v_1, \ldots, v_n$ is linearly independent in V. Prove that $Tv_1, \ldots, Tv_n$ is linearly independent in W.

10 Suppose $v_1, \ldots, v_n$ spans V and $T \in \mathcal{L}(V, W)$. Show that $Tv_1, \ldots, Tv_n$ spans $\operatorname{range} T$.

11 Suppose that V is finite-dimensional and that $T \in \mathcal{L}(V, W)$. Prove that there exists a subspace U of V such that

$$U \cap \operatorname{null} T = \{0\} \quad \text{and} \quad \operatorname{range} T = \{Tu : u \in U\}.$$

12 Suppose T is a linear map from $\mathbf{F}^4$ to $\mathbf{F}^2$ such that

$$\operatorname{null} T = \{(x_1, x_2, x_3, x_4) \in \mathbf{F}^4 : x_1 = 5x_2 \text{ and } x_3 = 7x_4\}.$$

Prove that T is surjective.

13 Suppose U is a three-dimensional subspace of $\mathbf{R}^8$ and that T is a linear map from $\mathbf{R}^8$ to $\mathbf{R}^5$ such that $\operatorname{null} T = U$. Prove that T is surjective.

14 Prove that there does not exist a linear map from $\mathbf{F}^5$ to $\mathbf{F}^2$ whose null space equals $\{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{F}^5 : x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5\}$.

15 Suppose there exists a linear map on V whose null space and range are both finite-dimensional. Prove that V is finite-dimensional.

16 Suppose V and W are both finite-dimensional. Prove that there exists an injective linear map from V to W if and only if $\dim V \leq \dim W$.

17 Suppose V and W are both finite-dimensional. Prove that there exists a surjective linear map from V onto W if and only if $\dim V \geq \dim W$.

18 Suppose V and W are finite-dimensional and that U is a subspace of V. Prove that there exists $T \in \mathcal{L}(V, W)$ such that $\operatorname{null} T = U$ if and only if $\dim U \geq \dim V - \dim W$.

19 Suppose W is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that T is injective if and only if there exists $S \in \mathcal{L}(W, V)$ such that ST is the identity operator on V.

20 Suppose W is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that T is surjective if and only if there exists $S \in \mathcal{L}(W, V)$ such that TS is the identity operator on W.

21 Suppose V is finite-dimensional, $T \in \mathcal{L}(V, W)$, and U is a subspace of W. Prove that $\{v \in V : Tv \in U\}$ is a subspace of V and

$$\dim\{v \in V : Tv \in U\} = \dim \operatorname{null} T + \dim(U \cap \operatorname{range} T).$$

22 Suppose U and V are finite-dimensional vector spaces and $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$. Prove that

$$\dim \operatorname{null} ST \leq \dim \operatorname{null} S + \dim \operatorname{null} T.$$

23 Suppose U and V are finite-dimensional vector spaces and $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$. Prove that

$$\dim \operatorname{range} ST \leq \min\{\dim \operatorname{range} S, \dim \operatorname{range} T\}.$$

24 (a) Suppose $\dim V = 5$ and $S, T \in \mathcal{L}(V)$ are such that $ST = 0$. Prove that $\dim \operatorname{range} TS \leq 2$.

(b) Give an example of $S, T \in \mathcal{L}(\mathbf{F}^5)$ with $ST = 0$ and $\dim \operatorname{range} TS = 2$.

25 Suppose that W is finite-dimensional and $S, T \in \mathcal{L}(V, W)$. Prove that $\operatorname{null} S \subseteq \operatorname{null} T$ if and only if there exists $E \in \mathcal{L}(W)$ such that $T = ES$.

26 Suppose that V is finite-dimensional and $S, T \in \mathcal{L}(V, W)$. Prove that $\operatorname{range} S \subseteq \operatorname{range} T$ if and only if there exists $E \in \mathcal{L}(V)$ such that $S = TE$.

27 Suppose $P \in \mathcal{L}(V)$ and $P^2 = P$. Prove that $V = \operatorname{null} P \oplus \operatorname{range} P$.

28 Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ is such that $\deg Dp = (\deg p) - 1$ for every nonconstant polynomial $p \in \mathcal{P}(\mathbf{R})$. Prove that D is surjective.

The notation D is used above to remind you of the differentiation map that sends a polynomial p to p'.

29 Suppose $p \in \mathcal{P}(\mathbf{R})$. Prove that there exists a polynomial $q \in \mathcal{P}(\mathbf{R})$ such that $5q'' + 3q' = p$.

This exercise can be done without linear algebra, but it's more fun to do it using linear algebra.

30 Suppose $\varphi \in \mathcal{L}(V, \mathbf{F})$ and $\varphi \neq 0$. Suppose $u \in V$ is not in $\operatorname{null} \varphi$. Prove that

$$V = \operatorname{null} \varphi \oplus \{au : a \in \mathbf{F}\}.$$

31 Suppose V is finite-dimensional, X is a subspace of V, and Y is a finite-dimensional subspace of W. Prove that there exists $T \in \mathcal{L}(V, W)$ such that $\operatorname{null} T = X$ and $\operatorname{range} T = Y$ if and only if $\dim X + \dim Y = \dim V$.

32 Suppose V is finite-dimensional with $\dim V > 1$. Show that if $\varphi \colon \mathcal{L}(V) \to \mathbf{F}$ is a linear map such that $\varphi(ST) = \varphi(S)\varphi(T)$ for all $S, T \in \mathcal{L}(V)$, then $\varphi = 0$.

Hint: The description of the two-sided ideals of $\mathcal{L}(V)$ given by Exercise 17 in Section 3A might be useful.

33 Suppose that V and W are real vector spaces and $T \in \mathcal{L}(V, W)$. Define $T_{\mathbf{C}} \colon V_{\mathbf{C}} \to W_{\mathbf{C}}$ by

$$T_{\mathbf{C}}(u + iv) = Tu + iTv$$

for all $u, v \in V$.

(a) Show that $T_{\mathbf{C}}$ is a (complex) linear map from $V_{\mathbf{C}}$ to $W_{\mathbf{C}}$.

(b) Show that $T_{\mathbf{C}}$ is injective if and only if T is injective.

(c) Show that $\operatorname{range} T_{\mathbf{C}} = W_{\mathbf{C}}$ if and only if $\operatorname{range} T = W$.

*See Exercise 8 in Section 1B for the definition of the complexification $V_{\mathbf{C}}$. The linear map $T_{\mathbf{C}}$ is called the **complexification** of the linear map T.*