

# 1 Abstract

Combinatorial topology has recently been successfully applied to data analysis. One approach is to compute homology groups of some persistence complex constructed by experimental data and try to classify some objects by them. However complexity of computation of persistent homology is high enough to limit practical usage of the approach.

There is a question of how to construct persistent complex with same homology as initially given with smaller dimension. In this paper we prove Quillen-McCord theorem (Quillen fiber lemma) in the setting of persistent homology. It can possibly be used to reduce persistence complex to smaller one if given complex is equivalent to a nerve of some partially ordered set.

## 2 Preliminaries

### 2.1 Persistence modules and $\varepsilon$ -interleavings

Basic definition is the following:

**Definition 1.** *Persistence complex* is a family of chain complexes  $C_\star^{i_0} \xrightarrow{f_{i_0}} C_\star^{i_1} \xrightarrow{f_{i_1}} C_\star^{i_2} \xrightarrow{f_{i_2}} \dots$  with  $I$  — some indexing set and  $f_i$  being chain maps.

These objects naturally arise in experiments. If we consider  $I$  as time, this is a structure we obtain while observing some dynamic structure which can at any time be represented by a chain complex.

**Definition 2.** Let  $R$  be a ring. *Persistence module* is a family of  $R$ -modules  $M^i$  with homomorphisms  $\phi_i :: M^i \rightarrow M^{i+1}$ .

Example of a persistent module is given by homology modules of persistence complex  $C_\star$  (*persistent homology*).  $H_i^j(C_\star) = H_i(C_{\star j})$ , maps  $\phi_j$  are induced by  $f_i$ .

**Definition 3.** Persistence complex (module) is of *finite type* over  $R$  if all component complexes (modules) are finitely-generated as  $R$ -modules and all  $f_i$  ( $\phi_i$ ) are isomorphisms for  $i > m$  for some  $m$ .

Since experiments usually take finite time and operate finite amount of data, we can safely consider only complexes of finite type. Note that by construction homology of complex of finite type is a module of finite type.

**Definition 4.** *Graded module* over  $I$ -graded ring  $R$  with graded components  $R_i$  is an  $R$ -module  $M$  together with a decomposition  $M = \bigoplus_{j \in I} M^j : \forall i \in I R_i \cdot M^j \subset M^{j+i}$ . For correctness of the definition it's enough for  $I$  to be a semigroup.

**Definition 5.** Assume  $I$  is a monoid with linear order. Then *non-negatively graded module* over  $I$ -graded ring  $R$  is an  $R$ -module  $M$  together with a decomposition  $M = \bigoplus_{j \in I} M^j : \forall i \in I_{\geq 0} R_i \cdot M^j \subset M^{j+i}$

**Proposition 1.** . *The following hold:*

1. *Graded modules over  $R$  form a category.*
2. *Non-negatively graded modules over  $R$  form a category.*

Morphisms in both these categories are morphisms  $\phi$  between modules  $M$  and  $N$  such that  $\phi(M^j) \subset N^j$  for all  $j$  in indexing set.

**Theorem 1.** [Zomorodian05, Theorem 3.1]

*Category of persistence modules of finite type over Noetherian ring with unity  $R$  is equivalent to category of non-negatively graded finitely generated  $R[t]$ -modules.*

For clarification of the reference see [Corbet18]

*Remark 1.* If  $R$  is a field then  $R[t]$  is a PID and graded ideals are exactly  $t^n$  for  $n \in \mathbb{I}_{\geq 0}$ .

We have introduced a setting to work with persistence complexes as with graded modules. However computations in a category of non-negatively graded modules are not fit for applications since experiment always comes with an error and we do not have a measure for similarity of two persistence modules. We need technique which allows to construct morphisms of deformed complexes and track error propagation. This technique is developed in [GS16], we will recall it without proofs.

**Definition 6.** [GS16, Definition 2.7]

Let  $M$  and  $N$  be persistent modules,  $f : M \rightarrow N$  be a homomorphism of modules. Then  $f$  is called  $\varepsilon$ -morphism if  $f(M^j) \subset N^{j+\varepsilon}$  for  $\varepsilon \geq 0$ .

*Remark 2.* Note that 0-morphism is a morphism in category of non-negatively graded modules over  $R[t]$ .

*Remark 3.* Let  $R$  be a field. Then for every persistence module  $M$  and any  $\varepsilon$  there exists morphism  $Id_\varepsilon$  such that  $Id_\varepsilon(a) = t^\varepsilon a$ .

*Remark 4.* Restriction on  $R$  to be a field is crucial for construction. It is a strong restriction, however, it is acceptable.

**Definition 7.** Persistence modules  $M$  and  $N$  are called  $\varepsilon$ -interleaved ( $M \overset{\varepsilon}{\sim} N$ ) if there exist pair of  $\varepsilon$ -morphisms  $(\phi : M \rightarrow N, \psi : N \rightarrow M)$  ( $\varepsilon$ -interleaving) such that  $\phi \circ \psi = Id_{2\varepsilon} : N \rightarrow N$  and  $\psi \circ \phi = Id_{2\varepsilon} : M \rightarrow M$ .

*Remark 5.* There follows that  $M \overset{\varepsilon}{\sim} N$  implies  $M \overset{\alpha}{\sim} N$  for any  $\alpha > \varepsilon$  since for  $\varepsilon$ -interleaving  $(\phi, \psi)$  we have  $\alpha$ -interleaving  $(Id_{\alpha-\varepsilon} \circ \phi, Id_{\alpha-\varepsilon} \circ \psi)$ .

*Remark 6.* Note that due to Theorem 1 notion of  $\varepsilon$ -interleavings applies to persistence complexes.

**Definition 8.** We denote as  $\varepsilon$ -equivalence relation with the following properties:

For any  $M, N, L, \varepsilon, \varepsilon_1, \varepsilon_2$

- $M \overset{0}{\sim} M$ .
- $M \overset{\varepsilon}{\sim} N$  is equivalent to  $N \overset{\varepsilon}{\sim} M$ .
- if  $M \overset{\varepsilon_1}{\sim} N$  and  $N \overset{\varepsilon_2}{\sim} L$  then  $M \overset{\varepsilon_1+\varepsilon_2}{\sim} L$ .

**Proposition 2.**  $\varepsilon$ -interleaved persistence modules are  $\varepsilon$ -equivalent.

**Definition 9.** Persistent module  $M$  is called  $\varepsilon$ -trivial if  $M \stackrel{\varepsilon}{\sim} 0$ . It is equivalent to  $t^{2\varepsilon}M = 0$ .

**Lemma 1.** Let  $0 \rightarrow M \rightarrow L \rightarrow N \rightarrow 0$  be a short exact sequence. Then the following properties hold:

- If  $M \stackrel{\varepsilon_1}{\sim} 0$  and  $N \stackrel{\varepsilon_2}{\sim} 0$  then  $L \stackrel{\varepsilon_1 + \varepsilon_2}{\sim} 0$ .
- If  $L \stackrel{\varepsilon}{\sim} 0$  then  $M \stackrel{\varepsilon}{\sim} 0$  and  $N \stackrel{\varepsilon}{\sim} 0$ .

## 2.2 Quillen-McCord theorem

We give a formulation of classical theorem and then descend from general categorical case to specific.

**Definition 10.** Simplex category  $\Delta$  is the category with objects — nonempty linearly ordered sets of the form  $[n] = \{0, 1, \dots, n\}$  with  $n \geq 0$  and morphisms — order-preserving functions.

All morphisms in the simplex category are compositions of injective maps  $\delta_i^n : \{0, 1, \dots, i-1, i+1, \dots, n\} \mapsto \{0, 1, \dots, i-1, i, i+1, \dots, n\}$  and surjective maps  $\sigma_i^n : \{0, 1, \dots, i, i+1, \dots, n\} \mapsto \{0, 1, \dots, i, i+1, \dots, n\}$ .

**Definition 11.** Simplicial set  $S$  is a contravariant functor  $\Delta \rightarrow \text{Set}$ . Objects of image of  $S$  are called simplices, cardinality of a simplex is called a dimension of a simplex. We will identify  $s$  with set of its simplices while keeping in mind general structure.

Simplicial sets form a category under natural transformations (category of presheaves on  $\Delta$ ).

**Definition 12.** Simplicial set  $S$  is called a *simplicial complex* if for any  $s \in S$  s.t.  $v \subset s$   $v \in s$  and  $S$  is faithful.

We will also use the following definitions:

**Definition 13.** Join  $A \star B$  of simplicial complexes  $A$  and  $B$  is the simplicial complex with simplices — all possible unions of simplices  $a \in A$  and  $b \in B$ .

**Definition 14.** Join of topological spaces  $A$  and  $B$  is defined as follows:  $A \star B := A \sqcup_{p_0} (A \times B \times [0, 1]) \sqcup_{p_1} B$ , where  $p$  are projections of the cylinder  $A \times B \times [0, 1]$  onto faces.

**Definition 15.** Star  $\text{st}(x)$  of simplex  $x \in A$  ( $A$  simplicial complex) is the simplicial complex with simplices — all simplices  $a \in A$  such that there exists inclusion  $x \hookrightarrow a$ .

**Definition 16.** Link  $\text{lk}(x)$  of simplex  $x \in A$  is defined as follows:  $\text{lk}(x) = \{v \in \text{st}(x) \mid x \not\subset v\}$ .

**Definition 17.** Functor  $\mathcal{K} : \Delta \rightarrow \text{Top}$  which maps simplices to geometric simplices of corresponding dimension and morphisms to inclusions of faces and restrictions to subcomplexes is called *geometric realization*.

**Proposition 3.**  $\text{st}(x) = \text{lk}(x) \star x$ . Hence  $\mathcal{K}(\text{st}(x))$  is a cone over  $\mathcal{K}(x)$ .

**Proposition 4.**  $\mathcal{K}(A \star B) = \mathcal{K}(A) \star \mathcal{K}(B)$

Let  $C$  denote a category. We can construct the simplicial set called *nerve of category*  $C$  as follows:

Let objects of  $C$  be the only 0-dimensional simplices. Then let all morphisms be 1-dimensional simplices, all composable pairs of morphisms be 2-dimensional simplices and so on.

This construction is functorial, we denote nerve functor as  $(N)$ .

**Definition 18.** Geometric realization  $BC$  of  $\mathcal{N}(C)$  is called *classifying space* of  $C$ .

We will denote composition of nerve and geometric realization as  $\mathcal{B}$ . It is obviously a functor and we only need this construction to avoid confusing notation  $Bf$  for induced morphism of classifying spaces. By definition  $\mathcal{B}(C) = BC$ .

**Definition 19.** Let  $f : C \rightarrow D$  be a functor and  $d$  — object in  $D$ . Then *comma category*  $d \downarrow f$  is a category with objects — pairs  $(s, i_s)$  of objects in  $C$  and morphisms  $i_s : d \rightarrow f(s)$  and morphisms — morphisms  $g$  in  $C$  such that triangle  $i_s, i_{g(s)}, f(g)$  commutes.

**Theorem 2.** [Quillen72, Theorem A]

If  $f : C \rightarrow D$  is a functor such that the classifying space  $B(d \downarrow f)$  of the comma category  $d \downarrow f$  is contractible for any object  $d \in D$ , then  $f$  induces a homotopy equivalence  $BC \rightarrow BD$ .

Every poset  $X$  with order relation  $R$  can be seen as a category if we set  $\text{Ob}(X) = X$  and  $\text{Hom}_X(a, b) = \{r_{ab}\}$  if  $(a, b) \in R$  and  $\emptyset$  otherwise. Note that map between posets is functorial if and only if it preserves order.

Nerve construction on a poset yields a simplicial complex called *order complex*. Application of Quillen A theorem to posets yields the following theorem:

**Theorem 3. Quillen-McCord theorem**

Assume  $X, Y$  are finite posets,  $f : X \rightarrow Y$  is order-preserving map.

If  $\forall y \in Y \mathcal{B}(f^{-1}(Y_{\leq y}))$  is contractible, then  $\mathcal{B}f$  is a homotopy equivalence between  $BX$  and  $BY$ .

There holds homological [Bar11, Corollary 5.5] versions of this theorem:

**Theorem 4. Homological Quillen-McCord theorem**

Assume  $X, Y$  are finite posets,  $f : X \rightarrow Y$  is order-preserving map,  $R$  is a PID.

If  $\forall y \in Y H_i(\mathcal{B}(f^{-1}(Y_{\leq y})), R) = 0$  for any  $i$ ,  $\mathcal{B}f$  induces isomorphisms of all homology groups with coefficients in  $R$  on  $BX$  and  $BY$ .

Proofs of both theorems are necessary for construction of a desired result and will be recalled in section 2.4 as well as a specific proof of Quillen-McCord theorem [Bar11, Proof of Theorem 1.1].

## 2.3 Statement of the theorem

Assume we conduct an experiment and observe dynamics of a simplicial complex  $S^{i_0} \xrightarrow{f_{i_0}} S^{i_1} \xrightarrow{f_{i_1}} S^{i_2} \xrightarrow{f_{i_2}} \dots$ . Denote the whole sequence as  $S$ . For each simplicial complex  $S^i$  we can assign chain complex  $C_\star^i$  via standard construction of simplicial chain complex used in the definition of simplicial homology.

I.e. we observe persistence chain complex  $C_\star^{i_0} \xrightarrow{f'_{i_0}} C_\star^{i_1} \xrightarrow{f'_{i_1}} C_\star^{i_2} \xrightarrow{f'_{i_2}} \dots$ . Each simplicial complex in a series has geometric realization with the same homology as its chain complex.

**Definition 20.** We will denote as *persistence topological space*  $X$  sequence of topological spaces  $X^{i_0} \xrightarrow{f_{i_0}} X^{i_1} \xrightarrow{f_{i_1}} X^{i_2} \xrightarrow{f_{i_2}} \dots$  with compatible triangulations (chain complexes formed by simplicial complexes)  $C_\star^i$  such that  $C_\star$  is a persistence complex with structure maps induced by  $f_i$ . Homology of  $X$  is a persistence module with components — homology modules of  $X^i$ .

We will denote as morphism  $g$  between persistence topological spaces  $X$  and  $Y$  collection of maps  $g_i : X^i \rightarrow Y^i$  between components such that all  $g_i$  commute with  $f_i$ .

Apparently  $\mathcal{KS}$  is a persistence topological space.

**Definition 21.** Persistence topological space  $X$  is called  $\varepsilon$ -acyclic over  $R$  if for all indices  $H_i(X, R) \stackrel{\varepsilon}{\sim} H_i(pt, R)$ .

**Definition 22.** We say that morphism  $f : X \rightarrow Y$  between persistence topological spaces induces  $\varepsilon$ -interleaving if there exists  $g : Y \rightarrow X$  such that induced maps of pair  $(f, g)$  on graded homology modules form  $\varepsilon$ -interleaving.

To formulate a theorem we have to move to terms of posets. Define *persistence poset* as series of posets with structure maps analogously to persistence complexes and map of persistence posets as collection of maps between posets commuting with structure maps. It induces map on persistence order complexes. We also analogously define persistence poset of finite type — it is a finite sequence of finite posets. Easy to see that nerve (as a functor) of persistence poset of finite type is a persistence complex of finite type.

### Theorem 5. *Approximate Quillen-McCord theorem, draft statement*

Assume  $X, Y$  are persistent posets of finite type,  $f : X \rightarrow Y$  is order-preserving map,  $R$  is a field or  $\mathbb{Z}$ .

If  $\forall y \in Y \mathcal{B}(f^{-1}(Y_{\leq y}))$  is  $\varepsilon$ -acyclic over  $R$ ,  $\mathcal{B}f$  induces  $e$ -interleavings of all homology modules over  $R$  on  $BX$  and  $BY$ .

Value of  $e$  will be specified during the proof and is critical to applications being measure of error of approximation.

## 2.4 Prerequisite results

### 2.4.1 Quillen-McCord theorem

This section contains the proof of Quillen-McCord theorem given by Barmak.

**Proposition 5.** [Bar11, Proposition 2.1] Let  $K_1$  and  $K_2$  be two geometric simplicial complexes and geometric simplicial complex  $K = K_1 \cup K_2$  (as space and simplicial complex). Then if  $K_1 \cup K_2 \hookrightarrow K_1$  is a homotopy equivalence, then so is  $K_2 \hookrightarrow K$ .

**Proposition 6.** [Bar11, Proposition 2.2] Let  $f, g : X \rightarrow Y$  be order-preserving maps between finite posets such that  $\forall x \ f(x) \leq g(x)$ . Then  $\mathcal{B}(f)$  is homotopy-equivalent to  $\mathcal{B}(g)$ .

**Proof.**  $X$  has a finite set  $M$  of maximal elements. Take any of them ( $m_1$ ) and define  $h_1 : X \rightarrow Y$  such that  $h_1(m_1) = g(m_1)$  and  $h_1 = f$  on all other elements of  $X$ . This is an order-preserving map due to maximality of  $m_1$ . Take  $M \setminus \{m_1\}$  and build  $h_2$  which is equal to  $h_1$  on complement to  $m_2$  and to  $g$  on  $m_2$  and so on. We have built a finite sequence of maps  $h_0 = f \leq h_1 \leq h_2 \leq \dots \leq g = h_n$ .

Elements  $h_i(m_i)$  and  $h_{i-1}(m_i)$  are comparable. Hence there exists simplex  $\{h_i(m_i); h_{i-1}(m_i)\}$  in  $\mathcal{N}(Y)$ . Since  $\mathcal{N}(Y)$  is a simplicial complex, for other elements between selected where exist no holes and thus where is a linear homotopy between  $h_{i-1}$  and  $h_i$  which contracts simplex  $\{h_i(m_i); h_{i-1}(m_i)\}$ .

Hence there is a homotopy between  $f$  and  $g$ . □

**Proposition 7.** Note that  $\text{lk}(\mathcal{N}(x)) = \mathcal{N}(X_{>x}) \star \mathcal{N}(X_{<x})$ . Therefore  $\mathcal{K}(\text{lk}(\mathcal{N}(x))) = \mathcal{B}(X_{>x}) \star \mathcal{B}(X_{<x})$ .

*Remark 7.*  $\mathcal{B}(X) = \mathcal{B}(X \setminus \{x\}) \cup \mathcal{K}(\text{st}(\mathcal{N}(x)))$  for any  $x \in X$ .

**Lemma 2.** Let  $X$  be a finite poset and for  $x \in X$  either  $\mathcal{B}(X_{>x})$  or  $\mathcal{B}(X_{<x})$  is contractible. Then embedding  $\mathcal{B}(X \setminus \{x\}) \hookrightarrow \mathcal{B}(X)$  is a homotopy equivalence.

**Proof.** By proposition  $\mathcal{K}(\text{lk}(\mathcal{N}(x)))$  is contractible. Hence its embedding to its cone  $\mathcal{K}(\text{st}(\mathcal{N}(x)))$  is homotopy equivalence. Lemma follows by Proposition 5. □

**Definition 23.** Let  $f : X \rightarrow Y$  be an order preserving map between posets. Denote orders ( $\leq$ ) on  $X$  and  $Y$  as  $R_X$  and  $R_Y$ . Then we define poset  $M(f) = X \cup Y$  with  $R = R_X \cup R_Y \cup R_f$  where  $(x, y) \in R_f$  if and only if  $(f(x), y) \in R_Y$ .

We will by analogy denote this poset a *mapping cylinder* of  $f$ . There are also defined canonical inclusions  $i : X \rightarrow M(f)$  and  $j : Y \rightarrow M(f)$ .

**Proof. Quillen-McCord theorem**

Let  $X, Y$  be finite posets with order-preserving map  $f : X \rightarrow Y$ .

Every poset has linear extension. Let  $x_1, x_2, \dots, x_n$  be enumeration of  $X$  in such linear order and  $Y^r = \{x_1, \dots, x_r\} \cup Y \subset M(f)$  for any  $r$ .

Consider  $Y_{>x_r}^r = Y_{\geq f(x_r)}$ .  $\mathcal{B}(Y_{\geq f(x_r)})$  is a cone over  $\mathcal{B}(f(x_r))$ . It is contractible, therefore  $\mathcal{B}(Y^{r-1}) \hookrightarrow \mathcal{B}(Y^r)$  is homotopy equivalence by lemma. By iteration  $\mathcal{B}(j) : \mathcal{B}(Y^0) = \mathcal{B}(Y) \hookrightarrow \mathcal{B}(M(f)) = \mathcal{B}(Y^n)$  is homotopy equivalence between  $\mathcal{B}Y$  and  $M(f)$ .

Then consider linear extension of  $Y$  with enumeration  $y_1, \dots, y_m$  and  $X^r = X \cup \{y_{r+1}, \dots, y_m\} \subset M(f)$ .  $X_{\leq y_r}^{r-1} = f^{-1}(Y_{\leq y_r})$ . Latter is contractible by condition of the

theorem. Hence  $\mathcal{B}(X^r) \hookrightarrow \mathcal{B}(X^{r-1})$  is homotopy equivalence and  $\mathcal{B}(i)$  is a homotopy equivalence between  $X$  and  $M(f)$ .

Note that  $i(x) \leq (j \circ f)(x)$ . By Proposition 6  $\mathcal{B}(i)$  is homotopic to  $\mathcal{B}(j \circ f) = \mathcal{B}(j) \circ \mathcal{B}(f)$ . Hence  $\mathcal{B}(f)$  is the homotopy equivalence between  $BX$  and  $BY$ .  $\square$

## 2.4.2 Homological Quillen-McCord theorem

To derive homological version of the theorem we can variate the proof of standard version in the following manner:

**Proposition 8.** [Milnor56, Lemma 2.1] *Reduced homology modules with coefficients in a principal ideal domain of a join satisfy the relation  $H_{r+1}(A \star B, R) \simeq \bigoplus_{i+j=r} (H_i(A, R) \otimes_R H_j(B, R)) \oplus \bigoplus_{i+j=r-1} \text{Tor}_1^R(H_i(A, R), H_j(B, R))$ .*

**Lemma 3.** *Let  $X$  be a finite poset and for  $x \in X$  either  $H_i(\mathcal{B}(X_{<x}))$  or  $H_i(\mathcal{B}(X_{>x}))$  with coefficients in a PID are equal to homology of a point. Then embedding  $\mathcal{B}(X \setminus \{x\}) \hookrightarrow \mathcal{B}(X)$  induces isomorphisms of all homology groups.*

**Proof.**

By Proposition 8  $H_i(\mathcal{K}(\text{lk}(\mathcal{N}(x)))) = H_i(\mathcal{B}(X_{>x}) \star \mathcal{B}(X_{<x}))$  are trivial for all indices  $i$  — *Tor*-functors vanish if any of their arguments is trivial. Application of Mayer-Vietoris long exact sequence to covering from Remark 7 yields the lemma.  $\square$

Proof of the theorem is similar to proof finishing previous section. However, we write it here in detail in order to be able to highlight differences. Changed parts are written in italic.

## Proof. Homological Quillen-McCord theorem

Let  $X, Y$  be finite posets with order-preserving map  $f : X \rightarrow Y$ .

Every poset has linear extension. Let  $x_1, x_2, \dots, x_n$  be enumeration of  $X$  in such linear order and  $Y^r = \{x_1, \dots, x_r\} \cup Y \subset M(f)$  for any  $r$ .

Consider  $Y_{>x_r}^r = Y_{\geq f(x_r)}$ .  $\mathcal{B}(Y_{\geq f(x_r)})$  is a cone over  $\mathcal{B}(f(x_r))$ . It is contractible, therefore  $\mathcal{B}(Y^{r-1}) \hookrightarrow \mathcal{B}(Y^r)$  is homotopy equivalence by lemma 2. By iteration  $\mathcal{B}(j) : \mathcal{B}(Y^0) = \mathcal{B}(Y) \hookrightarrow \mathcal{B}(M(f)) = \mathcal{B}(Y^n)$  is homotopy equivalence between  $BY$  and  $M(f)$ .

Then consider linear extension of  $Y$  with enumeration  $y_1, \dots, y_m$  and  $X^r = X \cup \{y_{r+1}, \dots, y_m\} \subset M(f)$ .  $X_{\leq y_r}^{r-1} = f^{-1}(Y_{\leq y_r})$ . *Latter is acyclic over  $R$  by condition of the theorem. Hence  $\mathcal{B}(X^r) \hookrightarrow \mathcal{B}(X^{r-1})$  induces isomorphisms of all homology groups and by functoriality of homology  $\mathcal{B}(i)$  induces isomorphisms of all homology groups between  $X$  and  $M(f)$ .*

Note that  $i(x) \leq (j \circ f)(x)$ . By Proposition 6  $\mathcal{B}(i)$  is homotopic to  $\mathcal{B}(j \circ f) = \mathcal{B}(j) \circ \mathcal{B}(f)$ . *Homotopic maps induce same maps on homology,  $j$  is a homotopy equivalence and induce isomorphisms. Hence  $\mathcal{B}(f)$  induce isomorphisms between  $H_i(BX, R)$  and  $H_i(BY, R)$ .*  $\square$

We see two updates. First one is essential, it requires Lemma 3 and operates some equivalence propagating in a chain of length equal to cardinality of  $Y$ . Second follows automatically from functoriality of all used constructions, but contains statement about composition of functions. We expect these two parts to be changed while constructing a proof for approximate version of the theorem.

### 3 Results

We start with two obvious observations ( $R$  is always a field):

**Proposition 9.** 1. Let  $A, B$  be two  $\varepsilon$ -acyclic graded  $R$ -modules. Then  $A \oplus B$  is  $\varepsilon$ -trivial.

2. Let  $A$  and  $B$  be two graded  $R$ -modules with at least one of them being  $\varepsilon$ -trivial. Then  $A \otimes_{R_\bullet} B$  is  $\varepsilon$ -trivial ( $\bullet$  points that this is a tensor product of graded modules).

**Proposition 10.** Let  $A$  and  $B$  be two persistence topological spaces with at least one of them being  $\varepsilon$ -acyclic over  $R$ . Then  $A \star B$  is  $\varepsilon$ -acyclic over  $R$ .

**Proof.** Over a field all  $Tor$ -functors vanish since every vector space is free module, hence flat. Hence right hand side of expression of Proposition 8 is  $\varepsilon$ -trivial by Proposition 9.  $\square$

**Proposition 11.** Let  $0 \rightarrow \ker f \hookrightarrow A \xrightarrow{f} B \xrightarrow{\phi} C \xrightarrow{g} D \rightarrow \text{coker } g \rightarrow 0$  be exact sequence. Then if  $A$  and  $D$  are  $\varepsilon$ -trivial, then  $B \stackrel{2\varepsilon}{\sim} C$  with  $f$  being left morphism in interleaving pair.

**Proof.** In s.e.s  $0 \rightarrow \ker f \hookrightarrow A \xrightarrow{f} \text{Im } f \rightarrow 0$   $\text{Im } f$  is  $\varepsilon$ -trivial. By exactness it is equal to  $K = \ker \phi$ . On the other side from  $0 \rightarrow \ker g \xrightarrow{g} D \rightarrow \text{coker } f \rightarrow 0$  there follows that  $I = \text{im } \phi$  is  $\varepsilon$ -trivial. Hence by transitivity  $K \stackrel{2\varepsilon}{\sim} I$ .

We obtain exact sequence  $0 \rightarrow K \rightarrow B \xrightarrow{\phi} C \rightarrow I \rightarrow 0$  with  $K$  and  $I$   $\varepsilon$ -trivial.  $B = K \oplus \text{coIm } \phi$ ,  $C = I \oplus \text{Im } \phi$ . Coimage and image are canonically isomorphic, hence by Proposition 9 proposition follows.  $\square$

**Lemma 4.** Let  $X$  be a persistence poset and for  $x \in X$  either  $\mathcal{B}(X_{<x})$  or  $\mathcal{B}(X_{>x})$  are  $\varepsilon$ -acyclic. Then embedding  $\mathcal{B}(X \setminus \{x\}) \hookrightarrow \mathcal{B}(X)$  induces  $2\varepsilon$ -interleavings of all homology modules.

Given propositions above the proof is similar to proof of Lemma 3:

**Proof.**

By Proposition 10  $\mathcal{K}(\text{lk}(\mathcal{N}(x)))$  is  $\varepsilon$ -acyclic. Application of Mayer-Vietoris long exact sequence to covering from Remark 7 and Proposition 11 yield the lemma.  $\square$

TODO: construction of persistence cylinder, conditions and proof of the theorem.

#### Theorem 6. Approximate Quillen-McCord theorem, final statement

Assume  $X, Y$  are persistent posets of finite type,  $f : X \rightarrow Y$  is order-preserving map,  $R$  is a field or  $\mathbb{Z}$ .

If  $\forall y \in Y$   $\mathcal{B}(f^{-1}(Y_{\leq y}))$  is  $\varepsilon$ -acyclic over  $R$ ,  $\mathcal{B}f$  induces  $2\#Y\varepsilon$ -interleavings of all homology modules over  $R$  on  $BX$  and  $BY$ .



## 4 Error propagation in Mayer-Vietoris spectral sequence

Value of error propagation multiple in the result may probably be decreased with alternative proof using spectral sequences. Here we outline results of Govc and Scraba on error propagation in one specific spectral sequence associated to cover.

Assume persistence simplicial complex  $S$  is a filtered complex (assuming either ascending or descending filtration with any compatible structure maps) and there exists open covering  $\mathcal{U}$  compatible with filtration. Then there exists a spectral sequence called Mayer-Vietoris spectral sequence which converges to  $H_*(S)$ . [GS16, Theorem 2.30]

Let all sets in filtered cover  $\mathcal{U}$  be  $\varepsilon$ -acyclic with all intersections between them be  $\varepsilon$ -acyclic. This is a representation of the definition of  $\varepsilon$ -acyclic cover. [GS16, Definition 3.2]

Sets and all their nonempty intersections in any covering form a poset with inclusion being an order relation. Nerve of this poset is called *nerve of a covering* and we shall denote it as  $\mathcal{N}(\mathcal{U})$ .

There hold the following propositions ( $R$  is a field):

**Proposition 12.** [GS16, Corollary 5.2]

If  $\mathcal{U}$  is an  $\varepsilon$ -acyclic cover of  $X$ , then for all  $i$   $E_{i,0}^2$  of Mayer-Vietoris spectral sequence and  $H_i(\mathcal{N}(\mathcal{U}), R)$  are  $2\varepsilon$ -interleaved as graded modules.

**Proposition 13.** [GS16, Theorem 7.1]

Let  $D$  be dimension of  $\mathcal{N}(\mathcal{U})$ . Then for all  $i$   $H_i(X, R) \stackrel{(4D+2)\varepsilon}{\sim} H_i(\mathcal{N}(\mathcal{U}), R)$ .

These results given a good  $\varepsilon$ -acyclic cover may be useful to provide proof of approximate Quillen-McCord theorem with better error multiple but with additional restrictions on structure maps.

## 5 References

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## **5. Доказательство теоремы 3**

Доказанных утверждений достаточно для построения доказательства теоремы 3.