Chapter 8

Operators on Complex Vector Spaces

In this chapter we delve deeper into the structure of operators, with most of the attention on complex vector spaces. Some of the results in this chapter apply to both real and complex vector spaces; thus we do not make a standing assumption that $\mathbf{F} = \mathbf{C}$. Also, an inner product does not help with this material, so we return to the general setting of a finite-dimensional vector space.

Even on a finite-dimensional complex vector space, an operator may not have enough eigenvectors to form a basis of the vector space. Thus we will consider the closely related objects called generalized eigenvectors. We will see that for each operator on a finite-dimensional complex vector space, there is a basis of the vector space consisting of generalized eigenvectors of the operator. The generalized eigenspace decomposition then provides a good description of arbitrary operators on a finite-dimensional complex vector space.

Nilpotent operators, which are operators that when raised to some power equal 0, have an important role in these investigations. Nilpotent operators provide a key tool in our proof that every invertible operator on a finite-dimensional complex vector space has a square root and in our approach to Jordan form.

This chapter concludes by defining the trace and proving its key properties.

standing assumptions for this chapter

- $\mathbf{F}$ denotes $\mathbf{R}$ or $\mathbf{C}$.
- V denotes a finite-dimensional nonzero vector space over $\mathbf{F}$.

The Long Room of the Old Library at the University of Dublin, where William Hamilton (1805–1865) was a student and then a faculty member. Hamilton proved a special case of what we now call the Cayley–Hamilton theorem in 1853.

8A *Generalized Eigenvectors and Nilpotent Operators*

Null Spaces of Powers of an Operator

We begin this chapter with a study of null spaces of powers of an operator.

8.1 *sequence of increasing null spaces*

Suppose $T \in \mathcal{L}(V)$. Then

$$\{0\} = \operatorname{null} T^0 \subseteq \operatorname{null} T^1 \subseteq \cdots \subseteq \operatorname{null} T^k \subseteq \operatorname{null} T^{k+1} \subseteq \cdots.$$

Proof Suppose k is a nonnegative integer and $v \in \operatorname{null} T^k$. Then $T^k v = 0$, which implies that $T^{k+1} v = T(T^k v) = T(0) = 0$. Thus $v \in \operatorname{null} T^{k+1}$. Hence $\operatorname{null} T^k \subseteq \operatorname{null} T^{k+1}$, as desired. ■

The following result states that if two consecutive terms in the sequence of subspaces above are equal, then all later terms in the sequence are equal.

For similar results about decreasing sequences of ranges, see Exercises 6, 7, and 8.

8.2 *equality in the sequence of null spaces*

Suppose $T \in \mathcal{L}(V)$ and m is a nonnegative integer such that

$$\operatorname{null} T^m = \operatorname{null} T^{m+1}.$$

Then

$$\operatorname{null} T^m = \operatorname{null} T^{m+1} = \operatorname{null} T^{m+2} = \operatorname{null} T^{m+3} = \cdots.$$

Proof Let k be a positive integer. We want to prove that

$$\operatorname{null} T^{m+k} = \operatorname{null} T^{m+k+1}.$$

We already know from 8.1 that $\operatorname{null} T^{m+k} \subseteq \operatorname{null} T^{m+k+1}$.

To prove the inclusion in the other direction, suppose $v \in \operatorname{null} T^{m+k+1}$. Then

$$T^{m+1}(T^k v) = T^{m+k+1} v = 0.$$

Hence

$$T^k v \in \operatorname{null} T^{m+1} = \operatorname{null} T^m.$$

Thus $T^{m+k} v = T^m(T^k v) = 0$, which means that $v \in \operatorname{null} T^{m+k}$. This implies that $\operatorname{null} T^{m+k+1} \subseteq \operatorname{null} T^{m+k}$, completing the proof. ■

The result above raises the question of whether there exists a nonnegative integer m such that $\operatorname{null} T^m = \operatorname{null} T^{m+1}$. The next result shows that this equality holds at least when m equals the dimension of the vector space on which T operates.

8.3 *null spaces stop growing*

Suppose $T \in \mathcal{L}(V)$. Then

$$\operatorname{null} T^{\dim V} = \operatorname{null} T^{\dim V+1} = \operatorname{null} T^{\dim V+2} = \cdots.$$

Proof We only need to prove that $\operatorname{null} T^{\dim V} = \operatorname{null} T^{\dim V+1}$ (by 8.2). Suppose this is not true. Then, by 8.1 and 8.2, we have

$$\{0\} = \operatorname{null} T^0 \subsetneq \operatorname{null} T^1 \subsetneq \cdots \subsetneq \operatorname{null} T^{\dim V} \subsetneq \operatorname{null} T^{\dim V+1},$$

where the symbol $\subsetneq$ means "contained in but not equal to". At each of the strict inclusions in the chain above, the dimension increases by at least 1. Thus $\dim \operatorname{null} T^{\dim V+1} \geq \dim V + 1$, a contradiction because a subspace of V cannot have a larger dimension than $\dim V$. ■

It is not true that $V = \operatorname{null} T \oplus \operatorname{range} T$ for every $T \in \mathcal{L}(V)$. However, the next result can be a useful substitute.

8.4 *V is the direct sum of* $\operatorname{null} T^{\dim V}$ *and* $\operatorname{range} T^{\dim V}$

Suppose $T \in \mathcal{L}(V)$. Then

$$V = \operatorname{null} T^{\dim V} \oplus \operatorname{range} T^{\dim V}.$$

Proof Let $n = \dim V$. First we show that

$$(\operatorname{null} T^n) \cap (\operatorname{range} T^n) = \{0\}. \tag{8.5}$$

Suppose $v \in (\operatorname{null} T^n) \cap (\operatorname{range} T^n)$. Then $T^n v = 0$, and there exists $u \in V$ such that $v = T^n u$. Applying T^n to both sides of the last equation shows that $T^n v = T^{2n} u$. Hence $T^{2n} u = 0$, which implies that $T^n u = 0$ (by 8.3). Thus $v = T^n u = 0$, completing the proof of 8.5.

Now 8.5 implies that $\operatorname{null} T^n + \operatorname{range} T^n$ is a direct sum (by 1.46). Also,

$$\dim(\operatorname{null} T^n \oplus \operatorname{range} T^n) = \dim \operatorname{null} T^n + \dim \operatorname{range} T^n = \dim V,$$

where the first equality above comes from 3.94 and the second equality comes from the fundamental theorem of linear maps (3.21). The equation above implies that $\operatorname{null} T^n \oplus \operatorname{range} T^n = V$ (see 2.39), as desired. ■

For an improvement of the result above, see Exercise 19.

8.6 example: $\mathbf{F}^3 = \operatorname{null} T^3 \oplus \operatorname{range} T^3$ *for* $T \in \mathcal{L}(\mathbf{F}^3)$

Suppose $T \in \mathcal{L}(\mathbf{F}^3)$ is defined by

$$T(z_1, z_2, z_3) = (4z_2, 0, 5z_3).$$

Then null $T = \{(z_1, 0, 0) : z_1 \in \mathbf{F}\}$ and range $T = \{(z_1, 0, z_3) : z_1, z_3 \in \mathbf{F}\}$. Thus null $T \cap$ range $T \neq \{0\}$. Hence null $T +$ range T is not a direct sum. Also note that null $T +$ range $T \neq \mathbf{F}^3$. However, we have $T^3(z_1, z_2, z_3) = (0, 0, 125z_3)$. Thus we see that

$$\operatorname{null} T^3 = \{(z_1, z_2, 0) : z_1, z_2 \in \mathbf{F}\} \quad \text{and} \quad \operatorname{range} T^3 = \{(0, 0, z_3) : z_3 \in \mathbf{F}\}.$$

Hence $\mathbf{F}^3 = \operatorname{null} T^3 \oplus \operatorname{range} T^3$, as expected by 8.4.

Generalized Eigenvectors

Some operators do not have enough eigenvectors to lead to good descriptions of their behavior. Thus in this subsection we introduce the concept of generalized eigenvectors, which will play a major role in our description of the structure of an operator.

To understand why we need more than eigenvectors, let's examine the question of describing an operator by decomposing its domain into invariant subspaces. Fix $T \in \mathcal{L}(V)$. We seek to describe T by finding a "nice" direct sum decomposition

$$V = V_1 \oplus \cdots \oplus V_n,$$

where each V_k is a subspace of V invariant under T. The simplest possible nonzero invariant subspaces are one-dimensional. A decomposition as above in which each V_k is a one-dimensional subspace of V invariant under T is possible if and only if V has a basis consisting of eigenvectors of T (see 5.55). This happens if and only if V has an eigenspace decomposition

8.7
$$V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T),$$

where $\lambda_1, \dots, \lambda_m$ are the distinct eigenvalues of T (see 5.55).

The spectral theorem in the previous chapter shows that if V is an inner product space, then a decomposition of the form 8.7 holds for every self-adjoint operator if $\mathbf{F} = \mathbf{R}$ and for every normal operator if $\mathbf{F} = \mathbf{C}$ because operators of those types have enough eigenvectors to form a basis of V (see 7.29 and 7.31).

However, a decomposition of the form 8.7 may not hold for more general operators, even on a complex vector space. An example was given by the operator in 5.57, which does not have enough eigenvectors for 8.7 to hold. Generalized eigenvectors and generalized eigenspaces, which we now introduce, will remedy this situation.

8.8 definition: *generalized eigenvector*

Suppose $T \in \mathcal{L}(V)$ and λ is an eigenvalue of T. A vector $v \in V$ is called a *generalized eigenvector* of T corresponding to λ if $v \neq 0$ and

$$(T - \lambda I)^k v = 0$$

for some positive integer k.

A nonzero vector $v \in V$ is a generalized eigenvector of T corresponding to λ if and only if

$$(T - \lambda I)^{\dim V} v = 0,$$

as follows from applying 8.1 and 8.3 to the operator $T - \lambda I$.

Generalized eigenvalues are not defined because doing so would not lead to anything new. Reason: if $(T - \lambda I)^k$ is not injective for some positive integer k, then $T - \lambda I$ is not injective, and hence λ is an eigenvalue of T.

As we know, an operator on a complex vector space may not have enough eigenvectors to form a basis of the domain. The next result shows that on a complex vector space there are enough generalized eigenvectors to do this.

8.9 *a basis of generalized eigenvectors*

Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$. Then there is a basis of V consisting of generalized eigenvectors of T.

Proof Let $n = \dim V$. We will use induction on n. To get started, note that the desired result holds if $n = 1$ because then every nonzero vector in V is an eigenvector of T.

Now suppose $n > 1$ and the desired result holds for all smaller values of $\dim V$. Let λ be an eigenvalue of T. Applying 8.4 to $T - \lambda I$ shows that

This step is where we use the hypothesis that $\mathbf{F} = \mathbf{C}$, because if $\mathbf{F} = \mathbf{R}$ then T may not have any eigenvalues.

$$V = \operatorname{null}(T - \lambda I)^n \oplus \operatorname{range}(T - \lambda I)^n.$$

If $\operatorname{null}(T - \lambda I)^n = V$, then every nonzero vector in V is a generalized eigenvector of T, and thus in this case there is a basis of V consisting of generalized eigenvectors of T. Hence we can assume that $\operatorname{null}(T - \lambda I)^n \neq V$, which implies that $\operatorname{range}(T - \lambda I)^n \neq \{0\}$.

Also, $\operatorname{null}(T - \lambda I)^n \neq \{0\}$, because λ is an eigenvalue of T. Thus we have

$$0 < \dim \operatorname{range}(T - \lambda I)^n < n.$$

Furthermore, $\operatorname{range}(T - \lambda I)^n$ is invariant under T [by 5.18 with $p(z) = (z - \lambda)^n$]. Let $S \in \mathcal{L}\big(\operatorname{range}(T - \lambda I)^n\big)$ equal T restricted to $\operatorname{range}(T - \lambda I)^n$. Our induction hypothesis applied to the operator S implies that there is a basis of $\operatorname{range}(T - \lambda I)^n$ consisting of generalized eigenvectors of S, which of course are generalized eigenvectors of T. Adjoining that basis of $\operatorname{range}(T-\lambda I)^n$ to a basis of $\operatorname{null}(T-\lambda I)^n$ gives a basis of V consisting of generalized eigenvectors of T. ■

If $\mathbf{F} = \mathbf{R}$ and $\dim V > 1$, then some operators on V have the property that there exists a basis of V consisting of generalized eigenvectors of the operator, and (unlike what happens when $\mathbf{F} = \mathbf{C}$) other operators do not have this property. See Exercise 11 for a necessary and sufficient condition that determines whether an operator has this property.

8.10 example: *generalized eigenvectors of an operator on* $\mathbf{C}^3$

Define $T \in \mathcal{L}(\mathbf{C}^3)$ by

$$T(z_1, z_2, z_3) = (4z_2, 0, 5z_3)$$

for each $(z_1, z_2, z_3) \in \mathbf{C}^3$. A routine use of the definition of eigenvalue shows that the eigenvalues of T are 0 and 5. Furthermore, the eigenvectors corresponding to the eigenvalue 0 are the nonzero vectors of the form $(z_1, 0, 0)$, and the eigenvectors corresponding to the eigenvalue 5 are the nonzero vectors of the form $(0, 0, z_3)$. Hence this operator does not have enough eigenvectors to span its domain $\mathbf{C}^3$.

We compute that $T^3(z_1, z_2, z_3) = (0, 0, 125z_3)$. Thus 8.1 and 8.3 imply that the generalized eigenvectors of T corresponding to the eigenvalue 0 are the nonzero vectors of the form $(z_1, z_2, 0)$.

We also have $(T - 5I)^3(z_1, z_2, z_3) = (-125z_1 + 300z_2, -125z_2, 0)$. Thus the generalized eigenvectors of T corresponding to the eigenvalue 5 are the nonzero vectors of the form $(0, 0, z_3)$.

The paragraphs above show that each of the standard basis vectors of $\mathbf{C}^3$ is a generalized eigenvector of T. Thus $\mathbf{C}^3$ indeed has a basis consisting of generalized eigenvectors of T, as promised by 8.9.

If v is an eigenvector of $T \in \mathcal{L}(V)$, then the corresponding eigenvalue λ is uniquely determined by the equation $Tv = \lambda v$, which can be satisfied by only one $\lambda \in \mathbf{F}$ (because $v \neq 0$). However, if v is a generalized eigenvector of T, then it is not obvious that the equation $(T - \lambda I)^{\dim V} v = 0$ can be satisfied by only one $\lambda \in \mathbf{F}$. Fortunately, the next result tells us that all is well on this issue.

8.11 *generalized eigenvector corresponds to a unique eigenvalue*

Suppose $T \in \mathcal{L}(V)$. Then each generalized eigenvector of T corresponds to only one eigenvalue of T.

Proof Suppose $v \in V$ is a generalized eigenvector of T corresponding to eigenvalues α and λ of T. Let m be the smallest positive integer such that $(T - \alpha I)^m v = 0$. Let $n = \dim V$. Then

$$\begin{aligned} 0 &= (T - \lambda I)^n v \\ &= \big((T - \alpha I) + (\alpha - \lambda) I\big)^n v \\ &= \sum_{k=0}^{n} b_k (\alpha - \lambda)^{n-k} (T - \alpha I)^k v, \end{aligned}$$

where $b_0 = 1$ and the values of the other binomial coefficients b_k do not matter. Apply the operator $(T - \alpha I)^{m-1}$ to both sides of the equation above, getting

$$0 = (\alpha - \lambda)^n (T - \alpha I)^{m-1} v.$$

Because $(T - \alpha I)^{m-1} v \neq 0$, the equation above implies that $\alpha = \lambda$, as desired. ■

We saw earlier (5.11) that eigenvectors corresponding to distinct eigenvalues are linearly independent. Now we prove a similar result for generalized eigenvectors, with a proof that roughly follows the pattern of the proof of that earlier result.

8.12 *linearly independent generalized eigenvectors*

Suppose that $T \in \mathcal{L}(V)$. Then every list of generalized eigenvectors of T corresponding to distinct eigenvalues of T is linearly independent.

Proof Suppose the desired result is false. Then there exists a smallest positive integer m such that there exists a linearly dependent list $v_1, \dots, v_m$ of generalized eigenvectors of T corresponding to distinct eigenvalues $\lambda_1, \dots, \lambda_m$ of T (note that $m \geq 2$ because a generalized eigenvector is, by definition, nonzero). Thus there exist $a_1, \dots, a_m \in \mathbf{F}$, none of which are 0 (because of the minimality of m), such that

$$a_1 v_1 + \cdots + a_m v_m = 0.$$

Let $n = \dim V$. Apply $(T - \lambda_m I)^n$ to both sides of the equation above, getting

8.13
$$a_1 (T - \lambda_m I)^n v_1 + \cdots + a_{m-1} (T - \lambda_m I)^n v_{m-1} = 0.$$

Suppose $k \in \{1, \dots, m-1\}$. Then

$$(T - \lambda_m I)^n v_k \neq 0$$

because otherwise v_k would be a generalized eigenvector of T corresponding to the distinct eigenvalues λ_k and λ_m, which would contradict 8.11. However,

$$(T - \lambda_k I)^n \bigl((T - \lambda_m I)^n v_k \bigr) = (T - \lambda_m I)^n \bigl((T - \lambda_k I)^n v_k \bigr) = 0.$$

Thus the last two displayed equations show that $(T - \lambda_m I)^n v_k$ is a generalized eigenvector of T corresponding to the eigenvalue λ_k. Hence

$$(T - \lambda_m I)^n v_1, \dots, (T - \lambda_m I)^n v_{m-1}$$

is a linearly dependent list (by 8.13) of $m-1$ generalized eigenvectors corresponding to distinct eigenvalues, contradicting the minimality of m. This contradiction completes the proof. ■

Nilpotent Operators

8.14 definition: *nilpotent*

An operator is called *nilpotent* if some power of it equals 0.

Thus an operator $T \in \mathcal{L}(V)$ is nilpotent if and only if every nonzero vector in V is a generalized eigenvector of T corresponding to the eigenvalue 0.

8.15 example: *nilpotent operators*

(a) The operator $T \in \mathcal{L}(\mathbf{F}^4)$ defined by

$$T(z_1, z_2, z_3, z_4) = (0, 0, z_1, z_2)$$

is nilpotent because $T^2 = 0$.

(b) The operator on $\mathbf{F}^3$ whose matrix (with respect to the standard basis) is

$$\begin{pmatrix} -3 & 9 & 0 \\ -7 & 9 & 6 \\ 4 & 0 & -6 \end{pmatrix}$$

is nilpotent, as can be shown by cubing the matrix above to get the zero matrix.

(c) The operator of differentiation on $\mathcal{P}_m(\mathbf{R})$ is nilpotent because the $(m+1)^{\text{th}}$ derivative of every polynomial of degree at most m equals 0. Note that on this space of dimension $m+1$, we need to raise the nilpotent operator to the power $m+1$ to get the 0 operator.

The next result shows that when raising a nilpotent operator to a power, we never need to use a power higher than the dimension of the space. For a slightly stronger result, see Exercise 18.

The Latin word ***nil*** *means nothing or zero; the Latin word* ***potens*** *means having power. Thus* ***nilpotent*** *literally means having a power that is zero.*

8.16 *nilpotent operator raised to dimension of domain is* 0

Suppose $T \in \mathcal{L}(V)$ is nilpotent. Then $T^{\dim V} = 0$.

Proof Because T is nilpotent, there exists a positive integer k such that $T^k = 0$. Thus $\operatorname{null} T^k = V$. Now 8.1 and 8.3 imply that $\operatorname{null} T^{\dim V} = V$. Thus $T^{\dim V} = 0$. ■

8.17 *eigenvalues of nilpotent operator*

Suppose $T \in \mathcal{L}(V)$.

(a) If T is nilpotent, then 0 is an eigenvalue of T and T has no other eigenvalues.

(b) If $\mathbf{F} = \mathbf{C}$ and 0 is the only eigenvalue of T, then T is nilpotent.

Proof

(a) To prove (a), suppose T is nilpotent. Hence there is a positive integer m such that $T^m = 0$. This implies that T is not injective. Thus 0 is an eigenvalue of T.

To show that T has no other eigenvalues, suppose λ is an eigenvalue of T. Then there exists a nonzero vector $v \in V$ such that

$$\lambda v = Tv.$$

Repeatedly applying T to both sides of this equation shows that

$$\lambda^m v = T^m v = 0.$$

Thus $\lambda = 0$, as desired.

(b) Suppose $\mathbf{F} = \mathbf{C}$ and 0 is the only eigenvalue of T. By 5.27(b), the minimal polynomial of T equals z^m for some positive integer m. Thus $T^m = 0$. Hence T is nilpotent. ■

Exercise 23 shows that the hypothesis that $\mathbf{F} = \mathbf{C}$ cannot be deleted in (b) of the result above.

Given an operator on V, we want to find a basis of V such that the matrix of the operator with respect to this basis is as simple as possible, meaning that the matrix contains many 0's. The next result shows that if T is nilpotent, then we can choose a basis of V such that the matrix of T with respect to this basis has more than half of its entries equal to 0. Later in this chapter we will do even better.

8.18 *minimal polynomial and upper-triangular matrix of nilpotent operator*

Suppose $T \in \mathcal{L}(V)$. Then the following are equivalent.

(a) T is nilpotent.

(b) The minimal polynomial of T is z^m for some positive integer m.

(c) There is a basis of V with respect to which the matrix of T has the form

$$\begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix},$$

where all entries on and below the diagonal equal 0.

Proof Suppose (a) holds, so T is nilpotent. Thus there exists a positive integer n such that $T^n = 0$. Now 5.29 implies that z^n is a polynomial multiple of the minimal polynomial of T. Thus the minimal polynomial of T is z^m for some positive integer m, proving that (a) implies (b).

Now suppose (b) holds, so the minimal polynomial of T is z^m for some positive integer m. This implies, by 5.27(a), that 0 (which is the only zero of z^m) is the only eigenvalue of T. This further implies, by 5.44, that there is a basis of V with respect to which the matrix of T is upper triangular. This also implies, by 5.41, that all entries on the diagonal of this matrix are 0, proving that (b) implies (c).

Now suppose (c) holds. Then 5.40 implies that $T^{\dim V} = 0$. Thus T is nilpotent, proving that (c) implies (a). ■

Exercises 8A

1 Suppose $T \in \mathcal{L}(V)$. Prove that if $\dim \operatorname{null} T^4 = 8$ and $\dim \operatorname{null} T^6 = 9$, then $\dim \operatorname{null} T^m = 9$ for all integers $m \geq 5$.

2 Suppose $T \in \mathcal{L}(V)$, m is a positive integer, $v \in V$, and $T^{m-1}v \neq 0$ but $T^m v = 0$. Prove that $v, Tv, T^2 v, \dots, T^{m-1}v$ is linearly independent.

The result in this exercise is used in the proof of 8.45.

3 Suppose $T \in \mathcal{L}(V)$. Prove that

$$V = \operatorname{null} T \oplus \operatorname{range} T \iff \operatorname{null} T^2 = \operatorname{null} T.$$

4 Suppose $T \in \mathcal{L}(V)$, $\lambda \in \mathbf{F}$, and m is a positive integer such that the minimal polynomial of T is a polynomial multiple of $(z - \lambda)^m$. Prove that

$$\dim \operatorname{null}(T - \lambda I)^m \geq m.$$

5 Suppose $T \in \mathcal{L}(V)$ and m is a positive integer. Prove that

$$\dim \operatorname{null} T^m \leq m \dim \operatorname{null} T.$$

Hint: Exercise 21 in Section 3B may be useful.

6 Suppose $T \in \mathcal{L}(V)$. Show that

$$V = \operatorname{range} T^0 \supseteq \operatorname{range} T^1 \supseteq \cdots \supseteq \operatorname{range} T^k \supseteq \operatorname{range} T^{k+1} \supseteq \cdots.$$

7 Suppose $T \in \mathcal{L}(V)$ and m is a nonnegative integer such that

$$\operatorname{range} T^m = \operatorname{range} T^{m+1}.$$

Prove that $\operatorname{range} T^k = \operatorname{range} T^m$ for all $k > m$.

8 Suppose $T \in \mathcal{L}(V)$. Prove that

$$\operatorname{range} T^{\dim V} = \operatorname{range} T^{\dim V + 1} = \operatorname{range} T^{\dim V + 2} = \cdots.$$

9 Suppose $T \in \mathcal{L}(V)$ and m is a nonnegative integer. Prove that

$$\operatorname{null} T^m = \operatorname{null} T^{m+1} \iff \operatorname{range} T^m = \operatorname{range} T^{m+1}.$$

10 Define $T \in \mathcal{L}(\mathbf{C}^2)$ by $T(w, z) = (z, 0)$. Find all generalized eigenvectors of T.

11 Suppose that $T \in \mathcal{L}(V)$. Prove that there is a basis of V consisting of generalized eigenvectors of T if and only if the minimal polynomial of T equals $(z - \lambda_1) \cdots (z - \lambda_m)$ for some $\lambda_1, \dots, \lambda_m \in \mathbf{F}$.

Assume $\mathbf{F} = \mathbf{R}$ because the case $\mathbf{F} = \mathbf{C}$ follows from 5.27(b) and 8.9.

This exercise states that the condition for there to be a basis of V consisting of generalized eigenvectors of T is the same as the condition for there to be a basis with respect to which T has an upper-triangular matrix (see 5.44).

Caution: *If T has an upper-triangular matrix with respect to a basis $v_1, \dots, v_n$ of V, then v_1 is an eigenvector of T but it is not necessarily true that $v_2, \dots, v_n$ are generalized eigenvectors of T.*

12 Suppose $T \in \mathcal{L}(V)$ is such that every vector in V is a generalized eigenvector of T. Prove that there exists $\lambda \in \mathbf{F}$ such that $T - \lambda I$ is nilpotent.

13 Suppose $S, T \in \mathcal{L}(V)$ and ST is nilpotent. Prove that TS is nilpotent.

14 Suppose $T \in \mathcal{L}(V)$ is nilpotent and $T \neq 0$. Prove T is not diagonalizable.

15 Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$. Prove that T is diagonalizable if and only if every generalized eigenvector of T is an eigenvector of T.

For $\mathbf{F} = \mathbf{C}$, *this exercise adds another equivalence to the list of conditions for diagonalizability in 5.55.*

16 (a) Give an example of nilpotent operators S, T on the same vector space such that neither $S + T$ nor ST is nilpotent.
(b) Suppose $S, T \in \mathcal{L}(V)$ are nilpotent and $ST = TS$. Prove that $S + T$ and ST are nilpotent.

17 Suppose $T \in \mathcal{L}(V)$ is nilpotent and m is a positive integer such that $T^m = 0$.
(a) Prove that $I - T$ is invertible and that $(I - T)^{-1} = I + T + \cdots + T^{m-1}$.
(b) Explain how you would guess the formula above.

18 Suppose $T \in \mathcal{L}(V)$ is nilpotent. Prove that $T^{1 + \dim \operatorname{range} T} = 0$.

If $\dim \operatorname{range} T < \dim V - 1$, *then this exercise improves 8.16.*

19 Suppose $T \in \mathcal{L}(V)$ is not nilpotent. Show that

$$V = \operatorname{null} T^{\dim V - 1} \oplus \operatorname{range} T^{\dim V - 1}.$$

For operators that are not nilpotent, this exercise improves 8.4.

20 Suppose V is an inner product space and $T \in \mathcal{L}(V)$ is normal and nilpotent. Prove that $T = 0$.

21 Suppose $T \in \mathcal{L}(V)$ is such that $\operatorname{null} T^{\dim V - 1} \neq \operatorname{null} T^{\dim V}$. Prove that T is nilpotent and that $\dim \operatorname{null} T^k = k$ for every integer k with $0 \leq k \leq \dim V$.

22 Suppose $T \in \mathcal{L}(\mathbf{C}^5)$ is such that $\operatorname{range} T^4 \neq \operatorname{range} T^5$. Prove that T is nilpotent.

23 Give an example of an operator T on a finite-dimensional real vector space such that 0 is the only eigenvalue of T but T is not nilpotent.

This exercise shows that the implication (*b*) $\Longrightarrow$ (*a*) *in 8.17 does not hold without the hypothesis that* $\mathbf{F} = \mathbf{C}$.

24 For each item in Example 8.15, find a basis of the domain vector space such that the matrix of the nilpotent operator with respect to that basis has the upper-triangular form promised by 8.18(c).

25 Suppose that V is an inner product space and $T \in \mathcal{L}(V)$ is nilpotent. Show that there is an orthonormal basis of V with respect to which the matrix of T has the upper-triangular form promised by 8.18(c).