Recall: Bootstrap Consistency: $\exists$ sequence $a_n \nearrow$

$$P(\underbrace{a_n(\hat{\theta}_n - \theta)} \le x) - P^*(a_n \underbrace{(\hat{\theta}_n^* - \hat{\theta}_n)} \le x) \xrightarrow{P} 0$$

if the distribution or variance (if we know it is Gaussian) is unknown, we need to bootstrap.

obtain this distribution by simulation (bootstrapping)

Note: θ is fixed and unknown
$\hat{\theta}_n$ is random

- When does bootstrap consistency hold? (difficult to say with full generality, but...)
  It holds if $a_n(\hat{\theta}_n - \theta)$ is asymptotically Gaussian.
- When is bootstrap inconsistent?
  - Be careful if $a_n \neq \sqrt{n}$ and we have a non-Gaussian limit.

Simulation is a means to get the variance of the Gaussian distribution

Convergence of $\hat{\theta}_n$ is usually $\sqrt{n}$.

ex. $X_1, \dots, X_n \overset{iid}{\sim} N(\mu, 1)$,

- We need $\mu \ge 0$, so the MLE for μ is $\hat{\mu}_n = \max(0, \bar{X}_n)$
- Then $\sqrt{n}(\hat{\mu}_n - \mu)$ is asymptotically distributed as

$$\begin{cases} Z & \text{if } \mu > 0 \quad \text{(max does not play a role asymp'ly)} \\ \max(0, Z) & \text{if } \mu = 0 \quad \text{(max continues to play role ")} \end{cases}$$

* if $\mu = 0$, we have a non-Gaussian limit $\Rightarrow$ bootstrap inconsistency.

R CODE · Scaling by std ($1/\sqrt{n}$)

mu=0 $\rightarrow$ for $\hat{\mu} > 0$ (datasets 1, 3), CDF limits is Gaussian for $x \ge 0$, but not for $x \le 0$.

for $\hat{\mu} \le 0$, convergence does not hold (ref. CDF (asymptotic: n=10 000))

(For data in line 168) using naive approach (direct y~x), can't trust stderr, t-stat, Pr(>|t|) as these are based on constant variance assumption; errors are non-zero

Using bootstrap function (reg y~x using index subset sampled from [50] with replacement), the bootstrap stats are much closer to the true values. (Gaussian env. iid but with unknown, non-constant Var.

(If X's are fixed in regression, y's are no longer iid. Errors are no longer iid, so be cautious about using residuals for bootstrap.)

Constructing Confidence Intervals :

"B": "bootstrap"

- (Methods of making) Confidence Intervals based on bootstrap.

1. Quantile BCI : does not necessarily have correct coverage; sometimes does with Gaussian limit (distr)

$$I = \left[q_{\hat{\theta}^*}(\alpha/2) \, , \, q_{\hat{\theta}^*}(1-\alpha/2) \right]$$

2. t-BCI $\quad I = \hat{\theta} \pm \underbrace{z_{1-\frac{\alpha}{2}}}_{\text{quantile of sd.}} \, sd(\hat{\theta})$

with $\bar{x} \pm z_{1-\frac{\alpha}{2}} \, sd(\bar{x}) = \frac{\sigma}{\sqrt{n}}$

$\bar{x} \pm t_{1-\frac{\alpha}{2}, n-1} \frac{\hat{\sigma}}{\sqrt{n}}$ $\quad \Rightarrow \quad I = \hat{\theta} \pm t_{1-\frac{\alpha}{2}, n-1} \sqrt{Var^*(\hat{\theta}^*)}$

3. Reversed Quantile BCI :- has asymptotic coverage

$$I = \left[\hat{\theta} - q_{\hat{\theta}^*-\hat{\theta}}\left(1-\frac{\alpha}{2}\right), \; \hat{\theta} - q_{\hat{\theta}^*-\hat{\theta}}\left(\frac{\alpha}{2}\right) \right] \quad , \text{ quantile as } q$$

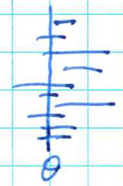
We have an estimator $\hat{\theta}_n$ for $\hat{\theta}$ based on n iid observations.

Goal : Confidence interval with coverage $1-\alpha$, ie.

$$Pr(\theta \in I) = 1-\alpha \, . \quad \alpha\,\% \text{ miss } \theta$$

$\equiv I \ni \theta \leftarrow \theta$ is fixed $\quad (1-\alpha)\%$ contain θ

$\uparrow$ I is random

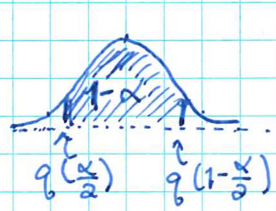


Consider proposal 3. We know that

$$1-\alpha = P\left(q_{\hat{\theta}^*-\hat{\theta}}\left(\frac{\alpha}{2}\right) \le \hat{\theta}^* - \hat{\theta} \le q_{\hat{\theta}^*-\hat{\theta}}\left(1-\frac{\alpha}{2}\right) \right)$$

$\uparrow$ ($\frac{\alpha}{2}$ quantile of $\hat{\theta}^* - \hat{\theta}$)

$$\underset{\text{bootstrap consistency}}{\approx} P\left(q_{\hat{\theta}^*-\hat{\theta}}\left(\frac{\alpha}{2}\right) \le \hat{\theta} - \theta \le q_{\hat{\theta}^*-\theta}\left(1-\frac{\alpha}{2}\right) \right)$$



$$= P\left(-q_{\hat{\theta}^*-\hat{\theta}}\left(\frac{\alpha}{2}\right) \ge \theta - \hat{\theta} \ge -q_{\hat{\theta}^*-\theta}\left(1-\frac{\alpha}{2}\right) \right)$$

$$= P\left(-q_{\hat{\theta}^*-\hat{\theta}}\left(1-\frac{\alpha}{2}\right) \le \theta - \hat{\theta} \le -q_{\hat{\theta}^*-\theta}\left(\frac{\alpha}{2}\right) \right)$$

$$= P\left(\hat{\theta} - q_{\hat{\theta}^*-\theta}\left(1-\frac{\alpha}{2}\right) \le \theta \le \hat{\theta} - q_{\hat{\theta}^*-\theta}\left(\frac{\alpha}{2}\right) \right)$$

So then $\Rightarrow I$ is in option 3.

(Constructing Confidence Intervals - Reverse Quantile BCI, cont'd)

- Alternatively, (equivalently),

$$q_{\hat{\theta}^* - \hat{\theta}}(\alpha) = q_{\hat{\theta}^*}(\alpha) - \hat{\theta}$$

(quantile shift/translation is monotonic; move by $\hat{\theta}$ to recenter around that ($\hat{\theta}$). will be $\oplus$

$$\Rightarrow \quad I = \left[2\hat{\theta} - q_{\hat{\theta}^*}(1-\tfrac{\alpha}{2}),\ 2\hat{\theta} - q_{\hat{\theta}^*}(\tfrac{\alpha}{2})\right]$$

▷ ie Proposal 3 is $\hat{\theta}^* - \hat{\theta} \approx \hat{\theta} - \theta$ (asymptotically)

- Proposal 4 - solution : scale by s.d. (like t-statistic, but we don't assume t distribution. We get & estimate distr by bootstrapping.

So how do we get $sd(\hat{\theta}^*)$? (get $sd(\hat{\theta})$ via the bootstrap)

▷ $$\frac{\hat{\theta}^* - \hat{\theta}}{sd(\hat{\theta}^*)} \approx \frac{\hat{\theta} - \theta}{sd(\hat{\theta})}$$

Use another/(additional) level of bootstrap!

$$\begin{tikzcd}
\theta \cdots & \hat{\theta} \cdots & \hat{\theta}^* \cdots \arrow[l, bend left, "sd(\hat{\theta})"] & \hat{\theta}^{**} \arrow[l, bend left, "sd(\hat{\theta}^*)"]
\end{tikzcd}$$

"double bootstrap"

▷ is the bootstrap CI close to $1-\alpha$?

Adjust α to get the appropriate coverage.