3C Matrices

Representing a Linear Map by a Matrix

We know that if $v_1, \dots, v_n$ is a basis of V and $T\colon V \to W$ is linear, then the values of $Tv_1, \dots, Tv_n$ determine the values of T on arbitrary vectors in V—see the linear map lemma (3.4). As we will soon see, matrices provide an efficient method of recording the values of the Tv_k's in terms of a basis of W.

3.29 definition: *matrix, $A_{j,k}$*

Suppose m and n are nonnegative integers. An *m-by-n matrix A* is a rectangular array of elements of $\mathbf{F}$ with m rows and n columns:

$$A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}.$$

The notation $A_{j,k}$ denotes the entry in row j, column k of A.

3.30 example: *$A_{j,k}$ equals entry in row j, column k of A*

Suppose $A = \begin{pmatrix} 8 & 4 & 5-3i \\ 1 & 9 & 7 \end{pmatrix}$.

Thus $A_{2,3}$ refers to the entry in the second row, third column of A, which means that $A_{2,3} = 7$.

When dealing with matrices, the first index refers to the row number; the second index refers to the column number.

Now we come to the key definition in this section.

3.31 definition: *matrix of a linear map, $\mathcal{M}(T)$*

Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \dots, v_n$ is a basis of V and $w_1, \dots, w_m$ is a basis of W. The *matrix of T* with respect to these bases is the m-by-n matrix $\mathcal{M}(T)$ whose entries $A_{j,k}$ are defined by

$$Tv_k = A_{1,k}w_1 + \cdots + A_{m,k}w_m.$$

If the bases $v_1, \dots, v_n$ and $w_1, \dots, w_m$ are not clear from the context, then the notation $\mathcal{M}\big(T, (v_1, \dots, v_n), (w_1, \dots, w_m)\big)$ is used.

The matrix $\mathcal{M}(T)$ of a linear map $T \in \mathcal{L}(V, W)$ depends on the basis $v_1, \dots, v_n$ of V and the basis $w_1, \dots, w_m$ of W, as well as on T. However, the bases should be clear from the context, and thus they are often not included in the notation.

To remember how $\mathcal{M}(T)$ is constructed from T, you might write across the top of the matrix the basis vectors $v_1, \dots, v_n$ for the domain and along the left the basis vectors $w_1, \dots, w_m$ for the vector space into which T maps, as follows:

$$
\mathcal{M}(T) = \begin{array}{c} \\ w_1 \\ \vdots \\ w_m \end{array} \begin{array}{ccccc} v_1 & \cdots & v_k & \cdots & v_n \\ \end{array}
$$

$$
\mathcal{M}(T) = \begin{array}{c|ccccc} & v_1 & \cdots & v_k & \cdots & v_n \\ \hline w_1 & & & A_{1,k} & & \\ \vdots & & & \vdots & & \\ w_m & & & A_{m,k} & & \end{array}.
$$

In the matrix above only the k^{th} column is shown. Thus the second index of each displayed entry of the matrix above is k. The picture above should remind you that Tv_k can be computed from $\mathcal{M}(T)$ by multiplying each entry in the k^{th} column by the corresponding w_j from the left column, and then adding up the resulting vectors.

The k^{th} column of $\mathcal{M}(T)$ consists of the scalars needed to write Tv_k as a linear combination of $w_1, \dots, w_m$:

$$Tv_k = \sum_{j=1}^{m} A_{j,k} w_j.$$

If T is a linear map from $\mathbf{F}^n$ to $\mathbf{F}^m$, then unless stated otherwise, assume the bases in question are the standard ones (where the k^{th} basis vector is 1 in the k^{th} slot and 0 in all other slots). If you think of elements of $\mathbf{F}^m$ as columns of m numbers, then you can think of the k^{th} column of $\mathcal{M}(T)$ as T applied to the k^{th} standard basis vector.

If T is a linear map from an n-dimensional vector space to an m-dimensional vector space, then $\mathcal{M}(T)$ is an m-by-n matrix.

3.32 example: *the matrix of a linear map from $\mathbf{F}^2$ to $\mathbf{F}^3$*

Suppose $T \in \mathcal{L}(\mathbf{F}^2, \mathbf{F}^3)$ is defined by

$$T(x, y) = (x + 3y, 2x + 5y, 7x + 9y).$$

Because $T(1, 0) = (1, 2, 7)$ and $T(0, 1) = (3, 5, 9)$, the matrix of T with respect to the standard bases is the 3-by-2 matrix below:

$$\mathcal{M}(T) = \begin{pmatrix} 1 & 3 \\ 2 & 5 \\ 7 & 9 \end{pmatrix}.$$

When working with $\mathcal{P}_m(\mathbf{F})$, use the standard basis $1, x, x^2, \dots, x^m$ unless the context indicates otherwise.

3.33 example: *matrix of the differentiation map from $\mathcal{P}_3(\mathbf{R})$ to $\mathcal{P}_2(\mathbf{R})$*

Suppose $D \in \mathcal{L}(\mathcal{P}_3(\mathbf{R}), \mathcal{P}_2(\mathbf{R}))$ is the differentiation map defined by $Dp = p'$. Because $(x^n)' = nx^{n-1}$, the matrix of D with respect to the standard bases is the 3-by-4 matrix below:

$$\mathcal{M}(D) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Addition and Scalar Multiplication of Matrices

For the rest of this section, assume that U, V, and W are finite-dimensional and that a basis has been chosen for each of these vector spaces. Thus for each linear map from V to W, we can talk about its matrix (with respect to the chosen bases).

Is the matrix of the sum of two linear maps equal to the sum of the matrices of the two maps? Right now this question does not yet make sense because although we have defined the sum of two linear maps, we have not defined the sum of two matrices. Fortunately, the natural definition of the sum of two matrices has the right properties. Specifically, we make the following definition.

3.34 definition: ***matrix addition***

The *sum of two matrices of the same size* is the matrix obtained by adding corresponding entries in the matrices:

$$\begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} + \begin{pmatrix} C_{1,1} & \cdots & C_{1,n} \\ \vdots & & \vdots \\ C_{m,1} & \cdots & C_{m,n} \end{pmatrix}$$
$$= \begin{pmatrix} A_{1,1}+C_{1,1} & \cdots & A_{1,n}+C_{1,n} \\ \vdots & & \vdots \\ A_{m,1}+C_{m,1} & \cdots & A_{m,n}+C_{m,n} \end{pmatrix}.$$

In the next result, the assumption is that the same bases are used for all three linear maps $S+T$, S, and T.

3.35 ***matrix of the sum of linear maps***

Suppose $S, T \in \mathcal{L}(V, W)$. Then $\mathcal{M}(S+T) = \mathcal{M}(S) + \mathcal{M}(T)$.

The verification of the result above follows from the definitions and is left to the reader.

Still assuming that we have some bases in mind, is the matrix of a scalar times a linear map equal to the scalar times the matrix of the linear map? Again, the question does not yet make sense because we have not defined scalar multiplication on matrices. Fortunately, the natural definition again has the right properties.

3.36 definition: ***scalar multiplication of a matrix***

The product of a scalar and a matrix is the matrix obtained by multiplying each entry in the matrix by the scalar:

$$\lambda \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} = \begin{pmatrix} \lambda A_{1,1} & \cdots & \lambda A_{1,n} \\ \vdots & & \vdots \\ \lambda A_{m,1} & \cdots & \lambda A_{m,n} \end{pmatrix}.$$

3.37 example: *addition and scalar multiplication of matrices*

$$2\begin{pmatrix} 3 & 1 \\ -1 & 5 \end{pmatrix} + \begin{pmatrix} 4 & 2 \\ 1 & 6 \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ -2 & 10 \end{pmatrix} + \begin{pmatrix} 4 & 2 \\ 1 & 6 \end{pmatrix} = \begin{pmatrix} 10 & 4 \\ -1 & 16 \end{pmatrix}$$

In the next result, the assumption is that the same bases are used for both the linear maps λT and T.

3.38 *the matrix of a scalar times a linear map*

Suppose $\lambda \in \mathbf{F}$ and $T \in \mathcal{L}(V, W)$. Then $\mathcal{M}(\lambda T) = \lambda \mathcal{M}(T)$.

The verification of the result above is also left to the reader.

Because addition and scalar multiplication have now been defined for matrices, you should not be surprised that a vector space is about to appear. First we introduce a bit of notation so that this new vector space has a name, and then we find the dimension of this new vector space.

3.39 notation: $\mathbf{F}^{m,n}$

For m and n positive integers, the set of all m-by-n matrices with entries in $\mathbf{F}$ is denoted by $\mathbf{F}^{m,n}$.

3.40 $\dim \mathbf{F}^{m,n} = mn$

Suppose m and n are positive integers. With addition and scalar multiplication defined as above, $\mathbf{F}^{m,n}$ is a vector space of dimension mn.

Proof The verification that $\mathbf{F}^{m,n}$ is a vector space is left to the reader. Note that the additive identity of $\mathbf{F}^{m,n}$ is the m-by-n matrix all of whose entries equal 0.

The reader should also verify that the list of distinct m-by-n matrices that have 0 in all entries except for a 1 in one entry is a basis of $\mathbf{F}^{m,n}$. There are mn such matrices, so the dimension of $\mathbf{F}^{m,n}$ equals mn. ■

Matrix Multiplication

Suppose, as previously, that $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ is a basis of W. Suppose also that $u_1, \ldots, u_p$ is a basis of U.

Consider linear maps $T: U \to V$ and $S: V \to W$. The composition ST is a linear map from U to W. Does $\mathcal{M}(ST)$ equal $\mathcal{M}(S)\mathcal{M}(T)$? This question does not yet make sense because we have not defined the product of two matrices. We will choose a definition of matrix multiplication that forces this question to have a positive answer. Let's see how to do this.

Suppose $\mathcal{M}(S) = A$ and $\mathcal{M}(T) = B$. For $1 \leq k \leq p$, we have

$$\begin{aligned}(ST)u_k &= S\Bigl(\sum_{r=1}^{n} B_{r,k}v_r\Bigr)\\ &= \sum_{r=1}^{n} B_{r,k}Sv_r\\ &= \sum_{r=1}^{n} B_{r,k}\sum_{j=1}^{m} A_{j,r}w_j\\ &= \sum_{j=1}^{m}\Bigl(\sum_{r=1}^{n} A_{j,r}B_{r,k}\Bigr)w_j.\end{aligned}$$

Thus $\mathcal{M}(ST)$ is the m-by-p matrix whose entry in row j, column k, equals

$$\sum_{r=1}^{n} A_{j,r}B_{r,k}.$$

Now we see how to define matrix multiplication so that the desired equation $\mathcal{M}(ST) = \mathcal{M}(S)\mathcal{M}(T)$ holds.

3.41 definition: *matrix multiplication*

Suppose A is an m-by-n matrix and B is an n-by-p matrix. Then AB is defined to be the m-by-p matrix whose entry in row j, column k, is given by the equation

$$(AB)_{j,k} = \sum_{r=1}^{n} A_{j,r}B_{r,k}.$$

Thus the entry in row j, column k, of AB is computed by taking row j of A and column k of B, multiplying together corresponding entries, and then summing.

Note that we define the product of two matrices only when the number of columns of the first matrix equals the number of rows of the second matrix.

You may have learned this definition of matrix multiplication in an earlier course, although you may not have seen this motivation for it.

3.42 example: *matrix multiplication*

Here we multiply together a 3-by-2 matrix and a 2-by-4 matrix, obtaining a 3-by-4 matrix:

$$\begin{pmatrix} 1 & 2\\ 3 & 4\\ 5 & 6\end{pmatrix}\begin{pmatrix} 6 & 5 & 4 & 3\\ 2 & 1 & 0 & -1\end{pmatrix} = \begin{pmatrix} 10 & 7 & 4 & 1\\ 26 & 19 & 12 & 5\\ 42 & 31 & 20 & 9\end{pmatrix}.$$

Matrix multiplication is not commutative—AB is not necessarily equal to BA even if both products are defined (see Exercise 10). Matrix multiplication is distributive and associative (see Exercises 11 and 12).

In the next result, we assume that the same basis of V is used in considering $T \in \mathcal{L}(U,V)$ and $S \in \mathcal{L}(V,W)$, the same basis of W is used in considering $S \in \mathcal{L}(V,W)$ and $ST \in \mathcal{L}(U,W)$, and the same basis of U is used in considering $T \in \mathcal{L}(U,V)$ and $ST \in \mathcal{L}(U,W)$.

3.43 *matrix of product of linear maps*

If $T \in \mathcal{L}(U,V)$ and $S \in \mathcal{L}(V,W)$, then $\mathcal{M}(ST) = \mathcal{M}(S)\mathcal{M}(T)$.

The proof of the result above is the calculation that was done as motivation before the definition of matrix multiplication.

In the next piece of notation, note that as usual the first index refers to a row and the second index refers to a column, with a vertically centered dot used as a placeholder.

3.44 notation: $A_{j,\cdot}$, $A_{\cdot,k}$

Suppose A is an m-by-n matrix.

- If $1 \le j \le m$, then $A_{j,\cdot}$ denotes the 1-by-n matrix consisting of row j of A.
- If $1 \le k \le n$, then $A_{\cdot,k}$ denotes the m-by-1 matrix consisting of column k of A.

3.45 example: *$A_{j,\cdot}$ equals j^{th} row of A and $A_{\cdot,k}$ equals k^{th} column of A*

The notation $A_{2,\cdot}$ denotes the second row of A and $A_{\cdot,2}$ denotes the second column of A. Thus if $A = \begin{pmatrix} 8 & 4 & 5 \\ 1 & 9 & 7 \end{pmatrix}$, then

$$A_{2,\cdot} = \begin{pmatrix} 1 & 9 & 7 \end{pmatrix} \quad \text{and} \quad A_{\cdot,2} = \begin{pmatrix} 4 \\ 9 \end{pmatrix}.$$

The product of a 1-by-n matrix and an n-by-1 matrix is a 1-by-1 matrix. However, we will frequently identify a 1-by-1 matrix with its entry. For example,

$$\begin{pmatrix} 3 & 4 \end{pmatrix}\begin{pmatrix} 6 \\ 2 \end{pmatrix} = \begin{pmatrix} 26 \end{pmatrix}$$

because $3 \cdot 6 + 4 \cdot 2 = 26$. However, we can identify $\begin{pmatrix} 26 \end{pmatrix}$ with 26, writing $\begin{pmatrix} 3 & 4 \end{pmatrix}\begin{pmatrix} 6 \\ 2 \end{pmatrix} = 26$.

The next result uses the convention discussed in the paragraph above to give another way to think of matrix multiplication. For example, the next result and the calculation in the paragraph above explain why the entry in row 2, column 1, of the product in Example 3.42 equals 26.

3.46 *entry of matrix product equals row times column*

Suppose A is an m-by-n matrix and B is an n-by-p matrix. Then

$$(AB)_{j,k} = A_{j,\cdot}\, B_{\cdot,k}$$

if $1 \le j \le m$ and $1 \le k \le p$. In other words, the entry in row j, column k, of AB equals (row j of A) times (column k of B).

Proof Suppose $1 \le j \le m$ and $1 \le k \le p$. The definition of matrix multiplication states that

$$(AB)_{j,k} = A_{j,1}B_{1,k} + \cdots + A_{j,n}B_{n,k}. \tag{3.47}$$

The definition of matrix multiplication also implies that the product of the 1-by-n matrix $A_{j,\cdot}$ and the n-by-1 matrix $B_{\cdot,k}$ is the 1-by-1 matrix whose entry is the number on the right side of the equation above. Thus the entry in row j, column k, of AB equals (row j of A) times (column k of B). ■

The next result gives yet another way to think of matrix multiplication. In the result below, $(AB)_{\cdot,k}$ is column k of the m-by-p matrix AB. Thus $(AB)_{\cdot,k}$ is an m-by-1 matrix. Also, $AB_{\cdot,k}$ is an m-by-1 matrix because it is the product of an m-by-n matrix and an n-by-1 matrix. Thus the two sides of the equation in the result below have the same size, making it reasonable that they might be equal.

3.48 *column of matrix product equals matrix times column*

Suppose A is an m-by-n matrix and B is an n-by-p matrix. Then

$$(AB)_{\cdot,k} = AB_{\cdot,k}$$

if $1 \le k \le p$. In other words, column k of AB equals A times column k of B.

Proof As discussed above, $(AB)_{\cdot,k}$ and $AB_{\cdot,k}$ are both m-by-1 matrices. If $1 \le j \le m$, then the entry in row j of $(AB)_{\cdot,k}$ is the left side of 3.47 and the entry in row j of $AB_{\cdot,k}$ is the right side of 3.47. Thus $(AB)_{\cdot,k} = AB_{\cdot,k}$. ■

Our next result will give another way of thinking about the product of an m-by-n matrix and an n-by-1 matrix, motivated by the next example.

3.49 example: *product of a* 3-*by*-2 *matrix and a* 2-*by*-1 *matrix*

Use our definitions and basic arithmetic to verify that

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}\begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 19 \\ 31 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} + 1\begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}.$$

Thus in this example, the product of a 3-by-2 matrix and a 2-by-1 matrix is a linear combination of the columns of the 3-by-2 matrix, with the scalars (5 and 1) that multiply the columns coming from the 2-by-1 matrix.

The next result generalizes the example above.

3.50 *linear combination of columns*

Suppose A is an m-by-n matrix and $b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$ is an n-by-1 matrix. Then

$$Ab = b_1 A_{\cdot,1} + \cdots + b_n A_{\cdot,n}.$$

In other words, Ab is a linear combination of the columns of A, with the scalars that multiply the columns coming from b.

Proof If $k \in \{1, \ldots, m\}$, then the definition of matrix multiplication implies that the entry in row k of the m-by-1 matrix Ab is

$$A_{k,1} b_1 + \cdots + A_{k,n} b_n.$$

The entry in row k of $b_1 A_{\cdot,1} + \cdots + b_n A_{\cdot,n}$ also equals the number displayed above.

Because Ab and $b_1 A_{\cdot,1} + \cdots + b_n A_{\cdot,n}$ have the same entry in row k for each $k \in \{1, \ldots, m\}$, we conclude that $Ab = b_1 A_{\cdot,1} + \cdots + b_n A_{\cdot,n}$. ■

Our two previous results focus on the columns of a matrix. Analogous results hold for the rows of a matrix. Specifically, see Exercises 8 and 9, which can be proved using appropriate modifications of the proofs of 3.48 and 3.50.

The next result is the main tool used in the next subsection to prove the column–row factorization (3.56) and to prove that the column rank of a matrix equals the row rank (3.57). To be consistent with the notation often used with the column–row factorization, including in the next subsection, the matrices in the next result are called C and R instead of A and B.

3.51 *matrix multiplication as linear combinations of columns or rows*

Suppose C is an m-by-c matrix and R is a c-by-n matrix.

(a) If $k \in \{1, \ldots, n\}$, then column k of CR is a linear combination of the columns of C, with the coefficients of this linear combination coming from column k of R.

(b) If $j \in \{1, \ldots, m\}$, then row j of CR is a linear combination of the rows of R, with the coefficients of this linear combination coming from row j of C.

Proof Suppose $k \in \{1, \ldots, n\}$. Then column k of CR equals $CR_{\cdot,k}$ (by 3.48), which equals the linear combination of the columns of C with coefficients coming from $R_{\cdot,k}$ (by 3.50). Thus (a) holds.

To prove (b), follow the pattern of the proof of (a) but use rows instead of columns and use Exercises 8 and 9 instead of 3.48 and 3.50. ■

Column–Row Factorization and Rank of a Matrix

We begin by defining two nonnegative integers associated with each matrix.

3.52 definition: *column rank, row rank*

Suppose A is an m-by-n matrix with entries in $\mathbf{F}$.

- The *column rank* of A is the dimension of the span of the columns of A in $\mathbf{F}^{m,1}$.
- The *row rank* of A is the dimension of the span of the rows of A in $\mathbf{F}^{1,n}$.

If A is an m-by-n matrix, then the column rank of A is at most n (because A has n columns) and the column rank of A is also at most m (because $\dim \mathbf{F}^{m,1} = m$). Similarly, the row rank of A is also at most $\min\{m, n\}$.

3.53 example: *column rank and row rank of a* 2-*by*-4 *matrix*

Suppose

$$A = \begin{pmatrix} 4 & 7 & 1 & 8 \\ 3 & 5 & 2 & 9 \end{pmatrix}.$$

The column rank of A is the dimension of

$$\operatorname{span}\left(\begin{pmatrix} 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 7 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 8 \\ 9 \end{pmatrix}\right)$$

in $\mathbf{F}^{2,1}$. Neither of the first two vectors listed above in $\mathbf{F}^{2,1}$ is a scalar multiple of the other. Thus the span of this list of length four has dimension at least two. The span of this list of vectors in $\mathbf{F}^{2,1}$ cannot have dimension larger than two because $\dim \mathbf{F}^{2,1} = 2$. Thus the span of this list has dimension two, which means that the column rank of A is two.

The row rank of A is the dimension of

$$\operatorname{span}\left(\begin{pmatrix} 4 & 7 & 1 & 8 \end{pmatrix}, \begin{pmatrix} 3 & 5 & 2 & 9 \end{pmatrix}\right)$$

in $\mathbf{F}^{1,4}$. Neither of the two vectors listed above in $\mathbf{F}^{1,4}$ is a scalar multiple of the other. Thus the span of this list of length two has dimension two, which means that the row rank of A is two.

We now define the transpose of a matrix.

3.54 definition: *transpose,* A^{t}

The *transpose* of a matrix A, denoted by A^{t}, is the matrix obtained from A by interchanging rows and columns. Specifically, if A is an m-by-n matrix, then A^{t} is the n-by-m matrix whose entries are given by the equation

$$(A^{\mathrm{t}})_{k,j} = A_{j,k}.$$

3.55 example: *transpose of a matrix*

If $A = \begin{pmatrix} 5 & -7 \\ 3 & 8 \\ -4 & 2 \end{pmatrix}$, then $A^{\mathrm{t}} = \begin{pmatrix} 5 & 3 & -4 \\ -7 & 8 & 2 \end{pmatrix}$.

Note that here A is a 3-by-2 matrix and A^{t} is a 2-by-3 matrix.

The transpose has nice algebraic properties: $(A+B)^{\mathrm{t}} = A^{\mathrm{t}} + B^{\mathrm{t}}$, $(\lambda A)^{\mathrm{t}} = \lambda A^{\mathrm{t}}$, and $(AC)^{\mathrm{t}} = C^{\mathrm{t}} A^{\mathrm{t}}$ for all m-by-n matrices A, B, all $\lambda \in \mathbf{F}$, and all n-by-p matrices C (see Exercises 14 and 15).

The next result will be the main tool used to prove that the column rank equals the row rank (see 3.57).

3.56 *column–row factorization*

Suppose A is an m-by-n matrix with entries in $\mathbf{F}$ and column rank $c \geq 1$. Then there exist an m-by-c matrix C and a c-by-n matrix R, both with entries in $\mathbf{F}$, such that $A = CR$.

Proof Each column of A is an m-by-1 matrix. The list $A_{\cdot,1}, \dots, A_{\cdot,n}$ of columns of A can be reduced to a basis of the span of the columns of A (by 2.30). This basis has length c, by the definition of the column rank. The c columns in this basis can be put together to form an m-by-c matrix C.

If $k \in \{1, \dots, n\}$, then column k of A is a linear combination of the columns of C. Make the coefficients of this linear combination into column k of a c-by-n matrix that we call R. Then $A = CR$, as follows from 3.51(a). ■

In Example 3.53, the column rank and row rank turned out to equal each other. The next result states that this happens for all matrices.

3.57 *column rank equals row rank*

Suppose $A \in \mathbf{F}^{m,n}$. Then the column rank of A equals the row rank of A.

Proof Let c denote the column rank of A. Let $A = CR$ be the column–row factorization of A given by 3.56, where C is an m-by-c matrix and R is a c-by-n matrix. Then 3.51(b) tells us that every row of A is a linear combination of the rows of R. Because R has c rows, this implies that the row rank of A is less than or equal to the column rank c of A.

To prove the inequality in the other direction, apply the result in the previous paragraph to A^{t}, getting

$$\begin{aligned} \text{column rank of } A &= \text{row rank of } A^{\mathrm{t}} \\ &\leq \text{column rank of } A^{\mathrm{t}} \\ &= \text{row rank of } A. \end{aligned}$$

Thus the column rank of A equals the row rank of A. ■

Because the column rank equals the row rank, the last result allows us to dispense with the terms "column rank" and "row rank" and just use the simpler term "rank".

3.58 definition: *rank*

The *rank* of a matrix $A \in \mathbf{F}^{m,n}$ is the column rank of A.

See 3.133 and Exercise 8 in Section 7A for alternative proofs that the column rank equals the row rank.

Exercises 3C

1 Suppose $T \in \mathcal{L}(V, W)$. Show that with respect to each choice of bases of V and W, the matrix of T has at least $\dim \operatorname{range} T$ nonzero entries.

2 Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that $\dim \operatorname{range} T = 1$ if and only if there exist a basis of V and a basis of W such that with respect to these bases, all entries of $\mathcal{M}(T)$ equal 1.

3 Suppose $v_1, \dots, v_n$ is a basis of V and $w_1, \dots, w_m$ is a basis of W.

(a) Show that if $S, T \in \mathcal{L}(V, W)$, then $\mathcal{M}(S + T) = \mathcal{M}(S) + \mathcal{M}(T)$.

(b) Show that if $\lambda \in \mathbf{F}$ and $T \in \mathcal{L}(V, W)$, then $\mathcal{M}(\lambda T) = \lambda \mathcal{M}(T)$.

This exercise asks you to verify 3.35 and 3.38.

4 Suppose that $D \in \mathcal{L}\big(\mathcal{P}_3(\mathbf{R}), \mathcal{P}_2(\mathbf{R})\big)$ is the differentiation map defined by $Dp = p'$. Find a basis of $\mathcal{P}_3(\mathbf{R})$ and a basis of $\mathcal{P}_2(\mathbf{R})$ such that the matrix of D with respect to these bases is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Compare with Example 3.33. The next exercise generalizes this exercise.

5 Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that there exist a basis of V and a basis of W such that with respect to these bases, all entries of $\mathcal{M}(T)$ are 0 except that the entries in row k, column k, equal 1 if $1 \le k \le \dim \operatorname{range} T$.

6 Suppose $v_1, \dots, v_m$ is a basis of V and W is finite-dimensional. Suppose $T \in \mathcal{L}(V, W)$. Prove that there exists a basis $w_1, \dots, w_n$ of W such that all entries in the first column of $\mathcal{M}(T)$ [with respect to the bases $v_1, \dots, v_m$ and $w_1, \dots, w_n$] are 0 except for possibly a 1 in the first row, first column.

In this exercise, unlike Exercise 5, you are given the basis of V instead of being able to choose a basis of V.

7 Suppose $w_1, \dots, w_n$ is a basis of W and V is finite-dimensional. Suppose $T \in \mathcal{L}(V, W)$. Prove that there exists a basis $v_1, \dots, v_m$ of V such that all entries in the first row of $\mathcal{M}(T)$ [with respect to the bases $v_1, \dots, v_m$ and $w_1, \dots, w_n$] are 0 except for possibly a 1 in the first row, first column.

In this exercise, unlike Exercise 5, you are given the basis of W instead of being able to choose a basis of W.

8 Suppose A is an m-by-n matrix and B is an n-by-p matrix. Prove that

$$(AB)_{j,\cdot} = A_{j,\cdot}\, B$$

for each $1 \leq j \leq m$. In other words, show that row j of AB equals (row j of A) times B.

This exercise gives the row version of 3.48.

9 Suppose $a = \begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix}$ is a 1-by-n matrix and B is an n-by-p matrix. Prove that

$$aB = a_1 B_{1,\cdot} + \cdots + a_n B_{n,\cdot}\,.$$

In other words, show that aB is a linear combination of the rows of B, with the scalars that multiply the rows coming from a.

This exercise gives the row version of 3.50.

10 Give an example of 2-by-2 matrices A and B such that $AB \neq BA$.

11 Prove that the distributive property holds for matrix addition and matrix multiplication. In other words, suppose A, B, C, D, E, and F are matrices whose sizes are such that $A(B+C)$ and $(D+E)F$ make sense. Explain why $AB + AC$ and $DF + EF$ both make sense and prove that

$$A(B+C) = AB + AC \quad \text{and} \quad (D+E)F = DF + EF.$$

12 Prove that matrix multiplication is associative. In other words, suppose A, B, and C are matrices whose sizes are such that $(AB)C$ makes sense. Explain why $A(BC)$ makes sense and prove that

$$(AB)C = A(BC).$$

Try to find a clean proof that illustrates the following quote from Emil Artin: "It is my experience that proofs involving matrices can be shortened by 50% if one throws the matrices out."

13 Suppose A is an n-by-n matrix and $1 \leq j, k \leq n$. Show that the entry in row j, column k, of A^3 (which is defined to mean AAA) is

$$\sum_{p=1}^{n} \sum_{r=1}^{n} A_{j,p} A_{p,r} A_{r,k}.$$

14 Suppose m and n are positive integers. Prove that the function $A \mapsto A^{\mathrm{t}}$ is a linear map from $\mathbf{F}^{m,n}$ to $\mathbf{F}^{n,m}$.

15 Prove that if A is an m-by-n matrix and C is an n-by-p matrix, then

$$(AC)^{\mathrm{t}} = C^{\mathrm{t}}A^{\mathrm{t}}.$$

This exercise shows that the transpose of the product of two matrices is the product of the transposes in the opposite order.

16 Suppose A is an m-by-n matrix with $A \neq 0$. Prove that the rank of A is 1 if and only if there exist $(c_1, \ldots, c_m) \in \mathbf{F}^m$ and $(d_1, \ldots, d_n) \in \mathbf{F}^n$ such that $A_{j,k} = c_j d_k$ for every $j = 1, \ldots, m$ and every $k = 1, \ldots, n$.

17 Suppose $T \in \mathcal{L}(V)$, and $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are bases of V. Prove that the following are equivalent.

(a) T is injective.

(b) The columns of $\mathcal{M}(T)$ are linearly independent in $\mathbf{F}^{n,1}$.

(c) The columns of $\mathcal{M}(T)$ span $\mathbf{F}^{n,1}$.

(d) The rows of $\mathcal{M}(T)$ span $\mathbf{F}^{1,n}$.

(e) The rows of $\mathcal{M}(T)$ are linearly independent in $\mathbf{F}^{1,n}$.

Here $\mathcal{M}(T)$ means $\mathcal{M}\big(T, (u_1, \ldots, u_n), (v_1, \ldots, v_n)\big)$.