## 1C Subspaces

By considering subspaces, we can greatly expand our examples of vector spaces.

**1.33 definition: *subspace***

A subset $U$ of $V$ is called a *subspace* of $V$ if $U$ is also a vector space with the same additive identity, addition, and scalar multiplication as on $V$.

The next result gives the easiest way to check whether a subset of a vector space is a subspace.

*Some people use the terminology **linear subspace**, which means the same as subspace.*

**1.34 *conditions for a subspace***

A subset $U$ of $V$ is a subspace of $V$ if and only if $U$ satisfies the following three conditions.

**additive identity**
$0 \in U$.

**closed under addition**
$u, w \in U$ implies $u + w \in U$.

**closed under scalar multiplication**
$a \in \mathbf{F}$ and $u \in U$ implies $au \in U$.

Proof If $U$ is a subspace of $V$, then $U$ satisfies the three conditions above by the definition of vector space.

Conversely, suppose $U$ satisfies the three conditions above. The first condition ensures that the additive identity of $V$ is in $U$. The second condition ensures that addition makes sense on $U$. The third condition ensures that scalar multiplication makes sense on $U$.

*The additive identity condition above could be replaced with the condition that $U$ is nonempty (because then taking $u \in U$ and multiplying it by $0$ would imply that $0 \in U$). However, if a subset $U$ of $V$ is indeed a subspace, then usually the quickest way to show that $U$ is nonempty is to show that $0 \in U$.*

If $u \in U$, then $-u$ [which equals $(-1)u$ by 1.32] is also in $U$ by the third condition above. Hence every element of $U$ has an additive inverse in $U$.

The other parts of the definition of a vector space, such as associativity and commutativity, are automatically satisfied for $U$ because they hold on the larger space $V$. Thus $U$ is a vector space and hence is a subspace of $V$. ■

The three conditions in the result above usually enable us to determine quickly whether a given subset of $V$ is a subspace of $V$. You should verify all assertions in the next example.

**1.35 example:** *subspaces*

(a) If $b \in \mathbf{F}$, then
$$\{(x_1, x_2, x_3, x_4) \in \mathbf{F}^4 : x_3 = 5x_4 + b\}$$
is a subspace of $\mathbf{F}^4$ if and only if $b = 0$.

(b) The set of continuous real-valued functions on the interval $[0, 1]$ is a subspace of $\mathbf{R}^{[0,1]}$.

(c) The set of differentiable real-valued functions on $\mathbf{R}$ is a subspace of $\mathbf{R}^{\mathbf{R}}$.

(d) The set of differentiable real-valued functions $f$ on the interval $(0, 3)$ such that $f'(2) = b$ is a subspace of $\mathbf{R}^{(0,3)}$ if and only if $b = 0$.

(e) The set of all sequences of complex numbers with limit 0 is a subspace of $\mathbf{C}^\infty$.

Verifying some of the items above shows the linear structure underlying parts of calculus. For example, (b) above requires the result that the sum of two continuous functions is continuous. As another example, (d) above requires the result that for a constant $c$, the derivative of $cf$ equals $c$ times the derivative of $f$.

*The set $\{0\}$ is the smallest subspace of $V$, and $V$ itself is the largest subspace of $V$. The empty set is not a subspace of $V$ because a subspace must be a vector space and hence must contain at least one element, namely, an additive identity.*

The subspaces of $\mathbf{R}^2$ are precisely $\{0\}$, all lines in $\mathbf{R}^2$ containing the origin, and $\mathbf{R}^2$. The subspaces of $\mathbf{R}^3$ are precisely $\{0\}$, all lines in $\mathbf{R}^3$ containing the origin, all planes in $\mathbf{R}^3$ containing the origin, and $\mathbf{R}^3$. To prove that all these objects are indeed subspaces is straightforward—the hard part is to show that they are the only subspaces of $\mathbf{R}^2$ and $\mathbf{R}^3$. That task will be easier after we introduce some additional tools in the next chapter.

## *Sums of Subspaces*

When dealing with vector spaces, we are usually interested only in subspaces, as opposed to arbitrary subsets. The notion of the sum of subspaces will be useful.

*The union of subspaces is rarely a subspace (see Exercise 12), which is why we usually work with sums rather than unions.*

**1.36 definition:** *sum of subspaces*

Suppose $V_1, \dots, V_m$ are subspaces of $V$. The *sum* of $V_1, \dots, V_m$, denoted by $V_1 + \cdots + V_m$, is the set of all possible sums of elements of $V_1, \dots, V_m$. More precisely,
$$V_1 + \cdots + V_m = \{v_1 + \cdots + v_m : v_1 \in V_1, \dots, v_m \in V_m\}.$$

Let's look at some examples of sums of subspaces.

**1.37 example:** *a sum of subspaces of* $\mathbf{F}^3$

Suppose $U$ is the set of all elements of $\mathbf{F}^3$ whose second and third coordinates equal 0, and $W$ is the set of all elements of $\mathbf{F}^3$ whose first and third coordinates equal 0:

$$U = \{(x, 0, 0) \in \mathbf{F}^3 : x \in \mathbf{F}\} \quad \text{and} \quad W = \{(0, y, 0) \in \mathbf{F}^3 : y \in \mathbf{F}\}.$$

Then

$$U + W = \{(x, y, 0) \in \mathbf{F}^3 : x, y \in \mathbf{F}\},$$

as you should verify.

**1.38 example:** *a sum of subspaces of* $\mathbf{F}^4$

Suppose

$$U = \{(x, x, y, y) \in \mathbf{F}^4 : x, y \in \mathbf{F}\} \quad \text{and} \quad W = \{(x, x, x, y) \in \mathbf{F}^4 : x, y \in \mathbf{F}\}.$$

Using words rather than symbols, we could say that $U$ is the set of elements of $\mathbf{F}^4$ whose first two coordinates equal each other and whose third and fourth coordinates equal each other. Similarly, $W$ is the set of elements of $\mathbf{F}^4$ whose first three coordinates equal each other.

To find a description of $U + W$, consider a typical element $(a, a, b, b)$ of $U$ and a typical element $(c, c, c, d)$ of $W$, where $a, b, c, d \in \mathbf{F}$. We have

$$(a, a, b, b) + (c, c, c, d) = (a + c, a + c, b + c, b + d),$$

which shows that every element of $U + W$ has its first two coordinates equal to each other. Thus

1.39 $$U + W \subseteq \{(x, x, y, z) \in \mathbf{F}^4 : x, y, z \in \mathbf{F}\}.$$

To prove the inclusion in the other direction, suppose $x, y, z \in \mathbf{F}$. Then

$$(x, x, y, z) = (x, x, y, y) + (0, 0, 0, z - y),$$

where the first vector on the right is in $U$ and the second vector on the right is in $W$. Thus $(x, x, y, z) \in U + W$, showing that the inclusion 1.39 also holds in the opposite direction. Hence

$$U + W = \{(x, x, y, z) \in \mathbf{F}^4 : x, y, z \in \mathbf{F}\},$$

which shows that $U + W$ is the set of elements of $\mathbf{F}^4$ whose first two coordinates equal each other.

The next result states that the sum of subspaces is a subspace, and is in fact the smallest subspace containing all the summands (which means that every subspace containing all the summands also contains the sum).

**1.40** *sum of subspaces is the smallest containing subspace*

Suppose $V_1, \dots, V_m$ are subspaces of $V$. Then $V_1 + \cdots + V_m$ is the smallest subspace of $V$ containing $V_1, \dots, V_m$.

**Proof** The reader can verify that $V_1 + \cdots + V_m$ contains the additive identity 0 and is closed under addition and scalar multiplication. Thus 1.34 implies that $V_1 + \cdots + V_m$ is a subspace of $V$.

The subspaces $V_1, \dots, V_m$ are all contained in $V_1 + \cdots + V_m$ (to see this, consider sums $v_1 + \cdots + v_m$ where all except one of the $v_k$'s are 0). Conversely, every subspace of $V$ containing $V_1, \dots, V_m$ contains $V_1 + \cdots + V_m$ (because subspaces must contain all finite sums of their elements). Thus $V_1 + \cdots + V_m$ is the smallest subspace of $V$ containing $V_1, \dots, V_m$. ∎

*Sums of subspaces in the theory of vector spaces are analogous to unions of subsets in set theory. Given two subspaces of a vector space, the smallest subspace containing them is their sum. Analogously, given two subsets of a set, the smallest subset containing them is their union.*

## *Direct Sums*

Suppose $V_1, \dots, V_m$ are subspaces of $V$. Every element of $V_1 + \cdots + V_m$ can be written in the form

$$v_1 + \cdots + v_m,$$

where each $v_k \in V_k$. Of special interest are cases in which each vector in $V_1 + \cdots + V_m$ can be represented in the form above in only one way. This situation is so important that it gets a special name (direct sum) and a special symbol ($\oplus$).

**1.41** definition: *direct sum*, $\oplus$

Suppose $V_1, \dots, V_m$ are subspaces of $V$.

- The sum $V_1 + \cdots + V_m$ is called a *direct sum* if each element of $V_1 + \cdots + V_m$ can be written in only one way as a sum $v_1 + \cdots + v_m$, where each $v_k \in V_k$.
- If $V_1 + \cdots + V_m$ is a direct sum, then $V_1 \oplus \cdots \oplus V_m$ denotes $V_1 + \cdots + V_m$, with the $\oplus$ notation serving as an indication that this is a direct sum.

**1.42** example: *a direct sum of two subspaces*

Suppose $U$ is the subspace of $\mathbf{F}^3$ of those vectors whose last coordinate equals 0, and $W$ is the subspace of $\mathbf{F}^3$ of those vectors whose first two coordinates equal 0:

$$U = \{(x, y, 0) \in \mathbf{F}^3 : x, y \in \mathbf{F}\} \quad \text{and} \quad W = \{(0, 0, z) \in \mathbf{F}^3 : z \in \mathbf{F}\}.$$

Then $\mathbf{F}^3 = U \oplus W$, as you should verify.

**1.43 example:** *a direct sum of multiple subspaces*

Suppose $V_k$ is the subspace of $\mathbf{F}^n$ of those vectors whose coordinates are all 0, except possibly in the $k^{\text{th}}$ slot; for example, $V_2 = \{(0, x, 0, \dots, 0) \in \mathbf{F}^n : x \in \mathbf{F}\}$. Then

*To produce ⊕ in TEX, type* `\oplus`.

$$\mathbf{F}^n = V_1 \oplus \cdots \oplus V_n,$$

as you should verify.

---

Sometimes nonexamples add to our understanding as much as examples.

**1.44 example:** *a sum that is not a direct sum*

Suppose

$$V_1 = \{(x, y, 0) \in \mathbf{F}^3 : x, y \in \mathbf{F}\},$$

$$V_2 = \{(0, 0, z) \in \mathbf{F}^3 : z \in \mathbf{F}\},$$

$$V_3 = \{(0, y, y) \in \mathbf{F}^3 : y \in \mathbf{F}\}.$$

Then $\mathbf{F}^3 = V_1 + V_2 + V_3$ because every vector $(x, y, z) \in \mathbf{F}^3$ can be written as

$$(x, y, z) = (x, y, 0) + (0, 0, z) + (0, 0, 0),$$

where the first vector on the right side is in $V_1$, the second vector is in $V_2$, and the third vector is in $V_3$.

However, $\mathbf{F}^3$ does not equal the direct sum of $V_1, V_2, V_3$, because the vector $(0, 0, 0)$ can be written in more than one way as a sum $v_1 + v_2 + v_3$, with each $v_k \in V_k$. Specifically, we have

$$(0, 0, 0) = (0, 1, 0) + (0, 0, 1) + (0, -1, -1)$$

and, of course,

$$(0, 0, 0) = (0, 0, 0) + (0, 0, 0) + (0, 0, 0),$$

where the first vector on the right side of each equation above is in $V_1$, the second vector is in $V_2$, and the third vector is in $V_3$. Thus the sum $V_1 + V_2 + V_3$ is not a direct sum.

---

The definition of direct sum requires every vector in the sum to have a unique representation as an appropriate sum. The next result shows that when deciding whether a sum of subspaces is a direct sum, we only need to consider whether 0 can be uniquely written as an appropriate sum.

*The symbol ⊕, which is a plus sign inside a circle, reminds us that we are dealing with a special type of sum of subspaces—each element in the direct sum can be represented in only one way as a sum of elements from the specified subspaces.*

**1.45** *condition for a direct sum*

Suppose $V_1, \dots, V_m$ are subspaces of $V$. Then $V_1 + \cdots + V_m$ is a direct sum if and only if the only way to write $0$ as a sum $v_1 + \cdots + v_m$, where each $v_k \in V_k$, is by taking each $v_k$ equal to $0$.

**Proof** First suppose $V_1 + \cdots + V_m$ is a direct sum. Then the definition of direct sum implies that the only way to write $0$ as a sum $v_1 + \cdots + v_m$, where each $v_k \in V_k$, is by taking each $v_k$ equal to $0$.

Now suppose that the only way to write $0$ as a sum $v_1 + \cdots + v_m$, where each $v_k \in V_k$, is by taking each $v_k$ equal to $0$. To show that $V_1 + \cdots + V_m$ is a direct sum, let $v \in V_1 + \cdots + V_m$. We can write

$$v = v_1 + \cdots + v_m$$

for some $v_1 \in V_1, \dots, v_m \in V_m$. To show that this representation is unique, suppose we also have

$$v = u_1 + \cdots + u_m,$$

where $u_1 \in V_1, \dots, u_m \in V_m$. Subtracting these two equations, we have

$$0 = (v_1 - u_1) + \cdots + (v_m - u_m).$$

Because $v_1 - u_1 \in V_1, \dots, v_m - u_m \in V_m$, the equation above implies that each $v_k - u_k$ equals $0$. Thus $v_1 = u_1, \dots, v_m = u_m$, as desired. ■

The next result gives a simple condition for testing whether a sum of two subspaces is a direct sum.

*The symbol $\Longleftrightarrow$ used below means "if and only if"; this symbol could also be read to mean "is equivalent to".*

**1.46** *direct sum of two subspaces*

Suppose $U$ and $W$ are subspaces of $V$. Then

$$U + W \text{ is a direct sum} \iff U \cap W = \{0\}.$$

**Proof** First suppose that $U + W$ is a direct sum. If $v \in U \cap W$, then $0 = v + (-v)$, where $v \in U$ and $-v \in W$. By the unique representation of $0$ as the sum of a vector in $U$ and a vector in $W$, we have $v = 0$. Thus $U \cap W = \{0\}$, completing the proof in one direction.

To prove the other direction, now suppose $U \cap W = \{0\}$. To prove that $U + W$ is a direct sum, suppose $u \in U$, $w \in W$, and

$$0 = u + w.$$

To complete the proof, we only need to show that $u = w = 0$ (by 1.45). The equation above implies that $u = -w \in W$. Thus $u \in U \cap W$. Hence $u = 0$, which by the equation above implies that $w = 0$, completing the proof. ■

The result above deals only with the case of two subspaces. When asking about a possible direct sum with more than two subspaces, it is not enough to test that each pair of the subspaces intersect only at 0. To see this, consider Example 1.44. In that nonexample of a direct sum, we have $V_1 \cap V_2 = V_1 \cap V_3 = V_2 \cap V_3 = \{0\}$.

*Sums of subspaces are analogous to unions of subsets. Similarly, direct sums of subspaces are analogous to disjoint unions of subsets. No two subspaces of a vector space can be disjoint, because both contain* 0. *So disjointness is replaced, at least in the case of two subspaces, with the requirement that the intersection equal* $\{0\}$.

## *Exercises 1C*

**1** For each of the following subsets of $\mathbf{F}^3$, determine whether it is a subspace of $\mathbf{F}^3$.

(a) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}$

(b) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 4\}$

(c) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 x_2 x_3 = 0\}$

(d) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 = 5x_3\}$

**2** Verify all assertions about subspaces in Example 1.35.

**3** Show that the set of differentiable real-valued functions $f$ on the interval $(-4, 4)$ such that $f'(-1) = 3f(2)$ is a subspace of $\mathbf{R}^{(-4,4)}$.

**4** Suppose $b \in \mathbf{R}$. Show that the set of continuous real-valued functions $f$ on the interval $[0, 1]$ such that $\int_0^1 f = b$ is a subspace of $\mathbf{R}^{[0,1]}$ if and only if $b = 0$.

**5** Is $\mathbf{R}^2$ a subspace of the complex vector space $\mathbf{C}^2$?

**6** (a) Is $\{(a, b, c) \in \mathbf{R}^3 : a^3 = b^3\}$ a subspace of $\mathbf{R}^3$?

(b) Is $\{(a, b, c) \in \mathbf{C}^3 : a^3 = b^3\}$ a subspace of $\mathbf{C}^3$?

**7** Prove or give a counterexample: If $U$ is a nonempty subset of $\mathbf{R}^2$ such that $U$ is closed under addition and under taking additive inverses (meaning $-u \in U$ whenever $u \in U$), then $U$ is a subspace of $\mathbf{R}^2$.

**8** Give an example of a nonempty subset $U$ of $\mathbf{R}^2$ such that $U$ is closed under scalar multiplication, but $U$ is not a subspace of $\mathbf{R}^2$.

**9** A function $f\colon \mathbf{R} \to \mathbf{R}$ is called *periodic* if there exists a positive number $p$ such that $f(x) = f(x + p)$ for all $x \in \mathbf{R}$. Is the set of periodic functions from $\mathbf{R}$ to $\mathbf{R}$ a subspace of $\mathbf{R}^{\mathbf{R}}$? Explain.

**10** Suppose $V_1$ and $V_2$ are subspaces of $V$. Prove that the intersection $V_1 \cap V_2$ is a subspace of $V$.

**11** Prove that the intersection of every collection of subspaces of $V$ is a subspace of $V$.

**12** Prove that the union of two subspaces of $V$ is a subspace of $V$ if and only if one of the subspaces is contained in the other.

**13** Prove that the union of three subspaces of $V$ is a subspace of $V$ if and only if one of the subspaces contains the other two.

*This exercise is surprisingly harder than Exercise 12, possibly because this exercise is not true if we replace* $\mathbf{F}$ *with a field containing only two elements.*

**14** Suppose

$$U = \{(x, -x, 2x) \in \mathbf{F}^3 : x \in \mathbf{F}\} \quad \text{and} \quad W = \{(x, x, 2x) \in \mathbf{F}^3 : x \in \mathbf{F}\}.$$

Describe $U + W$ using symbols, and also give a description of $U + W$ that uses no symbols.

**15** Suppose $U$ is a subspace of $V$. What is $U + U$?

**16** Is the operation of addition on the subspaces of $V$ commutative? In other words, if $U$ and $W$ are subspaces of $V$, is $U + W = W + U$?

**17** Is the operation of addition on the subspaces of $V$ associative? In other words, if $V_1, V_2, V_3$ are subspaces of $V$, is

$$(V_1 + V_2) + V_3 = V_1 + (V_2 + V_3)?$$

**18** Does the operation of addition on the subspaces of $V$ have an additive identity? Which subspaces have additive inverses?

**19** Prove or give a counterexample: If $V_1, V_2, U$ are subspaces of $V$ such that

$$V_1 + U = V_2 + U,$$

then $V_1 = V_2$.

**20** Suppose

$$U = \{(x, x, y, y) \in \mathbf{F}^4 : x, y \in \mathbf{F}\}.$$

Find a subspace $W$ of $\mathbf{F}^4$ such that $\mathbf{F}^4 = U \oplus W$.

**21** Suppose

$$U = \{(x, y, x + y, x - y, 2x) \in \mathbf{F}^5 : x, y \in \mathbf{F}\}.$$

Find a subspace $W$ of $\mathbf{F}^5$ such that $\mathbf{F}^5 = U \oplus W$.

**22** Suppose

$$U = \{(x, y, x + y, x - y, 2x) \in \mathbf{F}^5 : x, y \in \mathbf{F}\}.$$

Find three subspaces $W_1, W_2, W_3$ of $\mathbf{F}^5$, none of which equals $\{0\}$, such that $\mathbf{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$.

**23** Prove or give a counterexample: If $V_1, V_2, U$ are subspaces of $V$ such that

$$V = V_1 \oplus U \quad \text{and} \quad V = V_2 \oplus U,$$

then $V_1 = V_2$.

*Hint: When trying to discover whether a conjecture in linear algebra is true or false, it is often useful to start by experimenting in* $\mathbf{F}^2$.

**24** A function $f\colon \mathbf{R} \to \mathbf{R}$ is called *even* if

$$f(-x) = f(x)$$

for all $x \in \mathbf{R}$. A function $f\colon \mathbf{R} \to \mathbf{R}$ is called *odd* if

$$f(-x) = -f(x)$$

for all $x \in \mathbf{R}$. Let $V_{\mathrm{e}}$ denote the set of real-valued even functions on $\mathbf{R}$ and let $V_{\mathrm{o}}$ denote the set of real-valued odd functions on $\mathbf{R}$. Show that $\mathbf{R}^{\mathbf{R}} = V_{\mathrm{e}} \oplus V_{\mathrm{o}}$.