1 Abstract

Combinatorial topology has recently been successfully applied to data analysis. One approach is to compute homology groups of some persistence complex constructed by experimental data and try to classify some objects by them. However complexity of computation of persistent homology is high enough to limit practical usage of the approach.

There is a question of how to construct persistence complex with same homology as initially given with smaller dimension. Some of the complexes arise from dynamics of some partially ordered set by nerve construction. In this paper we prove Quillen-McCord theorem (Quillen fiber lemma) in the setting of persistent homology. It can possibly be used to reduce persistence complex built from poset to a smaller one.

While proving the theorem we develop a setting convenient to prove approximate statements about persistence objects. This setting is based on notion of interleaving distance and representation theorem of persistent homology.

2 Preliminaries

2.1 Persistence modules and interleaving distance

Basic definition is the following:

Definition 1. *Persistence complex* is a family of chain complexes $C_\star^{i_0} \xrightarrow{f_{i_0}} C_\star^{i_1} \xrightarrow{f_{i_1}} C_\star^{i_2} \xrightarrow{f_{i_2}} \ldots$ where I is the linearly ordered set of indices and f_i are chain maps. We call maps $f = (\ldots, f_i, \ldots)$ *the structure maps* of a persistence complex.

These objects naturally arise in experiments. If we consider I as a time, this is a structure we obtain while observing some dynamic structure which can at any time be represented by a chain complex.

Definition 2. Let R be a ring. *Persistence module* is a family of R-modules M^i with homomorpisms $\phi_i :: M^i \to M^{i+1}$ as the structure maps. If R is a field it is natural to use notion of *persistence vector space*.

Example of a persistence module is given by (by default simplicial) homology modules of persistence complex $C_\star$ (*persistent homology*). $H_i^j(C_\star) = H_i(C_j)$, maps ϕ_j are induced by f_i.

Definition 3. Persistence complex (module) is of *finite type* over R if all components of complexes (modules) are finitely-generated as R-modules and all f_i (ϕ_i) are isomorphisms for $i > m$ for some m.

Since experiments usually take finite time and operate finite amount of data, we can safely consider only complexes of finite type. Note that by construction homology of complex of finite type is a module of finite type.

Definition 4. Assume I is a monoid with linear order. Then *non-negatively graded module* over I-graded ring R is an R-module M together with a decomposition $M = \bigoplus_{j \in I} M^j$: $\forall i \in I_{\geqslant 0}\, R_i \cdot M^j \subset M^{j+i}$

Non-negatively graded modules over R form a category. Maps ϕ between modules M and N such that $\phi(M^j) \subset N^j$ for all j in indexing set are morphisms in this category.

There is a well-known theorem:

Theorem 1. *[Zomorodian05, Theorem 3.1]*
Category of persistence modules of finite type over Noetherian ring with unity R indexed by $\mathbb{N}$ is equivalent to category of non-negatively graded finitely generated $R[t]$-modules.

It is proven in [Corbet18]. Authors provide generalization [Corbet18, Theorem 2] which is more suitable for our needs.

Theorem 2. *[Corbet18, Case of Corollary 20 of Theorem 2]*
Let R be a Noetherian ring with unity and G be a monoid with linear order such that $R[G]$ is Noetherian. Then the category of finitely generated graded $R[G]$-modules is isomorphic to the category of G-indexed persistence modules over R of finite type.

Corollary 1. *Category of persistence modules of finite type over $\mathbb{Z}$ indexed by infinite additive subgroup $\Gamma \subset \mathbb{R}$ is equivalent to category of non-negatively graded finitely generated $Nov(\Gamma)$-modules. $Nov(\Gamma)$ is the Novikov ring.*

Corollary 2. *Category of persistence modules of finite type over Noetherian ring R indexed by $\mathbb{Z}$ is equivalent to category of non-negatively graded finitely generated $R[t^{-1}, t]$-modules.*

Remark 1.
If R is a field then $R[t]$ is a PID and graded ideals are exactly t^n for $n \in Z$.
Rings $Nov(\Gamma)$ are also PIDs with graded ideals t^γ for $\gamma \in \Gamma$.

The most practically reasonable examples are given by considering corollary 1 with G is either $\mathbb{R}$ or $\mathbb{Z}$. These examples represent continuous and discrete time in observation with ability to reason about the past.

This equivalence is required to introduce a measure of similarity between persistence modules. It is essential since in applications we have to accept error in experimental data and we must decide whether observed homology modules are close enough to initial hypothesis or not.

Definition 5. [GS16, Definition 2.7]
Let M and N be non-negatively graded $R[\Gamma]$-modules, $f : M \to N$ be a homomorphism of modules. Then f is called *ε-morphism* if $f(M^j) \subset N^{j+\varepsilon}$ for $\varepsilon \geqslant 0$.

Remark 2. Note that 0-morphism is a morphism in category of non-negatively graded modules over $R[\Gamma]$.

Remark 3. Let $\mathbb{F}$ be a field. Then for every $\mathbb{Z}$-graded vector space V over $\mathbb{F}$ and any $\varepsilon \in \mathbb{Z}$ there exists morphism Id_ε such that $Id_\varepsilon(a) = t^\varepsilon a$.

Remark 4. For every Γ-graded module V over $\mathbb{Z}$ and any $\varepsilon \in \Gamma$ there exists morphism Id_ε such that $Id_\varepsilon(a) = t^\varepsilon a$.

Definition 6. Non-negatively graded modules M and N are called *ε-interleaved* ($M \overset{\varepsilon}{\sim} N$) if there exists pair of ε-morphisms ($\phi : M \to N,\ \psi : N \to M$) (*$\varepsilon$-interleaving*) such that $\phi \circ \psi = Id_{2\varepsilon} : N \to N$ and $\psi \circ \phi = Id_{2\varepsilon} : M \to M$.

This definition depends on existence of distinguished Id_ε morphisms and we shall operate only cases given by remarks above.

Remark 5. There follows that $M \overset{\varepsilon}{\sim} N$ implies $M \overset{\alpha}{\sim} N$ for any $\alpha > \varepsilon$ since for ε-interleaving (ϕ, ψ) we have α-interleaving $(Id_{\alpha-\varepsilon} \circ \phi, Id_{\alpha-\varepsilon} \circ \psi)$.

Definition 7. We denote as *ε-equivalence* relation with the following properties:
For any M, N, L, ε, ε_1, ε_2

- $M \overset{0}{\sim} M$.
- $M \overset{\varepsilon}{\sim} N$ is equivalent to $N \overset{\varepsilon}{\sim} M$.
- if $M \overset{\varepsilon_1}{\sim} N$ and $N \overset{\varepsilon_2}{\sim} L$ then $M \overset{\varepsilon_1+\varepsilon_2}{\sim} L$.

Proposition 1. *ε-interleaved non-negatively graded modules are ε-equivalent.*

Remark 6. Condition $M \overset{\varepsilon}{\sim} 0$ is equivalent to condition $t^{2\varepsilon}M = 0$. [GS16, Proposition 2.13]

Definition 8. Any ε-equivalence induces an extended pseudometric on its domain. This pseudometric is defined as $d(X,Y) = min\{\varepsilon \in I \mid X \overset{\varepsilon}{\sim} Y\}$. For non-negatively graded modules this pseudometric is called interleaving distance. [GS16, Definition 2.12]
We shall refer to this general pseudometric as to approximation distance.

Lemma 1. *Let $0 \to M \to L \to N \to 0$ be a short exact sequence of non-negatively graded modules. Then the following properties hold:*

- *If $M \overset{\varepsilon_1}{\sim} 0$ and $N \overset{\varepsilon_2}{\sim} 0$ then $L \overset{\varepsilon_1+\varepsilon_2}{\sim} 0$. [GS16, Proposition 4.6]*
- *If $M \overset{\varepsilon_1}{\sim} 0$ then $L \overset{2\varepsilon}{\sim} N$. [GS16, Proposition 4.1]*
- *If $N \overset{\varepsilon_1}{\sim} 0$ then $M \overset{2\varepsilon}{\sim} L$. [GS16, Proposition 4.1]*

2.2 Quillen-McCord theorem

2.2.1 Statement

In order to give a general context we formulate general theorem and deduce Quillen-McCord theorem as a special case.

Definition 9. *Join* $A \star B$ of simplicial complexes A and B is the simplicial complex with simplices — all possible unions of simplices $a \in A$ and $b \in B$.

Definition 10. *Join* of topological spaces A and B is defined as follows: $A \star B := A \sqcup_{p_0} (A \times B \times [0,1]) \sqcup_{p_1} B$, where p are projections of the cylinder $A \times B \times [0,1]$ onto faces.

Definition 11. *Star* $\mathrm{st}(x)$ of simplex $x \in A$ (A is a simplicial complex) is the minimal simplicial complex containing all simplices $a \in A$ such that there exists inclusion $x \hookrightarrow a$.

Definition 12. *Link* $\mathrm{lk}(x)$ of simplex $x \in A$ is defined as follows: $\mathrm{lk}(x) = \{v \in \mathrm{st}(x) | \, x \notin v\}$.

Definition 13. Let Δ be a simplicial category. Functor $\mathcal{K} : \Delta \to Top$ which maps simplices to geometric simplices of corresponding dimension and morphisms to inclusions of faces and restrictions to subcomplexes is called *geometric realization*.

Proposition 2. $\mathrm{st}(x) = \mathrm{lk}(x) \star x$.

Proposition 3. $\mathcal{K}(A \star B) = \mathcal{K}(A) \star \mathcal{K}(B)$. *Hence* $\mathcal{K}(\mathrm{st}(x))$ *is a cone over* $\mathcal{K}(x)$.

Let $\mathcal{C}$ denote a small category. We can construct the simplicial set called the *nerve of category* $\mathcal{C}$ as follows:
Let objects of $\mathcal{C}$ be the only 0-dimensional simplices. Then let all morphisms be 1-dimensional simplices, all composable pairs of morphisms be 2-dimensional simplices and so on with morphisms — inclusions of compositions into longer ones and replacements of compositions $i \circ j$ with $f = i \circ j$.
This construction is functorial over category of posets Pos, we denote nerve functor from Pos to $Psh(\Delta)$ as $\mathcal{N}$.

Definition 14. Geometric realization $B\mathcal{C}$ of $\mathcal{N}(\mathcal{C})$ is called *classifying space* of $\mathcal{C}$.

We denote composition of nerve and geometric realization as $\mathcal{B}$. It is obviously a functor and we only need this construction to avoid confusing notation Bf for induced morphism of classifying spaces. By definition $\mathcal{B}(\mathcal{C}) = B\mathcal{C}$.

Definition 15. Let $f : \mathcal{C} \to \mathcal{D}$ be a functor and d — object in $\mathcal{D}$. Then *comma category* $d \downarrow f$ is a category with objects — pairs (s, i_s) of objects in $\mathcal{C}$ and morphisms $i_s : d \to f(s)$ and morphisms — morphisms g in $\mathcal{C}$ such that triangle $i_s, i_{g(s)}, f(g)$ is commutative.

Theorem 3. *[Quillen72, Theorem A]*
If $f : C \to D$ *is a functor such that the classifying space* $B(d \downarrow f)$ *of the comma category* $d \downarrow f$ *is contractible for any object* $d \in D$*, then* f *induces a homotopy equivalence* $BC \to BD$.

Every poset X with order relation R can be seen as a category if we set $\mathrm{Ob}(X) = X$ and $Hom_X(a,b) = \{r_{ab}\}$ if $(a,b) \in R$ and $\emptyset$ otherwise. Note that map between posets is functorial if and only if it preserves order.
Nerve construction on a poset yields a simplicial complex called *order complex*. Application of Quillen A theorem to posets yields the following theorem:

Theorem 4. ***Quillen-McCord theorem***
Assume X, Y *are finite posets,* $f : X \to Y$ *is order-preserving map.*
If $\forall y \in Y$ $\mathcal{B}(f^{-1}(Y_{\leqslant y}))$ *is contractible, then* $\mathcal{B}f$ *is a homotopy equivalence between* BX *and* BY.

Proof of the theorem is necessary for construction of a desired result.

2.2.2 Proof

This section contains the proof of Quillen-McCord theorem given by Barmak.

Proposition 4. *[Bar11, Proposition 2.1] Let X and Y be two simplicial complexes such that $X \cup Y$ is a simplicial complex. Then if $\mathcal{K}(X) \cup \mathcal{K}(Y) \hookrightarrow \mathcal{K}(X)$ is a homotopy equivalence, then so is $\mathcal{K}(Y) \hookrightarrow \mathcal{K}(X \cup Y)$.*

Proposition 5. *Variation of [Bar11, Proposition 2.2] Let $f, g : X \to Y$ be order-preserving maps between finite posets such that $\forall x\ f(x) \leqslant g(x)$. Then $\mathcal{B}(f)$ is homotopy-equivalent to $\mathcal{B}(g)$.*

Proof. X has a finite set M of maximal elements. Take any of them (m_1) and define $h_1 : X \to Y$ such that $h_1(m_1) = g(m_1)$ and $h_1 = f$ on all other elements of X. This is an order-preserving map due to maximality of m_1. Take $M \setminus \{m_1\}$ and build h_2 which is equal to h_1 on complement to m_2 and to g on m_2 and so on. We have built a finite sequence of maps $h_0 = f \leqslant h_1 \leqslant h_1 \leqslant \ldots \leqslant g = h_n$.

Elements $h_i(m_i)$ and $h_{i-1}(m_i)$ are comparable. Hence there exists simplex $\{h_i(m_i); h_{i-1}(m_i)\}$ in $\mathcal{N}(Y)$. Since $\mathcal{N}(Y)$ is a simplicial complex, for other elements between selected where exist no holes and thus where is a linear homotopy between h_{i-1} and h_i which contracts simplex $\{h_i(m_i); h_{i-1}(m_i)\}$.

Hence there is a homotopy between f and g. □

Proposition 6. *Note that $\mathrm{lk}(\mathcal{N}(x)) = \mathcal{N}(X_{>x}) \star \mathcal{N}(X_{<x})$. Therefore $\mathcal{K}(\mathrm{lk}(\mathcal{N}(x))) = \mathcal{B}(X_{>x}) \star \mathcal{B}(X_{<x})$.*

Remark 7. $\mathcal{B}(X) = \mathcal{B}(X \setminus \{x\}) \cup \mathcal{K}(\mathrm{st}(\mathcal{N}(x)))$ for any $x \in X$.

Lemma 2. *Let X be a finite poset and for $x \in X$ either $\mathcal{B}(X_{>x})$ or $\mathcal{B}(X_{<x})$ is contractible. Then embedding $\mathcal{B}(X \setminus \{x\}) \hookrightarrow \mathcal{B}(X)$ is a homotopy equivalence.*

Proof. By proposition $\mathcal{K}(\mathrm{lk}(\mathcal{N}(x)))$ is contractible. Hence its embedding to its cone $\mathcal{K}(\mathrm{st}(\mathcal{N}(x)))$ is homotopy equivalence by Whitehead theorem. Lemma follows by Proposition 4. □

Definition 16. Let $f : X \to Y$ be an order preserving map between posets. Denote orders $(\leqslant)$ on X and Y as R_X and R_Y. Then we define poset $M(f) = X \coprod Y$ with $R = R_X \cup R_Y \cup R_f$ where $(x, y) \in R_f$ if and only if $(f(x), y) \in R_Y$.

We will by analogy denote this poset a *mapping cylinder* of f. There are also defined canonical inclusions $i : X \to M(f)$ and $j : Y \to M(f)$.

Proof. Quillen-McCord theorem
Let X, Y be finite posets with order-preserving map $f : X \to Y$.

Every poset has linear extension. Let $x_1, x_2, \ldots, x_n$ be enumeration of X in such linear order and $Y^r = \{x_1, \ldots, x_r\} \cup Y \subset M(f)$ for any r.

Consider $Y^r_{>x_r} = Y_{\geqslant f(x_r)}$. $\mathcal{B}(Y_{\geqslant f(x_r)})$ is a cone over $\mathcal{B}(f(x_r))$. It is contractible, therefore $\mathcal{B}(Y^{r-1}) \hookrightarrow \mathcal{B}(Y^r)$ is homotopy equivalence by Lemma 2. By iteration $\mathcal{B}(j) : \mathcal{B}(Y^0) = \mathcal{B}(Y) \hookrightarrow \mathcal{B}(M(f)) = \mathcal{B}(Y^n)$ is homotopy equivalence between BY and $M(f)$.

Then consider linear extension of Y with enumeration $y_1, \ldots, y_m$ and $X^r = X \cup \{y_{r+1}, \ldots, y_m\} \subset M(f)$. $X^{r-1}_{\leqslant y_r} = f^{-1}(Y_{\leqslant y_r})$. Latter is contractible by condition of the theorem. Hence $\mathcal{B}(X^r) \hookrightarrow \mathcal{B}(X^{r-1})$ is homotopy equivalence and by transitivity $\mathcal{B}(i)$ is a homotopy equivalence between X and $M(f)$.

Note that $i(x) \leqslant (j \circ f)(x)$. By Proposition 5 $\mathcal{B}(i)$ is homotopic to $\mathcal{B}(j \circ f) = \mathcal{B}(j) \circ \mathcal{B}(f)$. Hence $\mathcal{B}(f)$ is the homotopy equivalence between BX and BY. □

2.2.3 Proof of homological version

There holds homological [Bar11, Corollary 5.5] versions of this theorem:

Theorem 5. ***Homological Quillen-McCord theorem***
Assume X, Y are finite posets, $f : X \to Y$ is order-preserving map, R is a PID.
If $\forall y \in Y$ $H_i(\mathcal{B}(f^{-1}(Y_{\leqslant y})), R) = 0$ for any i, $\mathcal{B}f$ induces isomorphisms of all homology groups with coefficients in R on BX and BY.

To derive the theorem we can variate the proof of standard version in the following manner:

Proposition 7. *[Milnor56, Lemma 2.1] Reduced homology modules with coefficients in a principal ideal domain of a join satisfy the relation $H_{r+1}(A \star B, R) \simeq \bigoplus_{i+j=r}(H_i(A, R) \otimes_R H_j(B, R)) \oplus \bigoplus_{i+j=r-1} \mathrm{Tor}_1^{\mathrm{R}}(H_i(A, R), H_j(B, R))$.*

Lemma 3. *Let X be a finite poset and for $x \in X$ either $H_i(\mathcal{B}(X_{<x}))$ or $H_i(\mathcal{B}(X_{>x}))$ with coefficients in a PID are equal to homology of a point. Then embedding $\mathcal{B}(X \backslash \{x\}) \hookrightarrow \mathcal{B}(X)$ induces isomorphisms of all homology groups.*

Proof.
By Proposition 7 $H_i(\mathcal{K}(\mathrm{lk}(\mathcal{N}(x)))) = H_i(\mathcal{B}(X_{>x}) \star \mathcal{B}(X_{<x}))$ are trivial for all indices i — Tor-functors vanish if any of their arguments is trivial. Application of Mayer-Vietoris long exact sequence to covering from Remark 7 yields the lemma. □

Proof of the theorem is similar to proof finishing previous section. However, we write it here in detail in order to be able to highlight differences. Changed parts are written in italic.

Proof. Homological Quillen-McCord theorem
Let X, Y be finite posets with order-preserving map $f : X \to Y$.

Every poset has linear extension. Let $x_1, x_2, \ldots, x_n$ be enumeration of X in such linear order and $Y^r = \{x_1, \ldots, x_r\} \cup Y \subset M(f)$ for any r.

Consider $Y^r_{>x_r} = Y_{\geqslant f(x_r)}$. $\mathcal{B}(Y_{\geqslant f(x_r)})$ is a cone over $\mathcal{B}(f(x_r))$. It is contractible, therefore $\mathcal{B}(Y^{r-1}) \hookrightarrow \mathcal{B}(Y^r)$ is homotopy equivalence by Lemma 2. By iteration $\mathcal{B}(j) : \mathcal{B}(Y^0) =$

$\mathcal{B}(Y) \hookrightarrow \mathcal{B}(M(f)) = \mathcal{B}(Y^n)$ is homotopy equivalence between BY and $M(f)$.

Then consider linear extension of Y with enumeration $y_1, \ldots, y_m$ and $X^r = X \cup \{y_{r+1}, \ldots, y_n\} \subset M(f)$. $X^{r-1}_{\leqslant y_r} = f^{-1}(Y_{\leqslant y_r})$. *Latter is acyclic over R by condition of the theorem. Hence $\mathcal{B}(X^r) \hookrightarrow \mathcal{B}(X^{r-1})$ induces isomorphisms of all homology groups and by functoriality of homology $\mathcal{B}(i)$ induces isomorphisms of all homology groups between X and $M(f)$.*

Note that $i(x) \leqslant (j \circ f)(x)$. By Proposition 5 $\mathcal{B}(i)$ is homotopic to $\mathcal{B}(j \circ f) = \mathcal{B}(j) \circ \mathcal{B}(f)$. *Homotopic maps induce same maps on homology, j is a homotopy equivalence and induce isomorphisms. Hence $\mathcal{B}(f)$ induce isomorphisms between $H_i(BX, R)$ and $H_i(BY, R)$.* □

We see two updates. First one is essential, it requires Lemma 3 and operates some equivalence propagating in a chain of length equal to cardinality of Y. Second follows automatically from functoriality of all used constructions.

3 Persistence objects and approximation distances

3.1 Persistence objects and related constructions

We strive to provide version of this theorem suitable for usage in the setting of persistence complexes and in computations. New theorem must respect interleaving distances and give an accurate measure of error propagation.
We are not yet ready to give a statement. Let's develop appropriate technique before.

We have two types of persistence objects with similar definitions. It is appealing to form general notion of persistence object such that these definitions fall into special cases.

Definition 17. Consider I — fixed linearly ordered set. There is a sequence category $Fun(I, \mathcal{C})$ of functors from I to some category $\mathcal{C}$. We call objects of this category *persistence objects over $\mathcal{C}$*

Example 1.

- Persistence complex is a persistence object over category of chain complexes;
- Persistence R-module is a persistence object over category of R-modules;
- Persistence simplicial set is a persistence object over category $Psh(\Delta)$;
- Persistence poset is a persistence object over Pos;
- Persistence topological space is a persistence object over Top.

Definition 18. We denote images of morphisms in I as *structure* maps of a persistence object over $\mathcal{C}$. It is generally enough to consider generating set of these morphisms, i.e. set of morphisms which cannot be written as a composition of two nontrivial morphisms.

Notation remark — we use notation (X, ϕ) for "Persistence object X with generating structure maps ϕ".

Consider $\mathcal{F}$ — functor from $\mathcal{C}$ to $\mathcal{D}$. It naturally extends to functor between $Fun(I, \mathcal{C})$ and $Fun(I, \mathcal{D})$. Let P be persistence poset. Apparently $\mathcal{B}(S)$ is a persistence topological space.

Definition 19. Persistence topological space X is called *ε-acyclic* over R if for all indices $H_i(X, R) \overset{\varepsilon}{\sim} H_i(pt, R)$.

Definition 20. We say that morphism $f : X \to Y$ between persistence topological spaces induces ε-interleaving if there exists $g : Y \to X$ such that induced maps of pair (f, g) on graded homology modules form ε-interleaving.

Definition 21. *Persistence poset of finite type* is a finite sequence of finite posets.

In terms of functors "finite sequence"means that only finite set of indices has nontrivial image. It's convenient to define "trivial object"as initial object where present.

Property of being of finite type is preserved by functors which map initial objects to initial.

Proposition 8. *Let X be a persistence poset of finite type. Then BX has homology modules of finite type.*

Proof. Nerve of the empty poset is an empty simplicial set, which is an initial object. Hence BX is a geometric realization of simplicial complex of finite type. This simplicial complex can be used as a triangulation of X. Standard construction of chain complex by given simplicial complex assigns zero complex to empty simplicial complex, therefore preserves property of being finite type. Hence homology modules of BX are of finite type as homology of complex of finite type. $\square$

There is a general fact — if some universal object exists in C, it can be constructed component-wise in $Fun(I, C)$. Let's inspect this component-wise construction of universal objects by useful example:

Definition 22. Let $f = (f_1, \ldots, f_n, \ldots)$ be map of persistence posets (X, ϕ) and (Y, ψ). Then mapping cylinders $M(f_i)$ form persistence poset $M(f)$ with structure maps arising from universal property of coproduct and structure maps of X and Y.

To be explicit consider the following diagram in Pos with i, j being series of canonical inclusions:

$$\begin{tikzcd}
X_i \arrow[r, "i_i"] \arrow[d, "\phi_i"] & X_i \coprod Y_i \arrow[d, "\zeta_i"] & Y_i \arrow[l, "j_i"'] \arrow[d, "\psi_i"] \\
X_{i+1} \arrow[r, "i_{i+1}"] & X_{i+1} \coprod Y_{i+1} & Y_{i+1} \arrow[l, "j_{i+1}"']
\end{tikzcd}$$

Notation i for inclusion is kept for consistency with definition of mapping cylinder of posets.

Existence of order-preserving maps ζ_i is guaranteed by universal property of coproducts, they are set to be structure maps of $M(f)$. $M(f)$ is a *mapping cylinder of map of* *persistence posets.*

We also have canonical inclusions i and j arising from the same diagram.

Other examples of such component-wise constructions are kernels, cokernes, tensor product and direct sum of persistence modules.

We can also define a subobject in $Fun(I, \mathcal{C})$:

Definition 23.
Consider persistence object (X, ϕ) and subobjects Y_i of X_i.

Then if structure maps ϕ admit all pullbacks $\phi^\star$ $(Y, \phi^\star)$ is a subobject of (X, ϕ).

Pullback of $f : A \to B$ onto subobjects $[i]$ and $[j]$ is a map $f^\star$ between preimages of subobjects — isomorphism classes S, T of objects, such that the diagram

$$\begin{array}{ccc} S & \xrightarrow{[i]} & A \\ \downarrow{\scriptstyle f^\star} & & \downarrow{\scriptstyle f} \\ T & \xrightarrow{[j]} & B \end{array}$$

commutes.

Illustrative example is given by definition in $Fun(I, Pos)$:

Definition 24.
Consider persistence poset (X, ϕ) and sets $Y_i \subset X_i$.

If there is no element y in any Y_i such that $\phi(y) \notin Y_{i+1}$ then component-wise inclusion $Y \to X$ commutes with structure maps and form embedding of persistence posets. In this case $Y = (\dots, Y_i, \dots)$ is a *persistence subposet* of X.

Finally we give a definition of persistence covering.

Definition 25. Assume poset X splits into subposets X_j. It is not a trivial assumption since arbitrary split can lose some structure maps. Then every subposet X_j has its own classifying space BX_j. If these spaces (or minimal open sets containing them) cover the whole BX, they are called *persistence covering.*

3.2 Approximation distances

Persistence modules over good enough ring (as always here and after) give us the first example of approximation distance. So we can infer some results about distances. Specificity of these results is that they depend on implementation of approximation distance over specific category. So we can only give results for persistence modules with technical conditions specified in Preliminaries.

Proposition 9.
Let A, B be two persistence modules such that $d(A, 0) \leqslant \varepsilon$ and $d(B, 0) \leqslant \varepsilon$. Then the following hold:

1. $d(A \oplus B, 0) \leqslant \varepsilon$.

2. $d(A \otimes B, 0) \leqslant \varepsilon$.

Proof. Both statements follow from properties of these universal constructions in category of non-negatively graded modules and from Remark 6. □

We also need a result about distances in exact sequences.

Proposition 10. *Let $A \xrightarrow{f} B \xrightarrow{\phi} C \xrightarrow{g} D$ be exact sequence in category of persistence modules. Then if $d(A,0) \leqslant \varepsilon$ and $d(D,0) \leqslant \varepsilon$, then $B \overset{4\varepsilon}{\sim} C$ with ϕ being left morphism in interleaving pair.*

Proof. Under conditions of theorems 1 and 2 we can identify the category of persistence modules and the category of non-negatively graded modules.

In s.e.s $0 \to \ker f \hookrightarrow A \xrightarrow{f} \operatorname{im} f \to 0$ $\operatorname{im} f$ is ε-trivial. By exactness it is equal to $K = \ker \phi$. On the other side from $0 \to \ker g \xrightarrow{g} D \to D \to 0$ there follows that $d(I = \operatorname{im} \phi, 0) \leqslant \varepsilon$. Hence by transitivity $K \overset{2\varepsilon}{\sim} I$.

We obtain exact sequence $0 \to K \to B \xrightarrow{\phi} C \to I \to 0$. This sequences decomposes into sequences $0 \to K \to B \to \operatorname{coIm} \phi$ and $0 \to \operatorname{Im} \phi \to C \to I \to 0$. By lemma 1 we have that $d(B, \operatorname{coIm} \phi) \leqslant 2\varepsilon$ and $d(C, \operatorname{Im} \phi) \leqslant 2\varepsilon$. $coIm$ and Im are pointwise canonically isomorphic by first isomorphism theorems for modules, hence $d(B,C) \leqslant 4\varepsilon$. □

4 Approximate Quillen-McCord theorem

We are now able to rely on apparatus of interleavings, hence are ready to establish the target theorem.

Theorem 6. ***Approximate Quillen-McCord theorem, draft statement***
Assume X, Y are persistence posets of finite type indexed by I, $f : X \to Y$ is order-preserving map, and one of the following holds:

1. R is a field, $I = \mathbb{Z}$

2. $R = \mathbb{Z}$, I is an infinite additive subgroup in $\mathbb{R}$

.

Then if $\forall y = (\ldots, y_i, \ldots) \in Y$ $\mathcal{B}(f^{-1}(Y_{\leqslant y}))$ is ε-acyclic over R, $\mathcal{B}f$ induces e-interleavings of all homology spaces over R on BX and BY.

Value of e is set later.

Proposition 11. *Let A and B be two persistence topological spaces with at least one of them being ε-acyclic over $\mathbb{F}$. Then $A \star B$ is ε-acyclic over $\mathbb{F}$.*

Proof. Over a field all Tor-functors from Proposition 7 vanish since every vector space is free module, hence flat. Hence right hand side of expression of Proposition 7 is ε-equivalent to 0. $\square$

Let $X = (\ldots, X_i, \ldots)$ be persistence poset. Take arbitrary $x_i \in X_i$ where possible, and special bottom element $x_i = \perp$ for empty X_i's. We call $x = (\ldots, x_i, \ldots)$ element of X. Sets of type $X_{<x}$ with different relations to x are all defined component-wise and are all empty in components with $x_i = \perp$.

Proposition 12. *For $x = (\ldots, x_i, \phi(x_i), \ldots)$ with all x_i except firsts after bottoms being images of structure maps covering from Remark 7 can be extended to persistence covering $\mathcal{U}$ with Y^1 – preimage of $st(\mathcal{N}(x))$ under nerve functor and $Y^2 = X \setminus \{x\}$.*

Proof. It suffices to check that $X \setminus \{x\}$ and preimage of $st(\mathcal{N}(x))$ are subposets.

It is evident for $X \setminus \{x\}$. Elements in the preimage of $\mathrm{st}(\mathcal{N}(x_i))$ are exactly elements comparable to x_i. Since structure maps preserve order, they do not move comparable elements to incomparable. Hence preimage also forms subposet. $\square$

Lemma 4. *Let (X, ϕ) be a persistence poset and for $x = (\ldots, x_i, \phi(x_i), \ldots) \in X$ either $\mathcal{B}(X_{<x})$ or $\mathcal{B}(X_{>x})$ is ε-acyclic. Then embedding $\mathcal{B}(X \setminus \{x\}) \hookrightarrow \mathcal{B}(X)$ induces 4ε-interleavings of all homology spaces.*

Proof.
By Proposition 11 $\mathcal{K}(\mathrm{lk}(\mathcal{N}(x)))$ is ε-acyclic.

Given persistent covering we can define Mayer-Vietoris exact sequence on persistence homology modules component-wise by gluing sequences for components over structure maps. Proposition 10 yield the lemma. $\square$

Definition 26. We will denote as *linear extension of persistence poset X* series of linear extensions of X_i such with the same structure maps as X.

Proposition 13. *Every persistence poset has linear extension.*

Proof. Let X be persistence poset with structure maps ϕ. We will denote map from X_i as ϕ_i and refer to composition of k structure maps as to ϕ^k to simplify notation.

For any incomparable pair a, b in X_i one of the following hold:

1. $a \in \mathrm{im}(\phi_{i-1})$, $b \in \mathrm{im}(\phi_{i-1})$. Then we know that preimages are incomparable since ϕ_{i-1} is order preserving.

2. $a \in \mathrm{im}(\phi_{i-1})$, $b \notin \mathrm{im}(\phi_{i-1})$ or vice versa.

3. Both a and b are not in image of ϕ_{i-1}.

Assume for some incomparable a, b in X_i $\phi^k(a)$ and $\phi^k(b)$ are comparable for some natural k. W.l.o.g $\phi^k(a) \leqslant \phi^k(b)$. Then we have to add pair (a, b) to order on X_i. There is no possible contradiction since pair (b, a) in order implies $\phi^k(b) \leqslant \phi^k(a)$.
This extension extends infinitely to the left since we never have ordered pair in iterated

preimages of a and b. If there is no such k we leave ordering on a, b to order extension principle. This extension can be safely extended to both sides.

Application of this reasoning to all poset components and incomparable pairs in them yields proposition. □

Remark 8. While working with persistence topological spaces we sometimes see properties which hold component-wise. If for persistence space X or map f_i there is a property which holds for any X_i or f_i we call it component-wise property of X. For instance, if map f is component-wise homotopy equivalence, it induces 0-interleaving of homology modules.

We are now ready to adapt known proof to Quillen-McCord theorem for persistence posets.

Theorem 7. ***Approximate Quillen-McCord theorem, final statement***
Assume X, Y are persistence posets of finite type indexed by I, $f : X \to Y$ is order-preserving map. Let $m = \max_i(|Y_i|)$ be the maximal cardinality of components of Y and one of the following holds:

1. *R is a field, $I = \mathbb{Z}$*

2. *$R = \mathbb{Z}$, I is an infinite additive subgroup in $\mathbb{R}$*

.

Then if $\forall y = (\ldots, y_i, \ldots) \in Y$ $\mathcal{B}(f^{-1}(Y_{\leqslant y}))$ is ε-acyclic over R, $\mathcal{B}f$ induces $4m\varepsilon$-interleavings of all homology spaces over R on BX and BY.

Proof. Approximate Quillen-McCord theorem
Let X, Y be persistence posets of finite type with order-preserving map $f : X \to Y$.

Let $\overline{X}$ be linear extension of X. Let $x^i_1, x^i_2, \ldots, x^i_{n_i}$ be enumeration of $\overline{X}_i$ in its linear order and $Y^r_i = \{x^i_1, \ldots, x^i_r\} \cup Y_i \subset M(f_i)$ for any r up to $\max_i(n_i)$ and i. $Y^r = (\ldots, Y^r_i, \ldots)$ is a persistence subposet of $M(f)$ with extended order on X.

There are posets such that $n_i < r$. In these cases notation x_r means that on positions with $n_i < r$ we take x_{n_i}. n_i can be undefined, in this case $x_r = \perp$.

Consider $Y^r_{>x_r} = Y_{\geqslant f(x_r)}$. $\mathcal{B}(Y_{\geqslant f(x_r)})$ is a component-wise cone over $\mathcal{B}(f(x_r))$. It is component-wise contractible, therefore $\mathcal{B}(Y^{r-1}) \hookrightarrow \mathcal{B}(Y^r)$ is component-wise homotopy equivalence by Lemma 2. By iteration $\mathcal{B}(j) : \mathcal{B}(Y^0) = \mathcal{B}(Y) \hookrightarrow \mathcal{B}(M(f)) = \mathcal{B}(Y^n)$ is component-wise homotopy equivalence between BY and $M(f)$. Note that persistence structure is not used here.

Then consider linear extension of Y with enumerations $y^i_1, \ldots, y^i_{m_i}$ and $X^r_i = X \cup \{y^i_{r+1}, \ldots, y^i_{m_i}\} \subset M_i(f)$. $X^{r-1}_{\leqslant y_r} = f^{-1}(Y_{\leqslant y_r})$. Latter is ε-acyclic over $\mathbb{F}$ by condition of the theorem. Hence $\mathcal{B}(X^r) \hookrightarrow \mathcal{B}(X^{r-1})$ induces 4ε-interleavings of all homology modules and by transitivity of ε-equivalence $\mathcal{B}(i)$ induces $4m\varepsilon$-interleavings between homology of X and $M(f)$.

Note that $i(x) \leqslant (j \circ f)(x)$. By Proposition 5 $\mathcal{B}(i)$ is component-wise homotopic to $\mathcal{B}(j \circ f) = \mathcal{B}(j) \circ \mathcal{B}(f)$. Homotopic maps induce same maps on homology, j is a homotopy equivalence and induce 0-interleavings. Hence $\mathcal{B}(f)$ induce $4m\varepsilon$-interleavings between $H_i(BX, R)$ and $H_i(BY, R)$. □

5 Appendix: Error propagation in Mayer-Vietoris spectral sequence

Value of error propagation multiple in the result may probably be decreased with alternative proof using spectral sequences at the cost of some restriction on structure maps of posets. Here we outline results of Govc and Scraba on error propagation in one specific spectral sequence associated to cover.

Assume persistence simplicial complex S is a filtered complex (assuming either ascending or descending filtration with any compatible structure maps) and there exists open covering $\mathcal{U}$ compatible with filtration. Then there exists a spectral sequence called Mayer-Vietoris spectral sequence which converges to $H_\star(S)$. [GS16, Theorem 2.30]

Let all sets in filtered cover $\mathcal{U}$ be ε-acyclic with all intersections between them be ε-acyclic. This is a representation of the definition of ε-acyclic cover. [GS16, Definition 3.2]

Sets and all their nonempty intersections in any covering form a poset with inclusion being an order relation. Nerve of this poset is called *nerve of a covering* and we shall denote it as $\mathcal{N}(\mathcal{U})$.

There hold the following propositions:

Proposition 14. *[GS16, Corollary 5.2]*
If $\mathcal{U}$ is an ε-acyclic cover of X, then for all i $E^2_{i,0}$ of Mayer-Vietoris spectral sequence and $H_i(\mathcal{N}(\mathcal{U}), \mathbb{F})$ are 2ε-interleaved as graded modules.

Proposition 15. *[GS16, Theorem 7.1]*
Let D be dimension of $\mathcal{N}(\mathcal{U})$. Then for all i $H_i(X, \mathbb{F}) \overset{(4D+2)\varepsilon}{\sim} H_i(\mathcal{N}(\mathcal{U}), \mathbb{F})$.

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