VIII. APPENDIX

A. Useful Lemmas

The following results are used in the proofs.

Lemma 3: [27] Let $\{u_k\}$, $\{v_k\}$, $\{w_k\}$ and $\{z_k\}$ be the nonnegative sequences of random variables. If they satisfy

$$\mathbb{E}[u_{k+1}] \leq (1+z_k)u_k - v_k + w_k,$$
$$\sum_{k=0}^{\infty} z_k < \infty \ a.s., \text{ and } \sum_{k=0}^{\infty} w_k < \infty \ a.s.,$$

then u_k converges almost surely (a.s.), and $\sum_{k=0}^{\infty} v_k < \infty$ a.s..

Lemma 4: [27] Let $\{u_k\}$ be a non-negative sequence satisfying the following relationship for all $k \geq 0$:

$$u_{k+1} \leq (1+\gamma_k)u_k + v_k, \tag{16}$$

where sequence $\gamma_k \geq 0$ and $v_k \geq 0$ satisfy $\sum_{k=0}^{\infty} \gamma_k < \infty$ and $\sum_{k=0}^{\infty} v_k < \infty$, respectively. Then the sequence $\{u_k\}$ will converge to a finite value $u > 0$.

Lemma 5: [28] Let $\{u_k\}$ be a non-negative sequence satisfying the following relationship for all $k \geq 0$:

$$u_{k+1} \leq (1-\gamma_{1,k})u_k + \gamma_{2,k} \tag{17}$$

where $\gamma_{1,k} \geq 0$ and $\gamma_{2,k} \geq 0$ satisfying

$$\frac{a_1}{(a_2 k+1)^{b_1}} \leq \gamma_{1,k} \leq 1, \ \frac{a_3}{(a_2 k+1)^{b_2}} \leq \gamma_{2,k} \leq 1,$$

for some $c_1, c_2, c_3 > 0$, $0 \leq b_1 < 1$, and $b_1 < b_2$. Then for all $0 \leq b_0 < b_2 - b_1$, we have $\lim_{k\to\infty}(k+1)^{b_0} u_k = 0$.

B. Proof of Lemma 1

Denote $\bar{\mathbf{y}} = \frac{1}{N}\mathbf{1}^\top \mathbf{y}$ and $\Theta = I - \frac{1}{N}\mathbf{1}\mathbf{1}^\top$, then we have $\|\Theta\| \leq 1$ and

$$\begin{aligned}\left\|\mathbf{y}_{k+1} - \mathbf{1}\bar{\mathbf{x}}_{k+1}\right\|^2 &= \left\|\Theta\mathbf{y}_{k+1} + \mathbf{1}\bar{\mathbf{y}}_{k+1} - \mathbf{1}\bar{\mathbf{x}}_{k+1}\right\|^2 \\ &\leq \left\|\mathbf{y}_{k+1}\right\|^2,\end{aligned} \tag{18}$$

where the inequality follows from (7). Therefore, we can complete the proof by showing $\sum_{k=0}^{\infty} \beta_k \mathbb{E}\left[\|\mathbf{y}_k\|^2\right] < \infty$. Since $\sum_{k=1}^{\infty} \beta_k^2 < \infty$, there obviously exists k_1 such that $\beta_k < 1/\lambda_2$, $\forall k > k_1$. We know that $\sum_{k=0}^{k_1} \beta_k \mathbb{E}\left[\|\mathbf{y}_k\|^2\right]$ is always bounded. Therefore, we only need to focus on proving $\sum_{k=k_1+1}^{\infty} \beta_k \mathbb{E}\left[\|\mathbf{y}_k\|^2\right] < \infty$

Based on (6b), we obtain

$$\begin{aligned}&\mathbb{E}\left[\|\mathbf{y}_{k+1}\|^2\right] \\ =&\mathbb{E}\left[\|A_k\mathbf{y}_k + \mathbf{x}_{k+1} - \mathbf{x}_k\|^2\right] + \beta_k^2\mathbb{E}\left[\|L\mathbf{e}_k\|^2\right] \\ &- \mathbb{E}\left[\beta_k(A_k\mathbf{y}_k + \mathbf{x}_{k+1} - \mathbf{x}_k)^\top L\mathbf{e}_k\right] \\ \leq&\mathbb{E}\left[(\|A_k\mathbf{y}_k\| + \|\mathbf{x}_{k+1} - \mathbf{x}_k\|)^2\right] + \beta_k^2\lambda_M^2 N\sigma^2 \\ \leq&(1+\nu_1)(1-\beta_k\lambda_2)^2\mathbb{E}\left[\|\mathbf{y}_k\|^2\right] + (1+\frac{1}{\nu_1})C^2N\alpha_k^2\beta_k^2 \\ &+ \lambda_M^2 N\sigma^2\beta_k^2,\end{aligned} \tag{19}$$

where the first inequality holds from $\mathbb{E}[\mathbf{e}_k] = 0$ and $\mathbb{E}\left[\|\mathbf{e}_k\|^2\right] \leq N\sigma^2$ in Assumption 6, the last inequality uses $(a+b)^2 \leq (1+\nu_1)a^2 + (1+\frac{1}{\nu_1})b^2$ for any $a, b \in \mathbb{R}$ and $\nu_1 > 0$, and λ_2 and λ_M are the second smallest eigenvalue and the eigenvalue with the largest magnitude of L [29], respectively. By setting $\nu_1 = \beta_k\lambda_2$, we further get

$$\begin{aligned}&\mathbb{E}\left[\|\mathbf{y}_{k+1}\|^2\right] \\ \leq&(1-\beta_k^2\lambda_2^2)(1-\beta_k\lambda_2)\mathbb{E}\left[\|\mathbf{y}_k\|^2\right] + (\alpha_k^2\beta_k^2 + \frac{\alpha_k^2\beta_k}{\lambda_2})C^2N \\ &+ \lambda_M^2 N\sigma^2\beta_k^2 \\ \leq&(1-\beta_k\lambda_2)\mathbb{E}\left[\|\mathbf{y}_k\|^2\right] + (\alpha_k^2\beta_k^2 + \frac{\alpha_k^2\beta_k}{\lambda_2})C^2N + \lambda_M^2 N\sigma^2\beta_k^2\end{aligned} \tag{20}$$

Since $\beta_k\lambda_2 > 0$, we can have the following relationship:

$$\begin{aligned}\mathbb{E}\left[\|\mathbf{y}_{k+1}\|^2\right] \leq&(1+\beta_k^2)\mathbb{E}\left[\|\mathbf{y}_k\|^2\right] - \lambda_2\beta_k\mathbb{E}\left[\|\mathbf{y}_k\|^2\right] \\ &+ (\alpha_k^2\beta_k^2 + \frac{\alpha_k^2\beta_k}{\lambda_2})C^2N + \lambda_M^2 N\sigma^2\beta_k^2\end{aligned} \tag{21}$$

Since $\sum_{k=k_1+1}^{\infty} \alpha_k^2\beta_k < \infty$ and $\sum_{k=k_1+1}^{\infty} \beta_k^2 < \infty$, it is guaranteed that $\sum_{k=k_1+1}^{\infty}[(\alpha_k^2\beta_k^2 + \frac{\alpha_k^2\beta_k}{\lambda_2})C^2N + \lambda_M^2 N\sigma^2\beta_k^2] < \infty$. Thus, we can conclude that $\sum_{k=k_1+1}^{\infty} \beta_k\mathbb{E}\left[\|\mathbf{y}_k\|^2\right] < \infty$ from Lemma 3.

C. Proof of Lemma 2

For $\mathbb{E}\left[\|\mathbf{x}_k\|^2\right]$, we can obtain

$$\begin{aligned}\mathbb{E}\left[\|\mathbf{x}_k\|^2\right] =&\mathbb{E}\left[\|\mathbf{x}_k - \mathbf{x}^* + \mathbf{x}^*\|^2\right] \\ \leq&2\mathbb{E}\left[\|\mathbf{x}_k - \mathbf{x}^*\|^2\right] + 2\left\|\mathbf{x}^*\right\|^2.\end{aligned}$$

Since $\mathbf{x}^*$ is a constant, we will only need to prove the boundedness of $\mathbb{E}\left[\|\mathbf{x}_k - \mathbf{x}^*\|^2\right]$.

According to the dynamics in (6a), it can be verified that the following relationship holds:

$$\begin{aligned}&\mathbb{E}\left[\|\mathbf{x}_{k+1} - \mathbf{x}^*\|^2\right] \\ =&\mathbb{E}\left[\|\mathbf{x}_k - \mathbf{x}^* - \alpha_k\beta_k(G(\mathbf{x}_k, \mathbf{y}_k) - G(\mathbf{x}^*, \mathbf{1}\bar{\mathbf{x}}^*))\|^2\right] \\ =&\mathbb{E}\left[\|\mathbf{x}_k - \mathbf{x}^*\|^2\right] + \alpha_k^2\beta_k^2\mathbb{E}\left[\|G(\mathbf{x}_k, \mathbf{y}_k) - G(\mathbf{x}^*, \mathbf{1}\bar{\mathbf{x}}^*)\|^2\right] \\ &- 2\alpha_k\beta_k\mathbb{E}\left[(\mathbf{x}_k - \mathbf{x}^*)^\top(G(\mathbf{x}_k, \mathbf{y}_k) - G(\mathbf{x}^*, \mathbf{1}\bar{\mathbf{x}}^*))\right] \\ \leq&\mathbb{E}\left[\|\mathbf{x}_k - \mathbf{x}^*\|^2\right] + 4C^2N\alpha_k^2\beta_k^2 \\ &- 2\alpha_k\beta_k\mathbb{E}[(\mathbf{x}_k - \mathbf{x}^*)^\top(G(\mathbf{x}_k, \mathbf{y}_k) - G(\mathbf{x}^*, \mathbf{1}\bar{\mathbf{x}}^*))],\end{aligned} \tag{22}$$

where the inequality is followed by Assumption 5. For the last term of (22), we have

$$\begin{aligned}&- 2\alpha_k\beta_k(\mathbf{x}_k - \mathbf{x}^*)^\top(G(\mathbf{x}_k, \mathbf{y}_k) - G(\mathbf{x}^*, \mathbf{1}\bar{\mathbf{x}}^*)) \\ =&- 2\alpha_k\beta_k(\mathbf{x}_k - \mathbf{x}^*)^\top(G(\mathbf{x}_k, \mathbf{y}_k) - G(\mathbf{x}_k, \mathbf{1}\bar{\mathbf{x}}) \\ &+ G(\mathbf{x}_k, \mathbf{1}\bar{\mathbf{x}}) - G(\mathbf{x}^*, \mathbf{1}\bar{\mathbf{x}}^*)) \\ =&- 2\alpha_k\beta_k(\mathbf{x}_k - \mathbf{x}^*)^\top(G(\mathbf{x}_k, \mathbf{y}_k) - G(\mathbf{x}_k, \mathbf{1}\bar{\mathbf{x}})) \\ &- 2\alpha_k\beta_k(\mathbf{x}_k - \mathbf{x}^*)^\top(\Phi(\mathbf{x}_k) - \Phi(\mathbf{x}^*)) \\ \leq&2L_g\alpha_k\beta_k\left\|\mathbf{x}_k - \mathbf{x}^*\right\|\left\|\mathbf{y}_k - \mathbf{1}\bar{\mathbf{x}}_k\right\| - 2m\alpha_k\beta_k\left\|\mathbf{x}_k - \mathbf{x}^*\right\|^2\end{aligned} \tag{23}$$

$$\leq\frac{1}{\nu_2}L_g^2\alpha_k^2\beta_k\left\|\mathbf{x}_k - \mathbf{x}^*\right\|^2 + \nu_2\beta_k\left\|\mathbf{y}_k - \mathbf{1}\bar{\mathbf{x}}_k\right\|^2,$$

where the first inequality is from Assumptions 2 and 3, and the last inequality uses Yong's inequality. By letting $\nu_2 = 1$, we get

$$\begin{aligned}
&\mathbb{E}\left[\|\mathbf{x}_{k+1}-\mathbf{x}^*\|^2\right] \\
\leq&(1+L_g^2\alpha_k^2\beta_k)\mathbb{E}\left[\|\mathbf{x}_k-\mathbf{x}^*\|^2\right] \\
&+4C^2N\alpha_k^2\beta_k^2+\beta_k\left\|\mathbf{y}_k-\mathbf{1}\bar{\mathbf{x}}_k\right\|^2.
\end{aligned} \tag{24}$$

If $\sum_{k=0}^{\infty}\alpha_k^2\beta_k < \infty$ and $\sum_{k=0}^{\infty}\beta_k^2 < \infty$, there is $\sum_{k=0}^{\infty}\alpha_k^2\beta_k^2 < \infty$ and $\sum_{k=0}^{\infty}\beta_k\mathbb{E}\left[\|\mathbf{y}_k-\mathbf{1}\bar{\mathbf{x}}_k\|^2\right] < \infty$. Thus, based on Lemma 4, $\mathbb{E}\left[\|\mathbf{x}_k\|^2\right]$ will converge to a finite value.

D. Proof of Corollary 1

If ω_1 and ω_2 satisfy the conditions in Corollary 1, then α_k and β_k will satisfy the conditions in Theorem 1 to ensure that $\mathbb{E}\left[\|\mathbf{x}_k-\mathbf{x}^*\|^2\right]$ converges to zero. According to (20), we have the following relationship when k is large enough:

$$\mathbb{E}\left[\|\mathbf{y}_{k+1}\|^2\right] \leq (1-\beta_k\lambda_2)\mathbb{E}\left[\|\mathbf{y}_k\|^2\right]+\eta_k,$$

where $\eta_k=\frac{c_4}{(c_2k+1)^{\omega_3}}$ for some $c_4>0$ and $\omega_3=\min\{2\omega_1+\omega_2, 2\omega_2\}$. Therefore, we have

$$\lim_{k\to\infty}(k+1)^{\omega_4}\mathbb{E}\left[\|\mathbf{y}_k\|^2\right]=0 \tag{25}$$

based on Lemma 5, where $0\leq\omega_4<\omega_3-\omega_2$.

Based on (11) and (18), we have

$$\begin{aligned}
&\frac{\sum_{k=0}^{T}m\alpha_k\beta_k\mathbb{E}\left[\|\mathbf{x}_k-\mathbf{x}^*\|^2\right]}{\sum_{k=0}^{T}\alpha_k\beta_k} &(26)\\
\leq&\frac{\|\mathbf{x}_0-\mathbf{x}^*\|^2-\mathbb{E}\left[\|\mathbf{x}_{T+1}-\mathbf{x}^*\|^2\right]}{\sum_{k=0}^{T}\alpha_k\beta_k}+\frac{4C^2N\sum_{k=0}^{T}\alpha_k^2\beta_k^2}{\sum_{k=0}^{T}\alpha_k\beta_k} \\
&+\frac{\frac{\bar{L}^2}{m}\sum_{k=0}^{T}\alpha_k\beta_k\mathbb{E}\left[\|\mathbf{y}_k\|^2\right]}{\sum_{k=0}^{T}\alpha_k\beta_k}. &(27)
\end{aligned}$$

Equation (25) indicates that $\mathbb{E}\left[\|\mathbf{y}_k\|^2\right]$ is in the same order of α_k^2 or β_k, and thus, the third term of (27) converges to zero with a rate $O\left(\frac{1}{(T+1)^{\omega}}\right)$, where $\omega=\min\{2\omega_1,\omega_2\}$. Moreover, the second term of (27) converges to zero with a rate $O\left(\frac{1}{(T+1)^{\omega_1+\omega_2}}\right)$. Since $\omega_1+\omega_2>\omega$, (26) will decay with a rate ω.

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