1 Abstract

Combinatorial topology has recently been successfully applied to data analysis. One approach is to compute homology groups of some persistence complex constructed by experimental data and try to classify some objects by them. However complexity of computation of persistent homology is high enough to limit practical usage of the approach.

There is a question of how to construct persistence complex with same homology as initially given with smaller dimension. Some of complexes arise from dynamics of some partially ordered set by nerve construction. In this paper we prove Quillen-McCord theorem (Quillen fiber lemma) in the setting of persistent homology. It can possibly be used to reduce persistence complex built from poset to smaller one.

2 Preliminaries

2.1 Persistence modules and ε-interleavings

Basic definition is the following:

Definition 1. *Persistence complex* is a family of chain complexes $C_\star^{i_0} \xrightarrow{f_{i_0}} C_\star^{i_1} \xrightarrow{f_{i_1}} C_\star^{i_2} \xrightarrow{f_{i_2}} \dots$ where I is the set of indices and f_i are chain maps. We will call $f = (\dots, f_i, \dots)$ *the structure maps* of a persistence complex.

These objects naturally arise in experiments. If we consider I as a measure of time, this is a structure we obtain while observing some dynamic structure which can at any time be represented by a chain complex.

Definition 2. Let R be a ring. *Persistence module* is a family of R-modules M^i with homomorpisms $\phi_i :: M^i \to M^{i+1}$ as the structure maps. If R is a field it is natural to use notion of *persistence vector space*.

Example of a persistence module is given by homology modules of persistence complex $C_\star$ (*persistent homology*). $H_i^j(C_\star) = H_i(C_j)$, maps ϕ_j are induced by f_i.

Definition 3. Persistence complex (module) is of *finite type* over R if all components of complexes (modules) are finitely-generated as R-modules and all f_i (ϕ_i) are isomorphisms for $i > m$ for some m.

Since experiments usually take finite time and operate finite amount of data, we can safely consider only complexes of finite type. Note that by construction homology of complex of finite type is a module of finite type.

Definition 4. *Graded module* over I-graded ring R with graded components R_i is an R-module M together with a decomposition $M = \bigoplus_{j\in I} M^j : \ \forall i \in I \, R_i \cdot M^j \subset M^{j+i}$. For correctness of the definition it's enough for I to be a semigroup.

Definition 5. Assume I is a monoid with linear order. Then *non-negatively graded module* over I-graded ring R is an R-module M together with a decomposition $M = \bigoplus_{j\in I} M^j$: $\forall i \in I_{\geqslant 0} \, R_i \cdot M^j \subset M^{j+i}$

Proposition 1.

1. *Graded modules over R form a category.*
2. *Non-negatively graded modules over R form a category.*

Morphisms in both these categories are morphisms ϕ between modules M and N such that $\phi(M^j) \subset N^j$ for all j in indexing set.

Theorem 1. *[Zomorodian05, Theorem 3.1]*
Category of persistence modules of finite type over Noetherian ring with unity R is equivalent to category of non-negatively graded finitely generated $R[t]$-modules.

For clarification of the reference see [Corbet18]

Remark 1. If R is a field then $R[t]$ is a PID and graded ideals are exactly t^n for $n \in I_{\geqslant 0}$.

We have introduced a setting to work with persistence complexes as with graded modules. However computations in a category of non-negatively graded modules are not fit for applications since experiment always comes with an error and we do not have a measure for similarity of two persistence modules. We need technique which allows to construct morphisms of deformed complexes and track error propagation. This technique is developed in [GS16], we recall it without proofs.

Definition 6. [GS16, Definition 2.7]
Let M and N be persistence modules, $f : M \to N$ be a homomorphism of modules. Then f is called *ε-morphism* if $f(M^j) \subset N^{j+\varepsilon}$ for $\varepsilon \geqslant 0$.

Remark 2. Note that 0-morphism is a morphism in category of non-negatively graded modules over $R[t]$.

Remark 3. Let $\mathbb{F}$ be a field. Then for every persistence vector space V over $\mathbb{F}$ and any ε there exists morphism Id_ε such that $Id_\varepsilon(a) = t^\varepsilon a$.

Definition 7. Persistence vector spaces M and N are called *ε-interleaved* ($M \overset{\varepsilon}{\sim} N$) if there exists pair of ε-morphisms ($\phi : M \to N,\ \psi : N \to M$) (*$\varepsilon$-interleaving*) such that $\phi \circ \psi = Id_{2\varepsilon} : N \to N$ and $\psi \circ \phi = Id_{2\varepsilon} : M \to M$.

Remark 4. There follows that $M \overset{\varepsilon}{\sim} N$ implies $M \overset{\alpha}{\sim} N$ for any $\alpha > \varepsilon$ since for ε-interleaving (ϕ, ψ) we have α-interleaving $(Id_{\alpha-\varepsilon} \circ \phi, Id_{\alpha-\varepsilon} \circ \psi)$.

Remark 5. Note that due to Theorem 1 notion of ε-interleavings applies to persistence complexes.

Definition 8. We denote as *ε-equivalence* relation with the following properties:
For any M, N, L, ε, ε_1, ε_2

- $M \overset{0}{\sim} M$.
- $M \overset{\varepsilon}{\sim} N$ is equivalent to $N \overset{\varepsilon}{\sim} M$.
- if $M \overset{\varepsilon_1}{\sim} N$ and $N \overset{\varepsilon_2}{\sim} L$ then $M \overset{\varepsilon_1+\varepsilon_2}{\sim} L$.

Proposition 2. *ε-interleaved persistence vector spaces are ε-equivalent.*

Definition 9. Persistence vector space M is called *ε-trivial* if $M \overset{\varepsilon}{\sim} 0$. This is equivalent to condition $t^{2\varepsilon}M = 0$.

Lemma 1. *Let $0 \to M \to L \to N \to 0$ be a short exact sequence of persistence vector spaces. Then the following properties hold:*

- *If $M \overset{\varepsilon_1}{\sim} 0$ and $N \overset{\varepsilon_2}{\sim} 0$ then $L \overset{\varepsilon_1+\varepsilon_2}{\sim} 0$.*
- *If $L \overset{\varepsilon}{\sim} 0$ then $M \overset{\varepsilon}{\sim} 0$ and $N \overset{\varepsilon}{\sim} 0$.*

2.2 Quillen-McCord theorem

In order to give a general context we formulate general theorem and deduce Quillen-McCord theorem as a special case.

Definition 10. *Simplex category* Δ is the category with objects — nonempty linearly ordered sets of the form $[n] = \{0, 1, \dots, n\}$ with $n \geqslant 0$ and morphisms — order-preserving functions.

All morphisms in the simplex category are compositions of injective maps $\delta_i^n : \{0, 1, \dots, i-1, i+1 \dots, n\} \mapsto \{0, 1, \dots, i-1, i, i+1, \dots, n\}$ and surjective maps $\sigma_i^n : \{0, 1, \dots, i, i+1, i+2 \dots, n\} \mapsto \{0, 1, \dots, i, i+1, \dots, n\}$.

Definition 11. *Simplicial set* S is a contravariant functor $\Delta \to Set$. Objects of image of S are called simplices, cardinality of a simplex is called a dimension of a simplex. We will identify s with set of its simplices while keeping in mind general structure.

Simplicial sets form a category under natural transformations (category $Psh(\Delta)$ of presheaves on Δ).

Definition 12. Simplicial set S is called a *simplicial complex* if for any $s \in S$ s.t. $v \subset s$ $v \in s$ and S is faithful as a functor.

We will also use the following definitions:

Definition 13. *Join* $A \star B$ of simplicial complexes A and B is the simplicial complex with simplices — all possible unions of simplices $a \in A$ and $b \in B$.

Definition 14. *Join* of topological spaces A and B is defined as follows: $A \star B \ := A \sqcup_{p_0} (A \times B \times [0, 1]) \sqcup_{p_1} B$, where p are projections of the cylinder $A \times B \times [0, 1]$ onto faces.

Definition 15. *Star* $\mathrm{st}(x)$ of simplex $x \in A$ (A simplicial complex) is the simplicial complex with simplices — all simplices $a \in A$ such that there exists inclusion $x \hookrightarrow a$.

Definition 16. *Link* $\mathrm{lk}(x)$ of simplex $x \in A$ is defined as follows: $\mathrm{lk}(x) = \{v \in \mathrm{st}(x) | \ x \notin v\}$.

Definition 17. Functor $\mathcal{K} : \Delta \to Top$ which maps simplices to geometric simplices of corresponding dimension and morphisms to inclusions of faces and restrictions to subcomplexes is called *geometric realization*.

Proposition 3. $\mathrm{st}(x) = \mathrm{lk}(x) \star x$.

Proposition 4. *$\mathcal{K}(A \star B) = \mathcal{K}(A) \star \mathcal{K}(B)$. Hence $\mathcal{K}(\mathrm{st}(x))$ is a cone over $\mathcal{K}(x)$.*

Let $\mathcal{C}$ denote a small category. We can construct the simplicial set called *nerve of category* $\mathcal{C}$ as follows:
Let objects of $\mathcal{C}$ be the only 0-dimensional simplices. Then let all morphisms be 1-dimensional simplices, all composable pairs of morphisms be 2-dimensional simplices and so on with morphisms — inclusions of compositions into longer ones and replacements of compositions $i \circ j$ with $f = i \circ j$.
This construction is functorial over category of small categories Cat, we denote nerve functor from Cat to $Psh(\Delta)$ as $\mathcal{N}$.

Definition 18. Geometric realization $B\mathcal{C}$ of $\mathcal{N}(\mathcal{C})$ is called *classifying space* of $\mathcal{C}$.

We denote composition of nerve and geometric realization as $\mathcal{B}$. It is obviously a functor and we only need this construction to avoid confusing notation Bf for induced morphism of classifying spaces. By definition $\mathcal{B}(\mathcal{C}) = B\mathcal{C}$.

Definition 19. Let $f : \mathcal{C} \to \mathcal{D}$ be a functor and d — object in $\mathcal{D}$. Then *comma category* $d \downarrow f$ is a category with objects — pairs (s, i_s) of objects in $\mathcal{C}$ and morphisms $i_s : d \to f(s)$ and morphisms — morphisms g in $\mathcal{C}$ such that triangle $i_s, i_{g(s)}, f(g)$ is commutative.

Theorem 2. *[Quillen72, Theorem A]*
If $f : C \to D$ is a functor such that the classifying space $B(d \downarrow f)$ of the comma category $d \downarrow f$ is contractible for any object $d \in D$, then f induces a homotopy equivalence $BC \to BD$.

Every poset X with order relation R can be seen as a category if we set $\mathrm{Ob}(X) = X$ and $Hom_X(a, b) = \{r_{ab}\}$ if $(a, b) \in R$ and $\emptyset$ otherwise. Note that map between posets is functorial if and only if it preserves order.
Nerve construction on a poset yields a simplicial complex called *order complex*. Application of Quillen A theorem to posets yields the following theorem:

Theorem 3. ***Quillen-McCord theorem***
Assume X, Y are finite posets, $f : X \to Y$ is order-preserving map.
If $\forall y \in Y$ $\mathcal{B}(f^{-1}(Y_{\leqslant y}))$ is contractible, then $\mathcal{B}f$ is a homotopy equivalence between BX and BY.

There holds homological [Bar11, Corollary 5.5] versions of this theorem:

Theorem 4. ***Homological Quillen-McCord theorem***
Assume X, Y are finite posets, $f : X \to Y$ is order-preserving map, R is a PID.
If $\forall y \in Y$ $H_i(\mathcal{B}(f^{-1}(Y_{\leqslant y})), R) = 0$ for any i, $\mathcal{B}f$ induces isomorphisms of all homology groups with coefficients in R on BX and BY.

Proofs of the theorem is necessary for construction of a desired result and will be recalled in section 2.4 as well as a specific proof of Quillen-McCord theorem [Bar11, Proof of Theorem 1.1].

2.3 Statement of the theorem

Assume we conduct an experiment and observe dynamics of a simplicial complex $S^{i_0} \xrightarrow{f_{i_0}} S^{i_1} \xrightarrow{f_{i_1}} S^{i_2} \xrightarrow{f_{i_2}} \ldots$. Denote the whole sequence as S. For each simplicial complex S^i we can assign chain complex $C^i_\star$ via standard construction of simplicial chain complex used in the definition of simplicial homology.

I.e. we observe persistence chain complex $C^{i_0}_\star \xrightarrow{f'_{i_0}} C^{i_1}_\star \xrightarrow{f'_{i_1}} C^{i_2}_\star \xrightarrow{f'_{i_2}} \ldots$. Each simplicial complex in a series has geometric realization with the same homology as its chain complex.

Definition 20. We will denote as *persistence topological space* X sequence of topological spaces $X^{i_0} \xrightarrow{f_{i_0}} X^{i_1} \xrightarrow{f_{i_1}} X^{i_2} \xrightarrow{f_{i_2}} \ldots$ with compatible triangulations (chain complexes formed by simplicial complexes) $C^i_\star$ such that $C_\star$ is a persistence complex with structure maps induced by f_i. Homology of X is a persistence module with components — homology modules of X^i.

We will denote as morphism g between persistence topological spaces X and Y collection of maps $g_i : X^i \to Y^i$ between components such that all g_i commute with f_i.

Apparently $\mathcal{K}S$ is a persistence topological space.

Definition 21. Persistence topological space X is called *ε-acyclic* over $\mathbb{F}$ if for all indices $H_i(X, \mathbb{F}) \overset{\varepsilon}{\sim} H_i(pt, R)$.

Definition 22. We say that morphism $f : X \to Y$ between persistence topological spaces induces ε-interleaving if there exists $g : Y \to X$ such that induced maps of pair (f, g) on graded homology modules form ε-interleaving.

Definition 23. We define *persistence poset* P as possibly infinite in both directions series of posets P_i with order-preserving structure maps.

Map of persistence posets X, Y is defined as collection of component-wise order-preserving maps commuting with structure maps.

Persistence poset of finite type is a finite sequence of finite posets.

It's easy to see that nerve (as a functor) of persistence poset of finite type is a persistence complex of finite type.

Theorem 5. ***Approximate Quillen-McCord theorem, draft statement***
Assume X, Y are persistent posets of finite type, $f : X \to Y$ is order-preserving map, $\mathbb{F}$ is a field.
If $\forall y = (\ldots, y_i, \ldots) \in Y$ $\mathcal{B}(f^{-1}(Y_{\leqslant y}))$ is ε-acyclic over $\mathbb{F}$, $\mathcal{B}f$ induces e-interleavings of all homology spaces over $\mathbb{F}$ on BX and BY.

Value of e will be specified during the proof and is critical to applications being measure of error of approximation.

2.4 Prerequisite results

2.4.1 Quillen-McCord theorem

This section contains the proof of Quillen-McCord theorem given by Barmak.

Proposition 5. *[Bar11, Proposition 2.1] Let X and Y be two simplicial complexes. Then if $\mathcal{K}(X) \cup \mathcal{K}(Y) \hookrightarrow \mathcal{K}(X)$ is a homotopy equivalence, then so is $\mathcal{K}(Y) \hookrightarrow \mathcal{K}(X \cup Y)$.*

Proposition 6. *[Bar11, Proposition 2.2] Let $f, g : X \to Y$ be order-preserving maps between finite posets such that $\forall x \; f(x) \leqslant g(x)$. Then $\mathcal{B}(f)$ is homotopy-equivalent to $\mathcal{B}(g)$.*

Proof. X has a finite set M of maximal elements. Take any of them (m_1) and define $h_1 : X \to Y$ such that $h_1(m_1) = g(m_1)$ and $h_1 = f$ on all other elements of X. This is an order-preserving map due to maximality of m_1. Take $M \setminus \{m_1\}$ and build h_2 which is equal to h_1 on complement to m_2 and to g on m_2 and so on. We have built a finite sequence of maps $h_0 = f \leqslant h_1 \leqslant h_1 \leqslant \ldots \leqslant g = h_n$.

Elements $h_i(m_i)$ and $h_{i-1}(m_i)$ are comparable. Hence there exists simplex $\{h_i(m_i); h_{i-1}(m_i)\}$ in $\mathcal{N}(Y)$. Since $\mathcal{N}(Y)$ is a simplicial complex, for other elements between selected where exist no holes and thus where is a linear homotopy between h_{i-1} and h_i which contracts simplex $\{h_i(m_i); h_{i-1}(m_i)\}$.

Hence there is a homotopy between f and g. □

Proposition 7. *Note that $\mathrm{lk}(\mathcal{N}(x)) = \mathcal{N}(X_{>x}) \star \mathcal{N}(X_{<x})$. Therefore $\mathcal{K}(\mathrm{lk}(\mathcal{N}(x))) = \mathcal{B}(X_{>x}) \star \mathcal{B}(X_{<x})$.*

Remark 6. $\mathcal{B}(X) = \mathcal{B}(X \setminus \{x\}) \cup \mathcal{K}(\mathrm{st}(\mathcal{N}(x)))$ for any $x \in X$.

Lemma 2. *Let X be a finite poset and for $x \in X$ either $\mathcal{B}(X_{>x})$ or $\mathcal{B}(X_{<x})$ is contractible. Then embedding $\mathcal{B}(X \setminus \{x\}) \hookrightarrow \mathcal{B}(X)$ is a homotopy equivalence.*

Proof. By proposition $\mathcal{K}(\mathrm{lk}(\mathcal{N}(x)))$ is contractible. Hence its embedding to its cone $\mathcal{K}(\mathrm{st}(\mathcal{N}(x)))$ is homotopy equivalence by Whitehead theorem. Lemma follows by Proposition 5. □

Definition 24. Let $f : X \to Y$ be an order preserving map between posets. Denote orders $(\leqslant)$ on X and Y as R_X and R_Y. Then we define poset $M(f) = X \coprod Y$ with $R = R_X \cup R_Y \cup R_f$ where $(x, y) \in R_f$ if and only if $(f(x), y) \in R_Y$.

We will by analogy denote this poset a *mapping cylinder* of f. There are also defined canonical inclusions $i : X \to M(f)$ and $j : Y \to M(f)$.

Proof. Quillen-McCord theorem
Let X, Y be finite posets with order-preserving map $f : X \to Y$.

Every poset has linear extension. Let $x_1, x_2, \ldots, x_n$ be enumeration of X in such linear order and $Y^r = \{x_1, \ldots, x_r\} \cup Y \subset M(f)$ for any r.

Consider $Y^r_{>x_r} = Y_{\geqslant f(x_r)}$. $\mathcal{B}(Y_{\geqslant f(x_r)})$ is a cone over $\mathcal{B}(f(x_r))$. It is contractible, therefore $\mathcal{B}(Y^{r-1}) \hookrightarrow \mathcal{B}(Y^r)$ is homotopy equivalence by lemma. By iteration $\mathcal{B}(j) : \mathcal{B}(Y^0) = \mathcal{B}(Y) \hookrightarrow \mathcal{B}(M(f)) = \mathcal{B}(Y^n)$ is homotopy equivalence between BY and $M(f)$.

Then consider linear extension of Y with enumeration $y_1, \dots, y_m$ and $X^r = X \cup \{y_{r+1}, \dots, y_m\} \subset M(f)$. $X^{r-1}_{\leqslant y_r} = f^{-1}(Y_{\leqslant y_r})$. Latter is contractible by condition of the theorem. Hence $\mathcal{B}(X^r) \hookrightarrow \mathcal{B}(X^{r-1})$ is homotopy equivalence and by transitivity $\mathcal{B}(i)$ is a homotopy equivalence between X and $M(f)$.

Note that $i(x) \leqslant (j \circ f)(x)$. By Proposition 6 $\mathcal{B}(i)$ is homotopic to $\mathcal{B}(j \circ f) = \mathcal{B}(j) \circ \mathcal{B}(f)$. Hence $\mathcal{B}(f)$ is the homotopy equivalence between BX and BY. $\square$

2.4.2 Homological Quillen-McCord theorem

To derive homological version of the theorem we can variate the proof of standard version in the following manner:

Proposition 8. *[Milnor56, Lemma 2.1] Reduced homology modules with coefficients in a principal ideal domain of a join satisfy the relation* $H_{r+1}(A \star B, R) \simeq \bigoplus_{i+j=r} (H_i(A, R) \otimes_R H_j(B, R)) \oplus \bigoplus_{i+j=r-1} \mathrm{Tor}_1^{\mathrm{R}}(H_i(A, R), H_j(B, R))$.

Lemma 3. *Let X be a finite poset and for $x \in X$ either $H_i(\mathcal{B}(X_{<x}))$ or $H_i(\mathcal{B}(X_{>x}))$ with coefficients in a PID are equal to homology of a point. Then embedding $\mathcal{B}(X \backslash \{x\}) \hookrightarrow \mathcal{B}(X)$ induces isomorphisms of all homology groups.*

Proof.
By Proposition 8 $H_i(\mathcal{K}(\mathrm{lk}(\mathcal{N}(x)))) = H_i(\mathcal{B}(X_{>x}) \star \mathcal{B}(X_{<x}))$ are trivial for all indices i — Tor-functors vanish if any of their arguments is trivial. Application of Mayer-Vietoris long exact sequence to covering from Remark 6 yields the lemma. $\square$

Proof of the theorem is similar to proof finishing previous section. However, we write it here in detail in order to be able to highlight differences. Changed parts are written in italic.

Proof. Homological Quillen-McCord theorem
Let X, Y be finite posets with order-preserving map $f : X \to Y$.

Every poset has linear extension. Let $x_1, x_2, \dots, x_n$ be enumeration of X in such linear order and $Y^r = \{x_1, \dots, x_r\} \cup Y \subset M(f)$ for any r.

Consider $Y^r_{>x_r} = Y_{\geqslant f(x_r)}$. $\mathcal{B}(Y_{\geqslant f(x_r)})$ is a cone over $\mathcal{B}(f(x_r))$. It is contractible, therefore $\mathcal{B}(Y^{r-1}) \hookrightarrow \mathcal{B}(Y^r)$ is homotopy equivalence by lemma 2. By iteration $\mathcal{B}(j) : \mathcal{B}(Y^0) = \mathcal{B}(Y) \hookrightarrow \mathcal{B}(M(f)) = \mathcal{B}(Y^n)$ is homotopy equivalence between BY and $M(f)$.

Then consider linear extension of Y with enumeration $y_1, \dots, y_m$ and $X^r = X \cup \{y_{r+1}, \dots, y_n\} \subset M(f)$. $X^{r-1}_{\leqslant y_r} = f^{-1}(Y_{\leqslant y_r})$. *Latter is acyclic over R by condition of the theorem. Hence $\mathcal{B}(X^r) \hookrightarrow \mathcal{B}(X^{r-1})$ induces isomorphisms of all homology groups and by functoriality of homology $\mathcal{B}(i)$ induces isomorphisms of all homology groups between X*

and $M(f)$.

Note that $i(x) \leqslant (j \circ f)(x)$. By Proposition 6 $\mathcal{B}(i)$ is homotopic to $\mathcal{B}(j \circ f) = \mathcal{B}(j) \circ \mathcal{B}(f)$. *Homotopic maps induce same maps on homology, j is a homotopy equivalence and induce isomorphisms. Hence $\mathcal{B}(f)$ induce isomorphisms between $H_i(BX, R)$ and $H_i(BY, R)$.* $\square$

We see two updates. First one is essential, it requires Lemma 3 and operates some equivalence propagating in a chain of length equal to cardinality of Y. Second follows automatically from functoriality of all used constructions.

3 Results

We start with two obvious observations:

Proposition 9.

1. *Let A, B be two ε-trivial graded vector spaces. Then $A \oplus B$ is ε-trivial.*

2. *Let A and B be two graded vector spaces with at least one of them being ε-trivial. Then $A \otimes_{R_\bullet} B$ is ε-trivial ($\bullet$ points that this is a tensor product of graded spaces).*

Proposition 10. *Let A and B be two persistence topological spaces with at least one of them being ε-acyclic over $\mathbb{F}$. Then $A \star B$ is ε-acyclic over $\mathbb{F}$.*

Proof. Over a field all Tor-functors from Proposition 8 vanish since every vector space is free module, hence flat. Hence right hand side of expression of Proposition 8 is ε-trivial by Proposition 9. $\square$

Proposition 11. *Let $0 \to \ker f \hookrightarrow A \xrightarrow{f} B \xrightarrow{\phi} C \xrightarrow{g} D \to \operatorname{coker} g \to 0$ be exact sequence in category of nonnegatively graded vector spaces. Then if A and D are ε-trivial, then $B \overset{2\varepsilon}{\sim} C$ with ϕ being left morphism in interleaving pair.*

Proof. In s.e.s $0 \to \ker f \hookrightarrow A \xrightarrow{f} \operatorname{im} f \to 0$ $\operatorname{im} f$ is ε-trivial. By exactness it is equal to $K = \ker \phi$. On the other side from $0 \to \ker g \xrightarrow{g} D \to D \to 0$ there follows that $I = \operatorname{im} \phi$ is ε-trivial. Hence by transitivity $K \overset{2\varepsilon}{\sim} I$.

We obtain exact sequence $0 \to K \to B \xrightarrow{\phi} C \to I \to 0$. $B = K \oplus \operatorname{coIm} \phi$, $C = I \oplus \operatorname{Im} \phi$. Coimage and image are canonically isomorphic, hence by Proposition 9 proposition follows. $\square$

Assume X is a persistence poset and $Y_i \subset X_i$. Then we can induce structure of persistence poset on $Y = (\ldots, Y_i, \ldots)$.
Let ϕ be structure maps of X. There is one case there construction of structure maps ψ of Y is nontrivial: image of ϕ_{i-1} of some $a \in Y_{i-1}$ is outside of Y_i. Then we set $\psi(a)$ to be maximal element of Y_i such that it is less or equal to $\phi(a)$.

Definition 25. We call Y with this induced structure *persistence subposet* of X.

Definition 26. Consider persistence poset $X = (\dots, X_i, \dots)$ with coverings $\mathcal{U}_i$ on each $\mathcal{B}(X_i)$. We call $\mathcal{U} = (\dots, \mathcal{U}_i, \dots)$ persistence covering if there exists collection of persistence subposets $Y_i^j \subset X$ such that $\mathcal{B}Y_i^j$ form the covering $\mathcal{U}_i$.

Proposition 12. *Covering from Remark 6 can be extended to persistence covering* $\mathcal{U}$ *with* Y^1 *– preimage of* $st(\mathcal{N}(X))$ *under nerve functor and* $Y^2 = X \setminus \{x\}$.

Lemma 4. *Let* X *be a persistence poset and for* $x = (\dots, x_i, \dots) \in X$ *either* $\mathcal{B}(X_{<x})$ *or* $\mathcal{B}(X_{>x})$ *are* ε*-acyclic. Then embedding* $\mathcal{B}(X \setminus \{x\}) \hookrightarrow \mathcal{B}(X)$ *induces* 2ε*-interleavings of all homology spaces.*

Proof.

By Proposition 10 $\mathcal{K}(\mathrm{lk}(\mathcal{N}(x)))$ is ε-acyclic.

Given persistence covering we can define persistence Mayer-Vietoris exact sequence on homology spaces component-wise by gluing sequences for components over structure maps on homology. By equivalence of categories it defines exact sequence *Seq* of persistence vector spaces.

Seq construction on $\mathcal{U}$ and Proposition 11 yield the lemma. □

Definition 27. Let $f = (f_1, \dots, f_n, \dots)$ be map of persistence posets X with structure maps ϕ and Y with structure maps ψ. Then mapping cylinders $M(f_i)$ form persistence poset $M(f)$ with structure maps arising from universal property of coproduct and structure maps of X and Y.

To be explicit consider the following diagram in *Cat* with i, j being series of canonical inclusions:

$$\begin{tikzcd}
X_i \arrow[r, "i_i"] \arrow[d, "\phi_i"] & X_i \coprod Y_i \arrow[d, "\zeta_i"] & Y_i \arrow[l, "j_i"'] \arrow[d, "\psi_i"] \\
X_{i+1} \arrow[r, "i_{i+1}"] & X_{i+1} \coprod Y_{i+1} & Y_{i+1} \arrow[l, "j_{i+1}"']
\end{tikzcd}$$

Notation i for inclusion is kept for consistency with definition of mapping cylinder of posets.

Existence of order-preserving maps ζ_i is guaranteed by universal property of coproducts, they are set to be structure maps of $M(f)$. $M(f)$ is a *mapping cylinder of map of persistence posets*.

We also have canonical inclusions i and j arising from the same diagram.

Definition 28. We will denote as *linear extension of persistence poset* X series of linear extensions of X_i such with the same structure maps as X.

Proposition 13. *Every persistence poset has linear extension.*

Proof. Let X be persistence poset with structure maps ϕ. We will denote map from X_i as ϕ_i and refer to composition of k structure maps as to ϕ^k to simplify notation.

For any incomparable pair a, b in X_i one of the following hold:

1. $a \in \mathrm{im}(\phi_{i-1})$, $b \in \mathrm{im}(\phi_{i-1})$. Then we know that preimages are incomparable since ϕ_{i-1} is order preserving.

2. $a \in \mathrm{im}(\phi_{i-1})$, $b \notin \mathrm{im}(\phi_{i-1})$ or vice versa.

3. Both a and b are not in image of ϕ_{i-1}.

Assume for some incomparable a, b in X_i $\phi^k(a)$ and $\phi^k(b)$ are comparable for some natural k. W.l.o.g $\phi^k(a) \leqslant \phi^k(b)$. Then we have to add pair (a, b) to order on X_i. There is no possible contradiction since pair (b, a) in order implies $\phi^k(b) \leqslant \phi^k(a)$.
This extension extends infinitely to the left since we never have ordered pair in iterated preimages of a and b. If there is no such k we leave ordering on a, b to order extension principle. This extension can be safely extended to both sides.

Application of this reasoning to all poset components and incomparable pairs in them yields proposition. □

Remark 7. While working with persistence topological spaces we sometimes see properties which hold component-wise. If for persistence space X or map f_i there is a property which holds for any X_i or f_i we call it component-wise property of X. For instance, if map f is component-wise homotopy equivalence, it induces 0-interleaving of homology modules.

We are now ready to adapt known proof to Quillen-McCord theorem for persistence posets.

Theorem 6. ***Approximate Quillen-McCord theorem, final statement***
Assume X, Y are persistent posets of finite type, $f : X \to Y$ is order-preserving map, $\mathbb{F}$ is a field. Let $m = \max_i(|Y_i|)$ — maximal cardinality of components of Y.
If $\forall y = (\dots, y_i, \dots) \in Y$ $\mathcal{B}(f^{-1}(Y_{\leqslant y}))$ is ε-acyclic over $\mathbb{F}$, $\mathcal{B}f$ induces $2m\varepsilon$-interleavings of all homology spaces over $\mathbb{F}$ on BX and BY.

Proof. Approximate Quillen-McCord theorem
Let X, Y be persistence posets of finite type with order-preserving map $f : X \to Y$.

Let $\overline{X}$ be linear extension of X. Let $x_1^i, x_2^i, \dots, x_{n_i}^i$ be enumeration of $\overline{X}_i$ in its linear order and $Y_i^r = \{x_1^i, \dots, x_r^i\} \cup Y_i \subset M(f_i)$ for any r up to $\max_i(n_i)$ and i. $Y^r = (\dots, Y_i^r, \dots)$ is a persistence subposet of $M(f)$ with extended order on X.

There are posets such that $n_i < r$. In these cases notation x_r means that on positions with $n_i < r$ we take x_{n_i}.

Consider $Y^r_{>x_r} = Y_{\geqslant f(x_r)}$. $\mathcal{B}(Y_{\geqslant f(x_r)})$ is a component-wise cone over $\mathcal{B}(f(x_r))$. It is component-wise contractible, therefore $\mathcal{B}(Y^{r-1}) \hookrightarrow \mathcal{B}(Y^r)$ is component-wise homotopy equivalence by lemma 2. By iteration $\mathcal{B}(j) : \mathcal{B}(Y^0) = \mathcal{B}(Y) \hookrightarrow \mathcal{B}(M(f)) = \mathcal{B}(Y^n)$ is component-wise homotopy equivalence between BY and $M(f)$. Note that persistence structure is not used here.

Then consider linear extension of Y with enumerations $y_1^i, \dots, y_{m_i}^i$ and $X_i^r = X \cup \{y_{r+1}^i, \dots, y_{m_i}^i\} \subset M_i(f)$. $X^{r-1}_{\leqslant y_r} = f^{-1}(Y_{\leqslant y_r})$. Latter is ε-acyclic over R by condition of the theorem. Hence $\mathcal{B}(X^r) \hookrightarrow \mathcal{B}(X^{r-1})$ induces 2ε-interleavings of all homology modules and by transitivity of ε-equivalence $\mathcal{B}(i)$ induces $2m\varepsilon$-interleavings between homology of X and $M(f)$.

Note that $i(x) \leqslant (j \circ f)(x)$. By Proposition 6 $\mathcal{B}(i)$ is homotopic to $\mathcal{B}(j \circ f) = \mathcal{B}(j) \circ \mathcal{B}(f)$. Homotopic maps induce same maps on homology, j is a homotopy equivalence and induce 0-interleavings. Hence $\mathcal{B}(f)$ induce $2m\varepsilon$-interleavings between $H_i(BX, R)$ and $H_i(BY, R)$. □

4 Error propagation in Mayer-Vietoris spectral sequence

Value of error propagation multiple in the result may probably be decreased with alternative proof using spectral sequences at the cost of some restriction on structure maps of posets. Here we outline results of Govc and Scraba on error propagation in one specific spectral sequence associated to cover.

Assume persistence simplicial complex S is a filtered complex (assuming either ascending or descending filtration with any compatible structure maps) and there exists open covering $\mathcal{U}$ compatible with filtration. Then there exists a spectral sequence called Mayer-Vietoris spectral sequence which converges to $H_\star(S)$. [GS16, Theorem 2.30]

Let all sets in filtered cover $\mathcal{U}$ be ε-acyclic with all intersections between them be ε-acyclic. This is a representation of the definition of ε-acyclic cover. [GS16, Definition 3.2]

Sets and all their nonempty intersections in any covering form a poset with inclusion being an order relation. Nerve of this poset is called *nerve of a covering* and we shall denote it as $\mathcal{N}(\mathcal{U})$.

There hold the following propositions:

Proposition 14. *[GS16, Corollary 5.2]*
If $\mathcal{U}$ is an ε-acyclic cover of X, then for all i $E^2_{i,0}$ of Mayer-Vietoris spectral sequence and $H_i(\mathcal{N}(\mathcal{U}), \mathbb{F})$ are 2ε-interleaved as graded modules.

Proposition 15. *[GS16, Theorem 7.1]*
Let D be dimension of $\mathcal{N}(\mathcal{U})$. Then for all i $H_i(X, \mathbb{F}) \overset{(4D+2)\varepsilon}{\sim} H_i(\mathcal{N}(\mathcal{U}), \mathbb{F})$.

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