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Quillen-McCord theorem and homological persistence in categories of functors

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Abstract

Quillen-McCord theorem gives a sufficient condition on a map between classifying spaces of posetal categories to be a homotopy equivalence. Jonathan Ariel Barmak in his paper [Bar11] gives an elementary topological proof and proves a homological version of the theorem.

We formulate and prove the homological Quillen-McCord theorem in the setting of persistent homology using the technique of interleaving distances (according to [GS16]) and generalized persistence theorem ([Corbet18]). To establish the technique we introduce persistence objects as objects in appropriate functor categories and prove several results, e.g. order extension principle for objects in Fun(I, Pos) and approximate triviality of left derived functors of approximately trivial objects in Fun(I, R-Mod).

Since the given proof gives explicit Lipschitz constant for the map of persistence classifying spaces, we expect this result to be useful in TDA for reducing the complexity of experimental data.

1 Introduction

Computation of the homotopy type of an arbitrary CW complex is an open problem. It is theoretically possible to compute homology groups in all cases, but in practice such computations are limited by resources. Compexity generally grows with a dimension of a complex. Hence there is an optimization problem — given CW complex B, construct CW complex A such that $\dim(A) < \dim(B)$ and $H_\star(A) \cong H_\star(B)$.

The Alexandrov-Čech theorem is proven [1, 2] to be a useful tool in applications. One can associate to a given covering $\mathcal{U}$ a partially ordered set of sets $U \in \mathcal{U}$ and all their intersections, ordered by inclusion. The classifying space of this poset (to be precise, its posetal category) is a barycentric subdivision of geometric realization of the nerve of $\mathcal{U}$, endowed with orientation, hence these two spaces are homeomorphic. Given this observation, one can formulate the theorem in terms of classifying spaces.

Jonathan Barmak proves [Bar11] the homological version of Quillen-McCord theorem (also known as Quillen fiber lemma or Quillen's theorem A for posets). It is stated as follows.

Theorem. *Assume X, Y are finite posets, $f : X \to Y$ is an order-preserving map, R is a PID.*
If $\forall y \in Y$ $H_i(\mathcal{B}(f^{-1}(Y_{\leqslant y})), R) = 0$ for any i, $\mathcal{B}f$ induces isomorphisms of all homology groups with coefficients in R on BX and BY.

Provided an algorithm for construction of X and a map by Y, this theorem may provide a partial solution of stated optimization problem, in particular, for oriented nerves of coverings.

Coverings we operate may come from series of observations, for instance, from long experiment. While considering the series in a whole, we are interested in persistence of nerves. To operate experimental data some stable version of the theorem is required.

In this paper, we formulate and prove the persistent homological version of Quillen-McCord theorem.

Theorem. *Assume X, Y are persistence posets of finite type indexed by very good monoid I, $f : X \to Y$ is an order-preserving map. Let m be the number of elements of Y and R is a PID.*

Then if $\forall y = (\dots, y_i, \dots) \in Y$ $\mathcal{B}(f^{-1}(Y_{\leqslant y}))$ is ε-acyclic over R, BX and BY are $4m\varepsilon$-interleaved over R.

The paper is structured as follows.

2. In preliminaries we give an outline of well-known notions used throughout the proposal.
   2.1. First preliminary subsection is devoted to a notion of interleaving distance (following [GS16]) which fits as a required measure of similarity between series. Usability of this notion is guaranteed by persistence theorem [Zomorodian05]. We try to keep some level of generality, in particular, we work with a generalized version of the theorem proven in [Corbet18] and formulate it. This generality is necessary for applications and allows us to operate both discrete and continuous models of time in an experiment.
   2.2. Second subsection gives a classical formulation of the Quillen-McCord theorem and necessary definitions. In order to bind the theorem to an abstract context in which the general Quillen A theorem [Quillen72] is used, we derive it from the general theorem. Additionally, we outline Barmak's ([Bar11]) proofs of classical and homological versions of the theorem.
3. In the apparatus section we systematize and sometimes introduce various results forming the toolchain to prove the target theorem. Its necessary set of definitions is close to well-established in a field (for instance, [Bubenik15]), but the whole toolchain is independent.
   3.1. At first, we reformulate several preliminary definitions in terms of functor categories and transfer some notions, e.g. classifying space of a category and covering, to appropriate functor categories.
   3.2. In the second section we formulate and prove the linear extension principle for functors to the category of posets. We give both finite and general versions with an AC-dependent proof of the latter.
   3.3. At the end of the section we give some technical stability results concerning objects of appropriate functor categories.
4. Finally, we formulate and prove the main result — stable (w.r.t. interleaving distances) homological Quillen-McCord theorem. We adapt a proof given in ([Bar11]) using developed apparatus.

2 Preliminaries

2.1 Persistence modules and interleaving distance

We have to start with the definition of a simplicial set.

Definition 2.1. The simplex category Δ is a category of non-empty totally-ordered sets of finite length with order-preserving functions as morphisms.

Definition 2.2. *Simplicial set* is a contravariant functor from Δ to *Set*.

Proposition 2.3. *Simplicial sets form the category sSet.*

The real initial definition, representing a series of simplicial sets, is the following.

Definition 2.4. *Persistence simplicial set* is a family of simplicial sets $C_0 \xrightarrow{f_0} C_1 \xrightarrow{f_1} C_2 \xrightarrow{f_2} \dots$ where f_i are natural transformations. We call the maps $f = (\dots, f_i, \dots)$ the *structure maps* of a persistence simplicial set.

Definition 2.5. Let R be a ring. *Persistence module over* R is a family of R-modules M_i with homomorphisms $\phi_i : M^i \to M^{i+1}$ as the structure maps. We denote composition of structure maps between M_i and M_j by ϕ_{ij}.

We only consider one-dimensional persistence.

An example of a persistence module is given by homology modules of persistence simplicial set $C_\star$ (*persistent homology*). We set $H_i^j(C_\star) := H_i(C_j)$, the maps ϕ_j are induced by f_i.

Definition 2.6. Persistence simplicial set (module) is of *finite type* over R if all its simplicial sets (modules) have finite sets of vertices (are finitely generated as R-modules) and all f_i (ϕ_i) are isomorphisms for $i > m$ for some m.

Definition 2.7. Persistence module is of *finitely presented type* over R if all its modules are finitely presented as R-modules and all f_i (ϕ_i) are isomorphisms for $i > m$ for some m.

Note that by construction homology of a persistence simplicial set of finite type is a persistence module of finite type. Hence, in particular, is a module of finitely presented type. We shall generally use terms of modules in this section.

One can study maps of persistence modules.

Definition 2.8. Let M and N be persistence modules. Then family f of maps $f_i : M_i \to N_i$ is called a morphism from M to N if all f_is commute with structure maps.

Definition 2.9. Let M and N be persistence modules, f is a collection of maps $f_i : M_i \to N_{i+\varepsilon}$. Then if all f_i commute with structure maps, f is called an *ε-morphism*.

There is a general notion of interleaving distance between persistence modules.

Definition 2.10. We denote by $Id_\varepsilon : M \to M$ the shift of persistence module, defined by compositions of structure maps $M_i \to M_{i+\varepsilon}$ for all i.

Definition 2.11. Persistence modules M and N are called *ε-interleaved* ($M \overset{\varepsilon}{\sim} N$) if there exists a pair of ε-morphisms ($\phi : M \to N$, $\psi : N \to M$) called an *ε-interleaving* such that $\phi \circ \psi = Id_{2\varepsilon} : N \to N$ and $\psi \circ \phi = Id_{2\varepsilon} : M \to M$.

Remark 2.12. There follows that $M \overset{\varepsilon}{\sim} N$ implies $M \overset{\alpha}{\sim} N$ for any $\alpha > \varepsilon$ since for ε-interleaving (ϕ, ψ) we have α-interleaving $(Id_{\alpha-\varepsilon} \circ \phi, Id_{\alpha-\varepsilon} \circ \psi)$.

Definition 2.13. The ε-interleaving induces an extended pseudometric on a set of persistence modules. This pseudometric is defined as $d(X, Y) = min\{\varepsilon \in I \mid X \overset{\varepsilon}{\sim} Y\}$. This pseudometric is called interleaving distance. [GS16, Definition 2.12]

There is a well-known theorem.

Theorem 1. *[Zomorodian05, Theorem 3.1]*
The category of persistence modules of finite type over Noetherian ring with unity R is equivalent to the category of graded finitely generated $R[t]$-modules.

It is proven in [Corbet18]. The authors provide a generalization that is more suitable for our needs.

Definition 2.14. Let $(G, \star)$ be a commutative monoid and $g_1, g_2 \in G$. We say that $g_1 \preceq g_2$ if $\exists h \in G: \ h \star g_1 = g_2$, h is not a neutral element.

Definition 2.15. following [Corbet18, Definition 11] Monoid $(G, \star)$ is called *good* if the following hold:

- $(G, \star)$ is commutative;
- $g_1 \star g_2 = g_1 \star g_3$ implies $g_2 = g_3$ (cancellation);
- $g_1 \preceq g_2$ and $g_2 \preceq g_1$ imply $g_1 = g_2$ (anti-symmetricity);
- For any finite $H \subseteq G$ there exists at most finitely many distinct elements (cm, common multiples) m such that $\forall h \in H: \ h \preceq m$ and there is no common multiple $m_1 \prec m$ satisfying cm (pl, partially least) (property of being weak plcm).

Definition 2.16. [Corbet18, Definition 12] Let R be a ring and G be a good monoid. *(Generalized) persistence module* is a family of R-modules M^i with homomorphisms $\phi_{ij}: M^i \to M^j$ satisfying $\phi_{ii} = Id$ and relation $\phi_{ij} \circ \phi_{jk} = \phi_{ik}$ for any $i < j < k \in G$ as the structure maps.

Theorem 2. *[Corbet18, Theorem 21]*
Let R be a ring with unity and G be a good monoid. Then the category of finitely presented graded $R[G]$-modules is isomorphic to the category of G-indexed persistence modules over R of finitely presented type.

Remark 2.17. Cancellative commutative monoid with $\preceq$ being a total order (totally ordered) is good.

Definition 2.18. We call cancellative commutative totally ordered monoids *very good*.

Example 2.19. Consider the monoid of non-negative real numbers $\mathbb{R}_{\geqslant 0}$ with addition as a monoidal operation. The addition of real numbers is cancellative and $\preceq$ is a total order. This monoid is very good.

For contrast, we can consider $\mathbb{R}$. Since $g_1 = h + g_2$ implies $g_2 = (-h) + g_1 \preceq$ is trivial and R is not a good monoid in sense of the given definition.

We shall use the multiplicative notation for monoids for compatibility with references in further constructions. This notation is inspired by standard notation in polynomial ring $R[t]$, which is a monoid ring over a free monoid on a single generator t.

We can examine how do notions related to interleaving distance look like in a category of graded modules under additional constraints on indexing set.

Definition 2.20. [GS16, Definition 2.7]
Let M and N be graded $R[G]$-modules, $f : M \to N$ be a homomorphism of modules. Then f is called *ε-morphism* if $f(M^j) \subset N^{j+\varepsilon}$.

Proposition 2.21. *Let $\mathbb{R}$ be a ring with unity and G be a very good monoid. Then for any graded $R[G]$-module M and any $\varepsilon \in \mathbb{G}$ there exists morphism $Id_\varepsilon(a) = t^\varepsilon a$, where t is a successor of identity.*

Proposition 2.22. *[GS16, Proposition 2.13] Condition $M \overset{\varepsilon}{\sim} 0$ is equivalent to condition $t^{2\varepsilon} M = 0$.*

Lemma 2.23. *Let $0 \to M \to L \to N \to 0$ be a short exact sequence of graded modules. Then the following properties hold.*

- *If $M \overset{\varepsilon_1}{\sim} 0$ and $N \overset{\varepsilon_2}{\sim} 0$ then $L \overset{\varepsilon_1+\varepsilon_2}{\sim} 0$. [GS16, Proposition 4.6]*
- *If $L \overset{\varepsilon}{\sim} 0$ then $M \overset{\varepsilon}{\sim} 0$ and $N \overset{\varepsilon}{\sim} 0$.*
- *If $M \overset{\varepsilon}{\sim} 0$ then $L \overset{2\varepsilon}{\sim} N$. [GS16, Proposition 4.1]*
- *If $N \overset{\varepsilon}{\sim} 0$ then $M \overset{2\varepsilon}{\sim} L$. [GS16, Proposition 4.1]*

Proof. The second statement of the lemma requires proof. Denote non-trivial maps in s.e.s as i and q.

Then $i(t^{2\varepsilon} a) = t^{2\varepsilon} i(a) = 0$ for any $a \in M$. i is injective. Hence by Proposition 2.22 $M \overset{\varepsilon}{\sim} 0$.

On the other side $0 = q(t^{2\varepsilon} a) = t^{2\varepsilon} q(a)$ where $q$is surjective. Hence $N \overset{\varepsilon}{\sim} 0$. $\square$

By equivalence of categories, the lemma holds for persistence modules over rings with unity indexed by very good monoids.

2.2 Quillen-McCord theorem

Definition 2.24. Let X be a set and $\mathcal{P}(X)$ be its powerset. Then a set $S \subset \mathcal{P}(X)$ is called a simplicial complex if it for any $V \in S$ and $W \subset V$ $W \in S$.

Definition 2.25. *Join* $A \star B$ of simplicial complexes A and B is the simplicial complex with simplices — all possible unions of simplices $a \in A$ and $b \in B$.

Let A be a simplicial complex.

Definition 2.26. *Star* $\mathrm{st}(x)$ of simplex $x \in A$ is the minimal by inclusion simplicial complex containing all simplices $a \in A$ such that there exists inclusion $x \hookrightarrow a$.

Definition 2.27. *Link* $\mathrm{lk}(x)$ of simplex $x \in A$ is defined as follows: $\mathrm{lk}(x) = \{v \in \mathrm{st}(x) |\ x \notin v\}$.

Proposition 2.28. $\mathrm{st}(x) = \mathrm{lk}(x) \star x$.

Definition 2.29. Functor $||: sSet \to Top$ which maps n-simplices to geometric n-simplices and morphisms to inclusions of faces and restrictions to them is called standard *geometric realization.*

Definition 2.30. *Join* of topological spaces A and B is defined as follows: $A \star B := A \sqcup_{p_0} (A \times B \times [0,1]) \sqcup_{p_1} B$, where p are projections of the cylinder $A \times B \times [0,1]$ onto faces.

The next proposition gives motivation for the definition of a join of simplicial complexes.

Proposition 2.31. $|A \star B| = |A| \star |B|$. *Hence* $|\mathrm{st}(x)|$ *is a cone over* $|x|$.

Definition 2.32. Let $\mathcal{C}$ be a small category. Then we can functorially (w.r.t. to category Cat) assign a simplicial set $\mathcal{N}(\mathcal{C})$ called *nerve of a category* to it.

Construction goes as follows:

- we assign to each object of $\mathcal{C}$ a 0-simplex and to each morphism in $\mathcal{C}$ a 1-simplex with order following corresponding arrow in a category;
- then we take the set of all morphisms as an alphabet and write all the words in it such that we can move from the first letter to the last following arrows in a category. We assign an l-simplex to a word of length l. Obviously, each commutative triangle $f, g, h = f \circ g$ in $\mathcal{C}$ gives rise to two morphisms between these words — one is replacing h with $f \circ g$ and maps to a degeneracy map, another replaces $f \circ g$ with h and maps to a face map.

By construction this object is a simplicial set.

Definition 2.33. Geometric realization $B\mathcal{C}$ of $\mathcal{N}(\mathcal{C})$ is called *classifying space* of $\mathcal{C}$.

We denote the composition of nerve and geometric realization as $\mathcal{B}$. It is a composition of functors, hence a functor. By definition $\mathcal{B}(\mathcal{C}) = B\mathcal{C}$ and we prefer notation $\mathcal{B}(f)$ to Bf.

Definition 2.34. Let $f : \mathcal{C} \to \mathcal{D}$ be a functor and d — object in $\mathcal{D}$. Then *comma category* $d \downarrow f$ is a category with objects — pairs (s, i_s) of objects in $\mathcal{C}$ and morphisms $i_s : d \to f(s)$ and morphisms — morphisms g in $\mathcal{C}$ such that triangle $i_s, i_{g(s)}, f(g)$ is commutative.

Theorem 3. *[Quillen72, Theorem A]*
If $f : C \to D$ *is a functor such that the classifying space* $B(d \downarrow f)$ *of the comma category* $d \downarrow f$ *is contractible for any object* $d \in D$*, then* f *induces a homotopy equivalence* $BC \to BD$.

The nerve construction on a posetal category yields a simplicial set with good properties that there is an unique simplex spanning each unordered set and that subsets of its simplices are also simplices. These simplicial sets form a subcategory $sCpx$ of $sSet$. There is a functor F from this subcategory to the category of simplicial complexes which forgets orientation.

Proposition 2.35. *Let X be an object of sCpx. Then $|X|$ is homeomorphic to realization of simplicial complex $[F(X)]$.*

This obvious (geometric realization forgets orientation) proposition allowed Barmak to operate simplicial complexes instead of simplicial sets in his paper.

Application of Quillen's A theorem to posets yields the following theorem (we identify poset with its posetal category).

Theorem 4. ***Quillen-McCord theorem***
Assume X, Y are finite posets, $f : X \to Y$ is an order-preserving map.
If $\forall y \in Y$ $\mathcal{B}(f^{-1}(Y_{\leqslant y}))$ is contractible, then $\mathcal{B}f$ is a homotopy equivalence between BX and BY.

Theorem 5. ***Homological Quillen-McCord theorem*** *[Bar11, Corollary 5.5]*
Assume X, Y are finite posets, $f : X \to Y$ is an order-preserving map, R is a PID.
If $\forall y \in Y$ $H_i(\mathcal{B}(f^{-1}(Y_{\leqslant y})), R) = 0$ for any i, $\mathcal{B}f$ induces isomorphisms of all homology groups with coefficients in R on BX and BY.

Proofs of both theorems are necessary to understand the text and we recall them in brief.

2.2.1 Barmak's proof of Quillen-McCord theorem

Proposition 2.36. *Variation of [Bar11, Lemma 2.2] Let $f, g : X \to Y$ be order-preserving maps between finite posets such that $\forall x$ $f(x) \leqslant g(x)$. Then $\mathcal{B}(f)$ is homotopy-equivalent to $\mathcal{B}(g)$.*

Proposition 2.37. *Note that* $\mathrm{lk}(F(\mathcal{N}(x))) = F(\mathcal{N}(X_{>x})) \star F(\mathcal{N}(X_{<x}))$. *Therefore* $|\mathrm{lk}(\mathcal{N}(x))| = \mathcal{B}(X_{>x}) \star \mathcal{B}(X_{<x})$.

The crucial observation is the existence of the following covering for any $x \in X$.

$$\mathcal{B}(X) = \mathcal{B}(X \setminus \{x\}) \cup |\mathrm{st}(\mathcal{N}(x))|\,. \tag{2.1}$$

Lemma 2.38. *Let X be a finite poset and for $x \in X$ either $\mathcal{B}(X_{>x})$ or $\mathcal{B}(X_{<x})$ is contractible. Then embedding $\mathcal{B}(X \setminus \{x\}) \hookrightarrow \mathcal{B}(X)$ is a homotopy equivalence.*

This lemma is a most important part of the proof for this paper and we recall a proof.
Proof. By hypothesis, $|\mathrm{lk}(\mathcal{N}(x))| = |\mathrm{st}(\mathcal{N}(x))| \cap \mathcal{B}(X \setminus \{x\})$ is contractible. Hence its embedding to its cone $|\mathrm{st}(\mathcal{N}(x))|$ is a homotopy equivalence by Whitehead theorem. Being a subcomplex, it is a strong deformation retract. Then, $\mathcal{B}(X \setminus \{x\})$ is a strong deformation retract of $\mathcal{B}X = |\mathrm{st}(\mathcal{N}(x))| \cup \mathcal{B}(X \setminus \{x\})$ □

Definition 2.39. Variation of [Bar11, Proposition 2.1] Let $f : X \to Y$ be an order-preserving map between posets. Denote orders ($\leqslant$) on X and Y as R_X and R_Y. Then we define poset $M(f) = X \coprod Y$ with $R = R_X \cup R_Y \cup R_f$ where $(x, y) \in R_f$ if and only if $(f(x), y) \in R_Y$.

We shall by analogy denote this poset a *mapping cylinder* of f. There are also defined canonical inclusions $i_X : X \to M(f)$ and $i_Y : Y \to M(f)$.

Proof. Quillen-McCord theorem
Let X, Y be finite posets with an order-preserving map $f : X \to Y$.

Every poset has a linear extension. Let $x_1, x_2, \dots, x_n$ be an enumeration of X in an arbitrary linear extension and $Y^r = \{x_1, \dots, x_r\} \cup Y \subset M(f)$ for any r.

Consider $Y^r_{>x_r} = Y_{\geqslant f(x_r)}$. $\mathcal{B}(Y_{\geqslant f(x_r)})$ is a cone over $\mathcal{B}(f(x_r))$. It is contractible, therefore $\mathcal{B}(Y^{r-1}) \hookrightarrow \mathcal{B}(Y^r)$ is a homotopy equivalence by Lemma 2.38. By iteration $\mathcal{B}(j) : \mathcal{B}(Y^0) = \mathcal{B}(Y) \hookrightarrow \mathcal{B}(M(f)) = \mathcal{B}(Y^n)$ is homotopy equivalence between BY and $M(f)$.

Then consider a linear extension of Y with enumeration $y_1, \dots, y_m$ and $X^r = X \cup \{y_{r+1}, \dots, y_m\} \subset M(f)$. $X^{r-1}_{\leqslant y_r} = f^{-1}(Y_{\leqslant y_r})$. The latter is contractible by the condition of the theorem. Hence $\mathcal{B}(X^r) \hookrightarrow \mathcal{B}(X^{r-1})$ is a homotopy equivalence and by transitivity $\mathcal{B}(i_X)$ is a homotopy equivalence between X and $M(f)$.

Note that $i(x) \leqslant (i_Y \circ f)(x)$. By Proposition 2.36 $\mathcal{B}(i_X)$ is homotopic to $\mathcal{B}(i_Y \circ f) = \mathcal{B}(i_Y) \circ \mathcal{B}(f)$. Hence $\mathcal{B}(f)$ is the homotopy equivalence between BX and BY. $\square$

2.2.2 Barmak's proof of homological Quillen-McCord theorem

Proposition 2.40. *[Milnor56, Lemma 2.1] Reduced homology modules with coefficients in a principal ideal domain of a join satisfy the relation* $H_{r+1}(A \star B, R) \simeq \bigoplus_{i+j=r} (H_i(A, R) \otimes_R H_j(B, R)) \oplus \bigoplus_{i+j=r-1} \operatorname{Tor}_1^{\mathrm{R}}(H_i(A, R), H_j(B, R))$.

Lemma 2.41. *Let X be a finite poset and for $x \in X$ either $H_i(\mathcal{B}(X_{<x}))$ or $H_i(\mathcal{B}(X_{>x}))$ with coefficients in a PID are equal to the homology of a point. Then embedding $\mathcal{B}(X \setminus \{x\}) \hookrightarrow \mathcal{B}(X)$ induces isomorphisms of all homology groups.*

Proof.
By Proposition 2.40 $H_i(|\mathrm{lk}(\mathcal{N}(x))|) = H_i(\mathcal{B}(X_{>x}) \star \mathcal{B}(X_{<x}))$ are trivial for all indices i. Application of Mayer-Vietoris long exact sequence to covering (2.1) yields the lemma. $\square$

Proof of the theorem is similar to the proof finishing the previous subsection. We write it here in detail in order to be able to highlight differences. Changed parts are written in italic.

Proof. Homological Quillen-McCord theorem
Let X, Y be finite posets with an order-preserving map $f : X \to Y$.

Every poset has a linear extension. Let $x_1, x_2, \dots, x_n$ be an enumeration of X in the fixed linear extension and $Y^r = \{x_1, \dots, x_r\} \cup Y \subset M(f)$ for any r.

Consider $Y^r_{>x_r} = Y_{\geqslant f(x_r)}$. $\mathcal{B}(Y_{\geqslant f(x_r)})$ is a cone over $\mathcal{B}(f(x_r))$. It is contractible, therefore $\mathcal{B}(Y^{r-1}) \hookrightarrow \mathcal{B}(Y^r)$ is a homotopy equivalence by Lemma 2.38. By iteration $\mathcal{B}(j) : \mathcal{B}(Y^0) = \mathcal{B}(Y) \hookrightarrow \mathcal{B}(M(f)) = \mathcal{B}(Y^n)$ is a homotopy equivalence between BY and $M(f)$.

Then consider a linear extension of Y with enumeration $y_1, \dots, y_m$ and $X^r = X \cup \{y_{r+1}, \dots, y_n\} \subset M(f)$. $X^{r-1}_{\leqslant y_r} = f^{-1}(Y_{\leqslant y_r})$. *The latter is acyclic over R by condition* *of the theorem. Hence $\mathcal{B}(X^r) \hookrightarrow \mathcal{B}(X^{r-1})$ induces isomorphisms of all homology groups and by the functoriality of homology $\mathcal{B}(i)$ induces isomorphisms of all homology groups between X and $M(f)$.*

Note that $i(x) \leqslant (j \circ f)(x)$. By Proposition 2.36 $\mathcal{B}(i)$ is homotopic to $\mathcal{B}(j \circ f) = \mathcal{B}(j) \circ \mathcal{B}(f)$. *Homotopic maps induce the same maps on homology, j is a homotopy equivalence and induces isomorphisms. Hence $\mathcal{B}(f)$ induces isomorphisms between $H_i(BX, R)$ and $H_i(BY, R)$.* □

We see two updates. The first one is essential, it requires Lemma 2.41 and operates some equivalence propagating in a chain of length equal to the cardinality of Y. Second follows automatically from the functoriality of all used constructions.

3 Apparatus for main results

3.1 Persistence objects and related constructions

We have two types of persistence objects with similar definitions. It is appealing to form a general notion of a persistence object such that these definitions fall into special cases.

Definition 3.1. Consider I — the poset category of a fixed linearly ordered set. There is a sequence category $Fun(I, \mathcal{C})$ of functors from I to some category $\mathcal{C}$. We call objects of this category *persistence objects over $\mathcal{C}$*

Example 3.2.

- Persistence complex is a persistence object over the category of chain complexes;
- Persistence R-module is a persistence object over the category of R-modules;
- Persistence simplicial set is a persistence object over the category $sSet$;
- Persistence poset is a persistence object over Pos;
- Persistence topological space is a persistence object over Top.

Definition 3.3. We denote images of morphisms in I as *structure maps* of a persistence object over $\mathcal{C}$. For countable I it is generally enough to consider the generating set of these morphisms, i.e. the set of indecomposable morphisms.

We use the notation (X, ϕ) for "Persistence object X with structure maps ϕ over fixed indexing category I". We use notation ϕ_{ij} for a structure map between X_i and X_j.

Consider $\mathcal{F}$ — a functor from $\mathcal{C}$ to $\mathcal{D}$. It naturally extends to a functor between $Fun(I, \mathcal{C})$ and $Fun(I, \mathcal{D})$. Let P be a persistence poset. Apparently $\mathcal{B}(P)$ is a persistence topological space.

Definition 3.4. Persistence topological space X is called *ε-acyclic* over R if for all indices $H_i(X, R) \overset{\varepsilon}{\sim} H_i(pt, R)$.

Definition 3.5. *Persistence poset of finite type* is a finite sequence of finite posets.

In terms of functors "finite sequence" means that only the finite set of indices has a nontrivial image. It's convenient to define "trivial object" as the initial object where present.

The property of being of finite type is preserved by functors that map initial objects to initial.

Proposition 3.6. *Let X be a persistence poset of finite type. Then BX has homology modules of finite type.*

Proof. The nerve of an empty poset is an empty simplicial set, which is an initial object. Hence BX is a geometric realization of a simplicial complex of finite type. The homology of BX can be computed as the homology of this simplicial complex. Hence homology modules of BX are of finite type. □

Corollary 3.7. *In particular, BX has finitely presented homology modules.*

There is a general fact — if some universal object exists in C, it can be constructed component-wise in $Fun(I, C)$. Let's inspect this component-wise construction of universal objects by example.

Definition 3.8. Let $f = (f_1, \ldots, f_n, \ldots)$ be a map of persistence posets (X, ϕ) and (Y, ψ). Then mapping cylinders $M(f_j)$ form persistence poset $M(f)$ with structure maps arising from the universal property of coproduct and structure maps of X and Y.

To be explicit consider the following diagram in Pos with i_X, j_Y being series of canonical inclusions.

$$\begin{tikzcd}
X_j \arrow[r, "i_{Xj}"] \arrow[d, "\phi_{i_X}"] & X_j \coprod Y_j \arrow[d, "\zeta_j"] & Y_j \arrow[l, "i_{Yj}"'] \arrow[d, "\psi_{i_Y}"] \\
X_{j+1} \arrow[r, "i_{Xj+1}"] & X_{j+1} \coprod Y_{j+1} & Y_{j+1} \arrow[l, "i_{Yj+1}"']
\end{tikzcd} \tag{3.1}$$

The existence of order-preserving maps ζ_j is guaranteed by the universal property of coproducts, they are set to be structure maps of $M(f)$. $(M(f), \zeta_j)$ is a *mapping cylinder of map of persistence posets*.

We also have canonical inclusions i_X and i_Y arising from the same diagram.

Other examples of such component-wise constructions are kernels, cokernels, and homology modules.

We can also define a subobject in $Fun(I, \mathcal{C})$.

Definition 3.9.
Consider persistence object (X, ϕ) and subobjects Y_i of X_i.

Then if structure maps ϕ admit all pullbacks $\phi^\star$, $(Y, \phi^\star)$ is a subobject of (X, ϕ).

Pullback of $f : A \to B$ onto subobjects $[i]$ and $[j]$ is a map $f^\star$ between preimages of subobjects — isomorphism classes S, T of objects such that the diagram

$$\begin{tikzcd} S \arrow[r, "{[i]}"] \arrow[d, "f^\star"] & A \arrow[d, "f"] \\ T \arrow[r, "{[j]}"] & B \end{tikzcd} \tag{3.2}$$

commutes.

Illustrative example is given by definition in $Fun(I, Pos)$.

Definition 3.10.
Consider persistence poset (X, ϕ) and sets $Y_i \subset X_i$.

If there is no element y in any Y_i such that $\phi(y) \notin Y_{i+1}$ then component-wise inclusion $Y \to X$ commutes with structure maps and form an embedding of persistence posets. In this case $Y = (\dots, Y_i, \dots)$ is a *persistence subposet* of X.

We can also define an *element* of a persistence poset.

Definition 3.11. Let (X, ϕ) be a persistence poset, $x_i \in X_i$ such that x_i has no preimages under structure maps. In all components with degrees above i there exist elements $\phi(x_i)$ in components. We define $x_j = \perp$ for $j < i$ for consistency of the definition.

Although this definition operates the contradiction it is eliminated by applications. For instance, we can consider the following example of a subposet.

Example 3.12. Let x be an element of the persistence subposet (X, ϕ). Then we can define $X_{<x}$ component-wise as the poset of elements less than x and as a trivial poset if $x_i = \perp$.

$X_{<x}$ contains component-wise only elements comparable with x. Since structure maps are order-preserving, $X_{<x}$ is a subposet.

Finally, we give a definition of *persistence covering*.

Definition 3.13. Assume a poset X splits into subposets X_j. It is not a trivial assumption since arbitrary split can lose some structure maps. Then every subposet X_j has its own classifying space BX_j. If these spaces (or minimal open sets containing them) cover the whole BX, they are called a *persistence covering*.

This definition gives an example of how structures in the category of persistence posets can be transferred to other persistence categories. It is possible to reformulate the definition as internal to the category of persistence topological spaces but we prefer to keep a more constructive way.

3.2 Order extension principle for persistence posets

In his proof of the Quillen-McCord theorem Barmak relies on the order extension principle. To be able to transfer Barmak's proof of Quillen-McCord theorem to the persistent case we have to stress a similar statement for persistence posets.

Definition 3.14. We denote as *extension of persistence poset* X series of partially-ordered extensions of X_i such that the structure maps of X are well-defined on these extensions. If extensions of all components are linear we call this series linear extension.

Proposition 3.15. ***Transfer of order.*** *Let f be a morphism between posets X and Y and $\overline{Y}$ be a linear extension of Y. Then f induces partially ordered extension $\hat{X}$ of X such that f is well-defined as map $\hat{X} \to \overline{Y}$.*

Proof. Consider two incomparable points $a, b \in X$ and map $f : X \to \overline{Y}$ which is obviously well-defined.
One of the following hold.

- f(b) $<$ f(a)
- f(a) $<$ f(b)
- f(a) $=$ f(b)

If strict inequality holds, we can impose a single relation on a and b — we inherit relation from images.
If equality holds we do not add any new relation. □

Proposition 3.16. ***Left propagation of linear extension.*** *Assume indexing set I^{op} of persistence poset (X, ϕ) is well-founded and there exists maximal index i such that $X_j = \emptyset$ for any $j > i$. Then (X, ϕ) has a linear extension.*

Proof. We can extend the order on the component X_i to linear. Given this order, we can transfer it to the left via all possible structure maps. We obtain an extension of (X, ϕ) because all preimages of incomparable elements were incomparable and we have equipped them with compatible orders. Now let's assume we obtained linear orders in components X_j for $j > j_0$ by this construction. Then X_{j_0} can be linearly extended. The proposition follows by transfinite induction and by simple induction if X is finite. □

This statement can also be seen as a corollary of a more general proposition. One can consider set $E(X, \phi)$ of extensions of persistence poset (X, ϕ) with partial order defined as follows: let Y, Z be extensions of X, then $Y \geqslant Z$ if and only if Y is an extension of Z. Since underlying sets of these extensions are always the same we can identify elements of $E(X, \phi)$ with tuples of order relations on components of X.

Proposition 3.17. *Every linearly ordered subset of $E(X, \phi)$ has an upper bound in $E(X, \phi)$.*

Proof. Let $\{R^s | \ s \in S\}$ be a linearly ordered subset of $E(X, \phi)$ indexed by set S. Consider $R = \bigcup R^s$ where the union is taken component-wise. Assume for some elements $a, b \in X_i$ for some i some ϕ_{ij} cannot be defined on them as an order-preserving map. Then there exists $s \in S$ such that a and b are comparable in extension R^s. But R^s is an extension, hence ϕ_{ij} is defined on both a and b. By contradiction, proposition follows. □

Proposition 3.18. ***Persistent order extension principle.*** *Every persistence poset (X, ϕ) has linear extension.*

Proof. By Zorn's lemma $E(X, \phi)$ has maximal element M. Assume this element is not a linear extension of (X, ϕ).

Then in some M_i there exists incomparable pair (a, b). Consider $\phi_{ij}(a)$ and $\phi_{ij}(b)$ for all $j > i$. Assume without loss of generality for $j_2 > j_1$ that $\phi_{ij_1}(a) > \phi_{ij_1}(b)$ and $\phi_{ij_2}(a) < \phi_{ij_2}(b)$. Then the map $\phi_{j_1 j_2}$ cannot be defined. Hence any j relation between images of a and b has the same sign if exists. If there exists such j that relation between $\phi_{ij}(a)$ and $\phi_{ij}(b)$ exists we define relation between a and b accordingly. Otherwise, we define it arbitrarily and propagate order to the left — all preimages of a and b were incomparable and are now equipped with compatible orders.

We have constructed an extension of M which is taken to be maximal. Hence M must be a linear extension of (X, ϕ). $\square$

3.3 Approximation distances

Persistence modules give us the first example of an approximation distance. We can infer some results.

Proposition 3.19.
Let A, B be two persistence modules such that $d(A, 0) \leqslant \varepsilon$ and $d(B, 0) \leqslant \varepsilon$. Then $d(A \oplus B, 0) \leqslant \varepsilon$.

Proof. $\alpha A \oplus \alpha B = \alpha(A \oplus B)$. The result follows by Proposition 2.22 via Theorems 1 and 2. $\square$

Proposition 3.20.
Let A, B be two persistence modules such that $d(A, 0) \leqslant \varepsilon$ and $d(B, 0) \leqslant \varepsilon$. Then $d(A \otimes B, 0) \leqslant \varepsilon$.

Proof. The result follows from bilinearity of the tensor product and Theorems 1 and 2. $\square$

Proposition 3.21. *Let $P = \ldots \to P_n \to P_{n-1} \to \ldots$ be a persistence complex such that for all i's $P_i \overset{\varepsilon}{\sim} 0$. Then the homology modules of P are ε-interleaved with 0.*

Proof. Assume d_i is a differential in a complex. We know that $0 \to \operatorname{im} d_{i+1} \to \ker d_i \to H_i(P) \to 0$ is exact and that $0 \to \ker d_i \to P_i \to P_{i-1} \to 0$ is exact. The result follows by the application of Lemma 1 twice. $\square$

Proposition 3.22. *[Mitchell81, Page 2]*
Category $Fun(I, R{-}Mod)$ has enough projectives.

Since we have enough projectives we can compute derived functors. We need the following proposition.

Proposition 3.23. *Let R be commutative ring, A and B — R-modules such that either A or B is ε-interleaved with 0. Then $Tor_i^R(A,B) \overset{\varepsilon}{\sim} 0$.*

Proof. Since R is commutative, $Tor_i^R(A,B) = Tor_i^R(B,A)$. Without loss of generality assume $B \overset{\varepsilon}{\sim} 0$. Let P be the projective resolution of A. After taking the tensor product we obtain by Proposition 3.20 sequence of modules ε-interleaved with 0. The proposition follows by Proposition 3.21. □

We can also derive the result about exact sequences.

Proposition 3.24. *Let $A \xrightarrow{f} B \xrightarrow{\phi} C \xrightarrow{g} D$ be an exact sequence in the category of persistence modules. Then if $d(A,0) \leqslant \varepsilon$ and $d(D,0) \leqslant \varepsilon$, then $B \overset{4\varepsilon}{\sim} C$.*

Proof. Under conditions of theorems 1 and 2, we can identify the category of persistence modules and the category of non-negatively graded modules.

In s.e.s $0 \to \ker f \hookrightarrow A \xrightarrow{f} \operatorname{im} f \to 0$ $\operatorname{im} f$ is ε-trivial. By exactness, it is equal to $K = \ker \phi$. On the other side from $0 \to \ker g \xrightarrow{g} D \to D \to 0$ there follows that $d(I = \operatorname{im} \phi, 0) \leqslant \varepsilon$. Hence by transitivity $K \overset{2\varepsilon}{\sim} I$.

We obtain an exact sequence $0 \to K \to B \xrightarrow{\phi} C \to I \to 0$. This sequence decomposes into sequences $0 \to K \to B \to \operatorname{coIm} \phi$ and $0 \to \operatorname{Im} \phi \to C \to I \to 0$. By lemma 2.23 we have that $d(B, \operatorname{coIm} \phi) \leqslant 2\varepsilon$ and $d(C, \operatorname{Im} \phi) \leqslant 2\varepsilon$. $coIm$ and Im are pointwise canonically isomorphic by the first isomorphism theorem for modules, hence $d(B,C) \leqslant 4\varepsilon$. □

4 Main results

Proposition 4.1. *Let A and B be two persistence topological spaces with at least one of them being ε-acyclic over R. Then $A \star B$ is ε-acyclic over R.*

Proof. All Tor-functors from Proposition 2.40 are ε-interleaved with 0 by Proposition 3.23. Hence by Proposition 3.19 the right hand side of expression of Proposition 2.40 is ε-equivalent to 0.

□

Proposition 4.2. *Let x be an element of (X, ϕ). Then coverings* (2.1) *of all components of X form the persistence covering $\mathcal{U}$ with covering sets U_1 — preimage of* $\operatorname{st}(\mathcal{N}(x))$ *under nerve functor and $U_2 = X \setminus \{x\}$.*

Proof. It suffices to check that $X \setminus \{x\}$ and preimage of $st(\mathcal{N}(x))$ are subposets.

It is evident for $X \setminus \{x\}$. Elements in the preimage of $\operatorname{st}(\mathcal{N}(x_i))$ are exactly elements comparable to x_i. Since structure maps preserve order, they do not move comparable elements to incomparable ones. Hence preimage also forms a subposet. □

Lemma 4.3. *Let (X, ϕ) be a persistence poset and for $x = (\dots, x_i, \phi(x_i), \dots) \in X$ either $\mathcal{B}(X_{<x})$ or $\mathcal{B}(X_{>x})$ is ε-acyclic. Then persistent homology of $\mathcal{B}(X \setminus \{x\})$ and $\mathcal{B}(X)$ are 4ε-interleaved.*

Proof.
By Proposition 4.1 $|\mathrm{lk}(\mathcal{N}(x))|$ is ε-acyclic.

Given persistence covering we can define Mayer-Vietoris exact sequence on persistence homology modules component-wise by gluing sequences for components over structure maps. Proposition 3.24 yields the lemma. □

Remark 4.4. If map f is a component-wise homotopy equivalence, it induces 0-interleaving of homology modules.

We are now ready to adapt known proof to Quillen-McCord theorem for persistence posets.

Theorem 6. ***Approximate Quillen-McCord theorem***
Assume X, Y are persistence posets of finite type indexed by very good monoid I, $f : X \to Y$ is an order-preserving map. Let m be the number of elements of Y and R is a PID.

Then if $\forall y = (\dots, y_i, \dots) \in Y$ $\mathcal{B}(f^{-1}(Y_{\leqslant y}))$ is ε-acyclic over R, BX and BY are $4m\varepsilon$-interleaved over R.

Proof. Approximate Quillen-McCord theorem
Let X, Y be persistence posets of finite type with an order-preserving map $f : X \to Y$.

Let $\overline{X}$ be a linear extension of X. Recall that an element of a persistence poset is an initial element $x \in X_i$ with its images under all structure maps. We can enumerate all elements of X by lexicographic order of pairs (i, r) with i — index of a component in which element is born and r — the number of its initial element in order on $\overline{X}$.

Let $Y^r \subset M(f)$ be a union of the first r elements of X and Y. It is a persistence subposet of $M(f)$ with extended order on X.

Consider $Y^r_{>x_r} = Y_{\geqslant f(x_r)}$. $\mathcal{B}(Y_{\geqslant f(x_r)})$ is a component-wise cone over $\mathcal{B}(f(x_r))$. It is component-wise contractible, therefore $\mathcal{B}(Y^{r-1}) \hookrightarrow \mathcal{B}(Y^r)$ is a component-wise homotopy equivalence by Lemma 2.38. By iteration $\mathcal{B}(i_Y) : \mathcal{B}(Y^0) = \mathcal{B}(Y) \hookrightarrow \mathcal{B}(M(f)) = \mathcal{B}(Y^n)$ is a component-wise homotopy equivalence between BY and $M(f)$. Note that persistence structure is not used here.

Then consider a linear extension of Y with the enumeration of elements and $X^r \subset M_i(f)$ constructed analogously. $X^{r-1}_{\leqslant y_r} = f^{-1}(Y_{\leqslant y_r})$. Latter is ε-acyclic over $\mathbb{F}$ by the condition of the theorem. Hence homology modules of $\mathcal{B}(X^r)$ and $\mathcal{B}(X^{r-1})$ are 4ε-interleaved. By transitivity of ε-equivalence homology of BX and $M(f)$ are $4m\varepsilon$-interleaved.

We have that $H_i(BX) \overset{4m\varepsilon}{\sim} H_i(M(f))$ and $H_i(M(f)) \overset{0}{\sim} H_i(BY)$ for all i. Hence $H_i(BX) \overset{4m\varepsilon}{\sim} H_i(BY)$. □

We expect the stronger statement with no conditions on R to be true with another error multiple and that it can be proved using the technique of this paper by considering error propagation in Kunneth spectral sequence.

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