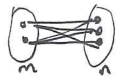
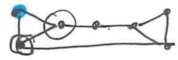
- Consider the graphs  (a)  (b) cut vertex $\kappa_1(b)=1$  (c)  (d)

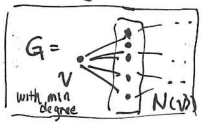
- Vertex connectivity ($\kappa(G)$)

- Def. (i) A vertex cut in G is a set S s.t. $G\setminus S$ is not connected. (ii) If $S=\{v\}$ then v is a cut vertex.
  (iii) A graph G : $|G|>k$ is called k-connected if no set S : $|S|<k$ can disconnect G , ie no cut set of size $<k$, ie no two vertices are separated by fewer than k other vertices
  (iv) $\kappa(G)$ is a maximal k s.t. G is k-connected . ie every 2 vertices are separated by at least k other vertices

- Assume we have a graph G with $\geq k+1$ vertices. How do we quantify connectivity?
- A maximally connected graph with n vertices is (n-1) connected. (ie a complete graph)
- Generally, (i) if we have a complete graph $G=K_n$ on n vertices, then $\kappa(G)=n-1$ ex. (a)
  (ii) for a complete bipartite graph $K_{m,n}$, $m<n$, then ex. (c)
  $\kappa(K_{m,n}) = \min\{m,n\} = m$ (the smaller)
  ie. we remove all the vertices on one side
  (To) determine the size of S ≡ connectivity of G ,
  if $S\subseteq K_{m,n}$ s.t $K_{m,n}\setminus S$ has vertices on both sides, then it is (still) connected .

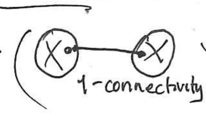
  (iii) Arbitrary graph We delete all the neighbors of a vertex
  (ex. $k(b)=1$; $k(a)=2$)
  $\forall\ x,y\in G$, $\exists$ 2 paths (that are internally disjoint) from x to y .

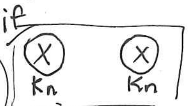
- Claim: $\forall$ graph G , $\kappa(G)\leq\delta(G)$. Let v be a vertex ($v\in G$) that has degree of $d(v)=\delta(G)$ - deleting N(v) disconnects v from the rest of G □

pf ex. 

(Trivially, for a complete graph, $|G|=d(v)+1=\delta(G)+1$, $(G=K_{\delta+1})\Rightarrow \boxed{\kappa(G)=\delta(G)=|G|-1}$ if G is complete

- Lemma: Since low $\delta\Rightarrow$ low connectivity , does high $\delta\overset{?}{\Rightarrow}$ high connectivity? (No)

ex. pf 

if 

$\delta(G)=n-1$ but since G is disconnected (ie subgraphs are disjoint) $\Rightarrow$ 0-connectivity. □

- Theorem (Mader 1972). Every graph with average degree $\geq 4k$ contains a k-connected (highly connected) subgraph.

ie. $\bar{d}(G)=\dfrac{\sum^{V(G)} d(v)}{|G|}=\dfrac{2e(G)}{|G|}\Rightarrow e(G)=\dfrac{\bar{d}}{2}|G|$.

Pf by induction, with a stronger statement on the number of vertices (n) .

Cases • (k=0, k=1 (0, single vertex) - trivial.)
• Assume $k\geq 2$. We shall prove the stronger
• statement : if G satisfies (i) $|G|=n\geq 2k-1$
and (ii) $|e(G)|=m\geq(2k-3)(n-k+1)+1$
$\Rightarrow$ then G has a k-connected subgraph.

Strategy: natural logic : throw out vertices with low degrees

• Procedure . if $\bar{d}(G)\geq 4k$, (i) and (ii) follow; (i) holds since $n>\Delta G\geq 4k$. ?
and (ii) from $m=\frac{1}{2}\bar{d}(G)n\geq 2kn$

Note that $\Delta G\geq\bar{d}(G)\geq 4k$, so there must exist a vertex with $\geq 4k$ neighbors .
(So, $n>4k+1>2k-1$). $\nearrow$ $e(G)\geq 2kn>(2k-3)(n-k+1)+1$.
By induction hypothesis,

4 Graph Theory Lecture 4: Connectivity cont'd 01 Mar 2018

e cont'd Thm (Mader 1972: Every graph with an average degree $\geq 4k$ contains a k-connected graph)

(Big idea: take a vertex with small degree. Look @ its left and right connectivity ...)
?

Pf Induction n. (For all k). given (i) $n \geq 2k-1$. and (ii) $m \geq (2k-3)(n-k+1)+1$, $m = |e(G)|$

Basis (i) $n = 2k-1$. A complete graph on $2k-1$ vertices is $(2k-2)$-connected
Then, $|e(G)| \geq (2k-3)(n-k+1)+1$. Using
$k = (n+1)/2$, $\Rightarrow$ $= (n-2)(n+1)/2 + 1$
$\Rightarrow |e(G)| \geq \frac{1}{2}n(n-1)$ = # of edges in K_{2k-1}.

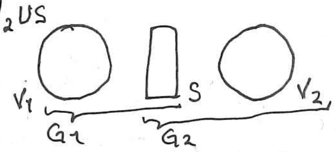
Induction step. We perform induction on n. Assume $n > 2k-1$, and that the statement holds for all $(n-1), (n-2), \ldots$.

- Look on vertex v of minimal degree.
  - Case 1 $\delta(v) \leq 2k-3$. Deleting v preserves (ii) precisely; we subtract $2k-3$ edges
    $\Rightarrow e(G \setminus v) = (2k-3)((n-k+1)-1)+1$

    a subgraph of a subgraph of the original G is also a subgraph of the original G

    Then take $G' = G \setminus v$, $|G| = n-1 \geq 2k-1$. We rewrite the above as
    $e(G') = (2k-3)(\underbrace{(n-1)}_{G \text{ with } n-1 \text{ vertices}} + k + 1)$
    . Completed by induction

  - Case 2: $\delta(G) \geq 2k-2$. If G is k-connected
    Otherwise, $\exists$ separating set $S \subseteq V(G)$, $|S| \leq k-1$ s.t $G \setminus S$ is not connected
    - Split the graph into multiple components. (ex. split $V(G) = V_1 \cup S \cup V_2$, these are disjoint sets and $S \neq \emptyset$.?.
      Let the two subgraphs $G_1 = V_1 \cup S$ and $G_2 = V_2 \cup S$



    - Both G_1 and G_2 have $\geq 2k-1$ vertices.

    (We want to show that when we cut the graph, G_i = all edges on $V_i \cup S$. we are left with subgraphs with many vertices and subgraphs with many edges.) - (Both G' have many vertices. take all edges in G_1, G_2 ⇝ don't lose any edge. in fact, we double count V-S ?)

    Pf $\exists$ vertex $v \leftarrow V_1$, then all neighbors of v are in G_1. Since $d(v) \geq 2k-2 \Rightarrow |G_1| \geq 2k-2+1$. Same applies for G_2

- Claim Either G_1 or G_2 have enough edges to satisfy the induction hypothesis.

  Pf by contradiction Suppose not (a) $e(G_1) < (2k-3)(|G_1|-k+1)+1$
  $< (2k-3)(|G_1|-k+1)$
  (applies also for G_2; since $e \in \mathbb{N}$) AND (b) $e(G_2) < (2k-3)(|G_2|-k+1)+1$
  $< (2k-3)(|G_2|-k+1)$

- Then, $e(G) \leq e(G_1) + e(G_2)$ since every edge is counted at least once.
  $\leq (2k-3)[(|G_1|-k+1)+(|G_2|-k+1)] = (2k-3)[(|G_1|+|G_2|-k+1)-k+1]$

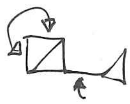
- To show that $[(|G_1|+|G_2|-k+1)-k+1] < n$, we prove here (on the right). $\leq (2k-3)[n-k+1] \geq e(G)$
  Contradiction.
  $|G_1|+|G_2| = n-|S| \leq n-k+1$
  $\Rightarrow$ at least one of G_1 or G_2 satisfy the induction hypothesis.
  We thus justify the above inequality $\Rightarrow$ proved. □

Vertex Connectivity

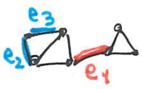
Mader Thm cont'd : Induction case 2 claim, cont'd ($\delta(G) \geq 2k-2$)

Big idea : If we have a graph with many edges and a small separating set, which splits G into two disjoint subgraphs,
↪ then one of the subgraphs G_i gets denser than the overall G.

iow, high δ graph : may have low connectivity but $\exists$ subgraph with high κ

- Edge Connectivity  " Edge connectivity is a notion of edge cuts " .

Def:

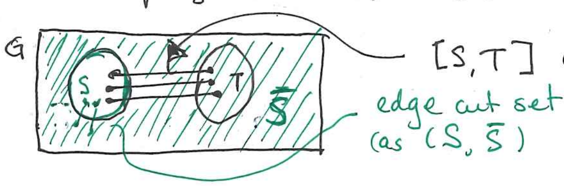
- $F \subseteq E(G)$ is a disconnecting set if $G \setminus F$ is not connected.
- G is κ' (edge)-connected if deleting any $k-1$ edges cannot disconnect G.
  Note: since we are not deleting any vertices, we don't need to assume anything about G, ...
- So, κ' = max k s.t. G is κ-edge connected.
- In a graph ex  , $\{e_1\}$ and $\{e_2, e_3\}$ are both disconnecting sets.
  e_1 is a bridge because deleting e_1 is sufficient for disconnecting the graph.

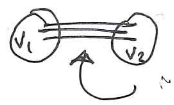
- Def Given two sets $S, T \subseteq V(G)$, the set $[S,T]$ is the collection of edges with one endpoint in S and another endpoint in T.

  The edge cut in G is a set of edges in the form $(S, \bar{S})$, where $\bar{S} \equiv V \setminus S$

- Remark : the minimal num. of edges required (to disconnect G) is always a cut.
  ⟹ no edges in the minimal set are within the set.

- Can we efficiently compute the minimal vertex, minimal edge graphs ?
  (ie in polynomial complexity, $O(n^c)$?)





- If $G = K_n$, then $\kappa'(G) = n-1$.
- For a graph G (ex)  $\kappa'(G) = 3$; $\kappa(G) = 2$. Observe that $\kappa' > \kappa$
- In fact, we claim that (statements) $\underbrace{\kappa(G) \leq \kappa'(G)}_{(ii)} \underbrace{\leq \delta(G)}_{(i)}$

Pf (i) If we have the graph  , $|N(v)| = \delta$, then deleting edges touching v, disconnects it from $G \setminus \{v\}$. ie $\kappa'(G) = \delta$ here

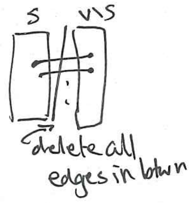
(ii) We show that we can disconnect the graph by deleting some x edges or by some x vertices

Edge Connectivity : Thm $\underbrace{\kappa(G) \le \kappa'(G)}_{(ii)} \underbrace{\le \delta(G)}_{(i)}$; claim (ii), cont'd.

- Def For $|G|>1$, let $[S, \bar{S}]$ be a min cut set whose number of edges in $\kappa'(G)$

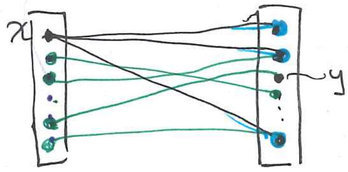
(ii) cont'd Consider two cases:

① S and $V \setminus S$ form a complete bipartite graph.
Then, $\kappa'(G) = |S|(n-|S|) \ge n-1 \ge \kappa(G)$. ✓
← which is a convex quadratic function (wrt $|S|$)



② $\exists \ge 1$ vertex on the left and 1 vertex on the right which are not adjacent; ie. $\exists$ vertices $x \in S$ and $y \in \bar{S}$ which are not connected.

1. Mark the neighbors of $x \in \bar{S}$.
2. Mark all vertices in $S \setminus \{x\}$ that have neighbors in $V \setminus S$
3. If we delete the marked vertices, then G is disconnected.



we usually have more edges than vertices.

Note that (i) x and y were not deleted and are now not connected.

(ii) $\forall$ deleted vertex, one may choose an edge in the cut which touches the vertex in such a way that no edge is chosen twice (or more), ie. unique edge (has distinct endpoints, compared/among $E(G)$).
→ num of vertices deleted $\le [S, \bar{S}]$

(iii) Edge connectivity is always an upper bound on vertex connectivity.
⇒ so vertex cuts are cheaper

- Def Bridge := an edge that connects G; deleting a bridge disconnects G, ie results in two connected components (since G was also connected)

- 2-connectivity. $\forall u, v \in G$, $\exists$ 2 paths $u \leadsto v$ on edges in $E(G)$
$\Downarrow$
κ-connectivity ≥ 2.

(But) does this hold for the reverse?
(Yes - Menger's theorem).

Thm Menger : (i) If $\kappa(G) = k$, then $\forall$ two vertices x and y, $\exists$ k internally disjoint paths from x to y.
$V(G)$

(ii) If $\kappa'(G) = k$, then $\forall$ x, and y, $\exists$ k edge-disjoint paths from x to y.
$E(G)$