Assignment 1 Chunlei Zhou.

Q1 (e) (i)

Let $Y = F(X)$

$Y \in [0,1]$

$P(Y \le y) = P(F(X) \le y) = P(X \le F^{-1}(y))$

$\because P(X \le x) = F(x)$

$\therefore P(X \le F^{-1}(y)) = F(F^{-1}(y)) = y$

$\therefore P(Y \le y) = y$

which is $Y \sim Unif(0,1)$

Let $Y = 1 - F(X)$

$Y \in [0,1]$

$P(Y \le y) = P(1 - F(X) \le y) = P(X \ge F^{-1}(1-y))$

$\because P(X \le x) = F(x) \Rightarrow P(X \ge x) = 1 - F(x)$

$\therefore P(X \ge F^{-1}(1-y)) = 1 - F(F^{-1}(1-y)) = y$

$\therefore P(Y \le y) = y$

which is $Y \sim Unif(0,1)$

$\therefore$ The P-value of the F-test should possess a uniform distribution on $[0,1]$

Q3 (a)

For $X = \begin{pmatrix} x_{11} & \cdots & x_{1q} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nq} \end{pmatrix} \Rightarrow X^T = \begin{pmatrix} x_{11} & \cdots & x_{n1} \\ \vdots & \ddots & \vdots \\ x_{1q} & \cdots & x_{nq} \end{pmatrix}$

$$\therefore X^TX = \begin{pmatrix} \sum_{i=1}^{n} x_{i1}^2 & \cdots & \sum_{i=1}^{n} x_{i1}x_{ik} & \cdots & \sum_{i=1}^{n} x_{i1}x_{iq} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \sum_{i=1}^{n} x_{iq}x_{i1} & \cdots & \sum_{i=1}^{n} x_{iq}x_{ik} & \cdots & \sum_{i=1}^{n} x_{iq}^2 \end{pmatrix}$$

if for each pair of $j \ne k$, $\sum_{i=1}^{n} x_{ij}x_{ik} = 0$

then X^TX is a diagonal matrix,

$\therefore (X^TX)^{-1}$ is a diagonal matrix

$\because \Sigma_{\hat{\beta}} = \delta^2 (X^TX)^{-1}$ $\therefore \Sigma_{\hat{\beta}}$ is a diagonal matrix

$\therefore$ the regression coefficients are mutually uncorrelated.

(b) $\hat{\beta} = (X^TX)^{-1}X^Ty$

$\hat{\beta}' = (X'^TX')^{-1}X'^Ty$

$\because X' = XA$ $\therefore \hat{\beta}' = (A^TX^TXA)^{-1}A^TX^Ty$

$\therefore \hat{y} = X(X^TX^{-1})X^Ty$

$\because XX^{-1} = (X^T)(X^T)^{-1} = 1$

$\therefore \hat{y} = y$

Also, $\hat{y} = XA(A^TX^TXA)^{-1}A^TX^Ty$

$\because A^T(A^T)^{-1} = XX^{-1} = (X^T)^{-1}X^T = AA^{-1} = 1$

$\therefore \hat{y} = y$

$\therefore$ The two models are equivalent.

$\hat{\beta}' = A^{-1}\hat{\beta}$

(c) $\Sigma_{\hat{\beta}'} = \delta^2 ((X')^T X')^{-1}$

$\because (X')^T X' = A^T X^T X A = \begin{pmatrix} n & 0 \\ 0 & \sum_{i=1}^{n} (x_i - \bar{x})^2 \end{pmatrix}$ is a diagonal matrix

$\therefore$ the components of $\hat{\beta}'$ are uncorrelated

$\because \hat{\beta}' = A^{-1} \hat{\beta}$

$\therefore$ the components of $\hat{\beta}$ are uncorrelated.

Q4 (a) Set $\frac{\partial SSE}{\partial \beta_j} = 0 \Rightarrow \sum_{i=1}^{n} x_{ij} (\beta_1 x_{i1} + \cdots + \beta_q x_{iq}) = \sum_{i=1}^{n} y_i x_{ij}$

$\because \left(\sum_{i=1}^{n} x_{ij} x_{i1} \cdots \sum_{i=1}^{n} x_{ij} x_{iq} \right) \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_q \end{pmatrix} = (x_{1j} \cdots x_{nj}) \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

$\Rightarrow (X^T X)\beta = X^T y$

$\Rightarrow \beta = (X^T X)^{-1} X^T y$

(b) When $\begin{cases} \frac{\partial \Lambda}{\partial \beta} = 0 \\ \frac{\partial \Lambda}{\partial \lambda} = 0 \\ d - C^T\beta = 0 \end{cases}$ Λ is the minimum.

$$\begin{cases} \frac{\partial \Lambda}{\partial \beta} = -2X^T y + 2X^T X \beta - C^T \lambda = 0 \\ \frac{\partial \Lambda}{\partial \lambda} = d - C^T \beta = 0 \end{cases}$$

$\Rightarrow \beta (X^T X)^{-1} X^T y + \frac{1}{2} (X^T X)^{-1} C^T \lambda$

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$$\Rightarrow d - C^T(X^TX)^{-1}X^Ty + \frac{1}{2}(X^TX)^{-1}C^T\lambda = 0$$

$$\therefore \lambda = 2[C(X^TX^{-1}C^T)]^{-1}(d - C(X^TX)^{-1}X^Ty)$$

$$\therefore \beta = (X^TX)^{-1}X^Ty + (X^TX)^{-1}C^T[C(X^TX^{-1}C^T)]^{-1}(d - C(X^TX)^{-1}X^Ty)$$

$$\Rightarrow \hat{\beta}_c = \hat{\beta}_u + (X^TX)^{-1}C^T[C(X^TX^{-1}C^T)]^{-1}(d - C\hat{\beta}_u)$$

$\hat{\beta}_c$ is the unconstrained least squares solution.