1 Abstract

Common approach in Topological Data Analysis is to compute homology groups of some persistence complex constructed from experimental data and try to classify some objects by them. Complexity of computation of persistent homology is high enough to limit practical usage of the approach. This concern raises a question of how to construct persistence complex with same homology as initially given with smaller dimension.

Quillen-McCord theorem (Quillen fiber lemma) gives sufficient condition when map between posets induces a homotopy equivalence in classifying spaces of their poset categories. In this paper we prove Quillen-McCord theorem in the setting of persistent homology. We believe it to be the tool to reduce complexity of observed persistence complexes.

While proving the theorem we develop a setting convenient to prove approximate statements about persistence objects and define general persistence object as object in appropriate functor category.

2 Preliminaries

2.1 Persistence modules and interleaving distance

Initial definition is the following:

Definition 1. [Zomorodian05, Definition 3.1]
Persistence complex is a family of chain complexes $C^0_\star \xrightarrow{f_0} C^1_\star \xrightarrow{f_1} C^2_\star \xrightarrow{f_2} \dots$ where f_i are chain maps. We call maps $f = (\dots, f_i, \dots)$ *the structure maps* of a persistence complex.

Definition 2. Let R be a ring. *Persistence module* is a family of R-modules M^i with homomorphisms $\phi_i :: M^i \to M^{i+1}$ as the structure maps. If R is a field it is natural to use notion of *persistence vector space*.

Example of a persistence module is given by homology modules of persistence complex $C_\star$ (*persistent homology*). $H^j_i(C_\star) = H_i(C_j)$, maps ϕ_j are induced by f_i.

Definition 3. Persistence complex (module) is of *finite type* over R if all components of complexes (modules) are finitely-generated as R-modules and all f_i (ϕ_i) are isomorphisms for $i > m$ for some m.

Definition 4. Persistence complex (module) is of *finitely presented type* over R if all components of complexes (modules) are finitely-presented as R-modules and all f_i (ϕ_i) are isomorphisms for $i > m$ for some m.

Note that by construction homology of complex of finite type is a module of finite type. It follows from the fact that quotient of finitely generated module by finitely generated module is finitely generated. Analogous statement fails for complexes of finitely presented type.

Definition 5. Assume I is a semigroup. Then *graded module* over I-graded ring R is an R-module M together with a decomposition $M = \bigoplus_{j \in I} M^j : \forall i \in I \; R_i \cdot M^j \subset M^{j+i}$

Graded modules over R form a category. Maps ϕ between modules M and N such that $\phi(M^j) \subset N^j$ for all j in indexing set are morphisms in this category.

There is a well-known theorem:

Theorem 1. *[Zomorodian05, Theorem 3.1]*
Category of persistence modules of finite type over Noetherian ring with unity R indexed by $\mathbb{N}$ is equivalent to category of graded finitely generated $R[t]$-modules.

It is proven in [Corbet18]. Authors provide generalization which is more suitable for our needs.

Definition 6. Let $(G, \star)$ be a commutative monoid and $g_1, g_2 \in G$. We say that $g_1 \preceq g_2$ if $\exists h \in G : \ h \star g_1 = g_2$, h is not neutral element.

Definition 7. following [Corbet18, Definition 11] Monoid $(G, \star)$ is called *good* if the following holds:

- $(G, \star)$ is commutative;
- $g_1 \star g_2 = g_1 \star g_3$ implies $g_2 = g_3$ (cancellation);
- $g_1 \preceq g_2$ and $g_2 \preceq g_1$ imply $g_1 = g_2$ (anti-symmetricity);
- For any finite $H \subseteq G$ there exists at most finitely many distinct elements (common multiples) m such that $\forall h \in H : \ h \preceq m$ and there is no common multiple $m_1 \prec m$ satisfying 1 (partially least) (property of being weak plcm).

Note that original definition does not require commutativity.

Definition 8. [Corbet18, Definition 12] Let R be a ring and G be a good monoid. *(Generalized) persistence module* is a family of R-modules M^i with homomorphisms $\phi_{ij} :: M^i \to M^j$ satisfying $\phi_{ii} = Id$ and relation $\phi_{ij} \circ \phi_{jk} = \phi_{ik}$ for any $i < j < k$ as the structure maps.

Theorem 2. *[Corbet18, Theorem 21]*
Let R be a ring with unity and G be a good monoid. Then the category of finitely presented graded $R[G]$-modules is isomorphic to the category of G-indexed persistence modules over R of finitely presented type.

Remark 1.
Our basic examples are given by considering this Theorem with G being either $(\mathbb{R}, +)$ or $(\mathbb{Z}, +)$. These examples represent continuous and discrete time in observation of some dynamic structure with ability to reason about the past. All techniques we describe are applicable for static data, e.g. for Čech complexes.

This equivalence is required to introduce a measure of similarity between persistence modules. It is essential since in applications we have to accept error in experimental data and we must decide whether observed homology modules are close enough to initial hypothesis or not.

Definition 9. [GS16, Definition 2.7]
Let M and N be graded $R[G]$-modules, $f : M \to N$ be a homomorphism of modules. Then f is called *ε-morphism* if $f(M^j) \subset N^{j+\varepsilon}$.

Remark 2. Note that 0-morphism is a morphism in category of graded modules over $R[G]$.

Remark 3. Cancellative commutative monoid with $\preceq$ being a total order (totally ordered) is good.

Definition 10. We shall call cancellative commutative totally ordered monoids *very good*.

In further constructions we shall use multiplicative notation for monoids.

Proposition 1. *Let $\mathbb{R}$ be a ring with unity and G be a very good monoid. Then for any graded $R[G]$-module M and any $\varepsilon \in \mathbb{G}$ there exists morphism $Id_\varepsilon(a) = t^\varepsilon a$ where t is a successor of identity.*

This proposition follows from definition of monoid ring and does not require any assumptions on R and ideals in $R[G]$.

Definition 11. Graded modules M and N are called *ε-interleaved* ($M \overset{\varepsilon}{\sim} N$) if there exists pair of ε-morphisms ($\phi : M \to N$, $\psi : N \to M$) (*ε-interleaving*) such that $\phi \circ \psi = Id_{2\varepsilon} : N \to N$ and $\psi \circ \phi = Id_{2\varepsilon} : M \to M$.

Remark 4. There follows that $M \overset{\varepsilon}{\sim} N$ implies $M \overset{\alpha}{\sim} N$ for any $\alpha > \varepsilon$ since for ε-interleaving (ϕ, ψ) we have α-interleaving $(Id_{\alpha-\varepsilon} \circ \phi, Id_{\alpha-\varepsilon} \circ \psi)$.

Definition 12. We denote as *ε-equivalence* relation with the following properties:
For any M, N, L, ε, ε_1, ε_2

- $M \overset{0}{\sim} M$.
- $M \overset{\varepsilon}{\sim} N$ is equivalent to $N \overset{\varepsilon}{\sim} M$.
- if $M \overset{\varepsilon_1}{\sim} N$ and $N \overset{\varepsilon_2}{\sim} L$ then $M \overset{\varepsilon_1+\varepsilon_2}{\sim} L$.

Proposition 2. *ε-interleaved non-negatively graded modules are ε-equivalent.*

Proposition 3. *[GS16, Proposition 2.13] Condition $M \overset{\varepsilon}{\sim} 0$ is equivalent to condition $t^{2\varepsilon}M = 0$.*

Definition 13. Any ε-equivalence induces an extended pseudometric on its domain. This pseudometric is defined as $d(X, Y) = min\{\varepsilon \in I \mid X \overset{\varepsilon}{\sim} Y\}$. For graded modules this pseudometric is called interleaving distance. [GS16, Definition 2.12]
We shall refer to this general pseudometric as to approximation distance since we do not want to overload the term, however we do not give another examples expect for interleaving distance. For deeper evaluation of this construction see [deSilva18].

Lemma 1. *Let $0 \to M \to L \to N \to 0$ be a short exact sequence of non-negatively graded modules. Then the following properties hold:*

- *If $M \overset{\varepsilon_1}{\sim} 0$ and $N \overset{\varepsilon_2}{\sim} 0$ then $L \overset{\varepsilon_1+\varepsilon_2}{\sim} 0$. [GS16, Proposition 4.6]*
- *If $L \overset{\varepsilon}{\sim} 0$ then $M \overset{\varepsilon}{\sim} 0$ and $N \overset{\varepsilon}{\sim} 0$.*
- *If $M \overset{\varepsilon}{\sim} 0$ then $L \overset{2\varepsilon}{\sim} N$. [GS16, Proposition 4.1]*
- *If $N \overset{\varepsilon}{\sim} 0$ then $M \overset{2\varepsilon}{\sim} L$. [GS16, Proposition 4.1]*

Proof. Second statement of the lemma requires the proof. Denote non-trivial maps in s.e.s as i and q.

Then $i(t^{2\varepsilon}a) = t^{2\varepsilon}i(a) = 0$ for any $a \in M$. i is injective. Hence by Proposition 3 $M \overset{\varepsilon}{\sim} 0$.

On the other side $0 = q(t^{2\varepsilon}a) = t^{2\varepsilon}q(a)$ where $q$is surjective. Hence $N \overset{\varepsilon}{\sim} 0$. □

2.2 Quillen-McCord theorem

2.2.1 Statement

Definition 14. *Join* $A \star B$ of simplicial complexes A and B is the simplicial complex with simplices — all possible unions of simplices $a \in A$ and $b \in B$.

Definition 15. *Join* of topological spaces A and B is defined as follows: $A \star B := A \sqcup_{p_0} (A \times B \times [0,1]) \sqcup_{p_1} B$, where p are projections of the cylinder $A \times B \times [0,1]$ onto faces.

Definition 16. *Star* $\mathrm{st}(x)$ of simplex $x \in A$ (A is a simplicial complex) is the minimal simplicial complex containing all simplices $a \in A$ such that there exists inclusion $x \hookrightarrow a$.

Definition 17. *Link* $\mathrm{lk}(x)$ of simplex $x \in A$ is defined as follows: $\mathrm{lk}(x) = \{v \in \mathrm{st}(x) |\ x \notin v\}$.

Definition 18. Let Δ be a simplicial category. Functor $||: \Delta \to Top$ which maps simplices to geometric simplices of corresponding dimension and morphisms to inclusions of faces and restrictions to subcomplexes is called standard *geometric realization*.

Proposition 4. $\mathrm{st}(x) = \mathrm{lk}(x) \star x$.

Proposition 5. $|A \star B| = |A| \star |B|$. *Hence* $|\mathrm{st}(x)|$ *is a cone over* $|x|$.

Let $\mathcal{C}$ denote a small category. We can construct the simplicial set called the *nerve of category* $\mathcal{C}$ as follows:
Let objects of $\mathcal{C}$ be the only 0-dimensional simplices. Then let all morphisms be 1-dimensional simplices, all composable pairs of morphisms be 2-dimensional simplices and so on with morphisms — inclusions of compositions into longer ones and replacements of compositions $i \circ j$ with $f = i \circ j$.
This construction is functorial over category of posets Pos, we denote nerve functor from Pos to category of simplicial sets over Δ as $\mathcal{N}$.

Definition 19. Geometric realization $B\mathcal{C}$ of $\mathcal{N}(\mathcal{C})$ is called *classifying space* of $\mathcal{C}$.

We denote composition of nerve and geometric realization as $\mathcal{B}$. It is obviously a functor and we only need this construction to avoid confusing notation Bf for induced morphism of classifying spaces. By definition $\mathcal{B}(\mathcal{C}) = B\mathcal{C}$.

Definition 20. Let $f: \mathcal{C} \to \mathcal{D}$ be a functor and d — object in $\mathcal{D}$. Then *comma category* $d \downarrow f$ is a category with objects — pairs (s, i_s) of objects in $\mathcal{C}$ and morphisms $i_s : d \to f(s)$ and morphisms — morphisms g in $\mathcal{C}$ such that triangle $i_s, i_{g(s)}, f(g)$ is commutative.

Theorem 3. *[Quillen72, Theorem A]*
If $f : C \to D$ is a functor such that the classifying space $B(d \downarrow f)$ of the comma category $d \downarrow f$ is contractible for any object $d \in D$, then f induces a homotopy equivalence $BC \to BD$.

Nerve construction on a poset category yields a simplicial complex called *order complex*. Application of Quillen A theorem to posets yields the following theorem (we identify poset with its poset category):

Theorem 4. ***Quillen-McCord theorem***
Assume X, Y are finite posets, $f : X \to Y$ is order-preserving map.
If $\forall y \in Y$ $\mathcal{B}(f^{-1}(Y_{\leqslant y}))$ is contractible, then $\mathcal{B}f$ is a homotopy equivalence between BX and BY.

Proof of the theorem is necessary for construction of a desired result.

2.2.2 Proof

Let us outline the proof of Quillen-McCord theorem given by Barmak.

Proposition 6. *[Bar11, Proposition 2.1] Let X and Y be two simplicial complexes such that $X \cup Y$ is a simplicial complex. Then if $|X| \cup |Y| \hookrightarrow |X|$ is a homotopy equivalence, then so is $|Y| \hookrightarrow |X \cup Y|$.*

Proposition 7. *Variation of [Bar11, Proposition 2.2] Let $f, g : X \to Y$ be order-preserving maps between finite posets such that $\forall x\ f(x) \leqslant g(x)$. Then $\mathcal{B}(f)$ is homotopy-equivalent to $\mathcal{B}(g)$.*

Proof. X has a finite set M of maximal elements. Take any of them (m_1) and define $h_1 : X \to Y$ such that $h_1(m_1) = g(m_1)$ and $h_1 = f$ on all other elements of X. This is an order-preserving map due to maximality of m_1. Take $M \setminus \{m_1\}$ and build h_2 which is equal to h_1 on complement to m_2 and to g on m_2 and so on. We have built a finite sequence of maps $h_0 = f \leqslant h_1 \leqslant h_1 \leqslant \ldots \leqslant g = h_n$.

Elements $h_i(m_i)$ and $h_{i-1}(m_i)$ are comparable. Hence there exists simplex $\{h_i(m_i); h_{i-1}(m_i)\}$ in $\mathcal{N}(Y)$. Since $\mathcal{N}(Y)$ is a simplicial complex, for other elements between selected where exist no holes and thus where is a linear homotopy between h_{i-1} and h_i which contracts simplex $\{h_i(m_i); h_{i-1}(m_i)\}$.

Hence there is a homotopy between f and g. □

Proposition 8. *Note that* $\mathrm{lk}(\mathcal{N}(x)) = \mathcal{N}(X_{>x}) \star \mathcal{N}(X_{<x})$. *Therefore* $|\mathrm{lk}(\mathcal{N}(x))| = \mathcal{B}(X_{>x}) \star \mathcal{B}(X_{<x})$.

Remark 5. $\mathcal{B}(X) = \mathcal{B}(X \setminus \{x\}) \cup |\mathrm{st}(\mathcal{N}(x))|$ for any $x \in X$.

Lemma 2. *Let X be a finite poset and for $x \in X$ either $\mathcal{B}(X_{>x})$ or $\mathcal{B}(X_{<x})$ is contractible. Then embedding $\mathcal{B}(X \setminus \{x\}) \hookrightarrow \mathcal{B}(X)$ is a homotopy equivalence.*

Proof. By proposition $|\mathrm{lk}(\mathcal{N}(x))|$ is contractible. Hence its embedding to its cone $|\mathrm{st}(\mathcal{N}(x))|$ is homotopy equivalence by Whitehead theorem. Lemma follows by Proposition 6. □

Definition 21. Variation of [Bar11, Proposition 2.1] Let $f : X \to Y$ be an order preserving map between posets. Denote orders $(\leqslant)$ on X and Y as R_X and R_Y. Then we define poset $M(f) = X \coprod Y$ with $R = R_X \cup R_Y \cup R_f$ where $(x, y) \in R_f$ if and only if $(f(x), y) \in R_Y$.

We shall by analogy denote this poset a *mapping cylinder* of f. There are also defined canonical inclusions $i_X : X \to M(f)$ and $i_Y : Y \to M(f)$.

Proof. Quillen-McCord theorem
Let X, Y be finite posets with order-preserving map $f : X \to Y$.

Every poset has linear extension. Let $x_1, x_2, \ldots, x_n$ be enumeration of X in such linear order and $Y^r = \{x_1, \ldots, x_r\} \cup Y \subset M(f)$ for any r.

Consider $Y^r_{>x_r} = Y_{\geqslant f(x_r)}$. $\mathcal{B}(Y_{\geqslant f(x_r)})$ is a cone over $\mathcal{B}(f(x_r))$. It is contractible, therefore $\mathcal{B}(Y^{r-1}) \hookrightarrow \mathcal{B}(Y^r)$ is homotopy equivalence by Lemma 2. By iteration $\mathcal{B}(j) : \mathcal{B}(Y^0) = \mathcal{B}(Y) \hookrightarrow \mathcal{B}(M(f)) = \mathcal{B}(Y^n)$ is homotopy equivalence between BY and $M(f)$.

Then consider linear extension of Y with enumeration $y_1, \ldots, y_m$ and $X^r = X \cup \{y_{r+1}, \ldots, y_m\} \subset M(f)$. $X^{r-1}_{\leqslant y_r} = f^{-1}(Y_{\leqslant y_r})$. Latter is contractible by condition of the theorem. Hence $\mathcal{B}(X^r) \hookrightarrow \mathcal{B}(X^{r-1})$ is homotopy equivalence and by transitivity $\mathcal{B}(i_X)$ is a homotopy equivalence between X and $M(f)$.

Note that $i(x) \leqslant (i_Y \circ f)(x)$. By Proposition 7 $\mathcal{B}(i_X)$ is homotopic to $\mathcal{B}(i_Y \circ f) = \mathcal{B}(i_Y) \circ \mathcal{B}(f)$. Hence $\mathcal{B}(f)$ is the homotopy equivalence between BX and BY. □

2.2.3 Proof of homological version

There holds homological [Bar11, Corollary 5.5] versions of this theorem:

Theorem 5. ***Homological Quillen-McCord theorem***
Assume X, Y are finite posets, $f : X \to Y$ is order-preserving map, R is a PID.
If $\forall y \in Y$ $H_i(\mathcal{B}(f^{-1}(Y_{\leqslant y})), R) = 0$ for any i, $\mathcal{B}f$ induces isomorphisms of all homology groups with coefficients in R on BX and BY.

To derive the theorem we can variate the proof of standard version in the following manner:

Proposition 9. *[Milnor56, Lemma 2.1] Reduced homology modules with coefficients in a principal ideal domain of a join satisfy the relation $H_{r+1}(A \star B, R) \simeq \bigoplus_{i+j=r}(H_i(A, R) \otimes_R H_j(B, R)) \oplus \bigoplus_{i+j=r-1} \mathrm{Tor}_1^{\mathrm{R}}(H_i(A, R), H_j(B, R))$.*

Lemma 3. *Let X be a finite poset and for $x \in X$ either $H_i(\mathcal{B}(X_{<x}))$ or $H_i(\mathcal{B}(X_{>x}))$ with coefficients in a PID are equal to homology of a point. Then embedding $\mathcal{B}(X\setminus\{x\}) \hookrightarrow \mathcal{B}(X)$ induces isomorphisms of all homology groups.*

Proof.
By Proposition 9 $H_i(|\mathrm{lk}(\mathcal{N}(x))|) = H_i(\mathcal{B}(X_{>x}) \star \mathcal{B}(X_{<x}))$ are trivial for all indices i — Tor-functors vanish if any of their arguments is trivial. Application of Mayer-Vietoris long exact sequence to covering from Remark 5 yields the lemma. □

Proof of the theorem is similar to proof finishing previous section. However, we write it here in detail in order to be able to highlight differences. Changed parts are written in italic.

Proof. Homological Quillen-McCord theorem
Let X, Y be finite posets with order-preserving map $f : X \to Y$.

Every poset has linear extension. Let $x_1, x_2, \dots, x_n$ be enumeration of X in such linear order and $Y^r = \{x_1, \dots, x_r\} \cup Y \subset M(f)$ for any r.

Consider $Y^r_{>x_r} = Y_{\geqslant f(x_r)}$. $\mathcal{B}(Y_{\geqslant f(x_r)})$ is a cone over $\mathcal{B}(f(x_r))$. It is contractible, therefore $\mathcal{B}(Y^{r-1}) \hookrightarrow \mathcal{B}(Y^r)$ is homotopy equivalence by Lemma 2. By iteration $\mathcal{B}(j) : \mathcal{B}(Y^0) = \mathcal{B}(Y) \hookrightarrow \mathcal{B}(M(f)) = \mathcal{B}(Y^n)$ is homotopy equivalence between BY and $M(f)$.

Then consider linear extension of Y with enumeration $y_1, \dots, y_m$ and $X^r = X \cup \{y_{r+1}, \dots, y_n\} \subset M(f)$. $X^{r-1}_{\leqslant y_r} = f^{-1}(Y_{\leqslant y_r})$. *Latter is acyclic over R by condition of the theorem. Hence $\mathcal{B}(X^r) \hookrightarrow \mathcal{B}(X^{r-1})$ induces isomorphisms of all homology groups and by functoriality of homology $\mathcal{B}(i)$ induces isomorphisms of all homology groups between X and $M(f)$.*

Note that $i(x) \leqslant (j \circ f)(x)$. By Proposition 7 $\mathcal{B}(i)$ is homotopic to $\mathcal{B}(j \circ f) = \mathcal{B}(j) \circ \mathcal{B}(f)$. *Homotopic maps induce same maps on homology, j is a homotopy equivalence and induce isomorphisms. Hence $\mathcal{B}(f)$ induce isomorphisms between $H_i(BX, R)$ and $H_i(BY, R)$.* □

We see two updates. First one is essential, it requires Lemma 3 and operates some equivalence propagating in a chain of length equal to cardinality of Y. Second follows automatically from functoriality of all used constructions.

3 Persistence objects and approximation distances

3.1 Persistence objects and related constructions

We strive to provide version of this theorem suitable for usage in the setting of persistence complexes and in computations. New theorem must respect interleaving distances and give an accurate measure of error propagation.
Let's develop appropriate technique before giving a statement.

We have two types of persistence objects with similar definitions. It is appealing to form general notion of persistence object such that these definitions fall into special cases.

Definition 22. Consider I — poset category of a fixed linearly ordered set. There is a sequence category $Fun(I, \mathcal{C})$ of functors from I to some category $\mathcal{C}$. We call objects of this category *persistence objects over $\mathcal{C}$*

Example 1.

- Persistence complex is a persistence object over category of chain complexes;
- Persistence R-module is a persistence object over category of R-modules;
- Persistence simplicial set is a persistence object over category $Psh(\Delta)$;
- Persistence poset is a persistence object over Pos;
- Persistence topological space is a persistence object over Top.

Definition 23. We denote images of morphisms in I as *structure* maps of a persistence object over $\mathcal{C}$. For countable I it is generally enough to consider generating set of these morphisms, i.e. set of morphisms which cannot be written as a composition of two nontrivial morphisms.

We use notation (X, ϕ) for "Persistence object X with structure maps ϕ over fixed indexing category I". We use notation ϕ_{ij} for a structure map between X_i and X_j.

Consider $\mathcal{F}$ — functor from $\mathcal{C}$ to $\mathcal{D}$. It naturally extends to functor between $Fun(I, \mathcal{C})$ and $Fun(I, \mathcal{D})$. Let P be persistence poset. Apparently $\mathcal{B}(P)$ is a persistence topological space.

Definition 24. Persistence topological space X is called *ε-acyclic* over R if for all indices $H_i(X, R) \overset{\varepsilon}{\sim} H_i(pt, R)$.

Definition 25. We say that morphism $f : X \to Y$ between persistence topological spaces induces ε-interleaving if there exists $g : Y \to X$ such that induced maps of pair (f, g) on graded homology modules form ε-interleaving.

Definition 26. *Persistence poset of finite type* is a finite sequence of finite posets.

In terms of functors "finite sequence"means that only finite set of indices has nontrivial image. It's convenient to define "trivial object"as initial object where present.

Property of being of finite type is preserved by functors which map initial objects to initial.

Proposition 10. *Let X be a persistence poset of finite type. Then BX has homology modules of finite type.*

Proof. Nerve of the empty poset is an empty simplicial set, which is an initial object. Hence BX is a geometric realization of simplicial complex of finite type. Homology of BX can be computed as homology of this simplicial comples. Hence homology modules of BX are of finite type. □

Corollary 1. *In particular BX has finitely presented homology modules.*

There is a general fact — if some universal object exists in C, it can be constructed component-wise in $Fun(I, C)$. Let's inspect this component-wise construction of universal objects by useful example:

Definition 27. Let $f = (f_1, \dots, f_n, \dots)$ be map of persistence posets (X, ϕ) and (Y, ψ). Then mapping cylinders $M(f_j)$ form persistence poset $M(f)$ with structure maps arising from universal property of coproduct and structure maps of X and Y.

To be explicit consider the following diagram in Pos with i_X, j_Y being series of canonical inclusions:

$$\begin{tikzcd} X_j \arrow[r, "i_{Xj}"] \arrow[d, "\phi_{i_X}"] & X_j \coprod Y_j \arrow[d, "\zeta_j"] & Y_j \arrow[l, "i_{Yj}"'] \arrow[d, "\psi_{i_Y}"] \\ X_{j+1} \arrow[r, "i_{Xj+1}"] & X_{j+1} \coprod Y_{j+1} & Y_{j+1} \arrow[l, "i_{Yj+1}"'] \end{tikzcd}$$

Existence of order-preserving maps ζ_j is guaranteed by universal property of coproducts, they are set to be structure maps of $M(f)$. $(M(f), \zeta_j)$ is a *mapping cylinder of map of persistence posets*.

We also have canonical inclusions i_X and i_Y arising from the same diagram.

Other examples of such component-wise constructions are kernels, cokernels and homology modules.

We can also define a subobject in $Fun(I, \mathcal{C})$:

Definition 28.
Consider persistence object (X, ϕ) and subobjects Y_i of X_i.

Then if structure maps ϕ admit all pullbacks $\phi^\star$ $(Y, \phi^\star)$ is a subobject of (X, ϕ).

Pullback of $f : A \to B$ onto subobjects $[i]$ and $[j]$ is a map $f^\star$ between preimages of subobjects — isomorphism classes S, T of objects such that the diagram

$$\begin{tikzcd} S \arrow[r, "{[i]}"] \arrow[d, "f^\star"] & A \arrow[d, "f"] \\ T \arrow[r, "{[j]}"] & B \end{tikzcd}$$ commutes.

Illustrative example is given by definition in $Fun(I, Pos)$:

Definition 29.
Consider persistence poset (X, ϕ) and sets $Y_i \subset X_i$.

If there is no element y in any Y_i such that $\phi(y) \notin Y_{i+1}$ then component-wise inclusion $Y \to X$ commutes with structure maps and form embedding of persistence posets. In this case $Y = (\dots, Y_i, \dots)$ is a *persistence subposet* of X.

We can also define *element* of a persistence poset.

Definition 30. Let (X, ϕ) be a persistence poset, $x_i \in X_i$. In all components with degree above i there exist elements $\phi(x_i)$ in components. In degrees below we take iterated preimages while possible and define $x_j = \perp$ for $j < i$ if there is no preimage.

Although this definition operates contradiction it is eliminated by applications.
For instance we can consider the following example of subposet:

Example 2. Let x be an element of persistence subposet (X, ϕ). Then we can define $X_{<x}$ component-wise as poset of elements less than x and as trivial poset if $x_i = \perp$.

$X_{<x}$ contains component-wise only elements comparable with x. Since structure maps are order preserving, $X_{<x}$ is a subposet.

Finally we give a definition of persistence covering.

Definition 31. Assume poset X splits into subposets X_j. It is not a trivial assumption since arbitrary split can lose some structure maps. Then every subposet X_j has its own classifying space BX_j. If these spaces (or minimal open sets containing them) cover the whole BX, they are called *persistence covering*.

This definition gives an example of how structures in category of persistence posets can be transferred to other persistence categories. It is possible to reformulate the definition as internal to category of persistence topological spaces but we prefer to keep more constructive way.

3.2 Order extension principle for persistence posets

In his proof of Quillen-McCord theorem Barmak relies on order extension principle. To be able to transfer Barmak's proof of Quillen-McCord theorem to persistent case we have to stress similar statement for persistence posets.

Definition 32. We denote as *extension of persistence poset* X series of partially-ordered extensions of X_i such that structure maps of X are well-defined on these extensions. If extensions of all components are linear we call this series linear extension.

Proposition 11. ***Transfer of order***. *Let f be a morphism between posets X and Y and $\overline{Y}$ be a linear extension of Y. Then f induces partially ordered extension $\hat{X}$ of X such that f is well-defined as map $\hat{X} \to \overline{Y}$.*

Proof. Consider two incomparable points $a, b \in X$ and map $f : X \to \overline{Y}$ which is obviously well-defined.
One of the following hold:

- f(b) $<$ f(a)
- f(a) $<$ f(b)
- f(a) $=$ f(b)

If strict inequality holds, we can impose a single relation on a and b — we inherit relation from images.
If equality holds we do not add any new relation. $\square$

Proposition 12. ***Left propagation of linear extension.*** *Assume indexing set I of persistence poset (X, ϕ) is well-founded with respect to inverted order and there exists maximal index i such that $X_j = \emptyset$ for any $j > i$. Then (X, ϕ) has linear extension.*

Proof. We can extend order on the component X_i to linear. Given this order we can transfer it to the left via all possible structure maps by proposition. We obtain extension of (X, ϕ) because all preimages of incomparable elements were imcomparable and we have equipped them with compatible orders. Now let's assume we obtained by this construction linear orders in components X_j for $j > j_0$. Then X_{j_0} can be linearly extended. Proposition follows by transfinite induction. $\square$

This statement can also be seen as a corollary of a more general proposition. One can consider set $E(X, \phi)$ of extensions of persistence poset (X, ϕ) with partial order defined as follows: let Y, Z be extensions of X, then $Y \geqslant Z$ if and only if Y is extension of Z. Since underlying sets of these extensions are always the same we can identify element of $E(X, \phi)$ with tuple of order relations on components of X.

Proposition 13. *Every linearly ordered subset of $E(X, \phi)$ has an upper bound in $E(X, \phi)$.*

Proof. Let $\{R^s |\ s \in S\}$ be a linearly ordered subset of $E(X, \phi)$ indexed by set S. Consider $R = \bigcup R^s$ where union is taken component-wise. Assume for some elements $a, b \in X_i$ for some i some ϕ_{ij} cannot be defined on them as order-preserving map. Then there exists $s \in S$ such that a and b are comparable in extension R^s. But R^s is an extension, hence ϕ_{ij} is defined on both a and b. By contradiction proposition follows. $\square$

Proposition 14. ***Persistent order extension principle.*** *Every persistence poset (X, ϕ) has linear extension.*

Proof. By Zorn's lemma $E(X, \phi)$ has maximal element M. Assume this element is not a linear extension of (X, ϕ).

Then in some M_i there exists incomparable pair (a, b). Consider $\phi_{ij}(a)$ and $\phi_{ij}(b)$ for all $j > i$. Assume without loss of generality for $j_2 > j_1$ that $\phi_{ij_1}(a) > \phi_{ij_1}(b)$ and $\phi_{ij_2}(a) < \phi_{ij_2}(b)$. Then map $\phi_{j_1 j_2}$ cannot be defined. Hence for any j relation between images of a and b has the same sign if exists. If there exists such j that relation between $\phi_{ij}(a)$ and $\phi_{ij}(b)$ exists we define relation between a and b accordingly. Otherwise we define it arbitrarily and propagate order to the left — all preimages of a and b were incomparable and are now equipped with compatible orders.

We have constructed an extension of M which is taken to be maximal. Hence M must be linear extension of (X, ϕ). $\square$

3.3 Approximation distances

Persistence modules give us the first example of approximation distance. So we can infer some results about distances.

Proposition 15.
Let A, B be two persistence modules such that $d(A,0) \leqslant \varepsilon$ and $d(B,0) \leqslant \varepsilon$. Then $d(A \oplus B, 0) \leqslant \varepsilon$.

Proof. $\alpha A \oplus \alpha B = \alpha(A \oplus B)$. Result follows by Proposition 3 via Theorems 1 and 2. □

Proposition 16.
Let A, B be two persistence modules such that $d(A,0) \leqslant \varepsilon$ and $d(B,0) \leqslant \varepsilon$. Then $d(A \otimes B, 0) \leqslant \varepsilon$.

Proof. Result follows from bilinearity of tensor product and Theorems 1 and 2. □

Proposition 17. *Let $P = \ldots \to P_n \to P_{n-1} \to \ldots$ be a persistence complex such that for all i's $P_i \overset{\varepsilon}{\sim} 0$. Then the homology modules of P are ε-interleaved with 0.*

Proof. Assume d_i is a differential in a complex. We know that $0 \to \operatorname{im} d_{i+1} \to \ker d_i \to H_i(P) \to 0$ is exact and that $0 \to \ker d_i \to P_i \to P_{i-1} \to 0$ is exact. Result follows by application of Lemma 1 twice. □

Proposition 18. *[Mitchell81, Page 2]*
Category Fun(I,R−Mod) has enough projectives.

Since we have enough projectives we can compute derived functors. We need the following proposition.

Proposition 19. *Let R be commutative ring, A and B – R-modules such that either A or B is ε-interleaved with 0. Then $Tor_i^R(A,B) \overset{\varepsilon}{\sim} 0$.*

Proof. Since R is commutative, $Tor_i^R(A,B) = Tor_i^R(B,A)$. Without loss of generality assume $B \overset{\varepsilon}{\sim} 0$. Let P be the projective resolution of A. After taking tensor product we obtain by Proposition 16 sequence of modules ε-interleaved with 0. Proposition follows by Proposition 17. □

We can also derive the result about exact sequences.

Proposition 20. *Let $A \xrightarrow{f} B \xrightarrow{\phi} C \xrightarrow{g} D$ be exact sequence in category of persistence modules. Then if $d(A,0) \leqslant \varepsilon$ and $d(D,0) \leqslant \varepsilon$, then $B \overset{4\varepsilon}{\sim} C$ with ϕ being left morphism in interleaving pair.*

Proof. Under conditions of theorems 1 and 2 we can identify the category of persistence modules and the category of non-negatively graded modules.

In s.e.s $0 \to \ker f \hookrightarrow A \xrightarrow{f} \operatorname{im} f \to 0$ $\operatorname{im} f$ is ε-trivial. By exactness it is equal to $K = \ker\phi$. On the other side from $0 \to \ker g \xrightarrow{g} D \to D \to 0$ there follows that $d(I = \operatorname{im}\phi, 0) \leqslant \varepsilon$. Hence by transitivity $K \overset{2\varepsilon}{\sim} I$.

We obtain exact sequence $0 \to K \to B \xrightarrow{\phi} C \to I \to 0$. This sequence decomposes into sequences $0 \to K \to B \to \operatorname{coIm}\phi$ and $0 \to \operatorname{Im}\phi \to C \to I \to 0$. By lemma 1 we have that $d(B, \operatorname{coIm}\phi) \leqslant 2\varepsilon$ and $d(C, \operatorname{Im}\phi) \leqslant 2\varepsilon$. *coIm* and *Im* are pointwise canonically isomorphic by first isomorphism theorems for modules, hence $d(B,C) \leqslant 4\varepsilon$. □

4 Approximate Quillen-McCord theorem

We are ready to establish the target theorem.

Theorem 6. ***Approximate Quillen-McCord theorem, draft statement***
Assume X, Y are persistence posets of finite type indexed by very good monoid I, $f : X \to Y$ is order-preserving map, and R is a PID.

Then if $\forall y = (\dots, y_i, \dots) \in Y$ $\mathcal{B}(f^{-1}(Y_{\leqslant y}))$ is ε-acyclic over R, $\mathcal{B}f$ induces e-interleavings of all homology spaces over R on BX and BY.

Value of e is set later.

Proposition 21. *Let A and B be two persistence topological spaces with at least one of them being ε-acyclic over R. Then $A \star B$ is ε-acyclic over R.*

Proof. All Tor-functors from Proposition 9 are ε-interleaved with 0 by Proposition 19. Hence by Proposition 15 right hand side of expression of Proposition 9 is ε-equivalent to 0.

□

Proposition 22. *Let x be an element of (X, ϕ). Then covering from Remark 5 can be extended to persistence covering $\mathcal{U}$ with covering sets U_1 – preimage of $\mathrm{st}(\mathcal{N}(x))$ under nerve functor and $U_2 = X \setminus \{x\}$.*

Proof. It suffices to check that $X \setminus \{x\}$ and preimage of $st(\mathcal{N}(x))$ are subposets.

It is evident for $X \setminus \{x\}$. Elements in the preimage of $\mathrm{st}(\mathcal{N}(x_i))$ are exactly elements comparable to x_i. Since structure maps preserve order, they do not move comparable elements to incomparable. Hence preimage also forms subposet. □

Lemma 4. *Let (X, ϕ) be a persistence poset and for $x = (\dots, x_i, \phi(x_i), \dots) \in X$ either $\mathcal{B}(X_{<x})$ or $\mathcal{B}(X_{>x})$ is ε-acyclic. Then embedding $\mathcal{B}(X \setminus \{x\}) \hookrightarrow \mathcal{B}(X)$ induces 4ε-interleavings of all homology spaces.*

Proof.
By Proposition 21 $|\mathrm{lk}(\mathcal{N}(x))|$ is ε-acyclic.

Given persistence covering we can define Mayer-Vietoris exact sequence on persistence homology modules component-wise by gluing sequences for components over structure maps. Proposition 20 yields the lemma. □

Remark 6. If map f is component-wise homotopy equivalence, it induces 0-interleaving of homology modules.

We are now ready to adapt known proof to Quillen-McCord theorem for persistence posets.

Theorem 7. ***Approximate Quillen-McCord theorem, final statement***
Assume X, Y are persistence posets of finite type indexed by very good monoid I, $f : X \to Y$ is order-preserving map. Let $m = \max_i(|Y_i|)$ be the maximal cardinality of components of Y and R is a PID.

Then if $\forall y = (\dots, y_i, \dots) \in Y$ $\mathcal{B}(f^{-1}(Y_{\leqslant y}))$ is ε-acyclic over R, $\mathcal{B}f$ induces $4m\varepsilon$-interleavings of all homology spaces over R on BX and BY.

Proof. Approximate Quillen-McCord theorem
Let X, Y be persistence posets of finite type with order-preserving map $f : X \to Y$.

Let $\overline{X}$ be linear extension of X. Let $x_1^i, x_2^i, \dots, x_{n_i}^i$ be enumeration of $\overline{X}_i$ in its linear order and $Y_i^r = \{x_1^i, \dots, x_r^i\} \cup Y_i \subset M(f_i)$ for any r up to $\max_i(n_i)$ and i. $Y^r = (\dots, Y_i^r, \dots)$ is a persistence subposet of $M(f)$ with extended order on X.

There are posets such that $n_i < r$. In these cases notation x_r means that on positions with $n_i < r$ we take x_{n_i}. n_i can be undefined, in this case $x_r = \perp$.

Consider $Y^r_{>x_r} = Y_{\geqslant f(x_r)}$. $\mathcal{B}(Y_{\geqslant f(x_r)})$ is a component-wise cone over $\mathcal{B}(f(x_r))$. It is component-wise contractible, therefore $\mathcal{B}(Y^{r-1}) \hookrightarrow \mathcal{B}(Y^r)$ is component-wise homotopy equivalence by Lemma 2. By iteration $\mathcal{B}(i_Y) : \mathcal{B}(Y^0) = \mathcal{B}(Y) \hookrightarrow \mathcal{B}(M(f)) = \mathcal{B}(Y^n)$ is component-wise homotopy equivalence between BY and $M(f)$. Note that persistence structure is not used here.

Then consider linear extension of Y with enumerations $y_1^i, \dots, y_{m_i}^i$ and $X_i^r = X \cup \{y_{r+1}^i, \dots, y_{m_i}^i\} \subset M_i(f)$. $X^{r-1}_{\leqslant y_r} = f^{-1}(Y_{\leqslant y_r})$. Latter is ε-acyclic over $\mathbb{F}$ by condition of the theorem. Hence $\mathcal{B}(X^r) \hookrightarrow \mathcal{B}(X^{r-1})$ induces 4ε-interleavings of all homology modules and by transitivity of ε-equivalence $\mathcal{B}(i_X)$ induces $4m\varepsilon$-interleavings between homology of X and $M(f)$.

Note that $i_X(x) \leqslant (i_Y \circ f)(x)$. By Proposition 7 $\mathcal{B}(i_X)$ is component-wise homotopic to $\mathcal{B}(i_Y \circ f) = \mathcal{B}(i_Y) \circ \mathcal{B}(f)$. Homotopic maps induce same maps on homology, i_Y is a homotopy equivalence and induce 0-interleavings. Hence $\mathcal{B}(f)$ induce $4m\varepsilon$-interleavings between $H_i(BX, R)$ and $H_i(BY, R)$. □

We expect the stronger statement with no conditions on R to be true with another error multiple and that it can be proved using the technique of this paper by considering error propagation in Kunneth spectral sequence.

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