Graph Theory Lecture 7 Eulerian & Hamiltonian Cycles

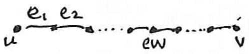
- **Def** An Euler trail is a walk which visits every edge of a graph exactly once.
  Euler tour ≡ Euler trail which starts and stops/ends at the same vertex

- **Thm (4.5)** A connected (multi)graph has an Euler tour iff all degrees are even
  A connected (multi)graph has an Euler trail iff it has 0 or 2 vertices of odd degree

  Remark: if all $d(v) \equiv$ even we call G an even graph.

- **Lemma (4.6)** Every maximal trail in an even graph is a closed trail

  Pf.: Let T be a maximal trail which starts at u and ends at v.
  - Look at any vertex w in the middle of the trail. There is a pair of edges incident to w and different appearances of w have disjoint pairs of edges



  - The only way to have this is if two appearances of w are consecutive, ie in a loop, but we don't allow this.

  - Look now at v. if $v \neq u$, then the degree of v is even (by def of even graph)
    All appearances of v in the middle of the trail ① use an even num. of edges,
    So, ② the total num of edges used is odd,

    Then ① and ② $\Rightarrow \exists$ edge e_v that we did not use, ~~so~~ which means that we can extend the trail, contradicting maximality.

    v may appear only once, at the end of the trail. □

Pf of Thm 4.5: There are two directions / cases:

(→) If G has an Euler tour; then $v \in G$, if v appears on the tour k times $\Rightarrow d(v) = 2k$

If G is even, let T be the longest trail in G. Observe by Lemma 4.6, that T is an Euler tour.
if T does not cover all edges of G, then it can be extended.
- ex. given a cycle with an extended edge from the cycle (ex. edge (u, w)) then when we look for the longest tour, we find/walk along the edges to try to get an Euler tour.

  If we do not cover all edges, and we have a vertex with degree ≥ 2, then there exists a walk around the graph to reach that edge (involving vertex that)
  However, this trail is longer than T, contradicting maximality of T.

(←) Take the longest trail → tour. If not all are covered, then there are edge(s) touching a vertex outside the cycle, so the trail can be extended
⇒ it contradicts maximality.

Corollary (4.7) A connected multigraph (allows multigraphs) has an Eulerian trail iff it has either 0 or 2 vertices of odd degree.

- Reduction: can add multiedge. Graph becomes even ⇒ has an Euler tour.
  If we delete an edge then we open it and it is an Euler trail.

Reduction: Let G be a graph with two odd-degreed vertices u and v. Let G' be $G + \{e\}$, and $e = (u, v)$ is a new edge in addition to all vertices between u and v, which $\notin G$.
Then G' has all even degrees ⇒ tour. Apply previous reduction to get a trail

- Other direction: if T is an Euler trail $\in G$, starting at u and ending at v, then add to G a new edge $e = (u, v)$ to obtain an Euler tour. (ie we have change the two vertices u, v from odd to even; in (G') there is an Euler tour.

- Hamiltonian cycles.
  - How do we travel through a graph, visiting all vertices, without backtracking, ie. revisiting vertices?
  - In real life, edges have weights; what is the cheapest way?

- A Hamiltonian path in G is a path which visits every vertex exactly once.

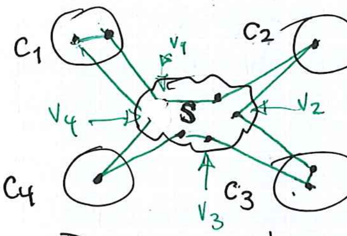
Def: A Hamiltonian cycle in G is a Hamiltonian path which starts and ends at the same vertex. G is a "Hamiltonian graph" if it has a Hamiltonian cycle.

ex: Take a 3D cube skeleton. The graph is Hamilton; we can prove this for any k dim skeleton

- Hamiltonian graphs are at least 2-connected ⇒ ≥2 disjoint paths; cannot disconnect G by deleting a single (k-1) vertex

Proposition (4.10) If G is Hamiltonian and $S \subseteq V(G)$, then $G \setminus S$ has $\leq |S|$ connected components.
ie. we must delete $\geq |S|$ vertices to obtain $|S|$ connected components

Pf. Consider the following graph, where C_i, $i=1,\dots,k \leq |S|$ are connected components separated by subset S, ie $C_1,\dots,C_k$ are components of $G \setminus S$



- Every C_i is connected and is directly connected to only S, ie. there are no edges between components.

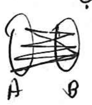
SIDE NOTE
(Conjecture: If we have a graph that is hard to disconnect, ie to break into k pieces, then we likely need ~100k, 1000k... vertices if it is a Hamiltonian. (but this is not really used in present day))

- Procedure: Walk along the Hamiltonian cycle. Let $v_i \in S$ be the first vertex that we visit when we leave a component C_i for the first time. (Can visit C_i multiple times.

- Note that all v_i are distinct by definition; else the cycle intersects itself.

(Let us consider G as having / with a directed Hamiltonian cycle. If $v_2 = v_0$, there is another edge coming into this point, which means that the cycle then intersects itself, which impossible.)

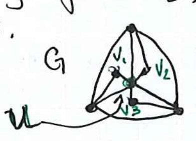
- Corollary. Let $H = (A \cup B, E)$ be a bipartite Hamiltonian graph. Then $|A| = |B|$.



Pf Since H is bipartite, $H \setminus A$ is not connected and has $|B|$ connected components of size 1 (ie an independent set of $|B|$ vertices) ⇒ $|A| \geq B$.

By symmetry, for $H \setminus B$, ⇒ $|B| \geq A$, which means $|A| = |B|$ □.

ex.



satisfies the above condition, but G is not Hamiltonian (we claim that)

- If $d(v) = 2$ then we know the behavior of Hamiltonian cycle around the vertex v. (for some vertex $v \in V(G)$).

Observe that if the HC ("= Hamiltonian cycle") exists and $d(v) = 2$, then both edges incident to v should be on the HC.

Here the vertices v_1, v_2, v_3 have degree of 2; their edges must be included. Then, the central vertex u has 3 edges ($\neq 2$) ⇒ G is not Hamiltonian.

(Hamiltonian cycles)

* **Thm. Dirac 1952**: Let G be an n-vertex graph, $n \geq 3$, s.t. $\delta(G) \geq n/2$. Then, G is Hamiltonian.

- Procedure. Start with a complete graph. Delete edges serially; any graph with $\frac{n}{2}$ edges remaining is a HC;

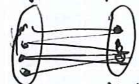
- (We will prove that) $\frac{n}{2}$ is a tight bound, ie. $\delta = \frac{n-1}{2}$ is insufficient. iow, the statement is precise.

ex. 4.14 (best-possible minimum degree bound).

- two connected graphs with k vertices each have exactly one common vertex v.

- $n = 2k-1$, $\frac{n-1}{2} = k-1 \Rightarrow \delta(G) = k-1$.
- v is a cut vertex $\Rightarrow$ G is not Hamiltonian!

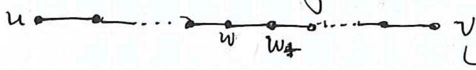
ex 2 Bipartite graph $G = (A \cup B, E)$, wherein $|A| \neq |B|$

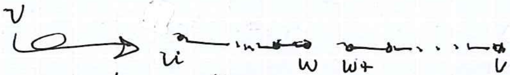


$n = 2k-1$, $\delta = k-1 = \frac{n-1}{2}$ H is not Hamiltonian because bipartite $|A| \neq |B|$

Remark: Why $n \geq 3$? If $n = 2$, then $\frac{n}{2} = 1$, ie G •—•

Pf. (Using maximality to shorten proof). Take the longest path P in G from u to v. The endpoints should have all neighbors $N(u)$, $N(v)$ in P. But what if



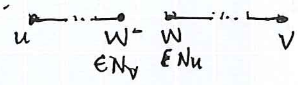


- Counterexample. Let G be the maximal counterexample on n vertices, ie no edge can be added without making it a HC (HG).

Since adding edges preserves $\delta(G) \geq \frac{n}{2}$, by doing this we can arrive at a non-Hamiltonian graph s.t. adding any new edges creates a HC.

ie. If G is maximal, take any pair u, v of non-adjacent vertices, then $G' = G + (u,v)$ is Hamiltonian
$\Rightarrow$ G has a path (from u to v) spanning all vertices.
ie longest path contains all vertices, including $N(u)$, $N(v)$
(fixing direction for sake of explanation),
$\forall w$ on the path, w^+ is the next vertex and w^- is the previous vertex.

- Claim $\exists$ a vertex w which is in $N(u)$ s.t. w^- is in $N(v)$, ie.



If we delete edge (w^-, w), we still have a path $u \to w \to v \to w^-$
This implies that G has a HC - contradiction.

Pf Let X be the set of vertices $X = \{w^- \mid \forall w \in N_u\}$. Note that $|X| = |N_u| \geq \frac{n}{2}$.
u does not have any neighbors to the left. Any vertex can be shifted to the right; also note $v \notin X$. Also note that $v \notin N_v \Rightarrow X, N(v) \subseteq G \setminus S$, which is set of size $n-1$.

Recall that $|A \cup B| = |A| + |B| - |A \cap B| \Rightarrow |A \cap B| = |A| + |B| - |A \cup B|$,
so $|X \cap N_v| = |X| + |N_v| - |X \cup N_v| \geq n/2 + n/2 - (n-1) = 1$. (sum of sets $\leq$ universe)
so X is N_v, and is (all N_u are) shifted by one.

(Hamiltonian Cycles, cont'd) Dirac, cont'd (Thm 4.13)

- The last vertex has an edge that goes somewhere internal, we can ... reroute HC to that neighbor as the new end vertex; keep going until we complete the HC.
  - * we care only about the sum $|N_u| + |N_v|$ and not the individual sets.
  - * We need to address pairs u, v which are not adjacent.

Thm 4.15 Ore (1960) Let G be an n-vertex graph, $n \geq 3$. s.that $\forall$ non adjacent $u, v \in G$ we have $d(u) + d(v) \geq |G| = n$ then G is Hamiltonian

Pf If this is not true, then let G be the maximal counterexample.

$G' = G + \{(u,v)\} \Rightarrow G'$ has HC by maximality in G. By def, $u \sim v$; u, v are not adjacent, so $\Rightarrow$ $d(u) + d(v) \geq n = |G|$.

ie we delete an edge (u,v) from the cycle of the Hamiltonian graph G' to produce a Hamiltonian path.

and $d(u) + d(v) \geq n \approx (\frac{n}{2} + \frac{n}{2})$ as per the previous claim in Dirac.
(that $\exists$ vertex $w \in N(u)$ s.t $w^- \in N(v)$, wherein $|N(v)| \approx \frac{n}{2}$ and $|N(u)| \approx \frac{n}{2}$
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