# 1 Abstract

Common approach in Topological Data Analysis is to compute homology groups of some persistence complex constructed from experimental data and try to classify some objects by them. Complexity of computation of persistent homology is high enough to limit practical usage of the approach.

It raises a question of how to construct persistence complex with same homology as initially given with smaller dimension. Some of the complexes arise from dynamics of some partially ordered set by nerve construction and Quillen-McCord theorem (Quillen fiber lemma) gives sufficient condition when map between posets induces a homotopy equivalence in classifying spaces of their poset categories. In this paper we prove Quillen-McCord theorem in the setting of persistent homology. We believe it to be the tool to reduce complexity of observed persistence complex.

While proving the theorem we develop a setting convenient to prove approximate statements about persistence objects and define general persistence object as object in appropriate functor category.

# 2 Preliminaries

## 2.1 Persistence modules and interleaving distance

Basic definition is the following:

**Definition 1.** [Zomorodian05, Definition 3.1]
*Persistence complex* is a family of chain complexes $C_\star^{i_0} \xrightarrow{f_{i_0}} C_\star^{i_1} \xrightarrow{f_{i_1}} C_\star^{i_2} \xrightarrow{f_{i_2}} \ldots$ where $I$ is the linearly ordered set of indices and $f_i$ are chain maps. We call maps $f = (\ldots, f_i, \ldots)$ *the structure maps* of a persistence complex.

These objects naturally arise in experiments. If we consider $I$ as a time, this is a structure we obtain while observing some dynamic structure which can at any time be represented by a chain complex.

**Definition 2.** Let $R$ be a ring. *Persistence module* is a family of $R$-modules $M^i$ with homomorpisms $\phi_i :: M^i \to M^{i+1}$ as the structure maps. If $R$ is a field it is natural to use notion of *persistence vector space*.

Example of a persistence module is given by homology modules of persistence complex $C_\star$ (*persistent homology*). $H_i^j(C_\star) = H_i(C_j)$, maps $\phi_j$ are induced by $f_i$.

**Definition 3.** Persistence complex (module) is of *finite type* over $R$ if all components of complexes (modules) are finitely-generated as $R$-modules and all $f_i$ ($\phi_i$) are isomorphisms for $i > m$ for some $m$.

**Definition 4.** Persistence complex (module) is of *finitely presented type* over $R$ if all components of complexes (modules) are finitely-presented as $R$-modules and all $f_i$ ($\phi_i$) are isomorphisms for $i > m$ for some $m$.

Note that by construction homology of complex of finite type is a module of finite type. It follows from the fact that quotient of finitely generated module by finitely generated module is finitely generated. Analogous statement fails for complexes of finitely presented type.

**Definition 5.** Assume $I$ is a semigroup. Then *graded module* over $I$-graded ring $R$ is an $R$-module $M$ together with a decomposition $M = \bigoplus_{j \in I} M^j : \forall i \in I \, R_i \cdot M^j \subset M^{j+i}$

Graded modules over $R$ form a category. Maps $\phi$ between modules $M$ and $N$ such that $\phi(M^j) \subset N^j$ for all $j$ in indexing set are morphisms in this category.

There is a well-known theorem:

**Theorem 1.** *[Zomorodian05, Theorem 3.1]*
*Category of persistence modules of finite type over Noetherian ring with unity $R$ indexed by $\mathbb{N}$ is equivalent to category of graded finitely generated $R[t]$-modules.*

It is proven in [Corbet18]. Authors provide generalization which is more suitable for our needs.

**Theorem 2.** *[Corbet18, Theorem 21]*
*Let $R$ be a ring with unity and $G$ be a monoid with linear order. Then the category of finitely presented graded $R[G]$-modules is isomorphic to the category of $G$-indexed persistence modules over $R$ of finitely presented type.*

*Remark* 1.
The most practically reasonable examples are given by considering corollary Theorem with $G$ is either $\mathbb{R}$ or $\mathbb{Z}$. These examples represent continuous and discrete time in observation with ability to reason about the past.

This equivalence is required to introduce a measure of similarity between persistence modules. It is essential since in applications we have to accept error in experimental data and we must decide whether observed homology modules are close enough to initial hypothesis or not.

**Definition 6.** [GS16, Definition 2.7]
Let $M$ and $N$ be graded $R[G]$-modules, $f : M \to N$ be a homomorphism of modules. Then $f$ is called *$\varepsilon$-morphism* if $f(M^j) \subset N^{j+\varepsilon}$.

*Remark* 2. Note that 0-morphism is a morphism in category of graded modules over $R[G]$.

**Proposition 1.** *Let $\mathbb{R}$ be a ring with unity and $G$ a monoid with linear order. Then for any graded $R[G]$-module $M$ and any $\varepsilon \in \mathbb{G}$ there exists morphism $Id_\varepsilon(a) = t^\varepsilon a$.*

This proposition follows from definition of monoid ring and does not require any assumptions on $R$ and ideals in $R[G]$.

**Definition 7.** Graded modules $M$ and $N$ are called *$\varepsilon$-interleaved* ($M \overset{\varepsilon}{\sim} N$) if there exists pair of $\varepsilon$-morphisms ($\phi : M \to N, \ \psi : N \to M$) (*$\varepsilon$-interleaving*) such that $\phi \circ \psi = Id_{2\varepsilon} : N \to N$ and $\psi \circ \phi = Id_{2\varepsilon} : M \to M$.

*Remark* 3. There follows that $M \overset{\varepsilon}{\sim} N$ implies $M \overset{\alpha}{\sim} N$ for any $\alpha > \varepsilon$ since for $\varepsilon$-interleaving $(\phi, \psi)$ we have $\alpha$-interleaving $(Id_{\alpha-\varepsilon} \circ \phi, Id_{\alpha-\varepsilon} \circ \psi)$.

**Definition 8.** We denote as *$\varepsilon$-equivalence* relation with the following properties:
For any $M$, $N$, $L$, $\varepsilon$, $\varepsilon_1$, $\varepsilon_2$

- $M \overset{0}{\sim} M$.
- $M \overset{\varepsilon}{\sim} N$ is equivalent to $N \overset{\varepsilon}{\sim} M$.
- if $M \overset{\varepsilon_1}{\sim} N$ and $N \overset{\varepsilon_2}{\sim} L$ then $M \overset{\varepsilon_1+\varepsilon_2}{\sim} L$.

**Proposition 2.** *$\varepsilon$-interleaved non-negatively graded modules are $\varepsilon$-equivalent.*

**Proposition 3.** *[GS16, Proposition 2.13] Condition $M \overset{\varepsilon}{\sim} 0$ is equivalent to condition $t^{2\varepsilon}M = 0$.*

**Definition 9.** Any $\varepsilon$-equivalence induces an extended pseudometric on its domain. This pseudometric is defined as $d(X, Y) = min\{\varepsilon \in I \mid X \overset{\varepsilon}{\sim} Y\}$. For graded modules this pseudometric is called interleaving distance. [GS16, Definition 2.12]
We shall refer to this general pseudometric as to approximation distance since we do not want to overload the term, however we do not give another examples expect for interleaving distance. For deeper evaluation of this construction see [deSilva18].

**Lemma 1.** *Let $0 \to M \to L \to N \to 0$ be a short exact sequence of non-negatively graded modules. Then the following properties hold:*

- *If $M \overset{\varepsilon_1}{\sim} 0$ and $N \overset{\varepsilon_2}{\sim} 0$ then $L \overset{\varepsilon_1+\varepsilon_2}{\sim} 0$. [GS16, Proposition 4.6]*
- *If $L \overset{\varepsilon}{\sim} 0$ then $M \overset{\varepsilon}{\sim} 0$ and $N \overset{\varepsilon}{\sim} 0$.*
- *If $M \overset{\varepsilon}{\sim} 0$ then $L \overset{2\varepsilon}{\sim} N$. [GS16, Proposition 4.1]*
- *If $N \overset{\varepsilon}{\sim} 0$ then $M \overset{2\varepsilon}{\sim} L$. [GS16, Proposition 4.1]*

**Proof.** Second statement of the lemma requires the proof. Denote non-trivial maps in s.e.s as $i$ and $q$.

Then $i(t^{2\varepsilon}a) = t^{2\varepsilon}i(a) = 0$ for any $a \in M$. $i$ is injective. Hence by Proposition 3 $M \overset{\varepsilon}{\sim} 0$.

On the other side $0 = q(t^{2\varepsilon}a) = t^{2\varepsilon}q(a)$ where $q$is surjective. Hence $N \overset{\varepsilon}{\sim} 0$. □

## 2.2 Quillen-McCord theorem

### 2.2.1 Statement

**Definition 10.** *Join* $A \star B$ of simplicial complexes $A$ and $B$ is the simplicial complex with simplices — all possible unions of simplices $a \in A$ and $b \in B$.

**Definition 11.** *Join* of topological spaces $A$ and $B$ is defined as follows: $A \star B := A \sqcup_{p_0} (A \times B \times [0, 1]) \sqcup_{p_1} B$, where $p$ are projections of the cylinder $A \times B \times [0, 1]$ onto faces.

**Definition 12.** *Star* $\mathrm{st}(x)$ *of simplex* $x \in A$ ($A$ is a simplicial complex) is the minimal simplicial complex containing all simplices $a \in A$ such that there exists inclusion $x \hookrightarrow a$.

**Definition 13.** *Link* $\mathrm{lk}(x)$ of simplex $x \in A$ is defined as follows: $\mathrm{lk}(x) = \{v \in \mathrm{st}(x) | \ x \notin v\}$.

**Definition 14.** Let $\Delta$ be a simplicial category. Functor $|| : \Delta \to Top$ which maps simplices to geometric simplices of corresponding dimension and morphisms to inclusions of faces and restrictions to subcomplexes is called standard *geometric realization*.

**Proposition 4.** $\mathrm{st}(x) = \mathrm{lk}(x) \star x$.

**Proposition 5.** $|A \star B| = |A| \star |B|$. *Hence* $|\mathrm{st}(x)|$ *is a cone over* $|x|$.

Let $\mathcal{C}$ denote a small category. We can construct the simplicial set called the *nerve of category* $\mathcal{C}$ as follows:
Let objects of $\mathcal{C}$ be the only 0-dimensional simplices. Then let all morphisms be 1-dimensional simplices, all composable pairs of morphisms be 2-dimensional simplices and so on with morphisms — inclusions of compositions into longer ones and replacements of compositions $i \circ j$ with $f = i \circ j$.
This construction is functorial over category of posets $Pos$, we denote nerve functor from $Pos$ to category of simplicial sets over $\Delta$ as $\mathcal{N}$.

**Definition 15.** Geometric realization $B\mathcal{C}$ of $\mathcal{N}(\mathcal{C})$ is called *classifying space* of $\mathcal{C}$.

We denote composition of nerve and geometric realization as $\mathcal{B}$. It is obviously a functor and we only need this construction to avoid confusing notation $Bf$ for induced morphism of classifying spaces. By definition $\mathcal{B}(\mathcal{C}) = B\mathcal{C}$.

**Definition 16.** Let $f : \mathcal{C} \to \mathcal{D}$ be a functor and $d$ — object in $\mathcal{D}$. Then *comma category* $d \downarrow f$ is a category with objects — pairs $(s, i_s)$ of objects in $\mathcal{C}$ and morphisms $i_s : d \to f(s)$ and morphisms — morphisms $g$ in $\mathcal{C}$ such that triangle $i_s, i_{g(s)}, f(g)$ is commutative.

**Theorem 3.** *[Quillen72, Theorem A]*
*If* $f : C \to D$ *is a functor such that the classifying space* $B(d \downarrow f)$ *of the comma category* $d \downarrow f$ *is contractible for any object* $d \in D$*, then* $f$ *induces a homotopy equivalence* $BC \to BD$*.*

Nerve construction on a poset category yields a simplicial complex called *order complex*. Application of Quillen A theorem to posets yields the following theorem (we identify poset with its poset category):

**Theorem 4.** ***Quillen-McCord theorem***
*Assume* $X, Y$ *are finite posets,* $f : X \to Y$ *is order-preserving map.*
*If* $\forall y \in Y$ $\mathcal{B}(f^{-1}(Y_{\leqslant y}))$ *is contractible, then* $\mathcal{B}f$ *is a homotopy equivalence between* $BX$ *and* $BY$*.*

Proof of the theorem is necessary for construction of a desired result.

### 2.2.2 Proof

Let us outline the proof of Quillen-McCord theorem given by Barmak.

**Proposition 6.** *[Bar11, Proposition 2.1] Let $X$ and $Y$ be two simplicial complexes such that $X \cup Y$ is a simplicial complex. Then if $|X| \cup |Y| \hookrightarrow |X|$ is a homotopy equivalence, then so is $|Y| \hookrightarrow |X \cup Y|$.*

**Proposition 7.** *Variation of [Bar11, Proposition 2.2] Let $f, g : X \to Y$ be order-preserving maps between finite posets such that $\forall x \; f(x) \leqslant g(x)$. Then $\mathcal{B}(f)$ is homotopy-equivalent to $\mathcal{B}(g)$.*

**Proof.** $X$ has a finite set $M$ of maximal elements. Take any of them $(m_1)$ and define $h_1 : X \to Y$ such that $h_1(m_1) = g(m_1)$ and $h_1 = f$ on all other elements of $X$. This is an order-preserving map due to maximality of $m_1$. Take $M \setminus \{m_1\}$ and build $h_2$ which is equal to $h_1$ on complement to $m_2$ and to $g$ on $m_2$ and so on. We have built a finite sequence of maps $h_0 = f \leqslant h_1 \leqslant h_1 \leqslant \ldots \leqslant g = h_n$.

Elements $h_i(m_i)$ and $h_{i-1}(m_i)$ are comparable. Hence there exists simplex $\{h_i(m_i); h_{i-1}(m_i)\}$ in $\mathcal{N}(Y)$. Since $\mathcal{N}(Y)$ is a simplicial complex, for other elements between selected where exist no holes and thus where is a linear homotopy between $h_{i-1}$ and $h_i$ which contracts simplex $\{h_i(m_i); h_{i-1}(m_i)\}$.

Hence there is a homotopy between $f$ and $g$. □

**Proposition 8.** *Note that* $\mathrm{lk}(\mathcal{N}(x)) = \mathcal{N}(X_{>x}) \star \mathcal{N}(X_{<x})$. *Therefore* $|\mathrm{lk}(\mathcal{N}(x))| = \mathcal{B}(X_{>x}) \star \mathcal{B}(X_{<x})$.

*Remark* 4. $\mathcal{B}(X) = \mathcal{B}(X \setminus \{x\}) \cup |\mathrm{st}(\mathcal{N}(x))|$ for any $x \in X$.

**Lemma 2.** *Let $X$ be a finite poset and for $x \in X$ either $\mathcal{B}(X_{>x})$ or $\mathcal{B}(X_{<x})$ is contractible. Then embedding $\mathcal{B}(X \setminus \{x\}) \hookrightarrow \mathcal{B}(X)$ is a homotopy equivalence.*

**Proof.** By proposition $|\mathrm{lk}(\mathcal{N}(x))|$ is contractible. Hence its embedding to its cone $|\mathrm{st}(\mathcal{N}(x))|$ is homotopy equivalence by Whitehead theorem. Lemma follows by Proposition 6. □

**Definition 17.** Variation of [Bar11, Proposition 2.1] Let $f : X \to Y$ be an order preserving map between posets. Denote orders $(\leqslant)$ on $X$ and $Y$ as $R_X$ and $R_Y$. Then we define poset $M(f) = X \coprod Y$ with $R = R_X \cup R_Y \cup R_f$ where $(x, y) \in R_f$ if and only if $(f(x), y) \in R_Y$.

We shall by analogy denote this poset a *mapping cylinder* of $f$. There are also defined canonical inclusions $i_X : X \to M(f)$ and $i_Y : Y \to M(f)$.

**Proof. Quillen-McCord theorem**
Let $X, Y$ be finite posets with order-preserving map $f : X \to Y$.

Every poset has linear extension. Let $x_1, x_2, \ldots, x_n$ be enumeration of $X$ in such linear order and $Y^r = \{x_1, \ldots, x_r\} \cup Y \subset M(f)$ for any $r$.

Consider $Y^r_{>x_r} = Y_{\geqslant f(x_r)}$. $\mathcal{B}(Y_{\geqslant f(x_r)})$ is a cone over $\mathcal{B}(f(x_r))$. It is contractible, therefore $\mathcal{B}(Y^{r-1}) \hookrightarrow \mathcal{B}(Y^r)$ is homotopy equivalence by Lemma 2. By iteration $\mathcal{B}(j) : \mathcal{B}(Y^0) = \mathcal{B}(Y) \hookrightarrow \mathcal{B}(M(f)) = \mathcal{B}(Y^n)$ is homotopy equivalence between $BY$ and $M(f)$.

Then consider linear extension of $Y$ with enumeration $y_1, \dots, y_m$ and $X^r = X \cup \{y_{r+1}, \dots, y_m\} \subset M(f)$. $X^{r-1}_{\leqslant y_r} = f^{-1}(Y_{\leqslant y_r})$. Latter is contractible by condition of the theorem. Hence $\mathcal{B}(X^r) \hookrightarrow \mathcal{B}(X^{r-1})$ is homotopy equivalence and by transitivity $\mathcal{B}(i_X)$ is a homotopy equivalence between $X$ and $M(f)$.

Note that $i(x) \leqslant (i_Y \circ f)(x)$. By Proposition 7 $\mathcal{B}(i_X)$ is homotopic to $\mathcal{B}(i_Y \circ f) = \mathcal{B}(i_Y) \circ \mathcal{B}(f)$. Hence $\mathcal{B}(f)$ is the homotopy equivalence between $BX$ and $BY$. $\square$

### 2.2.3 Proof of homological version

There holds homological [Bar11, Corollary 5.5] versions of this theorem:

**Theorem 5.** ***Homological Quillen-McCord theorem***
*Assume $X, Y$ are finite posets, $f : X \to Y$ is order-preserving map, $R$ is a PID.*
*If $\forall y \in Y$ $H_i(\mathcal{B}(f^{-1}(Y_{\leqslant y})), R) = 0$ for any $i$, $\mathcal{B}f$ induces isomorphisms of all homology groups with coefficients in $R$ on $BX$ and $BY$.*

To derive the theorem we can variate the proof of standard version in the following manner:

**Proposition 9.** *[Milnor56, Lemma 2.1] Reduced homology modules with coefficients in a principal ideal domain of a join satisfy the relation $H_{r+1}(A \star B, R) \simeq \bigoplus_{i+j=r} (H_i(A, R) \otimes_R H_j(B, R)) \oplus \bigoplus_{i+j=r-1} \mathrm{Tor}_1^{\mathrm{R}}(H_i(A, R), H_j(B, R))$.*

**Lemma 3.** *Let $X$ be a finite poset and for $x \in X$ either $H_i(\mathcal{B}(X_{<x}))$ or $H_i(\mathcal{B}(X_{>x}))$ with coefficients in a PID are equal to homology of a point. Then embedding $\mathcal{B}(X \backslash \{x\}) \hookrightarrow \mathcal{B}(X)$ induces isomorphisms of all homology groups.*

**Proof.**
By Proposition 9 $H_i(|\mathrm{lk}(\mathcal{N}(x))|) = H_i(\mathcal{B}(X_{>x}) \star \mathcal{B}(X_{<x}))$ are trivial for all indices $i$ — $Tor$-functors vanish if any of their arguments is trivial. Application of Mayer-Vietoris long exact sequence to covering from Remark 4 yields the lemma. $\square$

Proof of the theorem is similar to proof finishing previous section. However, we write it here in detail in order to be able to highlight differences. Changed parts are written in italic.

**Proof. Homological Quillen-McCord theorem**
Let $X, Y$ be finite posets with order-preserving map $f : X \to Y$.

Every poset has linear extension. Let $x_1, x_2, \dots, x_n$ be enumeration of $X$ in such linear order and $Y^r = \{x_1, \dots, x_r\} \cup Y \subset M(f)$ for any $r$.

Consider $Y^r_{>x_r} = Y_{\geqslant f(x_r)}$. $\mathcal{B}(Y_{\geqslant f(x_r)})$ is a cone over $\mathcal{B}(f(x_r))$. It is contractible, therefore $\mathcal{B}(Y^{r-1}) \hookrightarrow \mathcal{B}(Y^r)$ is homotopy equivalence by Lemma 2. By iteration $\mathcal{B}(j) : \mathcal{B}(Y^0) =$

$\mathcal{B}(Y) \hookrightarrow \mathcal{B}(M(f)) = \mathcal{B}(Y^n)$ is homotopy equivalence between $BY$ and $M(f)$.

Then consider linear extension of $Y$ with enumeration $y_1, \dots, y_m$ and $X^r = X \cup \{y_{r+1}, \dots, y_n\} \subset M(f)$. $X^{r-1}_{\leqslant y_r} = f^{-1}(Y_{\leqslant y_r})$. *Latter is acyclic over $R$ by condition of the theorem. Hence $\mathcal{B}(X^r) \hookrightarrow \mathcal{B}(X^{r-1})$ induces isomorphisms of all homology groups and by functoriality of homology $\mathcal{B}(i)$ induces isomorphisms of all homology groups between $X$ and $M(f)$.*

Note that $i(x) \leqslant (j \circ f)(x)$. By Proposition 7 $\mathcal{B}(i)$ is homotopic to $\mathcal{B}(j \circ f) = \mathcal{B}(j) \circ \mathcal{B}(f)$. *Homotopic maps induce same maps on homology, $j$ is a homotopy equivalence and induce isomorphisms. Hence $\mathcal{B}(f)$ induce isomorphisms between $H_i(BX, R)$ and $H_i(BY, R)$.* $\square$

We see two updates. First one is essential, it requires Lemma 3 and operates some equivalence propagating in a chain of length equal to cardinality of $Y$. Second follows automatically from functoriality of all used constructions.

# 3 Persistence objects and approximation distances

## 3.1 Persistence objects and related constructions

We strive to provide version of this theorem suitable for usage in the setting of persistence complexes and in computations. New theorem must respect interleaving distances and give an accurate measure of error propagation.
Let's develop appropriate technique before giving a statement.

We have two types of persistence objects with similar definitions. It is appealing to form general notion of persistence object such that these definitions fall into special cases.

**Definition 18.** Consider $I$ — poset category of a fixed linearly ordered set. There is a sequence category $Fun(I, \mathcal{C})$ of functors from $I$ to some category $\mathcal{C}$. We call objects of this category *persistence objects over* $\mathcal{C}$

**Example 1.**

- Persistence complex is a persistence object over category of chain complexes;
- Persistence $R$-module is a persistence object over category of $R$-modules;
- Persistence simplicial set is a persistence object over category $Psh(\Delta)$;
- Persistence poset is a persistence object over $Pos$;
- Persistence topological space is a persistence object over $Top$.

**Definition 19.** We denote images of morphisms in $I$ as *structure* maps of a persistence object over $\mathcal{C}$. For countable $I$ it is generally enough to consider generating set of these morphisms, i.e. set of morphisms which cannot be written as a composition of two nontrivial morphisms.

We use notation $(X, \phi)$ for "Persistence object X with structure maps $\phi$ over fixed indexing category $I$". Keeping in mind that indexing set must not be countable we generally use notation $\phi_i$ for a structure map between $X_i$ and $X_{i+1}$. We give both unit and indexation of structure maps no literal meaning but this notation is intuitive and incapsulates sequential nature of objects we consider.

Consider $\mathcal{F}$ — functor from $\mathcal{C}$ to $\mathcal{D}$. It naturally extends to functor between $Fun(I, \mathcal{C})$ and $Fun(I, \mathcal{D})$. Let $P$ be persistence poset. Apparently $\mathcal{B}(P)$ is a persistence topological space.

**Definition 20.** Persistence topological space $X$ is called *$\varepsilon$-acyclic* over $R$ if for all indices $H_i(X, R) \overset{\varepsilon}{\sim} H_i(pt, R)$.

**Definition 21.** We say that morphism $f : X \to Y$ between persistence topological spaces induces $\varepsilon$-interleaving if there exists $g : Y \to X$ such that induced maps of pair $(f, g)$ on graded homology modules form $\varepsilon$-interleaving.

**Definition 22.** *Persistence poset of finite type* is a finite sequence of finite posets.

In terms of functors "finite sequence"means that only finite set of indices has nontrivial image. It's convenient to define "trivial object"as initial object where present.

Property of being of finite type is preserved by functors which map initial objects to initial.

**Proposition 10.** *Let $X$ be a persistence poset of finite type. Then $BX$ has homology modules of finite type.*

**Proof.** Nerve of the empty poset is an empty simplicial set, which is an initial object. Hence $BX$ is a geometric realization of simplicial complex of finite type. Homology of $BX$ can be computed as homology of this simplicial comples. Hence homology modules of $BX$ are of finite type. □

**Corollary 1.** *In particular $BX$ has finitely presented homology modules.*

There is a general fact — if some universal object exists in $C$, it can be constructed component-wise in $Fun(I, C)$. Let's inspect this component-wise construction of universal objects by useful example:

**Definition 23.** Let $f = (f_1, \ldots, f_n, \ldots)$ be map of persistence posets $(X, \phi)$ and $(Y, \psi)$. Then mapping cylinders $M(f_j)$ form persistence poset $M(f)$ with structure maps arising from universal property of coproduct and structure maps of $X$ and $Y$.

To be explicit consider the following diagram in $Pos$ with $i_X$, $j_Y$ being series of canonical inclusions:

$$\begin{array}{ccccc} X_j & \xrightarrow{i_{Xj}} & X_j \coprod Y_j & \xleftarrow[i_{Yj}]{} & Y_j \\ \Big\downarrow \phi_{i_X} & & \Big\downarrow \zeta_j & & \Big\downarrow \psi_{i_Y} \\ X_{j+1} & \xrightarrow{i_{Xj+1}} & X_{j+1} \coprod Y_{j+1} & \xleftarrow[i_{Yj+1}]{} & Y_{j+1} \end{array}$$

Existence of order-preserving maps $\zeta_j$ is guaranteed by universal property of coproducts, they are set to be structure maps of $M(f)$. $(M(f), \zeta_j)$ is a *mapping cylinder of map of persistence posets.*

We also have canonical inclusions $i_X$ and $i_Y$ arising from the same diagram.

Other examples of such component-wise constructions are kernels, cokernels and homology modules.

We can also define a subobject in $Fun(I, \mathcal{C})$:

**Definition 24.**
Consider persistence object $(X, \phi)$ and subobjects $Y_i$ of $X_i$.

Then if structure maps $\phi$ admit all pullbacks $\phi^\star$ $(Y, \phi^\star)$ is a subobject of $(X, \phi)$.

Pullback of $f : A \to B$ onto subobjects $[i]$ and $[j]$ is a map $f^\star$ between preimages of subobjects — isomorphism classes $S$, $T$ of objects such that the diagram

$$\begin{array}{ccc} S & \xrightarrow{[i]} & A \\ \Big\downarrow f^\star & & \Big\downarrow f \\ T & \xrightarrow{[j]} & B \end{array}$$ commutes.

Illustrative example is given by definition in $Fun(I, Pos)$:

**Definition 25.**
Consider persistence poset $(X, \phi)$ and sets $Y_i \subset X_i$.

If there is no element $y$ in any $Y_i$ such that $\phi(y) \notin Y_{i+1}$ then component-wise inclusion $Y \to X$ commutes with structure maps and form embedding of persistence posets. In this case $Y = (\ldots, Y_i, \ldots)$ is a *persistence subposet* of $X$.

We can also define *element* of a persistence poset.

**Definition 26.** Let $(X, \phi)$ be a persistence poset, $x_i \in X_i$. In all components with degree above $i$ there exist elements $\phi(x_i)$ in components. In degrees below we take iterated preimages while possible and define $x_j = \perp$ for $j < i$ if there is no preimage.

Although this definition operates contradiction it is eliminated by applications.
For instance we can consider the following example of subposet:

**Example 2.** Let $x$ be an element of persistence subposet $(X, \phi)$. Then we can define $X_{<x}$ component-wise as poset of elements less than $x$ and as trivial poset if $x_i = \perp$.

$X_{<x}$ contains component-wise only elements comparable with $x$. Since structure maps are order preserving, $X_{<x}$ is a subposet.

Finally we give a definition of persistence covering.

**Definition 27.** Assume poset $X$ splits into subposets $X_j$. It is not a trivial assumption since arbitrary split can lose some structure maps. Then every subposet $X_j$ has its own classifying space $BX_j$. If these spaces (or minimal open sets containing them) cover the whole $BX$, they are called *persistence covering*.

This definition gives an example of how structures in category of persistence posets can be transferred to other persistence categories. It is possible to reformulate the definition as internal to category of persistence topological spaces but we prefer to keep more constructive way.

## 3.2 Order extension principle for persistence posets

In his proof of Quillen-McCord theorem Barmak relies on order extension principle. To be able to transfer Barmak's proof of Quillen-McCord theorem to persistent case we have to stress similar statement for persistence modules.

**Definition 28.** We denote as *linear extension of persistence poset* $X$ series of linear extensions of $X_i$ such that structure maps of $X$ are well-defined on these extensions.

Our target proposition (which we shall refer to as *persistent order extension principle*) is the following: every persistence poset has linear extension.

**Proposition 11.** ***Transfer of linear order***. *Let $f$ be a morphism between posets $X$ and $Y$ and $\overline{Y}$ be a linear extension of $Y$. Then $f$ induces at least one linear extension $\overline{X}$ of $X$ such that $f$ is well-defined as map $\overline{X} \to \overline{Y}$.*

**Proof.** Consider two incomparable points $a, b \in X$ and map $f : X \to \overline{Y}$ which is obviously well-defined.
One of the following hold:

- f(b) < f(a)
- f(a) < f(b)
- f(a) = f(b)

If strict inequality holds, we can impose a single relation on $a$ and $b$ — we inherit relation from images.
If equality holds we can take arbitrary order on a set $f^{-1}(f(a))$. □

We have the following corollary:

**Proposition 12.** *Every persistence poset $(X, \phi)$ with finite number of nonempty components has linear extension.*

**Proof.** We can extend order to linear on the component $X_i$ such that $X_j = \emptyset$ for $j > i$. Given this order we can extend it to the left by proposition. $\square$

This corollary is strong enough to be used in a proof of the target theorem. Proof of this corollary allows also to reasons about sequences infinite to the left. For sequences infinite to the right we can use the observation that *Pos* is cocomplete [Adamec94, Example 1.10 and Theorem 1.11, page 17].

**Proposition 13.** ***Persistent order extension principle****. Every persistence poset over well-founded indexing set has linear extension.*

**Proof.** Let $(X, \phi)$ be a persistence poset. It is by definition a diagram in *Pos*. *Pos* is cocomplete, hence there exists $\mathrm{colim}(X, \phi)$. In particular it is a co-cone over $(X, \phi)$ with maps $\psi_i : X_i \to \mathrm{colim}(X, \phi)$. For all $i$ $\psi_i$ transfers linear extension from $\overline{\mathrm{colim}(X, \phi)}$ to $X_i$.

Consider co-cone diagram

$$\begin{tikzcd} X_i \arrow[rr, "\phi_i"] \arrow[dr, "\psi_i"'] & & X_j \arrow[dl, "\psi_{i+1}"] \\ & \mathrm{colim}(X, \phi) & \end{tikzcd}$$

It is commutative, i.e. $\psi_i = \psi_{i+1} \circ \phi_i$. Extend colimit linearly and transfer this extension by $\psi$. It's enough to check that $\phi_i$ is well-defined as a morphism between $\overline{X_i}$ and $\overline{X_{i+1}}$.

Assume it is not. Then there exists pair of elements $a, b \in \overline{X_i}$ such that $a > b$ and $\phi_i(a) < \phi_i(b)$. There are three possibilities:

- $a > b$ is an element of order on $X_i$. It is impossible since $\phi_i$ preserves order on $X_i$.
- Relation $a > b$ is induced by relation $\psi_i(a) > \psi_i(b)$ on $\overline{\mathrm{colim}(X, \phi)}$. In this case from commutativity of the diagram we know that this relation induces relation $\phi_i(a) > \phi_i(b)$.
- $\psi_i(a) = \psi_i(b)$.

In last case we have to adjust our transferred extensions and make them less arbitrary. At first fix element $\alpha \in \overline{\mathrm{colim}(X, \phi)}$. In all components of $(X, \phi)$ order between preimages of $\alpha$ is not defined by extension of colimit. By diagram structure maps move preimages of $\alpha$ to preimages of $\alpha$, possibly not all.

For arbitrary $i \in I$ fix linear order on $\psi_i^{-1}(\alpha)$. Its back propagation is given by transfer along $\phi$, its forth propagation is given by co-cone diagram but it does not define linear order if structure maps are not surjective.

So we fix $i \in I$, fix orders on $\psi_i^{-1}$ for all elements of a colimit and perform an extension to the left. For any $i$ we have constructed persistence poset with $X_j$ linearly ordered for all $j \leqslant i$. $I$ is well-founded hence by transfinite recursion result follows. $\square$

Note that forth propagation of arbitrary linear order on any component frees us from usage of transfinite recursion. Hence we also have the following proposition:

**Proposition 14.** *Every persistence poset with surjective structure maps has linear extension.*

### 3.3 Approximation distances

Persistence modules over good enough ring (as always here and after) give us the first example of approximation distance. So we can infer some results about distances.

**Proposition 15.**
*Let $A$, $B$ be two persistence modules such that $d(A, 0) \leqslant \varepsilon$ and $d(B, 0) \leqslant \varepsilon$. Then $d(A \oplus B, 0) \leqslant \varepsilon$.*

**Proof.** $\alpha A \oplus \alpha B = \alpha(A \oplus B)$. Result follows by Proposition 3 via Theorems 1 and 2. □

**Proposition 16.**
*Let $A$, $B$ be two persistence modules such that $d(A, 0) \leqslant \varepsilon$ and $d(B, 0) \leqslant \varepsilon$. Then $d(A \otimes B, 0) \leqslant \varepsilon$.*

**Proof.** Result follows from bilinearity of tensor product and Theorems 1 and 2. □

**Proposition 17.** *Let $P = \ldots \to P_n \to P_{n-1} \to \ldots$ be a persistence complex such that for all $i$'s $P_i \overset{\varepsilon}{\sim} 0$. Then the homology modules of $P$ are $\varepsilon$-interleaved with $0$.*

**Proof.** Assume $d_i$ is a differential in a complex. We know that $0 \to \operatorname{im} d_{i+1} \to \ker d_i \to H_i(P) \to 0$ is exact and that $0 \to \ker d_i \to P_i \to P_{i-1} \to 0$ is exact. Result follows by application of Lemma 1 twice. □

**Proposition 18.** *[Mitchell81, Page 2]*
*Category $Fun(I,R{-}Mod)$ has enough projectives.*

Since we have enough projectives we can compute derived functors. We need the following proposition.

**Proposition 19.** *Let $R$ be commutative ring, $A$ and $B$ — $R$-modules such that either $A$ or $B$ is $\varepsilon$-interleaved with $0$. Then $Tor_i^R(A, B) \overset{\varepsilon}{\sim} 0$.*

**Proof.** Since $R$ is commutative, $Tor_i^R(A, B) = Tor_i^R(B, A)$. Without loss of generality assume $B \overset{\varepsilon}{\sim} 0$. Let $P$ be the projective resolution of $A$. After taking tensor product we obtain by Proposition 16 sequence of modules $\varepsilon$-interleaved with $0$. Proposition follows by Proposition 17. □

We can also derive the result about exact sequences.

**Proposition 20.** *Let $A \xrightarrow{f} B \xrightarrow{\phi} C \xrightarrow{g} D$ be exact sequence in category of persistence modules. Then if $d(A, 0) \leqslant \varepsilon$ and $d(D, 0) \leqslant \varepsilon$, then $B \overset{4\varepsilon}{\sim} C$ with $\phi$ being left morphism in interleaving pair.*

**Proof.** Under conditions of theorems 1 and 2 we can identify the category of persistence modules and the category of non-negatively graded modules.

In s.e.s $0 \to \ker f \hookrightarrow A \xrightarrow{f} \operatorname{im} f \to 0$ $\operatorname{im} f$ is $\varepsilon$-trivial. By exactness it is equal to $K = \ker \phi$. On the other side from $0 \to \ker g \xrightarrow{g} D \to D \to 0$ there follows that $d(I = \operatorname{im} \phi, 0) \leqslant \varepsilon$. Hence by transitivity $K \overset{2\varepsilon}{\sim} I$.

We obtain exact sequence $0 \to K \to B \xrightarrow{\phi} C \to I \to 0$. This sequence decomposes into sequences $0 \to K \to B \to \operatorname{coIm} \phi$ and $0 \to \operatorname{Im} \phi \to C \to I \to 0$. By lemma 1 we have that $d(B, \operatorname{coIm} \phi) \leqslant 2\varepsilon$ and $d(C, \operatorname{Im} \phi) \leqslant 2\varepsilon$. $coIm$ and $Im$ are pointwise canonically isomorphic by first isomorphism theorems for modules, hence $d(B, C) \leqslant 4\varepsilon$. □

# 4 Approximate Quillen-McCord theorem

We are ready to establish the target theorem.

**Theorem 6.** ***Approximate Quillen-McCord theorem, draft statement***
*Assume $X, Y$ are persistence posets of finite type indexed by totally ordered monoid $I$, $f : X \to Y$ is order-preserving map, and $R$ is a PID.*

*Then if $\forall y = (\ldots, y_i, \ldots) \in Y$ $\mathcal{B}(f^{-1}(Y_{\leqslant y}))$ is $\varepsilon$-acyclic over $R$, $\mathcal{B}f$ induces e-interleavings of all homology spaces over $R$ on $BX$ and $BY$.*

Value of $e$ is set later.

**Proposition 21.** *Let $A$ and $B$ be two persistence topological spaces with at least one of them being $\varepsilon$-acyclic over $R$. Then $A \star B$ is $\varepsilon$-acyclic over $R$.*

**Proof.** All $Tor$-functors from Proposition 9 are $\varepsilon$-interleaved with 0 by Proposition 19. Hence by Proposition 15 right hand side of expression of Proposition 9 is $\varepsilon$-equivalent to 0.

□

**Proposition 22.** *Let $x$ be an element of $(X, \phi)$. Then covering from Remark 4 can be extended to persistence covering $\mathcal{U}$ with covering sets $U_1$ – preimage of $\operatorname{st}(\mathcal{N}(x))$ under nerve functor and $U_2 = X \setminus \{x\}$.*

**Proof.** It suffices to check that $X \setminus \{x\}$ and preimage of $st(\mathcal{N}(x))$ are subposets.

It is evident for $X \setminus \{x\}$. Elements in the preimage of $\operatorname{st}(\mathcal{N}(x_i))$ are exactly elements comparable to $x_i$. Since structure maps preserve order, they do not move comparable elements to incomparable. Hence preimage also forms subposet. □

**Lemma 4.** *Let $(X, \phi)$ be a persistence poset and for $x = (\dots, x_i, \phi(x_i), \dots) \in X$ either $\mathcal{B}(X_{<x})$ or $\mathcal{B}(X_{>x})$ is $\varepsilon$-acyclic. Then embedding $\mathcal{B}(X \backslash \{x\}) \hookrightarrow \mathcal{B}(X)$ induces $4\varepsilon$-interleavings of all homology spaces.*

**Proof.**

By Proposition 21 $|\mathrm{lk}(\mathcal{N}(x))|$ is $\varepsilon$-acyclic.

Given persistence covering we can define Mayer-Vietoris exact sequence on persistence homology modules component-wise by gluing sequences for components over structure maps. Proposition 20 yields the lemma. □

*Remark* 5. If map $f$ is component-wise homotopy equivalence, it induces 0-interleaving of homology modules.

We are now ready to adapt known proof to Quillen-McCord theorem for persistence posets.

**Theorem 7.** ***Approximate Quillen-McCord theorem, final statement***
*Assume $X, Y$ are persistence posets of finite type indexed by $I$, $f : X \to Y$ is order-preserving map. Let $m = \max_i(|Y_i|)$ be the maximal cardinality of components of $Y$ and $R$ is a PID.*

*Then if $\forall y = (\dots, y_i, \dots) \in Y$ $\mathcal{B}(f^{-1}(Y_{\leqslant y}))$ is $\varepsilon$-acyclic over $R$, $\mathcal{B}f$ induces $4m\varepsilon$-interleavings of all homology spaces over $R$ on $BX$ and $BY$.*

**Proof. Approximate Quillen-McCord theorem**
Let $X, Y$ be persistence posets of finite type with order-preserving map $f : X \to Y$.

Let $\overline{X}$ be linear extension of $X$. Let $x_1^i, x_2^i, \dots, x_{n_i}^i$ be enumeration of $\overline{X}_i$ in its linear order and $Y_i^r = \{x_1^i, \dots, x_r^i\} \cup Y_i \subset M(f_i)$ for any $r$ up to $\max_i(n_i)$ and $i$. $Y^r = (\dots, Y_i^r, \dots)$ is a persistence subposet of $M(f)$ with extended order on $X$.

There are posets such that $n_i < r$. In these cases notation $x_r$ means that on positions with $n_i < r$ we take $x_{n_i}$. $n_i$ can be undefined, in this case $x_r = \perp$.

Consider $Y^r_{>x_r} = Y_{\geqslant f(x_r)}$. $\mathcal{B}(Y_{\geqslant f(x_r)})$ is a component-wise cone over $\mathcal{B}(f(x_r))$. It is component-wise contractible, therefore $\mathcal{B}(Y^{r-1}) \hookrightarrow \mathcal{B}(Y^r)$ is component-wise homotopy equivalence by Lemma 2. By iteration $\mathcal{B}(i_Y) : \mathcal{B}(Y^0) = \mathcal{B}(Y) \hookrightarrow \mathcal{B}(M(f)) = \mathcal{B}(Y^n)$ is component-wise homotopy equivalence between $BY$ and $M(f)$. Note that persistence structure is not used here.

Then consider linear extension of $Y$ with enumerations $y_1^i, \dots, y_{m_i}^i$ and $X_i^r = X \cup \{y_{r+1}^i, \dots, y_{m_i}^i\} \subset M_i(f)$. $X^{r-1}_{\leqslant y_r} = f^{-1}(Y_{\leqslant y_r})$. Latter is $\varepsilon$-acyclic over $\mathbb{F}$ by condition of the theorem. Hence $\mathcal{B}(X^r) \hookrightarrow \mathcal{B}(X^{r-1})$ induces $4\varepsilon$-interleavings of all homology modules and by transitivity of $\varepsilon$-equivalence $\mathcal{B}(i_X)$ induces $4m\varepsilon$-interleavings between homology of $X$ and $M(f)$.

Note that $i_X(x) \leqslant (i_Y \circ f)(x)$. By Proposition 7 $\mathcal{B}(i_X)$ is component-wise homotopic to $\mathcal{B}(i_Y \circ f) = \mathcal{B}(i_Y) \circ \mathcal{B}(f)$. Homotopic maps induce same maps on homology, $i_Y$ is a homotopy equivalence and induce 0-interleavings. Hence $\mathcal{B}(f)$ induce $4m\varepsilon$-interleavings between $H_i(BX, R)$ and $H_i(BY, R)$. □

# 5 Appendix: Error propagation in Mayer-Vietoris spectral sequence

Value of error propagation multiple in the result may probably be decreased with alternative proof using spectral sequences at the cost of some restriction on structure maps of posets. Here we outline results of Govc and Scraba on error propagation in one specific spectral sequence associated to cover.

Assume persistence simplicial complex $S$ is a filtered complex (assuming either ascending or descending filtration with any compatible structure maps) and there exists open covering $\mathcal{U}$ compatible with filtration. Then there exists a spectral sequence called Mayer-Vietoris spectral sequence which converges to $H_\star(S)$. [GS16, Theorem 2.30]

Let all sets in filtered cover $\mathcal{U}$ be $\varepsilon$-acyclic with all intersections between them be $\varepsilon$-acyclic. This is a representation of the definition of $\varepsilon$-acyclic cover. [GS16, Definition 3.2]

Sets and all their nonempty intersections in any covering form a poset with inclusion being an order relation. Nerve of this poset is called *nerve of a covering* and we shall denote it as $\mathcal{N}(\mathcal{U})$.

There hold the following propositions:

**Proposition 23.** *[GS16, Corollary 5.2]*
*If $\mathcal{U}$ is an $\varepsilon$-acyclic cover of $X$, then for all $i$ $E^2_{i,0}$ of Mayer-Vietoris spectral sequence and $H_i(\mathcal{N}(\mathcal{U}), \mathbb{F})$ are $2\varepsilon$-interleaved as graded modules.*

**Proposition 24.** *[GS16, Theorem 7.1]*
*Let $D$ be dimension of $\mathcal{N}(\mathcal{U})$. Then for all $i$ $H_i(X, \mathbb{F}) \overset{(4D+2)\varepsilon}{\sim} H_i(\mathcal{N}(\mathcal{U}), \mathbb{F})$.*

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