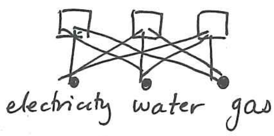
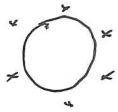
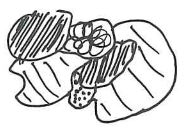
- Problem ex. : "Gas Electricity" Given 3 homes and 3 utilities, (how) can we build pipes on the plane (which do not intersect)?



- Problem ex. "Party with twenty guests"
  - Some guests do not get along with some other guests
  - each likes at least 10 other guests at the party
  - Can we put them around a round table s.t. everyone has neighbors that they like?



- Problem ex "Map of the World" Given the (planar) map of Europe, how do we / can we color the countries s.t. neighboring countries have distinct colors?



DEFINITIONS

- Def. A graph $G = (V, E)$ has vertex set V and a collection of pairs (u,v) (edges) E, $(u, v \in V)$
  - $|G| :=$ num of vertices in G
  - $|E(G)| :=$ num of edges in G.

- Multigraphs may have loops (a pair (u,u), $u \in V$) and may have repeated pairs (u,v) (multiedge) in $E(G)$

In this class we look at simple graphs, ie. no loops and every pair $(u,v) \in E(G)$ appears at most once.

- Def (Adjacency and Neighborhoods) (adjacency : $u \leftrightarrow v$; incidence : $e \leftrightarrow u, v$ or $e \leftrightarrow e'$)

Adjacency and Incidence: If $e = (u,v) \in E(G)$, we say that "u is adjacent to v." and that "edge e is incident to u and to v"

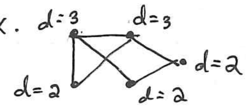
- Given two edges $e = (u,v)$ and $e' = (u,w)$, if $e \cap e' \neq \emptyset$ we say that e and e' are incident (e, e' share a vertex (at least one))

Neighbor
- If $(u,v) \in E(G)$, $u, v \in V(G) \equiv G$ (here), then we say that "v is a neighbor of u" and that "u is a neighbor of v" (∵ undirected) by symmetry.

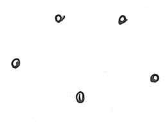
$N(u) = \{v : (u,v) \in E(G)\}$, $N(u) :=$ set of neighbors / neighborhood of u.

- Given a vertex $v \in G$, the degree $d(v) = |N(v)|$ = num. of neighbors of v in G. We say that "graph G is d-regular if all vertices have degree d.

ex.



- Types of graphs
  (a) Empty graph (no edges)

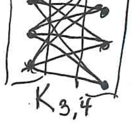


  (b) Complete graph (K_n : n = num of vertices, $\binom{n}{2}$ = num of edges)

ie every pair of vertices is connected

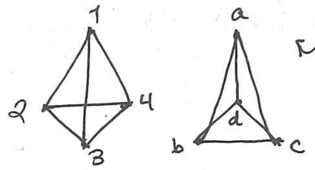
  (c) Complete bipartite ($K_{s,t}$)

The vertex set is a disjoint union of two sets of size s and t, and edges ≡ all s,t pairs intersecting both parts



$K_{3,4}$

- **Def**: (ISOMORPHISM)



are these the same graph?

$1 \to a,\ 2 \to c,\ 3 \to b,\ 4 \to d$

- Given $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ we say that "G_1 and G_2 are isomorphic"

iff $\Updownarrow$ $V_1 \to V_2$

s.t (1) it is a one-to-one mapping; and
(2) it preserves adjacency, ie (edges)
ie. $(f(u), f(v)) \in E_2$ iff $(u,v) \in E_1$

- Remarks ① Any graph is isomorphic to itself (wherein $f = I$ identity)

② Isomorphism is an equivalence relation, ie (reflexive, transitive, symmetric)

if G_1 is isomorphic to G_2 $\Leftrightarrow$ $f: v_1 \to v_2$ preserves the edges. Then $f^{-1}: v_2 \to v_1$ also preserves edges

Transitivity
③ If G_1 is isomorphic to G_2 and G_2 is isomorphic to G_3, then G_1 is isomorphic to G_3 $\Rightarrow$ G_2 is isomorphic to G_1

ie $f: v_1 \to v_2$, $g: v_2 \to v_3$ $\Rightarrow$ $g \circ f: v_1 \to v_3$ $\Rightarrow$ $f^{-1}: v_2 \to v_1$.

As we consider the vertices of a graph to be unlabelled, G is "insensitive" to the label.

REPRESENTATIONS OF GRAPHS

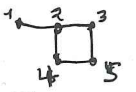
. $\forall$ vertex $v \in G$, we have a list of neighbors $N(v)$

- Adjacency Matrix: We can build an adjacency matrix A of dim. $n \times n$:

$A = (a_{ij})$ · rows/columns are indexed by vertex in G,

and $a_{ij} = \begin{cases} 1 & \text{if } (u,v) \in E \\ 0 & \text{if } (u,v) \notin E \end{cases}$

ex. for G



,

$$A_G = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

($a_{ii} = 1$ iff $(i,i) \in E$, ie a loop)
(if (u,v) has multiplicity (ex) then $a_{uv} > 1$)

· Observe that A_G is symmetric $\Rightarrow$ have eigenvalues

A_G has $\lambda_1, \dots, \lambda_n$ real eigenvalues with $x_1, \dots, x_n$ eigenvectors.

so $A x_1 = \lambda_1 x_1$ Note also that $a_{ij} = a_{ji}$

"Any adjacency matrix A is real and symmetric $\Leftrightarrow$ hence the spectral theorem proves that A has an orthogonal base of eigenvalues with real eigenvectors.

- Incidence matrix:
rows are indexed by vertices $u, v \in G$
columns are indexed by edges $e = (u,v) \in E(G)$ $\Rightarrow$ we can use spectral methods in graph theory.

ex.

$$B = \left(\begin{array}{c} 0 \\ 1 \\ 0 \\ 1 \end{array} \right) \begin{array}{l} \\ u \text{ @ } i \\ \\ v \text{ @ } i' \end{array}$$

$e = (u,v)$ @ j

ie (for $e = (u,v)$), $b_{ij} = \begin{cases} 1 & \text{if } u_i \in e_j \\ 0 & \text{otherwise} \end{cases}$

Remark: Both A and B are not unique because we don't consider labelling,

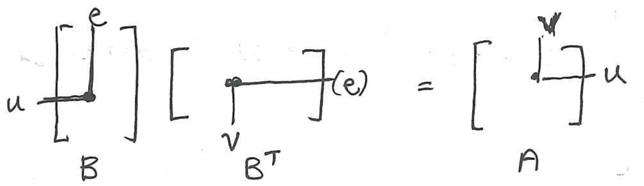
Nevertheless, A and B are equal to their isomorphisms up to permutation (ie. permuting is like relabelling vertices)

- Relations between adjacency (A) and incidence (B) matrices.

Suppose we have n vertices and m edges, so
$\underset{n\times n}{A} + \underset{n\times n}{D} = \underset{n\times m}{B}\,\underset{m\times n}{B^T}$, $\underset{n\times n}{D} = I\,\underset{n\times 1}{(d(v) : \forall v \in G)^T}$ (degree of $v := d(v)$)

$b_{ue} = 1$ if u in e ; $(b^T)_{ev} = 1$ if v in e, $e = (u,v)$

$b_{ue} = 1$ and $(b^T)_{ev} = 1$ for $a_{uv} = 1$. (0 otherwise), $u \neq v$ $\because$ simple graph



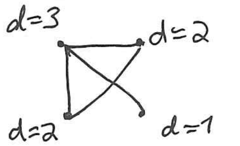
- Question: Can we connect 9 points in the plane to each other s.t. every point is connected to 3 other points?

No. Consider a graph $G = (V, E)$ with n vertices with degrees $d_1, \ldots, d_n$.
(The above graph has (a total of) degree sum of 27)
The sum of degrees of G is $\sum_{i=1}^{n} d_i = 2|E(G)|$ since each edge (u,v) is counted twice.
$\hookrightarrow$ must be even.

- Corollary
The sum of the degrees of a graph is even
$\Rightarrow$ num. of vertices of an odd degree is always even..

Def: $\Delta(G)$ = maximal degree in G
$\delta(G)$ = minimal degree in G
$\bar{d}(G)$ = average degree in G = $(\sum d(u))/|V(G)|$

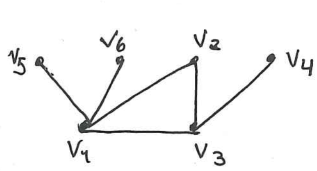
ex.



$\Delta = 3$
$\delta = 1$
$\bar{d} = 2$

- Walks, Paths, Angles.

Def: A walk is a sequence $v_1, \ldots, v_k$, $v_i \in V(G)$ s.t v_i and v_{i+1} are adjacent.
ex. $v_5\, v_1\, v_2\, v_3\, v_1\, v_6$



- A path is a walk in which every vertex is visited at most once
- A cycle (closed walk) is a path s.t., for a path $v_1, v_2, \ldots, v_l$, v_l is also connected to v_1

ex. P_4 3 edges

- length of path (walk) = # of edges in the walk.

C_4 4 edges

- Notation: P_k = path of length $k-1$, C_k = cycle of length k

- <u>Claim</u> Every walk from u to v in G contains a path from u to v in G

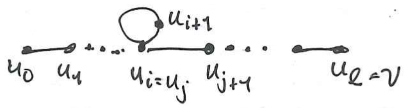
<u>Pf</u> by induction on length of walk.

- Let $u = u_1, \ldots, u_\ell = S$ be the walk. If $\ell = 1$, then $(u,v) \in E(G)$, i.e. $\overset{u}{\bullet}\!\!-\!\!\overset{v}{\bullet}$ is the path

- Induction step. Suppose $u_0, u_1, \ldots, u_\ell$ is not a path
  $\Rightarrow$ then there is a vertex on the walk that appears ≥ 2 times

Say $u_i = u_j$, $j > i$,

ex.



Then $u_0, u_1, \ldots, u_i = u_j, u_j, \ldots u_\ell$ is also a walk from u to v that is shorter than the original walk.

By induction, it contains a path from u to v.

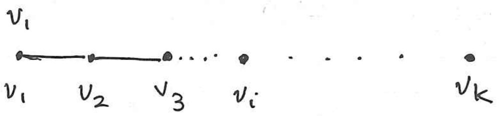
- <u>Claim</u> Every graph with minimal degree $\delta : \delta \geq 2$ has a path of length $\geq \delta$ (and a cycle of length $\geq \delta + 1$)

<u>Pf</u> Let, for $(v_1\ v_2 \ldots v_{k-1}\ v_k) = P$ be the longest path in G.
Note that all neighbors of v_k $(N(v_k))$ must be on the path.

$\Rightarrow$ there are at least δ vertices on the path that are in $N(v_k)$ and none of them are v_k.

$\Rightarrow$ num. of vertices on path $\geq \delta + 1$.

$\Rightarrow$ length of path $\geq \delta$



Consider the least (leftmost neighbor of v_k on the path is v_i. Then $v_i, v_{i+1}, \ldots, v_k$ is a cycle.