1 Abstract

Combinatorial topology has recently been successfully applied to data analysis. One approach is to compute homology groups of some persistence complex constructed by experimental data and try to classify some objects by them. However complexity of computation of persistent homology is high enough to limit practical usage of the approach.

There is a question of how to construct persistent complex with same homology as initially given with smaller dimension. In this paper we prove Quillen-McCord theorem (Quillen fiber lemma) in the setting of persistent homology. It can possibly be used to reduce persistence complex to smaller one if given complex is equivalent to a nerve of some partially ordered set.

2 Preliminaries

2.1 Persistence modules and ε-interleavings

Basic definition is the following:

Definition 1. *Persistence complex* is a family of chain complexes $C_\star^{i_0} \xrightarrow{f_{i_0}} C_\star^{i_1} \xrightarrow{f_{i_1}} C_\star^{i_2} \xrightarrow{f_{i_2}} \dots$ with I — some indexing set and f_i being chain maps.

These objects naturally arise in experiments. If we consider I as time, this is a structure we obtain while observing some dynamic structure which can at any time be represented by a chain complex.

Definition 2. Let R be a ring. *Persistence module* is a family of R-modules M^i with homomorpisms $\phi_i :: M^i \to M^{i+1}$.

Example of a persistent module is given by homology modules of persistence complex $C_\star$ (*persistent homology*). $H_i^j(C_\star) = H_i(C_{\star j})$, maps ϕ_j are induced by f_i.

Definition 3. Persistence complex (module) is of *finite type* over R if all component complexes (modules) are finitely-generated as R-modules and all f_i (ϕ_i) are isomorphisms for $i > m$ for some m.

Since experiments usually take finite time and operate finite amount of data, we can safely consider only complexes of finite type. Note that by construction homology of complex of finite type is a module of finite type.

Definition 4. *Graded module* over I-graded ring R with graded components R_i is an R-module M together with a decomposition $M = \bigoplus_{j \in I} M^j : \ \forall i \in I \; R_i \cdot M^j \subset M^{j+i}$. For correctness of the definition it's enough for I to be a semigroup.

Definition 5. Assume I is a monoid with linear order. Then *non-negatively graded module* over I-graded ring R is an R-module M together with a decomposition $M = \bigoplus_{j \in I} M^j$: $\forall i \in I_{\geqslant 0} \; R_i \cdot M^j \subset M^{j+i}$

Proposition 1. . *The following hold:*

1. *Graded modules over R form a category.*
2. *Non-negatively graded modules over R form a category.*

Morphisms in both these categories are morphisms ϕ between modules M and N such that $\phi(M^j) \subset N^j$ for all j in indexing set.

Theorem 1. *[Zomorodian05, Theorem 3.1]*
Category of persistence modules of finite type over R is equivalent to category of non-negatively graded finitely generated $R[t]$-modules.

Remark 1. If R is a field then $R[t]$ is a PID and graded ideals are exactly t^n for $n \in I_{\geqslant 0}$.

We have introduced a setting to work with persistence complexes as with graded modules. However computations in a category of non-negatively graded modules are not fit for applications since experiment always comes with an error and we do not have a measure for similarity of two persistence modules. We need technique which allows to construct morphisms of deformed complexes and track error propagation. This technique is developed in [GS16], we will recall it without proofs.

Definition 6. [GS16, Definition 2.7]
Let M and N be persistent modules, $f : M \to N$ be a homomorphism of modules. Then f is called *ε-morphism* if $f(M^j) \subset N^{j+\varepsilon}$ for $\varepsilon \geqslant 0$.

Remark 2. Note that 0-morphism is a morphism in category of non-negatively graded modules over $R[t]$.

Remark 3. Let R be a field. Then for every persistence module M and any ε there exists morphism Id_ε such that $Id_\varepsilon(a) = t^\varepsilon a$.

Remark 4. Restriction on R to be a field is crucial for construction. It is a strong restriction, however, it is acceptable.

Definition 7. Persistence modules M and N are called *ε-interleaved* ($M \overset{\varepsilon}{\sim} N$) if there exist pair of ε-morphisms ($\phi : M \to N,\ \psi : N \to M$) (*$\varepsilon$-interleaving*) such that $\phi \circ \psi = Id_{2\varepsilon} : N \to N$ and $\psi \circ \phi = Id_{2\varepsilon} : M \to M$.

Remark 5. There follows that $M \overset{\varepsilon}{\sim} N$ implies $M \overset{\alpha}{\sim} N$ for any $\alpha > \varepsilon$ since for ε-interleaving (ϕ, ψ) we have α-interleaving $(Id_{\alpha-\varepsilon} \circ \phi, Id_{\alpha-\varepsilon} \circ \psi)$.

Remark 6. Note that due to Theorem 1 notion of ε-interleavings applies to persistence complexes.

Definition 8. We denote as *ε-equivalence* relation with the following properties:
For any M, N, L, ε, ε_1, ε_2

- $M \overset{0}{\sim} M$.
- $M \overset{\varepsilon}{\sim} N$ is equivalent to $N \overset{\varepsilon}{\sim} M$.
- if $M \overset{\varepsilon_1}{\sim} N$ and $N \overset{\varepsilon_2}{\sim} L$ then $M \overset{\varepsilon_1+\varepsilon_2}{\sim} L$.

Proposition 2. *ε-interleaved persistence modules are ε-equivalent.*

Definition 9. Persistent module M is called *ε-trivial* if $M \overset{\varepsilon}{\sim} 0$. It is equivalent to $t^{2\varepsilon}M = 0$.

Lemma 1. *Let $0 \to M \to L \to N \to 0$ be a short exact sequence. Then the following properties hold:*

- *If $M \overset{\varepsilon_1}{\sim} 0$ and $N \overset{\varepsilon_2}{\sim} 0$ then $L \overset{\varepsilon_1+\varepsilon_2}{\sim} 0$.*
- *If $L \overset{\varepsilon}{\sim} 0$ then $M \overset{\varepsilon}{\sim} 0$ and $N \overset{\varepsilon}{\sim} 0$.*

2.2 Quillen-McCord theorem

We give a formulation of classical theorem and then descend from general categorical case to specific.

Definition 10. *Simplex category* Δ is the category with objects — nonempty linearly ordered sets of the form $[n] = \{0, 1, \dots, n\}$ with $n \geqslant 0$ and morphisms — order-preserving functions.

All morphisms in the simplex category are compositions of injective maps $\delta_i^n : \{0, 1, \dots, i-1, i+1 \dots, n\} \mapsto \{0, 1, \dots, i-1, i, i+1, \dots, n\}$ and surjective maps $\sigma_i^n : \{0, 1, \dots, i, i+1, i+2 \dots, n\} \mapsto \{0, 1, \dots, i, i+1, \dots, n\}$.

Definition 11. *Simplicial set* S is a contravariant functor $\Delta \to Set$. Objects of image of S are called simplices, cardinality of a simplex is called a dimension of a simplex. We will identify s with set of its simplices while keeping in mind general structure.

Simplicial sets form a category under natural transformations (category of presheaves on Δ).

Definition 12. Simplicial set S is called a *simplicial complex* if for any $s \in S$ s.t. $v \subset s$ $v \in s$ and S is faithful.

We will also use the following definitions:

Definition 13. *Join* $A \star B$ of simplicial complexes A and B is the simplicial complex with simplices — all possible unions of simplices $a \in A$ and $b \in B$.

Definition 14. *Join* of topological spaces A and B is defined as follows: $A \star B \ := A \sqcup_{p_0} (A \times B \times [0,1]) \sqcup_{p_1} B$, where p are projections of the cylinder $A \times B \times [0,1]$ onto faces.

Definition 15. *Star* $\mathrm{st}(x)$ of simplex $x \in A$ (A simplicial complex) is the simplicial complex with simplices — all simplices $a \in A$ such that there exists inclusion $x \hookrightarrow a$.

Definition 16. *Link* $\mathrm{lk}(x)$ of simplex $x \in A$ is defined as follows: $\mathrm{lk}(x) = \{v \in \mathrm{st}(x) | \ x \notin v\}$.

Definition 17. Functor $\mathcal{K} : \Delta \to Top$ which maps simplices to geometric simplices of corresponding dimension and morphisms to inclusions of faces and restrictions to subcomplexes is called *geometric realization*.

Proposition 3. $\mathrm{st}(x) = \mathrm{lk}(x) \star x$. *Hence* $\mathcal{K}(\mathrm{st}(x))$ *is a cone over* $\mathcal{K}(x)$.

Proposition 4. $\mathcal{K}(A \star B) = \mathcal{K}(A) \star \mathcal{K}(B)$

Let C denote a category. We can construct the simplicial set called *nerve of category* C as follows:
Let objects of C be the only 0-dimensional simplices. Then let all morphisms be 1-dimensional simplices, all composable pairs of morphisms be 2-dimensional simplices and so on.
This construction is functorial, we denote nerve functor as (N).

Definition 18. Geometric realization BC of $\mathcal{N}(C)$ is called *classifying space* of C.

We will denote composition of nerve and geometric realization as $\mathcal{B}$. It is obviously a functor and we only need this construction to avoid confusing notation Bf for induced morphism of classifying spaces. By definition $\mathcal{B}(C) = BC$.

Definition 19. Let $f : C \to D$ be a functor and d — object in D. Then *comma category* $d \downarrow f$ is a category with objects — pairs (s, i_s) of objects in C and morphisms $i_s : d \to f(s)$ and morphisms — morphisms g in C such that triangle $i_s, i_{g(s)}, f(g)$ commutes.

Theorem 2. *[Quillen72, Theorem A]*
If $f : C \to D$ *is a functor such that the classifying space* $B(d \downarrow f)$ *of the comma category* $d \downarrow f$ *is contractible for any object* $d \in D$*, then* f *induces a homotopy equivalence* $BC \to BD$.

Every poset X with order relation R can be seen as a category if we set $\mathrm{Ob}(X) = X$ and $Hom_X(a, b) = \{r_{ab}\}$ if $(a, b) \in R$ and $\emptyset$ otherwise. Note that map between posets is functorial if and only if it preserves order.
Nerve construction on a poset yields a simplicial complex called *order complex*. Application of Quillen A theorem to posets yields the following theorem:

Theorem 3. ***Quillen-McCord theorem***
Assume X, Y *are finite posets,* $f : X \to Y$ *is order-preserving map.*
If $\forall y \in Y$ $\mathcal{B}(f^{-1}(Y_{\leqslant y}))$ *is contractible, then* $\mathcal{B}f$ *is a homotopy equivalence between* BX *and* BY.

There holds homological [Bar11, Corollary 5.5] versions of this theorem:

Theorem 4. ***Homological Quillen-McCord theorem***
Assume X, Y *are finite posets,* $f : X \to Y$ *is order-preserving map,* R *is a PID.*
If $\forall y \in Y$ $H_i(\mathcal{B}(f^{-1}(Y_{\leqslant y})), R) = 0$ *for any* i*,* $\mathcal{B}f$ *induces isomorphisms of all homology groups with coefficients in* R *on* BX *and* BY.

Proofs of both theorems are necessary for construction of a desired result and will be recalled in section 2.4 as well as a specific proof of Quillen-McCord theorem [Bar11, Proof of Theorem 1.1].

2.3 Statement of the theorem

Assume we conduct an experiment and observe dynamics of a simplicial complex $S^{i_0} \xrightarrow{f_{i_0}} S^{i_1} \xrightarrow{f_{i_1}} S^{i_2} \xrightarrow{f_{i_2}} \ldots$. Denote the whole sequence as S. For each simplicial complex S^i we can assign chain complex $C^i_\star$ via standard construction of simplicial chain complex used in the definition of simplicial homology.

I.e. we observe persistence chain complex $C^{i_0}_\star \xrightarrow{f'_{i_0}} C^{i_1}_\star \xrightarrow{f'_{i_1}} C^{i_2}_\star \xrightarrow{f'_{i_2}} \ldots$. Each simplicial complex in a series has geometric realization with the same homology as its chain complex.

Definition 20. We will denote as *persistence topological space* X sequence of topological spaces $X^{i_0} \xrightarrow{f_{i_0}} X^{i_1} \xrightarrow{f_{i_1}} X^{i_2} \xrightarrow{f_{i_2}} \ldots$ with compatible triangulations (chain complexes formed by simplicial complexes) $C^i_\star$ such that $C_\star$ is a persistence complex with structure maps induced by f_i. Homology of X is a persistence module with components — homology modules of X^i.

We will denote as morphism g between persistence topological spaces X and Y collection of maps $g_i : X^i \to Y^i$ between components such that all g_i commute with f_i.

Apparently $\mathcal{K}S$ is a persistence topological space.

Definition 21. Persistence topological space X is called *ε-acyclic* over R if for all indices $H_i(X, R) \overset{\varepsilon}{\sim} H_i(pt, R)$.

Definition 22. We say that morphism $f : X \to Y$ between persistence topological spaces induces ε-interleaving if there exists $g : Y \to X$ such that induced maps of pair (f, g) on graded homology modules form ε-interleaving.

To formulate a theorem we have to move to terms of posets. Define *persistence poset* as series of posets with structure maps analogously to persistence complexes and map of persistence posets as collection of maps between posets commuting with structure maps. It induces map on persistence order complexes. We also analogously define persistence poset of finite type — it is a finite sequence of finite posets. Easy to see that nerve (as a functor) of persistence poset of finite type is a persistence complex of finite type.

In a final statement of the theorem there will also appear some technical conditions which we omit for now.

Theorem 5. ***Approximate Quillen-McCord theorem, draft statement***
Assume X, Y are persistent posets of finite type, $f : X \to Y$ is order-preserving map, R is a field or $\mathbb{Z}$.
If $\forall y \in Y$ $\mathcal{B}(f^{-1}(Y_{\leqslant y}))$ is ε-acyclic over R, $\mathcal{B}f$ induces e-interleavings of all homology modules over R on BX and BY.

Value of e will be specified during the proof and is critical to applications being measure of error of approximation.

2.4 Prerequisite results

2.4.1 Quillen-McCord theorem

This section contains the proof of Quillen-McCord theorem given by Barmak.

Proposition 5. *[Bar11, Proposition 2.1] Let K_1 and K_2 be two geometric simplicial complexes and geometric simplicial complex $K = K_1 \cup K_2$ (as space and simplicial complex). Then if $K_1 \cup K_2 \hookrightarrow K_1$ is a homotopy equivalence, then so is $K_2 \hookrightarrow K$.*

Proposition 6. *[Bar11, Proposition 2.2] Let $f, g : X \to Y$ be order-preserving maps between finite posets such that $\forall x\ f(x) \leqslant g(x)$. Then $\mathcal{B}(f)$ is homotopy-equivalent to $\mathcal{B}(g)$.*

Proof. X has a finite set M of maximal elements. Take any of them (m_1) and define $h_1 : X \to Y$ such that $h_1(m_1) = g(m_1)$ and $h_1 = f$ on all other elements of X. This is an order-preserving map due to maximality of m_1. Take $M \setminus \{m_1\}$ and build h_2 which is equal to h_1 on complement to m_2 and to g on m_2 and so on. We have built a finite sequence of maps $h_0 = f \leqslant h_1 \leqslant h_1 \leqslant \ldots \leqslant g = h_n$.

Elements $h_i(m_i)$ and $h_{i-1}(m_i)$ are comparable. Hence there exists simplex $\{h_i(m_i); h_{i-1}(m_i)\}$ in $\mathcal{N}(Y)$. Since $mathcal N(Y)$ is a simplicial complex, for other elements between selected where exist no holes and thus where is a linear homotopy between h_{i-1} and h_i which contracts simplex $\{h_i(m_i); h_{i-1}(m_i)\}$.

Hence there is a homotopy between f and g. □

Proposition 7. *Note that* $\mathrm{lk}(\mathcal{N}(x)) = \mathcal{N}(X_{>x}) \star \mathcal{N}(X_{<x})$. *Therefore* $\mathcal{K}(\mathrm{lk}(\mathcal{N}(x))) = \mathcal{B}(X_{>x}) \star \mathcal{B}(X_{<x})$.

Remark 7. $\mathcal{B}(X) = \mathcal{B}(X \setminus \{x\}) \cup \mathcal{K}(\mathrm{st}(\mathcal{N}(x)))$ for any $x \in X$.

Lemma 2. *Let X be a finite poset and for $x \in X$ either $\mathcal{B}(X_{>x})$ or $\mathcal{B}(X_{<x})$ is contractible. Then embedding $\mathcal{B}(X \setminus \{x\}) \hookrightarrow \mathcal{B}(X)$ is a homotopy equivalence.*

Proof. By proposition $\mathcal{K}(\mathrm{lk}(\mathcal{N}(x)))$ is contractible. Hence its embedding to its cone $\mathcal{K}(\mathrm{st}(\mathcal{N}(x)))$ is homotopy equivalence. Lemma follows by Proposition 5. □

Definition 23. Let $f : X \to Y$ be an order preserving map between posets. Denote orders $(\leqslant)$ on X and Y as R_X and R_Y. Then we define poset $M(f) = X \cup Y$ with $R = R_X \cup R_Y \cup R_f$ where $(x, y) \in R_f$ if and only if $(f(x), y) \in R_Y$.

We will by analogy denote this poset a *mapping cylinder* of f. There are also defined canonical inslusions $i : X \to M(f)$ and $j : Y \to M(f)$.

Proof. Quillen-McCord theorem
Let X, Y be finite posets with order-preserving map $f : X \to Y$.

Every poset has linear extension. Let $x_1, x_2, \ldots, x_n$ be enumeration of X in such linear order and $Y^r = \{x_1, \ldots, x_r\} \cup Y \subset M(f)$ for any r.

Consider $Y^r_{>x_r} = Y_{\geqslant f(x_r)}$. $\mathcal{B}(Y_{\geqslant f(x_r)})$ is a cone over $\mathcal{B}(f(x_r))$. It is contractible, therefore $\mathcal{B}(Y^{r-1}) \hookrightarrow \mathcal{B}(Y^r)$ is homotopy equivalence by lemma. By iteration $\mathcal{B}(j) : \mathcal{B}(Y^0) = \mathcal{B}(Y) \hookrightarrow \mathcal{B}(M(f)) = \mathcal{B}(Y^n)$ is homotopy equivalence between BY and $M(f)$.

Then consider linear extension of Y with enumeration $y_1, \dots, y_m$ and $X^r = X \cup \{y_{r+1}, \dots, y_n\} \subset M(f)$. $X^{r-1}_{\leqslant y_r} = f^{-1}(Y_{\leqslant y_r})$. Latter is contractible by condition of the theorem. Hence $\mathcal{B}(X^r) \hookrightarrow \mathcal{B}(X^{r-1})$ is homotopy equivalence and $\mathcal{B}(i)$ is a homotopy equivalence between X and $M(f)$.

Note that $i(x) \leqslant (j \circ f)(x)$. By Proposition 6 $\mathcal{B}(i)$ is homotopic to $\mathcal{B}(j \circ f) = \mathcal{B}(j) \circ \mathcal{B}(f)$. Hence $\mathcal{B}(f)$ is the homotopy equivalence between BX and BY. □

2.4.2 Homological Quillen-McCord theorem

To derive homological version of the theorem we can variate the proof of standard version in the following manner:

Proposition 8. *[Milnor56, Lemma 2.1] Reduced homology groups with coefficients in a principal ideal domain of a join satisfy the relation* $H_{r+1}(A \star B, R) \simeq \bigoplus_{i+j=r} (H_i(A, R) \otimes_R H_j(B, R)) \oplus \bigoplus_{i+j=r-1} \mathrm{Tor}_1^{\mathrm{R}}(H_i(A, R), H_j(B, R))$.

Lemma 3. *Let X be a finite poset and for $x \in X$ either $H_i(\mathcal{B}(X_{<x}))$ or $H_i(\mathcal{B}(X_{>x}))$ with coefficients in a PID are equal to homology of a point. Then embedding $\mathcal{B}(X \backslash \{x\}) \hookrightarrow \mathcal{B}(X)$ induces isomorphisms of all homology groups.*

Proof.
By Proposition 8 $H_i(\mathcal{K}(\mathrm{lk}(\mathcal{N}(x)))) = H_i(\mathcal{B}(X_{>x}) \star \mathcal{B}(X_{<x}))$ are trivial for all indices i — Tor-functors vanish if any of their arguments is trivial. Application of Mayer-Vietoris long exact sequence to covering from Remark 7 yields the lemma. □

Proof of the theorem is similar to proof finishing previous section. However, we write it here in detail in order to be able to highlight differences. Changed parts are written in italic.

Proof. Homological Quillen-McCord theorem
Let X, Y be finite posets with order-preserving map $f : X \to Y$.

Every poset has linear extension. Let $x_1, x_2, \dots, x_n$ be enumeration of X in such linear order and $Y^r = \{x_1, \dots, x_r\} \cup Y \subset M(f)$ for any r.

Consider $Y^r_{>x_r} = Y_{\geqslant f(x_r)}$. $\mathcal{B}(Y_{\geqslant f(x_r)})$ is a cone over $\mathcal{B}(f(x_r))$. It is contractible, therefore $\mathcal{B}(Y^{r-1}) \hookrightarrow \mathcal{B}(Y^r)$ is homotopy equivalence by lemma 2. By iteration $\mathcal{B}(j) : \mathcal{B}(Y^0) = \mathcal{B}(Y) \hookrightarrow \mathcal{B}(M(f)) = \mathcal{B}(Y^n)$ is homotopy equivalence between BY and $M(f)$.

Then consider linear extension of Y with enumeration $y_1, \dots, y_m$ and $X^r = X \cup \{y_{r+1}, \dots, y_n\} \subset M(f)$. $X^{r-1}_{\leqslant y_r} = f^{-1}(Y_{\leqslant y_r})$. *Latter is acyclic over R by condition of the theorem. Hence $\mathcal{B}(X^r) \hookrightarrow \mathcal{B}(X^{r-1})$ induces isomorphisms of all homology groups and by functoriality of homology $\mathcal{B}(i)$ induces isomorphisms of all homology groups between X*

and $M(f)$.

Note that $i(x) \leqslant (j \circ f)(x)$. By Proposition 6 $\mathcal{B}(i)$ is homotopic to $\mathcal{B}(j \circ f) = \mathcal{B}(j) \circ \mathcal{B}(f)$. *Homotopic maps induce same maps on homology, j is a homotopy equivalence and induce isomorphisms. Hence $\mathcal{B}(f)$ induce isomorphisms between $H_i(BX, R)$ and $H_i(BY, R)$.* $\square$

We see two updates. First one is essential, it requires Lemma 3 and operates some equivalence propagating in a chain of length equal to cardinality of Y. Second follows automatically from functoriality of all used constructions, but contains statement about composition of functions. We expect these two parts to be changed while constructing a proof for approximate version of the theorem.

2.5 Error propagation in Mayer-Vietoris spectral sequence

Assume persistence simplicial complex S is a filtered complex (assuming either ascending or descending filtration with any compatible structure maps) and there exists open covering $\mathcal{U}$ compatible with filtration. Then there exists a spectral sequence called Mayer-Vietoris spectral sequence which converges to $H_\star(S)$. [GS16, Theorem 2.30]

Let all sets in $\mathcal{U}$ be ε-acyclic.
TODO: cite convergence approximation results: we need convergence on second page and duplication of ε in trivial modules. This will require serious adaptation.

3 Results

Lemma 4. *Let X be a persistence poset of finite type such that $\mathcal{N}$ is a filtered simplicial complex and for $x \in X$ either $\mathcal{B}(X_{<x})$ or $\mathcal{B}(X_{>x})$ are ε-acyclic. Then embedding $\mathcal{B}(X \setminus \{x\}) \hookrightarrow \mathcal{B}(X)$ induces 2ε-interleavings of all homology modules.*

TODO: proof, construction of persistence cylinder, conditions and proof of the theorem.

Theorem 6. ***Approximate Quillen-McCord theorem, final statement***
Assume X, Y are persistent posets of finite type such that $\ldots$, $f : X \to Y$ is order-preserving map, R is a field or $\mathbb{Z}$.
If $\forall y \in Y$ $\mathcal{B}(f^{-1}(Y_{\leqslant y}))$ is ε-acyclic over R, $\mathcal{B}f$ induces e-interleavings of all homology modules over R on BX and BY.

4 References

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5. Доказательство теоремы 3

Приведём здесь утверждение леммы 2.1 из [5]:

Lemma 5. $H_{r+1}(A * B) \simeq \bigoplus_{i+j=r}(H_i(A) \otimes H_j(B)) \oplus \bigoplus_{i+j=r-1} \mathrm{Tor}(H_i(A), H_j(B))$.
Гомологии здесь рассматриваются приведённые с коэффициентами в области главных идеалов.

Аналог леммы 4 разбивается на два независимо доказываемых утверждения.

Lemma 6. *Пусть X — конечное частично упорядоченное множество, $x \in X$ — элемент такой, что или $\mathcal{K}(X_{>x})$, или $\mathcal{K}(X_{<x})$ m-ацикличен.*
Тогда $lk(x) = \mathcal{K}(X_{>x}) * \mathcal{K}(X_{<x})$ m-ацикличен.

Доказательство. Пусть для определённости в лемме 7 B выбран m-ацикличным. Этот выбор произволен в силу коммутативности операции джойна.
По линейности тензорного произведения из m-ацикличности B следует m-тривиальность $H_i(A) \otimes H_j(B)$ для любых индексов.
Из леммы 2 подмодули и фактормодули m-тривиальных модулей m-тривиальны.
В комплексе $\ldots \to P_n \otimes B \to \ldots \to P_1 \otimes B \to 0$ все модули m-тривиальны, следовательно, все его гомологии, то есть $Tor_\bullet(A, B)$ m-тривиальны.
Прямая сумма m-тривиальных модулей m-тривиальна, m-перемежения строятся покомпонентно. Следовательно, правая сторона выражения леммы 7 m-тривиальна, а значит, m-тривиальная и левая сторона, что и требовалось. □

Lemma 7. *Пусть $A \to B \xrightarrow{f} C \to D$ — подпоследовательность точной последовательности, A и D m-тривиальны. Тогда B и C $2m$-перемежены.*

Доказательство. Из точности следует, что $\mathrm{Ker}(f)$ и $\mathrm{coKer}(f)$ m-тривиальны, следовательно, по приближённой транзитивности $2m$-перемежены. При этом $\mathrm{coIm}(f) \overset{f}{\simeq} \mathrm{Im}(f)$.
$B = \mathrm{Ker}(f) \oplus \mathrm{coIm}(f)$; $C = \mathrm{coKer}(f) \oplus \mathrm{Im}(f)$.
$2m$-перемежение B и C строится покомпонентно. □

Lemma 8. *Приближённый вариант леммы 4*
Пусть X — конечное частично упорядоченное множество, $x \in X$ — элемент такой, что или $\mathcal{K}(X_{>x})$, или $\mathcal{K}(X_{<x})$ m-ацикличен. Тогда вложение $\mathcal{K}(X \setminus \{x\}) \hookrightarrow \mathcal{K}(X)$ является гомологической $2m$-эквивалентностью.

Доказательство. Лемма следует из лемм 8 и 9, в качестве точной последовательности в лемме 9 можно взять точную последовательность Майера-Вьеториса для того же разбиения, что и в лемме 6. □

Доказанных утверждений достаточно для построения доказательства теоремы 3.

6. Приложение: устойчивость в спектральных последовательностях

Мы будем рассматривать ограниченные в каком-то направлении гомологические спектральные последовательности, расположенные в первом квадранте.
Расположение в первом квадранте: $p < 0 \vee q < 0 \implies E_r^{p,q} = 0$.
Ограниченность: $\exists p_0: \ \forall p > p_0, \forall q \ E_r^{p,q} = 0$ или $\exists q_0: \ \forall q > q_0, \forall p \ E_r^{p,q} = 0$.
Эти ограничения означают, что на какой-то странице все дифференциалы становятся нулевыми, поскольку их образы лежат за пределами квадранта, то есть нулевые. Значит, с некоторой страницы n спектральная последовательность стабилизируется, т.е. $E_n = E_\infty$.
Гомологичность в этом контексте означает, что дифференциалы уменьшают сумму индексов.

Lemma 9. *Пусть все модули на r-й странице спектральной последовательности m-тривиальны. Тогда на $r+1$-й странице все модули m-тривиальны.*

Доказательство. Каждый модуль на $r+1$-й странице является модулем гомологий некоторого модуля с r-й страницы относительно дифференциалов r-й страницы. То есть он является фактор-модулем подмодуля m-тривиального модуля. Применяя дважды лемму 2, получаем, что он является фактор-модулем m-тривиального модуля, то есть m-тривиален. □

Definition 24. Будем говорить, что спектральная последовательность сходится к набору модулей $A_\bullet$, если для любого n на A_n определена убывающая фильтрация $\ldots \hookrightarrow F^p A \hookrightarrow F^{p-1}A \hookrightarrow \ldots \hookrightarrow F^0 A = A$ и для каждого n $E^{p,n-p} = F^{p-1}A/F^p A$.

Заметим, что по определению ограниченная спектральная последовательность в первом квадранте может сходится только к набору модулей, на каждом из которых определена фильтрация с конечным числом ненулевых членов, что будем записывать как $0 = F^n A \hookrightarrow F^{n-1}A \hookrightarrow \ldots \hookrightarrow F^0 A = A$.

Lemma 10. *Пусть $0 = F^n A \hookrightarrow F^{n-1}A \hookrightarrow \ldots \hookrightarrow F^0 A = A$ — фильтрация, при этом $\forall p \ F^p A/F^{p+1}A$ m-тривиален.*
Тогда A nm-тривиален.

Доказательство. $F^{n-1}A = F^{n-1}A/F^n A$, следовательно, m-тривиален.
Далее индукция по номеру члена фильтрации.
Пусть $F^{n-p}A$ pm-тривиален. $F^{n-(p+1)}A/F^{n-p}A$ m-тривиален по условию. Рассмотрим точную тройку $0 \to F^{n-p} \hookrightarrow F^{n-(p+1)} \to F^{n-(p+1)}A/F^{n-p}A \to 0$. По лемме 1 $F^{n-(p+1)}$ $pm + m = (p+1)m$-тривиален, что доказывает шаг индукции.

□

Следствием Леммы 4 является следующее общее утверждение:

Corollary 1. *Если все модули на первой странице спектральной последовательности m-тривиальны и последовательность стабилизируется на n-й странице, последовательность сходится к набору nm-тривиальных модулей.*