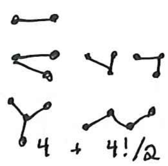
• Question: How many labelled trees are there on n vertices ?
(ie Count the num. of labelled spanning trees in K_n)

$n=2$
$n=3$
$n=4$



$4 + 4!/2 = 16$

$f(2)=1$
$f(3)=3$
$f(4)=16$
...

Cayley's formula.

Theorem: there are n^{n-2} labelled spanning trees on n vertices. (in K_n)

• Two ways to prove Cayley's formula : (I) Prüfer code : bijection btwn trees, sequences
(II) make and Count directed graphs, ("like counting formulae") Joyal.

(I) Prüfer Code

Build by bijection ← want to use 1-to-1 correspondence between T labelled trees (on n vertices) and sequences $(a_1, \dots, a_{n-2})$, $a_i \in [n] = \{1,\dots,n\}$ and to show/ have reversibility.

• (Useful to) Assume we have an ordered vertex set S.

To produce the Prüfer code of a tree; $(f(T))$.

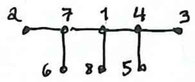
• Given a tree T on set S of size n which is ordered, ...

"we iteratively delete the leaf with the smallest label, and append the label of the neighbor to the sequence s.t. after $n-2$ generations, we are left with one edge (connecting two vertices)."

(a) Procedure : (1) Find the leaf with the minimal label, u.
(2) Delete u and put first symbol in the sequence the label of the unique neighbor of u (immediate ancestor of u)
(3) Continue (2) using tree $T' = T \setminus \{u\}$. (of size $n-1$)
(4) Stop when there are only two vertices (one edge) remaining.

• ex. so a Prüfer code has length $n-2$.

ex.



(7 4 4 1 7 1)

Step 1.	Delete 2,	add 7 to seq.
2.	" 3,	" 4
3	5,	4
4	4,	1
5	6	7
6	7,	1
7	end:	1—8 remains.

(b) Now show Prüfer code gives us 1-to-1 correspondence.

• If we can reverse the code, we can get exactly one code corresponding to one tree ($\forall$ trees) by bijection.

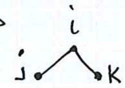
• Suppose $(a_1, \dots, a_{n-2})$ is a code / sequence of tree T on n vertices.

(b) cont'd (Showing 1 to 1 correspondence) Prop (we want to say that All elements that appear in the prüfer code are exactly the vertices that are not leaves of T)

→ Prove that (i) no leaves appear in the PC
(ii) all non-leaves appear in the PC.

Pf (i) Suppose i is a leaf with unique neighbor j in T. The only way i appears in PC is if j gets deleted. Then, however, we would have disconnected i from the rest of the graph (i is alone) (which contradicts def of tree and we must always have a tree).
⇒ No leaves appear in PC.

(ii) Let i be a vertex of degree ≥ 2 in T. Since in the end, there are exactly two vertices left, one of the vertices, j or k, will be deleted, and i will be added to PC.



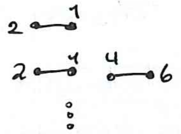
wherein $(i,j), (i,k) \in E(T)$, $j,k \in V(T)$

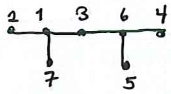
Cayley's Formula - Proof with Prüfer code, cont'd.

- Corollary. • Let $(a_1, \ldots, a_{n-2})$ be a Prüfer code of T.
  • Then, the minimal num. which is not $a_1, \ldots, a_{n-2}$ was the first vertex (leaf) deleted; this vertex was connected to a_1.

ex. Given 16631
(on $T = \{1,2,3,4,5,6,7\}$, Connect the min leaf l (at that time) to the vertex a in the sequence
ie add (l, a) to $E(T)$, $T' = T + (l,a)$.

Step 1. leaf 2 + 1
2. (4,6)
3. (5,6)
4. (6,3)
5. (3 1) → Step 6 : add an edge btwn the remaining vertices ((1,7)).





⟺ Process is reversible
⟹ (unique) / 1-to-1 correspondence.

(Idea of the proof : induction on size of tree)

(Pf2) Claim: For every code $(a_1, \ldots, a_{n-2})$, there is a unique tree T which produces this code
↳ by induction on the size of the vertex set S (an ordered ground set)

- Base : $|S| = 2$: Unique tree s_1 — s_2, with an empty Prüfer code.
- Induction step : Suppose that for any $|S| = n-1$ and for/with any code $(b_1, \ldots, b_{n-3})$ There a unique tree T' on S' with this code.

Pf. For n Let $(a_1, a_2, \ldots, a_{n-2})$ be a code (sequence) on ordered set S ; $|S| = n$
Let S_1 be the smallest label from S which is not in $a_1, \ldots, a_{n-2}$.
Then, we know (1) S_1 is a leaf connected to a_1
(2) $T \setminus S_1$ is a tree T', with labels $S \setminus \{S_1\} = S'$,
↳ $(a_2, \ldots, a_{n-2})$ is the Prüfer code for T'.

By induction, T' is unique, since we get T' from T, which is unique.
⟹ at any step of building, ~~any~~ the step is reversible and unique

* This can also be applied to counting the num. of spanning trees in any G.

⇓

Pf3 Joyal, 1981.

a In every tree, we label two vertices L and R (which can be the same vertex).
We count the num. of labelled n-vertex trees ~~for~~ with two marked vertices L, R.

- If the num. of trees is $f(n)$, then the num. of such objects is $n^2 f(n)$ (n options to choose L, and to choose R.)
Can we prove that the num. of such objects (num. of copies of the trees with different labelling) is $n^2 f(n) = n^n$?

Note that n^2 counts/includes all functions from $[n] \to [n]$ (ie with no conditions)
⟹ all objects are possible.

Main idea : Counting labels of different trees is like counting functions.

Pf. strategy : starting with a function, how can we get the tree ?.

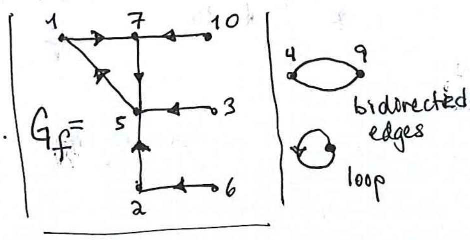
- **Def** A <u>directed graph</u> (aka. digraph) is a graph s.t. every edge has a direction $u \to v$.
  Edges can be bi<u>directional</u>, ie. ~~two edges~~ $(u \to v)$, $(v \to u) \in E(G)$.
  For this class <u>we assume no bidirectional</u> edges exist in G. (simple graph)

- <u>Joyal</u>: Given any f'n $h: [n] \to [n]$, we can build a directed graph on $[n]$ by placing edges $(i, h(i))$ in G.

<u>ex.</u> $(n=10)$

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 7 & 5 & 5 & 9 & 1 & 2 & 5 & 8 & 4 & 7 \end{pmatrix} \begin{matrix} L \\ \downarrow \\ R \end{matrix}$$

mapping.



- Observe that G_f is a digraph in which the out-degree of every vertex is exactly 1
  (ie $f(i)$ is the only out-neighbor of i). $\Rightarrow$ We have a unique one-to-one correspondence (degree).

ie Directed graphs on n vertices with outdegrees of 1 ($\forall$ vertex).

<u>Properties</u> • Looking at the connected components (not considering direction), we find (i) that $\forall$ each <u>component</u>, it consists of <u>exactly</u> <u>one cycle</u>

(ii) if the component has <u>x vertices</u>, Because of out degree 1, it has <u>x edges</u>.)

as per Lemma in which we proved that a path + 1 edge is exactly one cycle.

- Moreover, the cycle is directed.

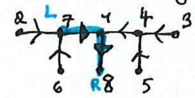
Note that the <u>union of cycles</u> gives set M, which is the <u>maximal</u> set by inclusion on which <u>h operates as a bijection</u>.
ie that works as a bijection h permute elements of M.

- If there is a bijection (one-to-one mapping), then M can be rewritten as the union of directed cycles. Bijection $\Longleftrightarrow$ all out- and in-degrees are exactly One
- Every component has exactly one cycle, so we can reduce the graph to only cycles
  $\hookrightarrow$ bijection with set M.

To prove reversibility in Joyal's theorem,
- Recall that a function is a unique mapping that works as a bijection on precisely the path with M

- The order of vertices on the <u>path</u> is exactly the order of the vertices in the <u>first line of M</u>
- Recall f'n is a unique mapping that works as a bijection precisely on the path P with maximal set M

In ex



- The vertices of path $P=(7,1,8)$ give M. Ordering these vertices of M gives us $M=\{1,7,8\}$, which gives us the first line in $f|_M$,
- and the second line from the order of vertices on the path from L to R, ie $P=(7,1,8)$.
- The remaining vertices, ~~which are~~ not bijection vertices, are oriented by the unique paths from each vertex to (the closest vertex (L vs. R)).

$$f|_M = \begin{matrix} M \\ P \end{matrix}\begin{pmatrix} 1 & 7 & 8 \\ 7 & 1 & 8 \end{pmatrix} \quad \text{ie} \to \quad f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 7 & 4 & 1 & 4 & 7 & 1 & 8 \end{pmatrix}$$

This is a convenient way to describe graphs, in general.

Big idea: instead of looking @ a particular <u>graph</u>, look @ a particular <u>function</u> and then see what kinds of graphs can be produced.