Nongaussian Isocurvature Perturbations from Inflation

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We present a class of very simple inflationary models of two scalar fields which leads to nongaussian isothermal perturbations with "blue" spectrum, $n > 1$. One of the models is inspired by supersymmetric theories where light scalar fields naturally acquire masses $\sim H$ during inflation. Another model presumes that one of the fields has a nonminimal interaction with gravity $\xi R \sigma^2$. By a slight modification of parameters of these models one can obtain either gaussian isothermal perturbations, or nongaussian adiabatic perturbations with $n > 1$.

I. INTRODUCTION

It is well known that quantum fluctuations of the inflaton field ϕ in the simplest inflationary models produce gaussian adiabatic perturbations of metric with nearly scale independent spectrum, $n \approx 1$ [1,2]. In the models involving several scalar fields one can also obtain significant isothermal perturbations [3]. However, such perturbations produce six times greater anisotropy of the microwave background radiation than adiabatic perturbations (for a review see [2,4]. To obtains small MBR anisotropy and still explain galaxy formation from isothermal perturbations one should strongly suppress entropy perturbations on the horizon scale while keeping them sufficiently large on the galaxy scale. This would be possible, for example, if entropy perturbations had "blue" spectrum decreasing at large scales with $n \gtrsim 1.5$. However, until now there were no inflationary models which would produce such entropy perturbations. In principle, it would be possible to obtain such perturbations in axion models [3], but in the same theories where one could have $n \gtrsim 1.5$ one would also have too many axion strings with domain walls attached to them [5]. Therefore typically the authors discussing possible observational consequences of inflationary entropy perturbations concentrated on the models with $n \leq 1$, where only a minor addition of entropy perturbation to adiabatic ones was admissible [6].

In this paper we will propose a class of very simple models where one can get either entropy or adiabatic perturbations with $n \gtrsim 1.5$. This possibility may be of particular interest in relation to open universe models [7]. The main idea is almost trivial: If one has a theory of a scalar field σ with a mass m, then the mode functions $\sigma_k(t)$ in the expansion of field operator $\hat{\sigma}(x,t)$

$$\hat{\sigma}(x,t) = \frac{1}{\sqrt{2}} \int \left[\sigma_k^*(t) e^{ikx} \hat{a}_k^- + \sigma_k(t) e^{-ikx} \hat{a}_k^\dagger \right] \frac{d^3k}{(2\pi)^3} \quad (1)$$

obey the differential equation

$$\ddot{\sigma}_k + 3\frac{\dot{a}}{a}\dot{\sigma}_k + (\frac{k^2}{a^2} + m^2)\sigma_k = 0 \ . \quad (2)$$

Here k is a comoving wavenumber. We will normalize the scale factor a in such a way that $a = 1$ at the end of inflation. At this time in the most interesting comoving wavenumber interval $k < H$ the mode functions $|\sigma_k(t)|$ for $m^2 \ll H^2$ become [8]

$$|\sigma_k^2(t)|k^3 = \frac{H^2}{2} \left(\frac{k}{H} \right)^{\frac{2m^2}{3H^2}}, \quad (3)$$

which corresponds to the spectral index $n \approx 1 + \frac{4m^2}{3H^2} > 1$. The estimate of energy density of fluctuations of the field σ can be obtained as follows:

$$\rho_\sigma \simeq \frac{m^2 \langle \sigma^2 \rangle}{2} = \frac{H^2 m^2}{8\pi^2} \int_0^H \frac{dk}{k} \left(\frac{k}{H} \right)^{\frac{2m^2}{3H^2}} = \frac{3H^4}{16\pi^2} \ . \quad (4)$$

The main contribution to the energy density is given by the modes with $k_0 < k < H$, where k_0 is exponentially small, $k_0 \sim H \exp\left(-\frac{H^2}{m^2}\right)$. Let us consider density perturbations of the field σ in the long wavelength limit, $k \ll k_0$. This limit is most interesting in the models with sufficiently large m/H, such that the wavelengths $k_0^{-1} \sim H^{-1} \exp\left(\frac{H^2}{m^2}\right)$ after their subsequent growth in an expanding universe still remain smaller than the galaxy length scale. In this case on the scale $k^{-1} \gg k_0^{-1}$ the fluctuating field σ wanders many times in the region $-H^2/m \lesssim \sigma \lesssim H^2/m$, so that its value averaged over the domain of a size k^{-1} vanishes. As a result, addition of a perturbation $\delta\sigma(k)$ with $k \ll k_0$ does not lead to usual density perturbations $m^2 \sigma \delta \sigma$ [9,10]. Perturbations of density will be nongaussian and quadratic in $\delta\sigma$; they can be very roughly estimated by

$$|\delta_k^\rho| \sim \frac{\delta\rho_\sigma(k)}{\rho_\sigma} \sim \frac{2\pi m^2 |\sigma_k^2(t)|k^3}{\rho_\sigma} \sim \frac{m^2}{H^2} \left(\frac{k}{H} \right)^{\frac{2m^2}{3H^2}}, \quad (5)$$

which implies that

$$\frac{\delta\rho_\sigma}{\rho_{\text{total}}} \sim |\delta_k^\rho| \frac{\rho_\sigma}{\rho_\phi} \sim \frac{m^2}{M_p^2} \left(\frac{k}{H}\right)^{\frac{2m^2}{3H^2}} . \qquad (6)$$

In (almost) all realistic models of inflation the ratio $\frac{m^2}{M_p^2}$ is extremely small [11]. Therefore initially perturbations of the field σ practically do not contribute to perturbations of metric (isocurvature perturbations). However, the inflaton field must decay to produce matter. If the products of its decay are ultrarelativistic particles, then their energy density subsequently decreases as a^{-4}, where a is the scale factor of the universe. Meanwhile, if the field σ does not decay, or decay very late, then for a long time its energy density decreases more slowly, as a^{-3}. As a result, perturbations of the field σ may give rise to significant perturbations of metric. This is the same mechanism which is responsible for isothermal perturbations in axion theory [3]. However, the axion mass during inflation vanishes. In our case m does not vanish, and eq. (6) implies that the perturbations have blue spectrum, which is exactly what we need.

It is not very easy to implement this idea. First of all, Hubble constant can be very large in the beginning of inflation, and during the last stages of inflation it may change significantly. For example, in the chaotic inflation scenario in the theory $\lambda\phi^4$ inflation begins at $H \sim M_p$, and stops at $H \sim \sqrt{\lambda}M_p$. One must fine-tune the parameters to ensure that H becomes comparable to m exactly at the epoch corresponding to the last 60 e-folds, when the perturbations responsible for the large scale structure of the observable part of the universe have been formed. Moreover, prior to this epoch H remains much greater than m. But then there is no reason to assume that the field σ initially was at the point $\sigma = 0$: it did not have any chance to roll down there at $H \gg m$.

In this paper we will propose a simple way to overcome these problems. It is based on the observation that masses of many scalars in supersymmetric theories (except, possibly, the mass of the inflaton field) acquire corrections $\Delta m^2 = \alpha H^2$ in the early universe [12]. Here α is some parameter; typically $\alpha = O(1)$, but exceptions are possible. To study perturbations of metric which may appear in such models we will consider here a simplest toy model where the scalar field σ acquires the mass $\sim H$ during inflation. It describes fields ϕ and σ with the effective potential

$$V(\phi, \sigma) = \frac{1}{2}M^2\phi^2 + \frac{1}{2}m^2\sigma^2 + \frac{1}{2}g^2\phi^2\sigma^2 . \qquad (7)$$

This potential has two valleys, at $\sigma = 0$ and at $\phi = 0$. If initially $|\phi| > |\sigma|$, the field σ rapidly rolls to the valley $\sigma = 0$ and stays there. In this regime the energy density during inflation becomes equal to $\frac{M^2\phi^2}{2}$, and the Hubble constant is $H^2 = \frac{4\pi M^2\phi^2}{3M_p^2}$. Then the potential (7) can be written as follows:

$$V(\phi, \sigma) = \frac{1}{2}M^2\phi^2 + \frac{1}{2}(m^2 + \alpha H^2)\sigma^2 , \qquad (8)$$

where $\alpha = \frac{3g^2 M_p^2}{4\pi M^2}$. Thus this simple model can be considered as a toy model exhibiting generation of an effective mass squared αH^2 of the field σ during inflation.

Similar mass term appears during inflation even if the fields ϕ and σ do not interact with each other ($g = 0$), but the field σ is nonminimally coupled to gravity. Indeed, the term $\frac{\xi}{2}R\sigma^2$ does not lead to any corrections to the rate of expansion of the universe in the regime $\sigma = 0$, but it gives a contribution $12\xi H^2$ to the effective mass of the field σ during inflation [14]. In what follows it will not be very important which mechanism gives a contribution $\sim H^2$ to the mass of the field σ; we will just assume that during inflation the field σ has an effective mass squared $m^2(H) = m^2 + \alpha H^2$. Note that if H changes slowly, then the field σ at large H approaches the state $\sigma = 0$ as follows: $\sigma = \sigma_0 \exp\left(-\frac{m^2(H)\,t}{3H}\right) \sim \sigma_0 \exp\left(-\frac{\alpha H\,t}{3}\right)$. Thus, unlike in the situations where the mass is constant, in our case the field σ within the time $\sim (\alpha H)^{-1}$ rolls down to $\sigma = 0$, and we may study its quantum fluctuations near this point. Note also, that now the relation $m^2(H) = \alpha H^2$ is satisfied at all large H, so that after fixing the parameter $\alpha = O(1)$ one does not need to fine-tune m to be of the same order as H at the end of inflation. Eq. (5) suggests that at the end of inflation with $\alpha H^2 \gg m^2$ we will have perturbations $\frac{\delta\rho_\sigma}{\rho_{\text{total}}} = |\delta_k^\rho| \frac{\rho_\sigma}{\rho_{\text{total}}}$ with blue spectrum, $|\delta_k^\rho| \sim \left(\frac{k}{H}\right)^{2\alpha/3}$. Further evolution of these perturbations is strongly model-dependent. The main point is that under certain conditions the factor $\frac{\rho_\sigma}{\rho_{\text{total}}}$ may grow up to 1, whereas the shape of the potential may not change considerably. In what follows we will study one of these possibilities. We will skip the numerical coefficients of the order of one since we are interested only in order of magnitude estimates.

According to [11], at the stage when the energy density is dominated by $\frac{1}{2}M^2\phi^2$ the scale factor is given by $a \simeq \exp(-2\pi\phi^2/M_p^2)$. Consider the evolution of the modes of field σ which have the scales bigger than the horizon scale H^{-1} during inflation. The comoving wavenumber k for these modes satisfies the condition: $k \ll Ha$. Neglecting the spatial derivatives in the equation for scalar field σ and assuming that during inflation $\alpha H^2 \gg m^2$ we obtain the following equation for σ_k: $3H\dot\sigma_k = -\alpha H^2\sigma_k$. The solution of this equation is $\sigma_k \propto a^{-\frac{\alpha}{3}}$. At the moment of horizon crossing ($k \sim H_k a_k$) the amplitude of quantum fluctuations is $|\sigma_k|^2 k^3 \sim H_k^2$, where H_k and a_k are correspondingly the values of the Hubble constant and scale factor taken at the moment when perturbation with the wavenumber k crosses the horizon. This together with the relation $\sigma_k \propto a^{-\frac{\alpha}{3}}$ yields the relation

$$|\sigma_k|^2 k^3 \sim H_k^2 \left(\frac{a_k}{a}\right)^{\frac{2\alpha}{3}} \sim H_k^2 \left(\frac{k}{H_k a_k}\right)^{\frac{2\alpha}{3}} \qquad (9)$$

for $k \ll Ha$. Taking into account that $H_k \sim M\left(\ln(H_k/k)\right)^{1/2}$, we find that at the end of inflation

$$|\sigma_k|^2\, k^3 \sim M^2 \left(\ln \frac{H_k}{k} \right)^{1-\frac{\alpha}{3}} \left(\frac{k}{M} \right)^{\frac{2\alpha}{3}} \quad \text{for } k \ll M. \quad (10)$$

Here we took the following normalization for the scale factor: $a = 1$ at the end of the inflation when $\phi \sim 1$.

Detailed evolution of these perturbations after inflation is model-dependent. For simplicity we assume here that the correction αH^2 to the mass of the field σ disappears soon after the end of inflation. This is the case for both of our models, but it may not be the case for supersymmetric models studied in [12], where some corrections to our results will appear. We will also assume that the particles σ are either stable or decay very late, and that the inflaton field decays into ultrarelativistic particles with the energy density $\rho_r(t_f) \sim M^2$ immediately after inflation. For a long time these particles dominate. Hubble constant decays as $H \propto M/a^2$ during radiation dominated epoch. The modes with $k \gg \max\{ma, Ha\}$ oscillate and their amplitudes decay as a^{-1}. The modes with $k \ll \max\{ma, Ha\}$ have practically constant amplitude at the beginning and then start to oscillate with the frequency $\sim m$ when the Hubble constant $H(a)$ drops below m. Meanwhile their amplitudes decay as $a^{-3/2}$. Using these facts one can easily calculate the spectrum of scalar field at $a > M/m$:

$$|\sigma_k|^2\, k^3 \sim a^{-3} M^2 \left(\ln \frac{H_k}{k} \right)^{1-\frac{2\alpha}{3}} \left(\frac{k}{M} \right)^{\frac{2\alpha}{3}} F(k) ,$$

$$F(k) = \begin{cases} \frac{M^2}{mk} & \text{for } M > k > \sqrt{Mm} , \\ \left(\frac{M}{m} \right)^{3/2} & \text{for } k < \sqrt{Mm} . \end{cases} \quad (11)$$

To make an estimate of the energy density of fluctuations of the field σ one may calculate the potential" energy density

$$\rho_\sigma \sim \langle \rho_\sigma^{pot} \rangle = \frac{1}{2} m^2 \langle \sigma^2 \rangle . \quad (12)$$

It is easy to check that the contributions of the other terms to the total energy either the same order of magnitude or smaller than (12). When H drops below m term (12) correctly counts the leading contribution to the energy density. Taking into account that σ_k depends only on $k \equiv \left| \vec{k} \right|$ and integrating over the angles one can write (12) in the following manner:

$$\langle \rho_\sigma \rangle \sim m^2 \int |\sigma_k|^2\, k^3 d(\ln k) . \quad (13)$$

Substituting (11) in (13) and integrating over $k < aM$, one obtains, for $\alpha \ll 1$,

$$\langle \rho_\sigma \rangle \simeq \frac{M^{\frac{7}{2}} m^{\frac{1}{2}}}{a^3 \alpha} \left(\frac{m}{M} \right)^{\frac{\alpha}{3}} \left(\ln \frac{M}{m} + \frac{3}{\alpha} \right) . \quad (14)$$

The produced σ-particles are not distributed homogeneously. If the scalar field is relatively stable and starts to dominate in some time after inflation then their energy density fluctuations can become very relevant for cosmology. As we already mentioned, to estimate these fluctuations one cannot simply apply the standard methods developed in [1] because in our case the homogeneous component of the field σ vanishes. A correlation function characterizing energy density fluctuations is

$$\xi(r,t) = \frac{1}{\langle \rho_\sigma \rangle^2} \left(\left\langle \rho_\sigma(\vec{x},t) \rho_\sigma(\vec{x}+\vec{r},t) \right\rangle - \langle \rho_\sigma \rangle^2 \right) . \quad (15)$$

The power spectrum $|\delta_k^\rho|^2$ is defined as follows:

$$\xi(r,t) = \int \frac{dk}{k} \frac{\sin kr}{kr} |\delta_k^\rho|^2 .$$

To make an estimate of $|\delta_k^\rho|$ we will take into account only the contribution of the potential term in the energy. After simple calculations one finds that

$$|\delta_k^\rho|^2 \sim \frac{m^4 k^3}{\langle \rho_\sigma \rangle^2} \int \left| \sigma_{\vec{k'}} \right|^2 \left| \sigma_{\vec{k}-\vec{k'}} \right|^2 d^3 k . \quad (16)$$

Because we are only interested in large scale fluctuations which have been produced at inflation, we can make a cut off at $k \sim M$. By substituting the spectrum (11) in (16) and taking into account (14) one finally gets:

$$|\delta_k^\rho| \sim C(k) \left(\frac{k}{M} \right)^{\frac{2\alpha}{3}} \quad (17)$$

for $k \ll \min\left(M \left(\frac{m}{M} \right)^{1/2}, \; M \exp(-1/2\alpha) \right)$. Here $C(k) = \sqrt{\alpha} \left(\ln \frac{M}{m} + \frac{3}{\alpha} \right)^{-1} \left(\ln \frac{M}{k} \right)^{1-\frac{\alpha}{3}} \left(\frac{M}{m} \right)^{\frac{2\alpha}{3}}$. Strictly speaking, the above expression for $|\delta_k^\rho|$ refers to the relative energy density fluctuations in $\langle \rho_\sigma \rangle$ after inflation, at $a \gg M/m$, but before the field σ starts to dominate.

Let us suppose that the particles σ at some stage began dominating the energy density of the universe. This does not necessarily imply that they dominate the energy density at present, because they could decay later on. However, as soon as they give the dominant contribution to density at some moment, perturbations of metric at all subsequent moments will be determined by $|\delta_k^\rho|$. Here one should distinguish between two possibilities. If the field σ does not decay (as in the axion theory) or decays in a hidden sector without changing the amount of photons in the MBR, then the corresponding perturbations can be called isothermal, or entropy perturbations. However, if the field σ dominates and decays sufficiently early, then we essentially have a secondary reheating, and the perturbations (6) should be considered as adiabatic perturbations with blue spectrum. In a more general case by tuning the moment of decay one can get a mixture of adiabatic and isothermal perturbations.

To evaluate possible implications of this result one should first of all estimate the desirable value of the coefficient α. Suppose that, as usual, perturbations of

metric on the galaxy scale have the wavelength which was about 10^{25} times greater than $H^{-1} \sim M^{-1}$ at the end of inflation (our final result will not be very sensitive to this and the subsequent assumptions). Suppose also that $C(k) \sim 10^2$ for $k \sim 10^{-25}M$. Then the requirement that $|\delta_k^\rho| \sim 10^{-5}$ on the galaxy scale gives us the estimate $\frac{2\alpha}{3} \sim 0.3$, which corresponds to the spectral index $n \sim 1.6$. One can easily check that this result is consistent with our assumption that $C(k) \sim 10^2$ for a broad range of relations between M and m. This means that an exact value of α is not very much sensitive to our assumptions concerning other parameters of our model. (We will specify the possible range of values of M and m in [15].) On the other hand, with $\frac{2\alpha}{3} \sim 0.3$ the amplitude of density perturbations falls more than 10 times on the way from the galaxy scale to the scale of horizon. This strongly suppresses the microwave background anisotropy, thus resolving the problem of the MBR anisotropy produced by isothermal perturbations being 6 times greater than that produced by adiabatic perturbations of the same magnitude. Of course it does not mean automatically that this model can easily pass all cosmological tests, maybe in the end it will be desirable to consider combined effects produced by adiabatic and isothermal perturbations. This should be a topic of a separate investigation. The main purpose of our work is to point out that there exists a very simple but overlooked class of models where the perturbations of metric have rather unusual properties: They are nongaussian, they can be either isothermal or adiabatic, and they have blue spectrum with $n > 1$.

Note that the nongaussian nature of perturbations in these models is especially interesting on scales smaller than k_0^{-1}. Indeed, on such scales one can introduce an effective "classical" quasi-homogeneous field σ. This field, however, takes different values in different points. The calculation of density perturbations in each small region of such type can be performed in the standard way [1,2], but the final result will depend on our position in the universe [3,15]. This introduces an additional type of the large-scale structure of the universe. For $\frac{2\alpha}{3} \sim 0.3$ the corresponding scale $\sim k_0^{-1}$ is too small to be of any real importance. It would be interesting, however, to study the corresponding effects for $\alpha \ll 1$.

The models considered above can be easily generalized. For example, in supersymmetric theories the terms proportional to H^2 appear not only in the quadratic parts of the effective potential but to other parts as well. A more general toy model to study this effect is [13]

$$ V = -\frac{1}{2}(m^2 + \frac{\alpha}{2}H^2)\sigma^2 + \frac{1}{4M_p^2}(\lambda m^2 + \frac{\beta}{2}H^2)\sigma^4 \ . \quad (18) $$

For $H \ll m_\sigma$, the effective mass squared at the minimum of the effective potential is $2m^2$. Meanwhile in the early universe at $H \gg m_\sigma$ the minimum is at $\sigma = \sqrt{\alpha/\beta} \, M_p$, and the effective mass squared of the field σ at that stage is αH^2. If $\alpha \ll 1$, then at the stage of inflation pertur-

bations of the field σ are generated. The main difference between this model and the one considered before is that in this model during inflation there exists a large homogeneous field σ. Therefore density perturbations can be calculated in a more standard way [1,2]. They are gaussian, with blue spectrum growing at large k as $k^{\alpha/3}$. In the Affleck-Dine model of baryogenesis [13] this may lead to isothermal perturbations of baryon density with the spectral index $n \approx 1 + 2\alpha/3$. We hope to return to this model and other issues discussed above in a separate publication [15].

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