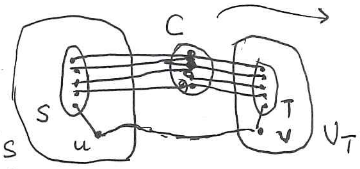
Recall Menger's theorem (1927) that stated

the maximum number of S-T disjoint paths in G equals
= the size of a minimal S-T separating set.

Pf by induction on num. of edges. Cases:

① Induction step assuming $S \cap T = \emptyset$. How many vertices are needed to separate S from T?



$|C| = k-1$.

- observation: C does not separate $S, T \in G$.
  $\Rightarrow \exists$ path directly from S to T that uses edge (u,v)
  → if we delete C, there is still a path.

- any set that separates S and $C \cup \{u\}$, or T and $C \cup \{v\}$ in G' or G has size k, where $G' = G \setminus \{(u,v)\}$.

  This holds because we still separate S from T when we delete such a set

→ each path in S-C is disjoint and each path in T-C is disjoint and S-C and T-C are disjoint.

Keypoint: going from S to T, we must go through either C or (u,v).
If we go through (u,v), it means there is a direct S-T path and so $S \cap T \neq \emptyset$, ie can go u-T and v-S. (contradiction).

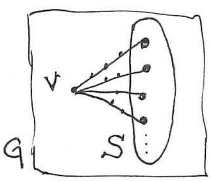
② General induction step
- Suppose $S \cap T \neq \emptyset$ and let $X = S \cap T$, so
- If C is the minimal set separating $S' = S \setminus X$ and $T' = T \setminus X$ in $G' = G \setminus X$, then $C \cup X$ is the set separating S and T in G. (and vice versa if/then)

- Suppose the minimal set separating S and T in G has size k.
  Then, the minimal set separating S' and T' in G' has size $k - |X|$.

- Apply the induction step to S' and T', (i) we have $k-|X|$ disjoint (vertex) paths from S' to T' in G', and together with X, (ii) every vertex in X is a path from S to T in G.
  Then (by (ii)) there are $k - |X| + |X| = k$ such paths in G

(General idea: it's not about connectivity. It is instead about sets:
Given two sets in G (a dense graph), how much does it cost to separate them?
If we can't separate S and T with <k vertices, then there are k disjoint paths between S and T.)

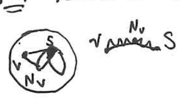
Corollary 3.22. " How many vertices does it cost to separate v from S, where $v \in S$? "



The size of the minimal "separating" set for v and S. equals the maximum number of paths from v to S which share only vertex v.

Note that $v \notin$ Separating set; else one can simply delete v to separate.

Pf Look at set $T = N_v$. Claim: Separating v from S is the same as separating $N_v = T$ from S.



If, after deleting some set of vertices C, there is a path from N_v to S and therefore $\exists$ also a path from v to S. How?
(since we don't delete v)

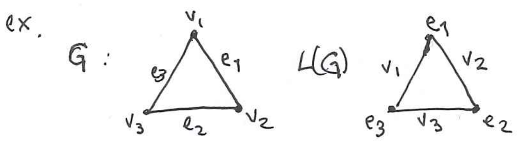
Apply Menger's theorem with $T = N_v$ and S

- Pf for Corollary 3.22 : "Min size of separating v-S set = max # paths from v to S disjoint except v"
  cont'd

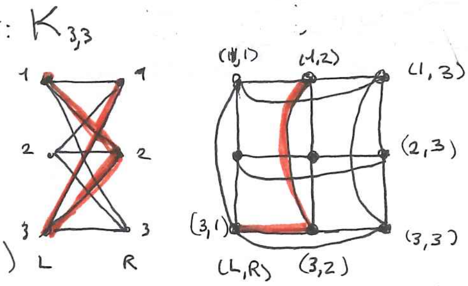
- Edge version of proof

- Def Given a graph $G=(V,E)$, in a line graph $L(G)$, vertices of G are edges of L(G) and edges of G are vertices of L(G)
  ie. $e \sim e'$ iff $e \cap e' = \emptyset$.

ex.



ex: $K_{3,3}$



- If ~~vertices~~ are connected by edges in G then edges are connected by vertices in L(G)

  ie. If (two) edges $e_1, e_2 \in E(G)$ are incident (share a vertex) in G, then they are connected (adjacent) by an edge in L(G). $(e_1, e_2) \in E(L(G))$

- Cor. 3.25 (i) If $u, v \in G$, $(u,v) \notin E(G)$) ie we delete (u,v) edge then the minimum number of vertices distinct from u and v and separate u and v is equal to the maximum number of internally disjoint u-v paths.

  (ii) the minimum number of edges separating u and v equals the maximum number of edge-disjoint paths from u to v.

Pf for (i) : Use Menger for N_u and N_v to show that separating u and v is the same as separating N_u and N_v

(ii) In graph L(G), let S be all the edges touching u and T be all the edges touching v.
Use Menger on L(G) for S and T.

3.26 : Global Version of Menger's theorem

G is k-connected iff $\forall u,v \in G$, there are k internally disjoint u-v paths.
G is k-edge-connected iff $\forall u,v \in G$, there are k edge-disjoint u-v paths.

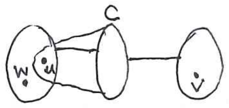
(ii) follows from Corollary 3.25.

Let G be k-connected, and $u, v \in G$.

- If $(u,v) \notin E(G) \Rightarrow$ then $\exists$ k internally disjoint paths from u to v. as per Cor 3.25 (i)
- Else, suppose $(u,v) \in E(G)$ and consider $G' = G \setminus \{(u,v)\}$

Claim G' is k-1 connected. Pf by contradiction

- Let C be a minimal set of vertices s.t. $G' \setminus C$ is not connected.
- If u or $v \in C$, then $G \setminus C = G' \setminus C \Rightarrow |C| = k$
- Elif both u and $v \notin C$, then
  - if $|C| < k$,
    $\to$ G with k-conn must have k+1 vertices.



- if $|C| < k-2$ (C is too small), then $\exists$ an extra vertex $w \in G$

Thm 3.26 Global Version of Menger, cont'd.

$\Rightarrow$ If w is in a set with u, then $C \cup u$ is a separating set in G' and in G of size $k-1$; contradiction.

Thus, there are (in G') $k-1$ disjoint paths, plus (u,v), gives us k disjoint paths from u to v.
G' is $(k-1)$-connected and thus.

* Size of separating vertex sets are the same as # of disjoint paths (either edge- or internally (vertex)-).

◦ Euler trails (Euler - 18th Century): "How / when can you walk on the edges of a graph s.t. each edge is traversed exactly once?"

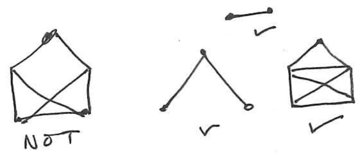
Def.
An (Euler) trail is a walk which uses every edge ~~at most~~ exactly once.
An (Euler) tour is a trail which is closed (i.e. starts and ends at the same vertex).

Thm. A connected multigraph G has an Euler tour iff all degrees are even.
(simple to prove for a simple graph.

Cor. A connected multigraph G has an Euler trail iff the num. of odd-degreed vertices is 2.

Remark. We start and finish at a vertex of odd degree.

eg



If all vertices are of odd degree, it is impossible to have a trail.