Homological Quillen's fiber lemma for persistence posets

Vitalii Guzeev

HSE faculty of mathematics

arXiv:2307.00177

Motivation

Complexity of persistent homology computation is a concern.
It would be nice to have an algorithm to reproducibly reduce a persistence complex (as experimental data) to a smaller one.

Quillen's fiber lemma

Let $X, Y \in Pos$ be finite, $f : X \to Y$ be a morphism.
If $\forall y \in Y$ $B(f^{-1}(Y_{\leqslant y}))$ is contractible, then Bf is a homotopy equivalence between BX and BY.

Homological Quillen's fiber lemma

Let $X, Y \in Pos$ be finite, $f : X \to Y$ be a morphism, R be a PID.
If $\forall y \in Y$ $H_i(B(f^{-1}(Y_{\leqslant y})), R) = 0$ for any i, then Bf induces isomorphisms of all homology modules with coefficients in R on BX and BY.

Jonathan Barmak's proof of homological Quillen's fiber lemma

Covering

Let x be an element in a poset X.
Then $BX = B(X \setminus \{x\}) \cup |\mathrm{st}(\mathcal{N}(x))|$

Mapping cylinder

The set $M(f) = X \coprod Y$ with the order $R = R_X \cup R_Y \cup \{(x, f(x)) \mid \forall x \in X\}$.

Major lemma

Under conditions of the theorem, the inclusion $B(X \setminus \{x\}) \to BX$ induces isomorphisms of all homology modules.

Proof scheme

For the lemma: Kunneth formula for a join and Mayer-Vietoris l.e.s.
For the theorem: $B(M(f))$ retracts to BY. Extend order on Y to linear and iterate the lemma, adding elements of Y to X one by one.

Persistence objects

$Fun_{Cat}(I, \mathcal{C})$ for posetal category I and given $\mathcal{C}$.
Many constructions work index-wise, e.g. mapping cylinder.

Persistence subposet

$X \subset Y$ if for each $i \in I$ X_i embeds to Y_i and everything commutes.

Element of a persistence poset

$x \in X$ if $x \subset Y$ and for each i $\# x_i \leq 1$.
Equivalently, it is an element $x_i \in X_i$ for some i and its images under structure maps.

Persistence covering of BX

$X = \bigcup X_i$ and $BX \sim \bigcup BX_i$
Barmak's covering extends to a persistence one.

Object of finite type

Persistence poset of finite type — finite sequence of finite posets; module — of finitely generated modules.

MAIN RESULT

Assume X, Y are persistence posets of finite type indexed by a totally ordered good monoid I, $f : X \to Y$ is an order-preserving map. Let m be the number of elements of Y and R is a PID.
Then if $\forall y \in Y$ $B(f^{-1}(Y_{\leqslant y}))$ is ε-acyclic over R, BX and BY are $4m\varepsilon$-interleaved over R.

Persistent order extension principle

An order on elements is defined by a first non-empty component. Assume I is totally ordered. Then elements are ordered by pair (birth index i, order of $x_i \in X_i$).
Series X_i' of extensions of X_i is an extension of a persistence poset X if structure maps of X are well-defined on X'. An extension that is a series of linear extensions is called a linear extension.
Consider the set of extensions of X ordered by inclusion and apply Zorn's lemma.
No need in Zorn's lemma for a finite-type case.

Interleaving distances

$$\begin{tikzcd}
\cdots \arrow[r] & M_0 \arrow[r] \arrow[dr] & M_1 \arrow[r] \arrow[dr] & M_2 \arrow[r] \arrow[dr] & M_3 \arrow[r] \arrow[dr] & M_4 \arrow[r] \arrow[dr] & M_5 \arrow[r] & \cdots \\
\cdots \arrow[r] & N_0 \arrow[r] \arrow[ur] & N_1 \arrow[r] \arrow[ur] & N_2 \arrow[r] \arrow[ur] & N_3 \arrow[r] \arrow[ur] & N_4 \arrow[r] \arrow[ur] & N_5 \arrow[r] & \cdots
\end{tikzcd}$$

(wiki)

$d(X, Y) = min\{\varepsilon \in I \mid X \overset{\varepsilon}{\sim} Y\}$

Correspondence theorem (Corbet, Kerber 2018)

Let R be a ring with unity and G be a good monoid (Definition 2.16, follow the QR code). Then the category of finitely presented graded $R[G]$-modules is isomorphic to the category of G-indexed persistence modules over R of finitely presented type.

Stability in G-indexed persistence modules

All results rely on stability in short exact sequences (Lemma 2.24).

- $d(A, 0) \leq \varepsilon$; $d(B, 0) \leq \varepsilon$ imply $d(A \oplus B, 0) \leq \varepsilon$; $d(A \otimes B, 0) \leq \varepsilon$
- Let P be a chain complex with $d(P_i, 0) \leq \varepsilon$. Then $d(H_i(P), 0) \leq \varepsilon$ (P is ε-acyclic).
- $d(A, 0) \leq \varepsilon$ or $d(B, 0) \leq \varepsilon$ imply $d(Tor_i(A, B), 0) \leq \varepsilon$.
- Let $A \to B \to C \to D$ be exact. Then $d(A, 0) \leq \varepsilon$ and $d(D, 0) \leq \varepsilon$ imply $B \overset{4\varepsilon}{\sim} C$.

Major lemma

Let $BX_{<x}$ or $BX_{>x}$ be ε-acyclic. Then persistent homology of $B(X \setminus \{x\})$ and $B(X)$ are 4ε-interleaved.

Proof scheme

Applies.

Further work

An algorithm? Relax a PID requirement using the Kunneth s.s.? Something else?

The work was conducted in the Laboratory for Applied Geometry and Topology at HSE (preceding the ATA Lab) under the supervision of Anton Ayzenberg.

viviag@yandex.ru, viviag.io