21-366 PROJECT REPORT

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Abstract. This paper explores the application of numerical methods to partial differential equations (PDEs), which model how physical systems evolve over space and time. Due to the complexity of many PDEs, analytical solutions are often infeasible, necessitating numerical approximations. We investigate several approaches, including *finite difference*, *finite element*, *spectral*, *Hermite radial basis function*, and *Lax-Wendroff* methods. Our results show that spectral methods excel with smooth periodic functions, while finite difference and finite element methods produce accurate and intuitive solutions for time-evolution and boundary value problems. Advanced techniques like RBF and Lax-Wendroff display strengths in specific contexts, but also highlight challenges related to nonlinearities and complex geometries.

1. Project Scope

One of the most widely used approaches for numerically solving PDEs is the **finite difference method** (FDM), which approximates derivatives by using differences between function values at grid points. This method is relatively easy to implement and is particularly effective for problems with simple geometries and boundary conditions. Another common approach is the **finite element method** (FEM), which divides the domain into smaller subdomains, or elements, and approximates the solution with piecewise functions. FEM is especially powerful in handling complex geometries and varying material properties, making it popular in engineering and structural analysis. **Spectral methods** take a different approach by expanding the solution in terms of global basis functions, such as Fourier or Chebyshev polynomials. These methods are known for their high accuracy, especially when the solution is smooth, but they typically require more computational effort and are best suited for problems with periodic or regular domains.

In addition to these classical methods, several modern or specialized techniques have been developed to tackle specific challenges in PDEs. For instance, **Hermite radial basis function** (RBF) methods are mesh-free approaches that use radial basis functions and their derivatives to approximate solutions. They offer high accuracy and flexibility, particularly in irregular domains, but can be computationally intensive. Another significant method is the **Lax-Wendroff scheme**, which is used for hyperbolic PDEs and provides second-order accuracy in both space and time by incorporating Taylor series expansions and flux terms. This method is valuable for problems involving wave propagation or shock formation, but may suffer from numerical dispersion or instability under certain conditions. Each method brings a different balance of complexity, accuracy, and stability, and the choice often depends on the specific characteristics of the PDE and the domain in which it is defined.

2. Results

We began by applying **1D spectral methods** to the function $f(x) = \sin(x)$, which yielded accurate results with a relatively low error of approximately 0.01 when using $N = 1000$ points. Increasing N further reduced the error, showcasing the method's strength in handling smooth periodic functions. However, performance decreased when we tested the more complex function $f(x) = e^{\sin(x)}$. In this case, the error plateaued at around 0.15 even with larger values of N, indicating limited improvement with resolution and suggesting that spectral methods are more sensitive to the analytic properties of the target function. Extending the approach to two dimensions, we applied spectral methods to the function $f(x, y) = e^{\sin(x)+\cos(y)}$. With $N = 1000$, the method produced a qualitatively correct shape but a relatively high error of 0.45, reflecting the increased computational complexity and challenges of spectral methods in higher dimensions.

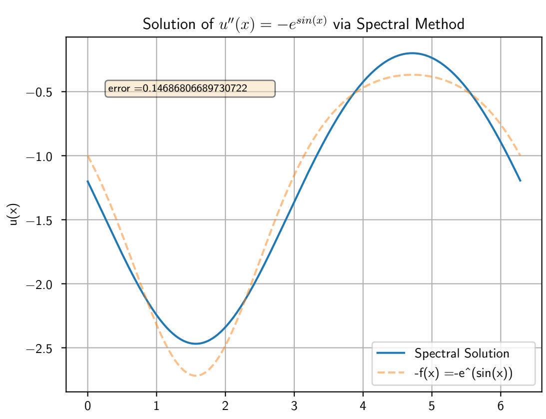
The **finite difference method (FDM)** was tested on the 1D wave and heat equations. For the wave equation with fixed endpoints, we simulated the solution over time steps from 0 to 1 second in intervals of 0.1 seconds. The numerical solution displayed the expected wave behavior, although it appeared somewhat rough—likely due to the simplicity of the discretization scheme. For the 1D heat equation, we used an explicit FDM approach and observed diffusion radiating outward over time, consistent with the physical behavior of heat distribution. Using the **finite element method (FEM)**, we modeled the 1D Poisson equation with $f(x) = 1$, employing 20 equally spaced nodes. The resulting solution had the characteristic inverted parabolic shape, matching the expected analytical solution. For **2D Hermite radial basis function (RBF)** methods and **differential quadrature**, we simulated the diffusion of e^{-r^2} over time using L1 weights to incorporate the time-fractional Caputo derivative. The diffusion profiles at midpoint and final time steps visually matched expected theoretical behavior. Finally, the **Lax-Wendroff scheme** was applied to both the linear advection equation $u_t + au_x = 0$ and the nonlinear **Burgers equation** $u_t + \left(\frac{u^2}{2}\right)_x = 0$. For the linear case, the method successfully captured the propagation of the initial sine wave $\sin(2\pi x)$ using only 200 points. In contrast, the nonlinear Burgers equation posed more difficulties, and the numerical results deviated significantly from the expected solution, highlighting the challenges of applying Lax-Wendroff to nonlinear problems.

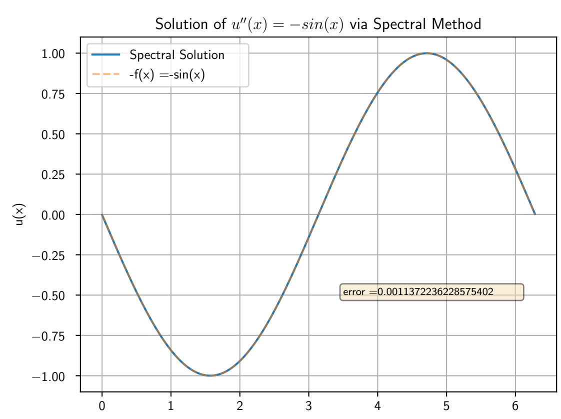
3. Final Thoughts

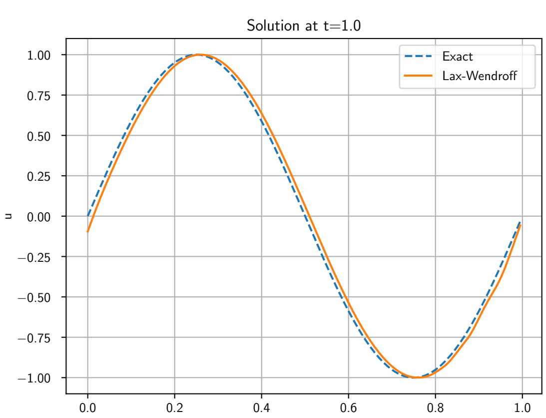
The study of numerical methods for solving partial differential equations reveals the diversity and trade-offs inherent in different approaches. Spectral methods demonstrated high accuracy for smooth, periodic functions in one dimension but struggled with more complex or non-analytic forms, especially in higher dimensions. Finite difference methods offered intuitive, grid-based solutions that effectively captured the expected behavior of both the wave and heat equations, though sometimes with reduced smoothness or accuracy. Finite element methods proved to be well-suited for problems like the Poisson equation, producing results that aligned closely with theoretical expectations even with a modest number of nodes.

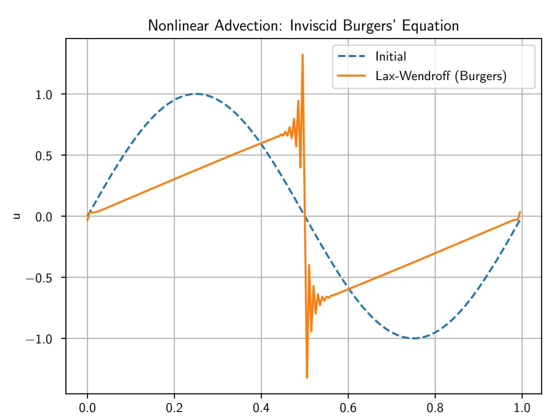
Advanced methods such as Hermite radial basis functions and Lax-Wendroff schemes showed both the potential of and limitations of more specialized techniques. The RBF approach, in conjunction with fractional derivatives, allowed for nuanced diffusion modeling, while Lax-Wendroff was effective for linear advection but less so for nonlinear dynamics like those found in the Burgers equation. Ultimately, the effectiveness of each method depends heavily on the nature of the PDE being solved, the smoothness and geometry of the solution, and the computational resources available. Continued exploration and hybridization of numerical strategies remain promising directions for future research in the field.

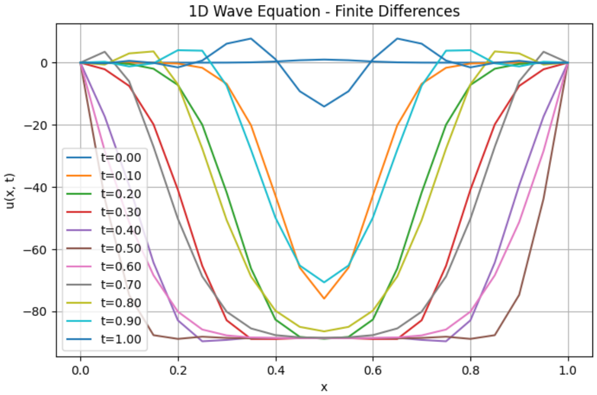
4. Figures

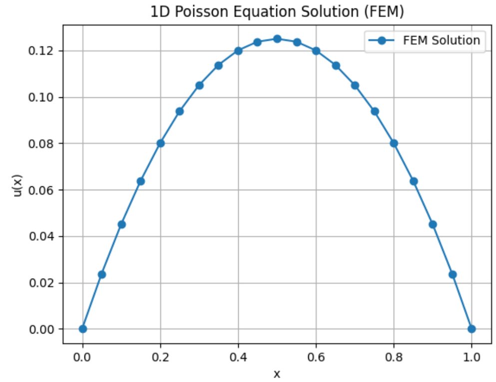


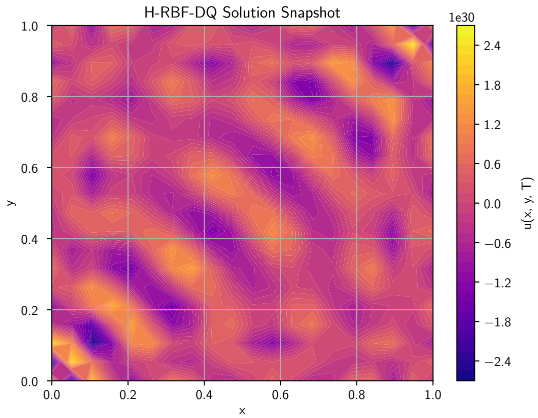


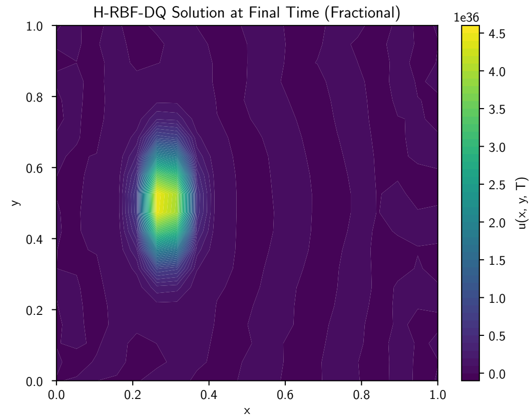


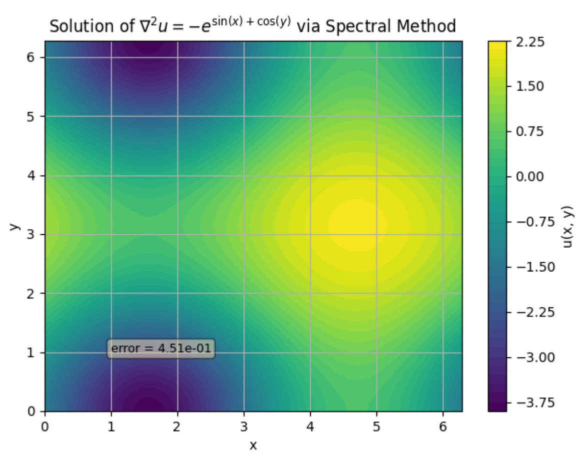


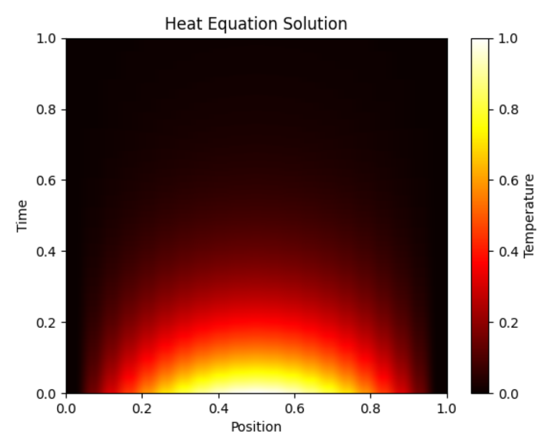












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