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Homework 4 ISLR Questions

4.8.3 Prove that Bayes' classifier is not linear, but quadratic.

Bayes Theorem: $Pr(Y=k \mid X=x) = \dfrac{\pi_k f_k(x)}{\sum_{l=1}^{K} \pi_l f_l(x)} = P_k(x)$

Take natural log on both side of equation $\log(\pi_k) + \log(f_k(x)) - \log\left(\sum_{l=1}^{K} \pi_l f_l(x)\right) = \log(P_k(x))$

Since $f_k(x)$ by definition is the density function (ISLR 4.16)

$$\log(f_k(x)) = -\log(\sigma_k\sqrt{2\pi}) - \underbrace{\frac{(x-\mu_k)^2}{2\sigma_k^2}}_{\text{Simplify}} \rightarrow \frac{x^2 - 2x\mu_k + \mu_k^2}{2\sigma_k^2}$$

$$= \frac{x^2}{2\sigma_k^2} - \frac{x\mu_k}{\sigma_k^2} + \frac{\mu_k^2}{2\sigma_k^2}$$

$$= -\log(\sigma_k\sqrt{2\pi}) - \frac{x^2}{2\sigma_k^2} + \frac{x\mu_k}{\sigma_k^2} - \frac{\mu_k^2}{2\sigma_k^2}$$

Put $\log(f_k(x))$ back into original equation:

$$\log(p_k(x)) = \log(\pi_k) + \left(-\log(\sigma_k\sqrt{2\pi}) - \frac{x^2}{2\sigma_k^2} + \frac{x\mu_k}{\sigma_k^2} - \frac{\mu_k^2}{2\sigma_k^2}\right) - \log\left(\sum_{l=1}^{K} \pi_l f_l(x)\right)$$

Move terms around and it's a quadratic equation:

$$\log(p_k(x)) = \left(-\frac{1}{2\sigma_k^2}\right)x^2 + \left(\frac{\mu_k}{\sigma_k^2}\right)x - \frac{\mu_k^2}{2\sigma_k^2} - \log\sigma_k + \log\sqrt{2\pi} + \log(\pi_k) - \log\left(\sum_{l=1}^{K} \pi_l f_l(x)\right)$$

4.8.7 Mean value for "Yes" = 10, mean value for "No" = 0. $\sigma^2 = 36$. 80% issue dividend. Predict probability that a company will issue dividend this year that its percentage profit was $X=4$ last year.

Assume $k=1$ for "Yes", $\mu_1 = 10$, $\sigma^2 = 36$, $\pi_1 = 0.8$

$k=2$ for "No", $\mu_2 = 0$, $\sigma^2 = 36$, $\pi_2 = 1-0.8 = 0.2$

$Pr(Y=1 \mid X=4) = P_1(4)$

Plugging $f_k(x)$ equation into Bayes gives: (ISLR 4.17)

$$p_k(x) = \frac{\pi_k \left(\frac{1}{\sqrt{2\pi}\sigma}\right) e^{\left(-\frac{1}{2\sigma^2}\right)(x-\mu_k)^2}}{\sum_{l=1}^{K} \pi_l \left(\frac{1}{\sqrt{2\pi}\sigma}\right) e^{\left(-\frac{1}{2\sigma^2}\right)(x-\mu_l)^2}}$$

$$P_1(4) = \frac{(\pi_1)\left(\frac{1}{\sqrt{2\pi}\sigma}\right) e^{\left(-\frac{1}{2\sigma^2}\right)(x-\mu_1)^2}}{(\pi_1)\left(\frac{1}{\sqrt{2\pi}\sigma}\right) e^{\left(-\frac{1}{2\sigma^2}\right)(x-\mu_1)^2} + (\pi_2)\left(\frac{1}{\sqrt{2\pi}\sigma}\right) e^{\left(-\frac{1}{2\sigma^2}\right)(x-\mu_2)^2}}$$

Plugging in numbers:

$$P_1(4) = \frac{(0.8)\left(\frac{1}{\sqrt{2\pi}\cdot 6}\right) e^{\left(-\frac{1}{2(36)}\right)(4-10)^2}}{(0.8)\left(\frac{1}{\sqrt{2\pi}\cdot 6}\right) e^{\left(-\frac{1}{2(36)}\right)(4-10)^2} + 0.2\left(\frac{1}{\sqrt{2\pi}\times 6}\right) e^{\left(-\frac{1}{2(36)}\right)(4-0)^2}}$$

$$P_1(4) \approx \frac{(0.8)(0.06649)(0.60653)}{(0.8)(0.06649)(0.60653) + (0.2)(0.06649)(0.80073)} \approx 0.75185 \approx 0.7519$$

There is a 75.19% chance that a company will issue dividend given X=4 profit.