Factorization of Polynomials over **R**

A polynomial with real coefficients may have no real zeros. For example, the polynomial $1 + x^2$ has no real zeros.

To obtain a factorization theorem over **R**, we will use our factorization theorem over **C**. We begin with the next result.

The failure of the fundamental theorem of algebra for **R** *accounts for the differences between linear algebra on real and complex vector spaces, as we will see in later chapters.*

4.14 *polynomials with real coefficients have nonreal zeros in pairs*

Suppose $p \in \mathcal{P}(\mathbf{C})$ is a polynomial with real coefficients. If $\lambda \in \mathbf{C}$ is a zero of p, then so is $\bar{\lambda}$.

Proof Let

$$p(z) = a_0 + a_1 z + \cdots + a_m z^m,$$

where $a_0, \ldots, a_m$ are real numbers. Suppose $\lambda \in \mathbf{C}$ is a zero of p. Then

$$a_0 + a_1 \lambda + \cdots + a_m \lambda^m = 0.$$

Take the complex conjugate of both sides of this equation, obtaining

$$a_0 + a_1 \bar{\lambda} + \cdots + a_m \bar{\lambda}^m = 0,$$

where we have used basic properties of the complex conjugate (see 4.4). The equation above shows that $\bar{\lambda}$ is a zero of p. ■

We want a factorization theorem for polynomials with real coefficients. We begin with the following result.

Think about the quadratic formula in connection with the result below.

4.15 *factorization of a quadratic polynomial*

Suppose $b, c \in \mathbf{R}$. Then there is a polynomial factorization of the form

$$x^2 + bx + c = (x - \lambda_1)(x - \lambda_2)$$

with $\lambda_1, \lambda_2 \in \mathbf{R}$ if and only if $b^2 \geq 4c$.

Proof Notice that

$$x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 + \left(c - \frac{b^2}{4}\right).$$

First suppose $b^2 < 4c$. Then the right side of the equation above is positive for every $x \in \mathbf{R}$. Hence the polynomial $x^2 + bx + c$ has no real zeros and thus cannot be factored in the form $(x - \lambda_1)(x - \lambda_2)$ with $\lambda_1, \lambda_2 \in \mathbf{R}$.

The equation above is the basis of the technique called completing the square.

Conversely, now suppose $b^2 \geq 4c$. Then there is a real number d such that $d^2 = \frac{b^2}{4} - c$. From the displayed equation above, we have

$$\begin{aligned} x^2 + bx + c &= \left(x + \frac{b}{2}\right)^2 - d^2 \\ &= \left(x + \frac{b}{2} + d\right)\left(x + \frac{b}{2} - d\right), \end{aligned}$$

which gives the desired factorization. ■

The next result gives a factorization of a polynomial over $\mathbf{R}$. The idea of the proof is to use the second version of the fundamental theorem of algebra (4.13), which gives a factorization of p as a polynomial with complex coefficients. Complex but nonreal zeros of p come in pairs; see 4.14. Thus if the factorization of p as an element of $\mathcal{P}(\mathbf{C})$ includes terms of the form $(x - \lambda)$ with λ a nonreal complex number, then $(x - \overline{\lambda})$ is also a term in the factorization. Multiplying together these two terms, we get

$$\left(x^2 - 2(\operatorname{Re}\lambda)x + |\lambda|^2\right),$$

which is a quadratic term of the required form.

The idea sketched in the paragraph above almost provides a proof of the existence of our desired factorization. However, we need to be careful about one point. Suppose λ is a nonreal complex number and $(x - \lambda)$ is a term in the factorization of p as an element of $\mathcal{P}(\mathbf{C})$. We are guaranteed by 4.14 that $(x - \overline{\lambda})$ also appears as a term in the factorization, but 4.14 does not state that these two factors appear the same number of times, as needed to make the idea above work. However, the proof works around this point.

In the next result, either m or M may equal 0. The numbers $\lambda_1, \dots, \lambda_m$ are precisely the real zeros of p, for these are the only real values of x for which the right side of the equation in the next result equals 0.

4.16 *factorization of a polynomial over* $\mathbf{R}$

Suppose $p \in \mathcal{P}(\mathbf{R})$ is a nonconstant polynomial. Then p has a unique factorization (except for the order of the factors) of the form

$$p(x) = c(x - \lambda_1)\cdots(x - \lambda_m)\left(x^2 + b_1x + c_1\right)\cdots\left(x^2 + b_Mx + c_M\right),$$

where $c, \lambda_1, \dots, \lambda_m, b_1, \dots, b_M, c_1, \dots, c_M \in \mathbf{R}$, with $b_k^2 < 4c_k$ for each k.

Proof First we will prove that the desired factorization exists, and after that we will prove the uniqueness.

Think of p as an element of $\mathcal{P}(\mathbf{C})$. If all (complex) zeros of p are real, then we have the desired factorization by 4.13. Thus suppose p has a zero $\lambda \in \mathbf{C}$ with $\lambda \notin \mathbf{R}$. By 4.14, $\overline{\lambda}$ is a zero of p. Thus we can write

$$\begin{aligned} p(x) &= (x-\lambda)\big(x-\overline{\lambda}\big)q(x) \\ &= \big(x^2 - 2(\operatorname{Re}\lambda)x + |\lambda|^2\big)q(x) \end{aligned}$$

for some polynomial $q \in \mathcal{P}(\mathbf{C})$ of degree two less than the degree of p. If we can prove that q has real coefficients, then using induction on the degree of p completes the proof of the existence part of this result.

To prove that q has real coefficients, we solve the equation above for q, getting

$$q(x) = \frac{p(x)}{x^2 - 2(\operatorname{Re}\lambda)x + |\lambda|^2}$$

for all $x \in \mathbf{R}$. The equation above implies that $q(x) \in \mathbf{R}$ for all $x \in \mathbf{R}$. Writing

$$q(x) = a_0 + a_1 x + \cdots + a_{n-2}x^{n-2},$$

where $n = \deg p$ and $a_0, \dots, a_{n-2} \in \mathbf{C}$, we thus have

$$0 = \operatorname{Im} q(x) = (\operatorname{Im} a_0) + (\operatorname{Im} a_1)x + \cdots + (\operatorname{Im} a_{n-2})x^{n-2}$$

for all $x \in \mathbf{R}$. This implies that $\operatorname{Im} a_0, \dots, \operatorname{Im} a_{n-2}$ all equal 0 (by 4.8). Thus all coefficients of q are real, as desired. Hence the desired factorization exists.

Now we turn to the question of uniqueness of our factorization. A factor of p of the form $x^2 + b_k x + c_k$ with $b_k^2 < 4c_k$ can be uniquely written as $(x-\lambda_k)\big(x-\overline{\lambda_k}\big)$ with $\lambda_k \in \mathbf{C}$. A moment's thought shows that two different factorizations of p as an element of $\mathcal{P}(\mathbf{R})$ would lead to two different factorizations of p as an element of $\mathcal{P}(\mathbf{C})$, contradicting 4.13. ■

Exercises 4

1 Suppose $w, z \in \mathbf{C}$. Verify the following equalities and inequalities.

(a) $z + \overline{z} = 2\operatorname{Re} z$

(b) $z - \overline{z} = 2(\operatorname{Im} z)i$

(c) $z\overline{z} = |z|^2$

(d) $\overline{w+z} = \overline{w} + \overline{z}$ and $\overline{wz} = \overline{w}\,\overline{z}$

(e) $\overline{\overline{z}} = z$

(f) $|\operatorname{Re} z| \le |z|$ and $|\operatorname{Im} z| \le |z|$

(g) $|\overline{z}| = |z|$

(h) $|wz| = |w|\,|z|$

The results above are the parts of 4.4 that were left to the reader.

2 Prove that if $w, z \in \mathbf{C}$, then $\big||w| - |z|\big| \le |w - z|$.

*The inequality above is called the **reverse triangle inequality**.*