5B The Minimal Polynomial

Existence of Eigenvalues on Complex Vector Spaces

Now we come to one of the central results about operators on finite-dimensional complex vector spaces.

5.19 *existence of eigenvalues*

Every operator on a finite-dimensional nonzero complex vector space has an eigenvalue.

Proof Suppose V is a finite-dimensional complex vector space of dimension $n > 0$ and $T \in \mathcal{L}(V)$. Choose $v \in V$ with $v \neq 0$. Then

$$v, Tv, T^2v, \ldots, T^nv$$

is not linearly independent, because V has dimension n and this list has length $n + 1$. Hence some linear combination (with not all the coefficients equal to 0) of the vectors above equals 0. Thus there exists a nonconstant polynomial p of smallest degree such that

$$p(T)v = 0.$$

By the first version of the fundamental theorem of algebra (see 4.12), there exists $\lambda \in \mathbf{C}$ such that $p(\lambda) = 0$. Hence there exists a polynomial $q \in \mathcal{P}(\mathbf{C})$ such that

$$p(z) = (z - \lambda)q(z)$$

for every $z \in \mathbf{C}$ (see 4.6). This implies (using 5.17) that

$$0 = p(T)v = (T - \lambda I)\big(q(T)v\big).$$

Because q has smaller degree than p, we know that $q(T)v \neq 0$. Thus the equation above implies that λ is an eigenvalue of T with eigenvector $q(T)v$. ■

The proof above makes crucial use of the fundamental theorem of algebra. The comment following Exercise 16 helps explain why the fundamental theorem of algebra is so tightly connected to the result above.

The hypothesis in the result above that $\mathbf{F} = \mathbf{C}$ cannot be replaced with the hypothesis that $\mathbf{F} = \mathbf{R}$, as shown by Example 5.9. The next example shows that the finite-dimensional hypothesis in the result above also cannot be deleted.

5.20 example: *an operator on a complex vector space with no eigenvalues*

Define $T \in \mathcal{L}\big(\mathcal{P}(\mathbf{C})\big)$ by $(Tp)(z) = zp(z)$. If $p \in \mathcal{P}(\mathbf{C})$ is a nonzero polynomial, then the degree of Tp is one more than the degree of p, and thus Tp cannot equal a scalar multiple of p. Hence T has no eigenvalues.

Because $\mathcal{P}(\mathbf{C})$ is infinite-dimensional, this example does not contradict the result above.

Eigenvalues and the Minimal Polynomial

In this subsection we introduce an important polynomial associated with each operator. We begin with the following definition.

5.21 definition: ***monic polynomial***

A *monic polynomial* is a polynomial whose highest-degree coefficient equals 1.

For example, the polynomial $2 + 9z^2 + z^7$ is a monic polynomial of degree 7.

5.22 ***existence, uniqueness, and degree of minimal polynomial***

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Then there is a unique monic polynomial $p \in \mathcal{P}(\mathbf{F})$ of smallest degree such that $p(T) = 0$. Furthermore, $\deg p \leq \dim V$.

Proof If $\dim V = 0$, then I is the zero operator on V and thus we take p to be the constant polynomial 1.

Now use induction on $\dim V$. Thus assume that $\dim V > 0$ and that the desired result is true for all operators on all vector spaces of smaller dimension. Let $u \in V$ be such that $u \neq 0$. The list $u, Tu, \ldots, T^{\dim V}u$ has length $1 + \dim V$ and thus is linearly dependent. By the linear dependence lemma (2.19), there is a smallest positive integer $m \leq \dim V$ such that $T^m u$ is a linear combination of $u, Tu, \ldots, T^{m-1}u$. Thus there exist scalars $c_0, c_1, c_2, \ldots, c_{m-1} \in \mathbf{F}$ such that

5.23 $$c_0 u + c_1 Tu + \cdots + c_{m-1}T^{m-1}u + T^m u = 0.$$

Define a monic polynomial $q \in \mathcal{P}_m(\mathbf{F})$ by

$$q(z) = c_0 + c_1 z + \cdots + c_{m-1}z^{m-1} + z^m.$$

Then 5.23 implies that $q(T)u = 0$.

If k is a nonnegative integer, then

$$q(T)\big(T^k u\big) = T^k\big(q(T)u\big) = T^k(0) = 0.$$

The linear dependence lemma (2.19) shows that $u, Tu, \ldots, T^{m-1}u$ is linearly independent. Thus the equation above implies that $\dim \operatorname{null} q(T) \geq m$. Hence

$$\dim \operatorname{range} q(T) = \dim V - \dim \operatorname{null} q(T) \leq \dim V - m.$$

Because $\operatorname{range} q(T)$ is invariant under T (by 5.18), we can apply our induction hypothesis to the operator $T|_{\operatorname{range} q(T)}$ on the vector space $\operatorname{range} q(T)$. Thus there is a monic polynomial $s \in \mathcal{P}(\mathbf{F})$ with

$$\deg s \leq \dim V - m \quad \text{and} \quad s(T|_{\operatorname{range} q(T)}) = 0.$$

Hence for all $v \in V$ we have

$$\big((sq)(T)\big)(v) = s(T)\big(q(T)v\big) = 0$$

because $q(T)v \in \operatorname{range} q(T)$ and $s(T)|_{\operatorname{range} q(T)} = s(T|_{\operatorname{range} q(T)}) = 0$. Thus sq is a monic polynomial such that $\deg sq \leq \dim V$ and $(sq)(T) = 0$.

The paragraph above shows that there is a monic polynomial of degree at most $\dim V$ that when applied to T gives the 0 operator. Thus there is a monic polynomial of smallest degree with this property, completing the existence part of this result.

Let $p \in \mathcal{P}(\mathbf{F})$ be a monic polynomial of smallest degree such that $p(T) = 0$. To prove the uniqueness part of the result, suppose $r \in \mathcal{P}(\mathbf{F})$ is a monic polynomial of the same degree as p and $r(T) = 0$. Then $(p - r)(T) = 0$ and also $\deg(p - r) < \deg p$. If $p - r$ were not equal to 0, then we could divide $p - r$ by the coefficient of the highest-order term in $p - r$ to get a monic polynomial (of smaller degree than p) that when applied to T gives the 0 operator. Thus $p - r = 0$, as desired. ■

The previous result justifies the following definition.

5.24 definition: ***minimal polynomial***

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Then the *minimal polynomial* of T is the unique monic polynomial $p \in \mathcal{P}(\mathbf{F})$ of smallest degree such that $p(T) = 0$.

To compute the minimal polynomial of an operator $T \in \mathcal{L}(V)$, we need to find the smallest positive integer m such that the equation

$$c_0 I + c_1 T + \cdots + c_{m-1} T^{m-1} = -T^m$$

has a solution $c_0, c_1, \ldots, c_{m-1} \in \mathbf{F}$. If we pick a basis of V and replace T in the equation above with the matrix of T, then the equation above can be thought of as a system of $(\dim V)^2$ linear equations in the m unknowns $c_0, c_1, \ldots, c_{m-1} \in \mathbf{F}$. Gaussian elimination or another fast method of solving systems of linear equations can tell us whether a solution exists, testing successive values $m = 1, 2, \ldots$ until a solution exists. By 5.22, a solution exists for some smallest positive integer $m \le \dim V$. The minimal polynomial of T is then $c_0 + c_1 z + \cdots + c_{m-1} z^{m-1} + z^m$.

Even faster (usually), pick $v \in V$ with $v \neq 0$ and consider the equation

5.25 $$c_0 v + c_1 T v + \cdots + c_{\dim V - 1} T^{\dim V - 1} v = -T^{\dim V} v.$$

Use a basis of V to convert the equation above to a system of $\dim V$ linear equations in $\dim V$ unknowns $c_0, c_1, \ldots, c_{\dim V - 1}$. If this system of equations has a unique solution $c_0, c_1, \ldots, c_{\dim V - 1}$ (as happens most of the time), then the scalars $c_0, c_1, \ldots, c_{\dim V - 1}, 1$ are the coefficients of the minimal polynomial of T (because 5.22 states that the degree of the minimal polynomial is at most $\dim V$).

These estimates are based on testing millions of random matrices.

Consider operators on $\mathbf{R}^4$ (thought of as 4-by-4 matrices with respect to the standard basis), and take $v = (1, 0, 0, 0)$ in the paragraph above. The faster method described above works on over 99.8% of the 4-by-4 matrices with integer entries in the interval $[-10, 10]$ and on over 99.999% of the 4-by-4 matrices with integer entries in $[-100, 100]$.

The next example illustrates the faster procedure discussed above.

5.26 example: *minimal polynomial of an operator on* $\mathbf{F}^5$

Suppose $T \in \mathcal{L}(\mathbf{F}^5)$ and

$$\mathcal{M}(T) = \begin{pmatrix} 0 & 0 & 0 & 0 & -3 \\ 1 & 0 & 0 & 0 & 6 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

with respect to the standard basis e_1, e_2, e_3, e_4, e_5. Taking $v = e_1$ for 5.25, we have

$$\begin{aligned} Te_1 &= e_2, & T^4e_1 &= T(T^3e_1) = Te_4 = e_5, \\ T^2e_1 &= T(Te_1) = Te_2 = e_3, & T^5e_1 &= T(T^4e_1) = Te_5 = -3e_1 + 6e_2. \\ T^3e_1 &= T(T^2e_1) = Te_3 = e_4, & & \end{aligned}$$

Thus $3e_1 - 6Te_1 = -T^5e_1$. The list $e_1, Te_1, T^2e_1, T^3e_1, T^4e_1$, which equals the list e_1, e_2, e_3, e_4, e_5, is linearly independent, so no other linear combination of this list equals $-T^5e_1$. Hence the minimal polynomial of T is $3 - 6z + z^5$.

Recall that by definition, eigenvalues of operators on V and zeros of polynomials in $\mathcal{P}(\mathbf{F})$ must be elements of $\mathbf{F}$. In particular, if $\mathbf{F} = \mathbf{R}$, then eigenvalues and zeros must be real numbers.

5.27 *eigenvalues are the zeros of the minimal polynomial*

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$.

(a) The zeros of the minimal polynomial of T are the eigenvalues of T.

(b) If V is a complex vector space, then the minimal polynomial of T has the form

$$(z - \lambda_1) \cdots (z - \lambda_m),$$

where $\lambda_1, \ldots, \lambda_m$ is a list of all eigenvalues of T, possibly with repetitions.

Proof Let p be the minimal polynomial of T.

(a) First suppose $\lambda \in \mathbf{F}$ is a zero of p. Then p can be written in the form

$$p(z) = (z - \lambda)q(z),$$

where q is a monic polynomial with coefficients in $\mathbf{F}$ (see 4.6). Because $p(T) = 0$, we have

$$0 = (T - \lambda I)(q(T)v)$$

for all $v \in V$. Because $\deg q = (\deg p) - 1$ and p is the minimal polynomial of T, there exists at least one vector $v \in V$ such that $q(T)v \neq 0$. The equation above thus implies that λ is an eigenvalue of T, as desired.

To prove that every eigenvalue of T is a zero of p, now suppose $\lambda \in \mathbf{F}$ is an eigenvalue of T. Thus there exists $v \in V$ with $v \neq 0$ such that $Tv = \lambda v$. Repeated applications of T to both sides of this equation show that $T^k v = \lambda^k v$ for every nonnegative integer k. Thus

$$p(T)v = p(\lambda)v.$$

Because p is the minimal polynomial of T, we have $p(T)v = 0$. Hence the equation above implies that $p(\lambda) = 0$. Thus λ is a zero of p, as desired.

(b) To get the desired result, use (a) and the second version of the fundamental theorem of algebra (see 4.13). ■

A nonzero polynomial has at most as many distinct zeros as its degree (see 4.8). Thus (a) of the previous result, along with the result that the minimal polynomial of an operator on V has degree at most $\dim V$, gives an alternative proof of 5.12, which states that an operator on V has at most $\dim V$ distinct eigenvalues.

Every monic polynomial is the minimal polynomial of some operator, as shown by Exercise 16, which generalizes Example 5.26. Thus 5.27(a) shows that finding exact expressions for the eigenvalues of an operator is equivalent to the problem of finding exact expressions for the zeros of a polynomial (and thus is not possible for some operators).

5.28 example: *An operator whose eigenvalues cannot be found exactly*

Let $T \in \mathcal{L}(\mathbf{C}^5)$ be the operator defined by

$$T(z_1, z_2, z_3, z_4, z_5) = (-3z_5, z_1 + 6z_5, z_2, z_3, z_4).$$

The matrix of T with respect to the standard basis of $\mathbf{C}^5$ is the 5-by-5 matrix in Example 5.26. As we showed in that example, the minimal polynomial of T is the polynomial

$$3 - 6z + z^5.$$

No zero of the polynomial above can be expressed using rational numbers, roots of rational numbers, and the usual rules of arithmetic (a proof of this would take us considerably beyond linear algebra). Because the zeros of the polynomial above are the eigenvalues of T [by 5.27(a)], we cannot find an exact expression for any eigenvalue of T in any familiar form.

Numeric techniques, which we will not discuss here, show that the zeros of the polynomial above, and thus the eigenvalues of T, are approximately the following five complex numbers:

$$-1.67, \quad 0.51, \quad 1.40, \quad -0.12 + 1.59i, \quad -0.12 - 1.59i.$$

Note that the two nonreal zeros of this polynomial are complex conjugates of each other, as we expect for a polynomial with real coefficients (see 4.14).

The next result completely characterizes the polynomials that when applied to an operator give the 0 operator.

5.29 $q(T) = 0 \iff$ *q is a polynomial multiple of the minimal polynomial*

Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $q \in \mathcal{P}(\mathbf{F})$. Then $q(T) = 0$ if and only if q is a polynomial multiple of the minimal polynomial of T.

Proof Let p denote the minimal polynomial of T.

First suppose $q(T) = 0$. By the division algorithm for polynomials (4.9), there exist polynomials $s, r \in \mathcal{P}(\mathbf{F})$ such that

5.30 $$q = ps + r$$

and $\deg r < \deg p$. We have

$$0 = q(T) = p(T)s(T) + r(T) = r(T).$$

The equation above implies that $r = 0$ (otherwise, dividing r by its highest-degree coefficient would produce a monic polynomial that when applied to T gives 0; this polynomial would have a smaller degree than the minimal polynomial, which would be a contradiction). Thus 5.30 becomes the equation $q = ps$. Hence q is a polynomial multiple of p, as desired.

To prove the other direction, now suppose q is a polynomial multiple of p. Thus there exists a polynomial $s \in \mathcal{P}(\mathbf{F})$ such that $q = ps$. We have

$$q(T) = p(T)s(T) = 0\,s(T) = 0,$$

as desired. ■

The next result is a nice consequence of the result above.

5.31 *minimal polynomial of a restriction operator*

Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and U is a subspace of V that is invariant under T. Then the minimal polynomial of T is a polynomial multiple of the minimal polynomial of $T|_U$.

Proof Suppose p is the minimal polynomial of T. Thus $p(T)v = 0$ for all $v \in V$. In particular,

$$p(T)u = 0 \text{ for all } u \in U.$$

Thus $p(T|_U) = 0$. Now 5.29, applied to the operator $T|_U$ in place of T, implies that p is a polynomial multiple of the minimal polynomial of $T|_U$. ■

See Exercise 25 for a result about quotient operators that is analogous to the result above.

The next result shows that the constant term of the minimal polynomial of an operator determines whether the operator is invertible.

5.32 *T not invertible $\Longleftrightarrow$ constant term of minimal polynomial of T is 0*

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Then T is not invertible if and only if the constant term of the minimal polynomial of T is 0.

Proof Suppose $T \in \mathcal{L}(V)$ and p is the minimal polynomial of T. Then

$$\begin{aligned} T \text{ is not invertible} &\Longleftrightarrow 0 \text{ is an eigenvalue of } T \\ &\Longleftrightarrow 0 \text{ is a zero of } p \\ &\Longleftrightarrow \text{the constant term of } p \text{ is } 0, \end{aligned}$$

where the first equivalence holds by 5.7, the second equivalence holds by 5.27(a), and the last equivalence holds because the constant term of p equals $p(0)$. ■

Eigenvalues on Odd-Dimensional Real Vector Spaces

The next result will be the key tool that we use to show that every operator on an odd-dimensional real vector space has an eigenvalue.

5.33 *even-dimensional null space*

Suppose $\mathbf{F} = \mathbf{R}$ and V is finite-dimensional. Suppose also that $T \in \mathcal{L}(V)$ and $b, c \in \mathbf{R}$ with $b^2 < 4c$. Then $\dim \operatorname{null}(T^2 + bT + cI)$ is an even number.

Proof Recall that $\operatorname{null}(T^2 + bT + cI)$ is invariant under T (by 5.18). By replacing V with $\operatorname{null}(T^2 + bT + cI)$ and replacing T with T restricted to $\operatorname{null}(T^2 + bT + cI)$, we can assume that $T^2 + bT + cI = 0$; we now need to prove that $\dim V$ is even.

Suppose $\lambda \in \mathbf{R}$ and $v \in V$ are such that $Tv = \lambda v$. Then

$$0 = (T^2 + bT + cI)v = (\lambda^2 + b\lambda + c)v = \left(\left(\lambda + \tfrac{b}{2}\right)^2 + c - \tfrac{b^2}{4}\right)v.$$

The term in large parentheses above is a positive number. Thus the equation above implies that $v = 0$. Hence we have shown that T has no eigenvectors.

Let U be a subspace of V that is invariant under T and has the largest dimension among all subspaces of V that are invariant under T and have even dimension. If $U = V$, then we are done; otherwise assume there exists $w \in V$ such that $w \notin U$.

Let $W = \operatorname{span}(w, Tw)$. Then W is invariant under T because $T(Tw) = -bTw - cw$. Furthermore, $\dim W = 2$ because otherwise w would be an eigenvector of T. Now

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W) = \dim U + 2,$$

where $U \cap W = \{0\}$ because otherwise $U \cap W$ would be a one-dimensional subspace of V that is invariant under T (impossible because T has no eigenvectors).

Because $U + W$ is invariant under T, the equation above shows that there exists a subspace of V invariant under T of even dimension larger than $\dim U$. Thus the assumption that $U \neq V$ was incorrect. Hence V has even dimension. ■

The next result states that on odd-dimensional vector spaces, every operator has an eigenvalue. We already know this result for finite-dimensional complex vectors spaces (without the odd hypothesis). Thus in the proof below, we will assume that $\mathbf{F} = \mathbf{R}$.

5.34 *operators on odd-dimensional vector spaces have eigenvalues*

Every operator on an odd-dimensional vector space has an eigenvalue.

Proof Suppose $\mathbf{F} = \mathbf{R}$ and V is finite-dimensional. Let $n = \dim V$, and suppose n is an odd number. Let $T \in \mathcal{L}(V)$. We will use induction on n in steps of size two to show that T has an eigenvalue. To get started, note that the desired result holds if $\dim V = 1$ because then every nonzero vector in V is an eigenvector of T.

Now suppose that $n \geq 3$ and the desired result holds for all operators on all odd-dimensional vector spaces of dimension less than n. Let p denote the minimal polynomial of T. If p is a polynomial multiple of $x - \lambda$ for some $\lambda \in \mathbf{R}$, then λ is an eigenvalue of T [by 5.27(a)] and we are done. Thus we can assume that there exist $b, c \in \mathbf{R}$ such that $b^2 < 4c$ and p is a polynomial multiple of $x^2 + bx + c$ (see 4.16).

There exists a monic polynomial $q \in \mathcal{P}(\mathbf{R})$ such that $p(x) = q(x)\big(x^2 + bx + c\big)$ for all $x \in \mathbf{R}$. Now

$$0 = p(T) = \big(q(T)\big)\big(T^2 + bT + cI\big),$$

which means that $q(T)$ equals 0 on $\operatorname{range}\big(T^2 + bT + cI\big)$. Because $\deg q < \deg p$ and p is the minimal polynomial of T, this implies that $\operatorname{range}\big(T^2 + bT + cI\big) \neq V$.

The fundamental theorem of linear maps (3.21) tells us that

$$\dim V = \dim \operatorname{null}\big(T^2 + bT + cI\big) + \dim \operatorname{range}\big(T^2 + bT + cI\big).$$

Because $\dim V$ is odd (by hypothesis) and $\dim \operatorname{null}\big(T^2 + bT + cI\big)$ is even (by 5.33), the equation above shows that $\dim \operatorname{range}\big(T^2 + bT + cI\big)$ is odd.

Hence $\operatorname{range}\big(T^2 + bT + cI\big)$ is a subspace of V that is invariant under T (by 5.18) and that has odd dimension less than $\dim V$. Our induction hypothesis now implies that T restricted to $\operatorname{range}\big(T^2 + bT + cI\big)$ has an eigenvalue, which means that T has an eigenvalue. ■

See Exercise 23 in Section 8B and Exercise 10 in Section 9C for alternative proofs of the result above.

Exercises 5B

1 Suppose $T \in \mathcal{L}(V)$. Prove that 9 is an eigenvalue of T^2 if and only if 3 or -3 is an eigenvalue of T.

2 Suppose V is a complex vector space and $T \in \mathcal{L}(V)$ has no eigenvalues. Prove that every subspace of V invariant under T is either $\{0\}$ or infinite-dimensional.

3 Suppose n is a positive integer and $T \in \mathcal{L}(\mathbf{F}^n)$ is defined by

$$T(x_1, \ldots, x_n) = (x_1 + \cdots + x_n, \ldots, x_1 + \cdots + x_n).$$

(a) Find all eigenvalues and eigenvectors of T.
(b) Find the minimal polynomial of T.

The matrix of T with respect to the standard basis of $\mathbf{F}^n$ consists of all 1's.

4 Suppose $\mathbf{F} = \mathbf{C}$, $T \in \mathcal{L}(V)$, $p \in \mathcal{P}(\mathbf{C})$ is a nonconstant polynomial, and $\alpha \in \mathbf{C}$. Prove that α is an eigenvalue of $p(T)$ if and only if $\alpha = p(\lambda)$ for some eigenvalue λ of T.

5 Give an example of an operator on $\mathbf{R}^2$ that shows the result in Exercise 4 does not hold if $\mathbf{C}$ is replaced with $\mathbf{R}$.

6 Suppose $T \in \mathcal{L}(\mathbf{F}^2)$ is defined by $T(w, z) = (-z, w)$. Find the minimal polynomial of T.

7 (a) Give an example of $S, T \in \mathcal{L}(\mathbf{F}^2)$ such that the minimal polynomial of ST does not equal the minimal polynomial of TS.
(b) Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that if at least one of S, T is invertible, then the minimal polynomial of ST equals the minimal polynomial of TS.

Hint: Show that if S is invertible and $p \in \mathcal{P}(\mathbf{F})$, then $p(TS) = S^{-1}p(ST)S$.

8 Suppose $T \in \mathcal{L}(\mathbf{R}^2)$ is the operator of counterclockwise rotation by $1°$. Find the minimal polynomial of T.

Because $\dim \mathbf{R}^2 = 2$, the degree of the minimal polynomial of T is at most 2. Thus the minimal polynomial of T is not the tempting polynomial $x^{180} + 1$, even though $T^{180} = -I$.

9 Suppose $T \in \mathcal{L}(V)$ is such that with respect to some basis of V, all entries of the matrix of T are rational numbers. Explain why all coefficients of the minimal polynomial of T are rational numbers.

10 Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $v \in V$. Prove that

$$\operatorname{span}(v, Tv, \ldots, T^m v) = \operatorname{span}(v, Tv, \ldots, T^{\dim V - 1} v)$$

for all integers $m \geq \dim V - 1$.

11 Suppose V is a two-dimensional vector space, $T \in \mathcal{L}(V)$, and the matrix of T with respect to some basis of V is $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$.

(a) Show that $T^2 - (a + d)T + (ad - bc)I = 0$.
(b) Show that the minimal polynomial of T equals

$$\begin{cases} z - a & \text{if } b = c = 0 \text{ and } a = d, \\ z^2 - (a + d)z + (ad - bc) & \text{otherwise.} \end{cases}$$

12 Define $T \in \mathcal{L}(\mathbf{F}^n)$ by $T(x_1, x_2, x_3, \dots, x_n) = (x_1, 2x_2, 3x_3, \dots, nx_n)$. Find the minimal polynomial of T.

13 Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $p \in \mathcal{P}(\mathbf{F})$. Prove that there exists a unique $r \in \mathcal{P}(\mathbf{F})$ such that $p(T) = r(T)$ and $\deg r$ is less than the degree of the minimal polynomial of T.

14 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$ has minimal polynomial $4 + 5z - 6z^2 - 7z^3 + 2z^4 + z^5$. Find the minimal polynomial of T^{-1}.

15 Suppose V is a finite-dimensional complex vector space with $\dim V > 0$ and $T \in \mathcal{L}(V)$. Define $f \colon \mathbf{C} \to \mathbf{R}$ by

$$f(\lambda) = \dim \operatorname{range}(T - \lambda I).$$

Prove that f is not a continuous function.

16 Suppose $a_0, \dots, a_{n-1} \in \mathbf{F}$. Let T be the operator on $\mathbf{F}^n$ whose matrix (with respect to the standard basis) is

$$\begin{pmatrix} 0 & & & & & -a_0 \\ 1 & 0 & & & & -a_1 \\ & 1 & \ddots & & & -a_2 \\ & & \ddots & & & \vdots \\ & & & & 0 & -a_{n-2} \\ & & & & 1 & -a_{n-1} \end{pmatrix}.$$

Here all entries of the matrix are 0 except for all 1's on the line under the diagonal and the entries in the last column (some of which might also be 0). Show that the minimal polynomial of T is the polynomial

$$a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n.$$

The matrix above is called the ***companion matrix*** *of the polynomial above. This exercise shows that every monic polynomial is the minimal polynomial of some operator. Hence a formula or an algorithm that could produce exact eigenvalues for each operator on each* $\mathbf{F}^n$ *could then produce exact zeros for each polynomial [by 5.27(a)]. Thus there is no such formula or algorithm. However, efficient numeric methods exist for obtaining very good approximations for the eigenvalues of an operator.*

17 Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and p is the minimal polynomial of T. Suppose $\lambda \in \mathbf{F}$. Show that the minimal polynomial of $T - \lambda I$ is the polynomial q defined by $q(z) = p(z + \lambda)$.

18 Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and p is the minimal polynomial of T. Suppose $\lambda \in \mathbf{F} \backslash \{0\}$. Show that the minimal polynomial of λT is the polynomial q defined by $q(z) = \lambda^{\deg p}\, p\left(\dfrac{z}{\lambda}\right)$.

19 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Let $\mathcal{E}$ be the subspace of $\mathcal{L}(V)$ defined by

$$\mathcal{E} = \{q(T) : q \in \mathcal{P}(\mathbf{F})\}.$$

Prove that $\dim \mathcal{E}$ equals the degree of the minimal polynomial of T.

20 Suppose $T \in \mathcal{L}(\mathbf{F}^4)$ is such that the eigenvalues of T are $3, 5, 8$. Prove that $(T - 3I)^2(T - 5I)^2(T - 8I)^2 = 0$.

21 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that the minimal polynomial of T has degree at most $1 + \dim \operatorname{range} T$.

If $\dim \operatorname{range} T < \dim V - 1$, *then this exercise gives a better upper bound than 5.22 for the degree of the minimal polynomial of T.*

22 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T is invertible if and only if $I \in \operatorname{span}(T, T^2, \dots, T^{\dim V})$.

23 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Let $n = \dim V$. Prove that if $v \in V$, then $\operatorname{span}(v, Tv, \dots, T^{n-1}v)$ is invariant under T.

24 Suppose V is a finite-dimensional complex vector space. Suppose $T \in \mathcal{L}(V)$ is such that 5 and 6 are eigenvalues of T and that T has no other eigenvalues. Prove that $(T - 5I)^{\dim V - 1}(T - 6I)^{\dim V - 1} = 0$.

25 Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and U is a subspace of V that is invariant under T.

(a) Prove that the minimal polynomial of T is a polynomial multiple of the minimal polynomial of the quotient operator T/U.

(b) Prove that

$$(\text{minimal polynomial of } T|_U) \times (\text{minimal polynomial of } T/U)$$

is a polynomial multiple of the minimal polynomial of T.

The quotient operator T/U was defined in Exercise 38 in Section 5A.

26 Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and U is a subspace of V that is invariant under T. Prove that the set of eigenvalues of T equals the union of the set of eigenvalues of $T|_U$ and the set of eigenvalues of T/U.

27 Suppose $\mathbf{F} = \mathbf{R}$, V is finite-dimensional, and $T \in \mathcal{L}(V)$. Prove that the minimal polynomial of $T_{\mathbf{C}}$ equals the minimal polynomial of T.

The complexification $T_{\mathbf{C}}$ *was defined in Exercise 33 of Section 3B.*

28 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that the minimal polynomial of $T' \in \mathcal{L}(V')$ equals the minimal polynomial of T.

The dual map T′ was defined in Section 3F.

29 Show that every operator on a finite-dimensional vector space of dimension at least two has an invariant subspace of dimension two.

Exercise 6 in Section 5C will give an improvement of this result when $\mathbf{F} = \mathbf{C}$.