Final Exam

Due: 27 May 3:00pm. No late submissions.
"Computers are useless. They can only give you answers." Pablo Picasso

1 (10 points) Multiplicative Linear Congruential Methods

Give an example of a linear congruential sequence with $m = 10$ for with X_0 never appears again in the sequence. Remember that a linear congruence sequence is given by

$$X_{n+1} = (aX_n + c) \ \ mod \ m$$

for $n \geq 0$.

2 (10 points) Theoretical aspects of the Metropolis-Hastings Algorithm

Why does the Metropolis-Hastings Algorithm works? We can try to grasp the answer by studying the detailed balance i.e., proving that

$$\begin{aligned} \pi(a) P(a \to b) &= \pi(b) P(b \to a) \\ \pi(a) P(a \to c) &= \pi(c) P(c \to a) \end{aligned}$$

where $\pi(n)$ is the stationary probability for the system to be at the n configuration and $P(a \to b)$ is the probability of the algorithm to move from the state a to the state b, for the game we discussed in class. Refer to Fig. (2.1) to refresh your mind.

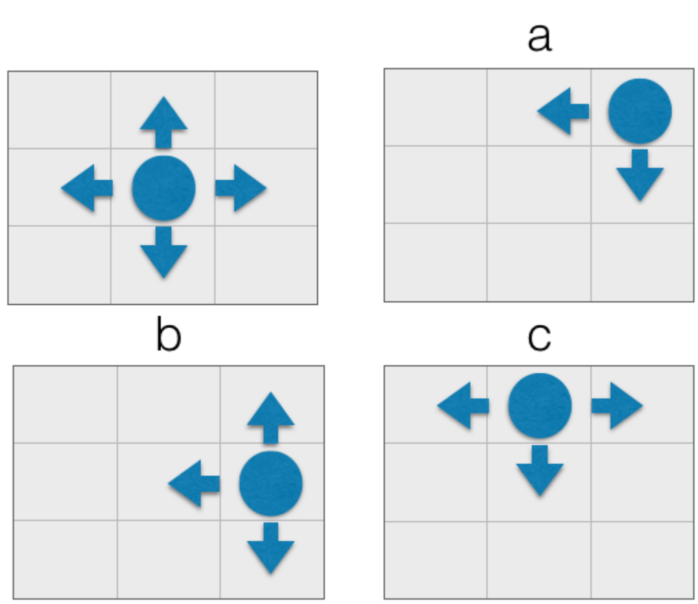

Figure 2.1: Discrete 3×3 case discussed in class.

3 (30 points) Markov-Chain Monte Carlo Implementation

Let us considerer a multidimensional unit bubble (*n-sphere*), i.e., the set of points $\{x_1, x_2, \ldots, x_n\} \in \mathbb{R}^n$ such that

$$x_1^2 + x_2^2 + \ldots + x_n^2 \leq 1,$$

where n is the dimension of the space. What is the volume of these bubbles? Well, let us discover it. We know that, since the radius is 1, the volumes V_n are of the form

$$\begin{array}{rcl} V_1 & = & 2 \\ V_2 & = & \pi \\ V_3 & = & \frac{4}{3}\pi \\ \vdots & \vdots & \vdots \\ V_n & = & ?. \end{array}$$

At first sight this problem may seem very complicated, and perhaps it is if we do not tackle it correctly... A way to approximate the result, which works decently, is by implementing a Markov-chain Monte Carlo program as follows:

- Define a size for the exploration, i.e., a δ, not too small, not too big as we learned in one of the homeworks.
- Set a number of trials.
- Start from the origin $(0, 0, \ldots, 0)$ to sample the n-dimensional space in steps of size δ.
- Move around the n-dimensional space and decide if it satisfies the criteria of being inside the bubble.
- Count the number of times your trials remains inside the bubble.
- Use that number to compute the volume.

Question. What is the volume of the unit bubble in **four** and **seven** dimensions? [1] Comment on your results and upload your code.

[1] You can check your answer by comparing with the analytical result:

```
import math
def V_bubble(dim):
    return math.pi ** (dim / 2.0) / math.gamma(dim / 2.0 + 1.0)
```