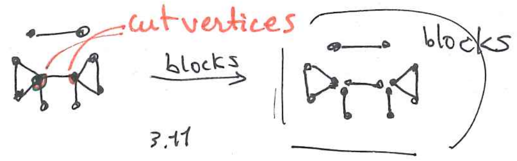
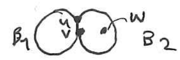
- 2-connected "components" of a graph
- Def a block in graph G is a maximal subgraph which does not have a cut vertex.
- Remark. A block is either an edge (+ two vertices) or a two-connected subgraph

[1] Prop 3.13 • Claim (observation) Any two blocks in a graph share at most 1 vertex; if they do, then this vertex is a cut vertex in G

ex.


Pf by contradiction. A single vertex deletion cannot disconnect either block.
Let $B_1, B_2 \subseteq G$ be two blocks s.t. $B_1 \cap B_2 = \{u, v\}$, ie share two vertices.,
Then deleting any vertex does not disconnect the two (path from any vertex in one block to remaining in other block)
ie $B_1 \cup B_2$ is a subgraph with no cut vertex.
This contradicts the maximality of the original blocks. □ (ie $B_1 \cup B_2 > B_1, B_2$)

- Let w be some vertex in $B_1 \cup B_2$. Then, if $w \neq u, v$ (some other vertices) ⇒ then $B_1 \cup B_2 \setminus \{w\}$ contains u.
- By definition of block connectivity, $B_1 - w$ and $B_2 - w$ are connected, so any vertex in $B_2 - w$ has a path to u, and any vertex in $B_1 - w$ has a path to u ⇒ $B_1 \cup B_2 \setminus w$ is connected → contradiction.

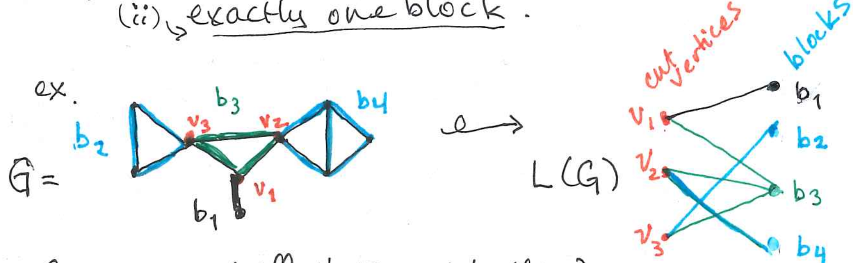


[2] Let G be connected and $B_1, B_2 \subseteq G$ be blocks s.t. $B_1 \cap B_2 = \{u\}$. Then we claim
Proposition: $G \setminus u$ is not connected, ie u is a cut vertex of G.

Pf by contradiction. Suppose not that there is a path P from $B_1 \setminus u$ to $B_2 \setminus u$. This means that $B_1 \cup B_2 \cup$ path has no cut vertex; this union is larger than either B_1 or B_2, so it contradicts maximality.

(Idea * the B_i share ≤ vertex which is a cut vertex. ∗ use this!)

[3] Corollary. The blocks in G (i) cover all edges of G; and each edge belongs to (ii) exactly one block.

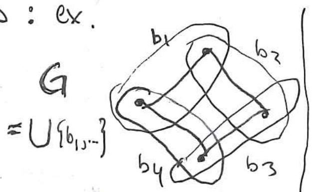
ex.


- the line graph L(G) of G is bipartite

⇒ Blocks do not share edges (as the blocks are an independent set in L(G))
⇒ each edge in the blocks belongs to exactly one block. (ii) proven

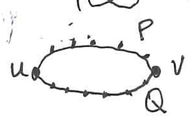
- G is connected iff the bipartite (line). graph is also connected.
Here the graph is connected.

→ Observe that (for (ii)), we cannot have cycles: ex.
In the example graph, the graph $G = \cup \{b_1, b_2, b_3, b_4\}$ has no cut vertex.



G is 2-connected
⇕ iff
$\forall u, v \in G$, ∃ 2 paths from u to v

Def Two paths are internally disjoint if the only vertices they share are the end vertices. ex. two paths (going $u \rightsquigarrow v$) P and Q are internally disjoint. Then,



Then, deleting any single vertex in $P \cup Q \setminus \{u, v\}$ (any internal vertex) cannot disconnect and cannot separate any pair u, v

Graph Theory. Lecture 5. 2 connectivity, cont'd — 07 March 2018

- 2-connected graphs, cont'd

Thm. (Whitney 1932) A graph G of $|G| \geq 3$ is 2-connected iff $\forall u, v \in G$, there are ≥ 2 internally disjoint paths from u to v,

* Idea: Consider graphs as matrix spaces (to develop algorithms)

G is 2-connected $\Rightarrow \geq 2$ paths

Pf by induction on distance $d(u,v) :=$ length of the shortest path from u to v.
(Thm 3.18)

Basis: • $d(u,v) = 1 \Rightarrow (u,v) \in E(G)$ u P v

• Since $\kappa'(G) \geq \kappa(G) \geq 2$, deleting edge (u,v) leaves $G - (u,v)$ connected

$\Rightarrow \exists$ path $P(u \rightsquigarrow v)$ that does not use edge (u,v)

Induction step. Suppose that $d(u,v) = k > 1$. (we want to show that $\exists w \in P$ where $d(u,w) = k-1$)

• By looking at the shortest path $u \rightsquigarrow v$, we label the penultimate vertex w. We find:

u ..... w v (i) $(w, v) \in E(G)$ and (ii) $d(u,w) = k-1$

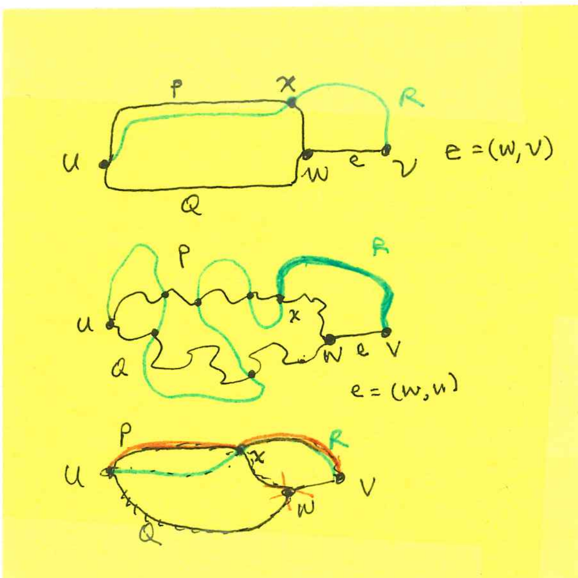
• Recall/note that $G \setminus \{w\}$ is 2-connected $\Rightarrow \exists$ path R from u to v in $G \setminus w$

• R can share vertices with $P \cup Q$ (internal vertices).

• and x is the last ~~time~~ vertex at which R and $P \cup Q$ intersect on the path, iow. "last vertex on R which belongs to $P \cup Q$"

• $x \neq w$.

• So, in a graph $G \setminus w$, we can take the path for ex. $(u, \underline{P(u \to x)}, x, \underline{R(x \to v)}, v)$



• For any k-connected graph G, there are k internally disjoint paths from u to v for any vertex pair $u, v \in V(G)$

• The idea is "How much does it cost to separate u, v?

(We will prove wrt vertices and wrt edges

MENGER'S THEOREM (1927).

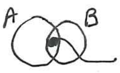
Def - Let $A, B \subseteq V$. An A-B path is a path with • one endpoint in A, and • the other endpoint in B and • all interior vertices outside $A \cup B$

- Any vertex in $A \cap B$ is a trivial A-B path.

Def ① vertices { $\subseteq V$ is a cut separating A and B if every AB path contains

② edges $\subseteq E(G)$ is a cut separating A and B if every A-B path contains an edge from ..

Remark (There are k edge-disjoint paths A-B for a k-connected G)



vertex (x) in $A \cap B$ is an A-B path

$\Downarrow$

" X separates A from B "

$\equiv X \supset A \cap B$.

Thm - Menger's Theorem (1927)

- Let G be a graph and $S, T \subseteq V(G)$, $G = (V, E)$. Then, the maximum num. of ST-disjoint paths is equal ($\equiv$) to the minimal $|X|$ separating S and T.
  (X is ~~the~~ an ST sep. vertex set)

Consider this:
If we delete something from the graph, is it separated?. How much does it cost (ie) how many vertices are required to separate S and T?
What is the largest collection of disjoint paths between S and T?
(ie max number of ST-vertex disjoint paths.)
$\leq$ min $|X|$

- Assume that if $|K| = k$, then ($(k-1)$ vertices) cannot be a separating set.
  $\Rightarrow$ $|X| = k \Rightarrow k$ vertex disjoint paths.

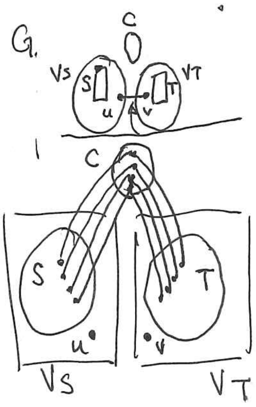
(For this proof, note that we make no assumptions about S, T disjointness).
(deal with both cases)

Pf. by induction on k vertices (in sep set)

- Suppose $S \cap T = \emptyset$. We prove by performing induction on the number of edges in G.
  - Basis $E(G) = \emptyset$ and $0 = 0$ ✓
  - Inductive step For $S, T \subseteq V(G)$, we let $S \cap T = \emptyset$. Thus we need to delete k vertices to separate S from T.

  - Pick an arbitrary edge $e = (u, v) \in E(G)$ and let $G' = G \setminus \{e\}$. Cases:
    (i) still Need to delete k vertices to separate S from T.
    $\Rightarrow$ by induction, there are k disjoint paths.

    (ii) Need to delete $< k$ vertices to separate S from T,



- Let C be the minimal separating set for $S-T$ in G'. We show that there are still k S-T disjoint paths. Note that both $C \cup \{u\}$ and $C \cup \{v\}$ are separated from S and from T, respectively, in G'.
  $\Rightarrow$ $|C| = k-1$.
  All paths in G from S to T use edge (u, v).

  Look on the components in $G' \setminus C$ and let
  $V_S = \bigcup$ {components containing any vertex of S} and
  $V_T = \bigcup$ { " " " " T} . Consider the left graph.

  If we delete C and keep (u,v), then there is still a S-T path. ($u, v \notin C$)
  In G', $V_S \cup V_T$ = union of components, and no edge in G' connects them.
  We claim :
  $C \cup u \equiv k$ and $C \cup v \equiv k$ (sep. from S-T).

- It "costs" k disjoint paths to separate S from $C \cup u$ and T from $C \cup v$. By definition, S and T share the same vertices in C.
  $\Rightarrow$ $\exists$ path $u \rightsquigarrow S \to C \to T \rightsquigarrow v$. ie. every path $S \to T$ must go through C
  iow, no p is $S \to \underline{T} \to \underline{C}$ as then C must be disjoint.

- We claim we can't separate S from $C \cup u$ with $< k$ vertices in G'.
  $C \cup u$ has precisely k vertices $\Rightarrow$ so there is one path to each vertex since paths are disjoint.

- Then $S-C$ and $T-C$ paths must be disjoint as they all go through C; else C would not be necessary, ie direct ST path and S, T would not be disjoint. Contradiction.

Graph Theory Lecture 5 : Menger's Theorem, cont'd 07 March 2018.

If we (try to) take $C \cup u$ and try to separate it from S, it still costs k vertices.

Claim: One needs k vertices to separate S from $C \cup u$ in G'.

Pf. Suppose $B \subset V(G')$ is a set separating S from $C \cup u$ $\in G'$
We want to claim that B separates S and T in G. Then, B should be of size $\geq k$.

If not, then $\exists$ ST path in $G \setminus B$. If this path does not have edge (u,v), then there is also a path in G', but all paths in G' from S to T contain a vertex in C! $\Rightarrow$ This is a path from S to C. Contradiction

This contradicts that B separates S and $C \cup u$ in G'. Otherwise, the path uses (u,v), in particular, to connect S to u.
$\Rightarrow$ contradiction.

$S \rightsquigarrow u$ does not use (u,v) ie it does not go through v first because S and T are disjoint.