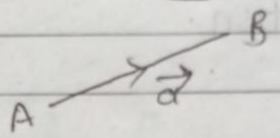
Vectors

Scalar Quantity :→ A quantity which has only magnitude and not related to any direction is called a scalar quantity.
Ex → Mass, length.

Vector Quantity :→ A quantity which has magnitude and also a direction in space is called a vector quantity.
Ex :- Displacement, Velocity.

Representation of Vectors :→

$\overrightarrow{AB}$



magnitude of vector $\vec{a}$ = AB
& direction is represented by a line.

Types of Vectors :→

1) **Null Vector or Zero Vector :→** If initial & terminal points of a vector coincide then it is called a zero vector denoted by $\vec{0}$ or **0**.
· Magnitude = 0 , direction = indeterminate.

2) **Unit Vector :→** A vector whose magnitude is of unit length along any vector $\vec{a}$ is called a unit vector in direction of $\vec{a}$ & denoted by $\hat{a}$.
* $|\hat{a}| = 1$
* Two unit vectors are equal if they have same direction.
* Unit vectors ∥ to x-axis, y-axis & z-axis are denoted by $\hat{i}, \hat{j}$ & $\hat{k}$ respectively.

3) **Reciprocal Vector:-** A vectar whose direction is same as that of a given vector $\vec{a}$ but its magnitude is the reciprocal of the magnitude of given vector $\vec{a}$ is called reciprocal of $\vec{a}$ and is denoted by a^{-1}.

if $\vec{a} = a \cdot \hat{a}$ then $a^{-1} = \frac{1}{a} \hat{a}$

4) **Equal Vector:-** Two non-zero vectors are said to be equal vectors if their magnitude are equal & directions are same.

5) **Negative Vector:-** Same Magnitude but opposite direction.

Ex: $\vec{a} = \vec{PQ}$

$-\vec{a} = \vec{QP}$

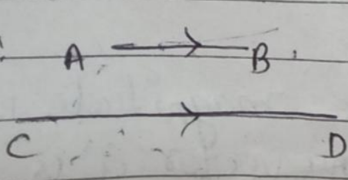
6) **Collinear Vector:-** Two or more non zero vectors are said to be collinear vectors if these are parallel to same line.

7) **Like & Unlike Vectors:-**

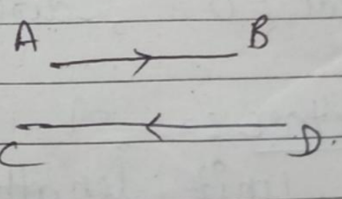
Like ↓ having same direction

Unlike ↓ having opp direction.

Ex:



Ex:



8) **Copalanar Vectors:-** non-zero vectors parallel to the same plane.

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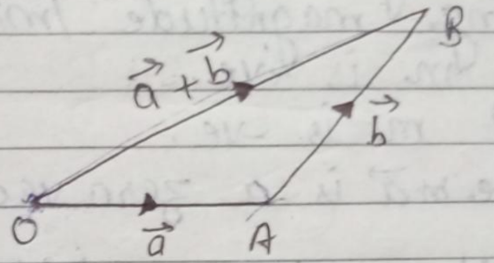
9) Localised Vector ↦ Free Vector

vector drawn ∥ to a given vector through a specified point as initial point. If initial point is not specified.

10) Position Vector: Let O be the origin and A be a point such that $\vec{OA} = \vec{a}$, then we say that the position vector of A is $\vec{a}$.

Addition of vectors ↦



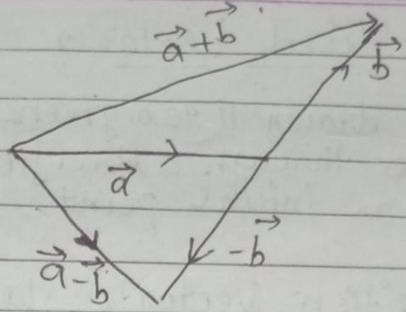
Properties ↦

* Vector addition is commutative i.e. $\vec{a}+\vec{b} = \vec{b}+\vec{a}$
* Vector addition is associative i.e. $\vec{a}+(\vec{b}+\vec{c}) = (\vec{a}+\vec{b})+\vec{c}$.
* $\vec{0}+\vec{a} = \vec{a}+\vec{0} = \vec{a}$. So, the zero vector is additive identity.
* $\vec{a}+(-\vec{a}) = \vec{0} = (-\vec{a})+\vec{a}$. So additive inverse of $\vec{a}$ is $-\vec{a}$.

Addition of Number of Vectors:-

$\vec{a}+\vec{b}+\vec{c}+\vec{d}+\vec{e}+\vec{f}$ ... So on
if $\vec{a}, \vec{b}, \vec{c}$ ... are vectors.

Difference of Vectors:-



Multiplication of a Vector by a scalar:→

Let $\vec{a}$ be any vector & m any scalar, then multiplication of $\vec{a}$ by m is defined as a vector having magnitude $|m||\vec{a}|$ & direction is same if m is +ve.
is different if m is -ve.
If m=0 the $m\vec{a}$ is a zero vector.

$$|2\vec{a}| = |2||\vec{a}| = 2|\vec{a}|$$

magnitude $|-3\vec{a}| = 3|\vec{a}|$ & direction is opp. to $\vec{a}$.

Properties:→

1) $\vec{a}$ & $\vec{b}$ are $\parallel$ iff $\vec{a} = m\vec{b}$ for some non-zero scalar m.

2) $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$ or $\vec{a} = |\vec{a}|\hat{a}$ where $\hat{a}$ is unit vector.

3) $m(n\vec{a}) = (mn)\vec{a} = n(m\vec{a})$

4) Distributive laws:- $(m+n)\vec{a} = m\vec{a} + n\vec{a}$ &
$n(\vec{a}+\vec{b}) = n\vec{a} + n\vec{b}$

5) $\vec{r}, \vec{a}, \vec{b}$ are coplanar iff $\vec{r} = x\vec{a} + y\vec{b}$ for some scalar x, y.

6) $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then $|\vec{r}| = \sqrt{x^2+y^2+z^2}$

7) If $\vec{a}$ & $\vec{b}$ are position vectors & point that divide the line AB in ratio $m:n$

internally $\dfrac{m\vec{b} + n\vec{a}}{m+n}$

externally $\dfrac{m\vec{b} - n\vec{a}}{m-n}$

8) Middle point of AB $= \frac{1}{2}(\vec{a}+\vec{b})$

9) $\vec{AB} = \vec{b} - \vec{a}$

10). Vectors $a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$, $a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$ & $a_3\hat{i} + b_3\hat{j} + c_3\hat{k}$ are coplanar if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

11) $|\vec{a}+\vec{b}| \leq |\vec{a}| + |\vec{b}|$

$|\vec{a}+\vec{b}| \geq |\vec{a}| - |\vec{b}|$

$|\vec{a}-\vec{b}| \leq |\vec{a}| + |\vec{b}|$

$|\vec{a}-\vec{b}| \geq |\vec{a}| - |\vec{b}|$

Scalar Product / Dot Product :→

1) $\vec{a}\cdot\vec{b} = ab\cos\theta$ where $0 \le \theta \le \pi$

2) projection of $\vec{b}$ along $\vec{a}$ $= \dfrac{\vec{b}\cdot\vec{a}}{|\vec{a}|}$

3) Vector $\perp^r$ to both $\vec{a}$ & $\vec{b}$ is given by $\vec{a}\times\vec{b}$

The unit vector $\perp^r$ to both $\vec{a}$ & $\vec{b}$ is given by

$$\hat{n} = \frac{(\vec{a}\times\vec{b})}{|\vec{a}\times\vec{b}|}$$

4) $\vec{a}\cdot\vec{b} = 0 \Rightarrow \vec{a} = 0$ or $\vec{b} = 0$.

5) Component of a vector $\vec{r}$ in direction of $\vec{a}$ $= \left(\dfrac{\vec{r}\cdot\vec{a}}{|\vec{a}|^2}\right)\vec{a}$

$\perp^r$ to $\vec{a}$ $= \vec{r} - \left\{\dfrac{(\vec{r}\cdot\vec{a})}{|\vec{a}|^2}\right\}\vec{a}$

6) If $\vec{a}$ & $\vec{b}$ are non-zero vectors, then
$\vec{a}\cdot\vec{b} = 0 \iff \vec{a} \perp \vec{b}$

7) $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \hat{a}\cdot\hat{b}$

8) $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

$$\left.\begin{aligned} \hat{i}\cdot\hat{j} &= \hat{j}\cdot\hat{i} \\ \hat{j}\cdot\hat{k} &= \hat{k}\cdot\hat{j} \\ \hat{i}\cdot\hat{k} &= \hat{k}\cdot\hat{i} \end{aligned}\right\} = 0$$

9) $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$.

* $\vec{a}$ & $\vec{b}$ are $\perp^r$ if $a_1b_1 + a_2b_2 + a_3b_3 = 0$.

If $\frac{a_1}{a_2} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$ then $\vec{a}$ & $\vec{b}$ are $\parallel$.

Vector or Cross Product of two vectors :→

1) $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$ ($\hat{n}$ → shows direction)

2) $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$

3) If $\vec{a} = \vec{b}$ or if $\vec{a}$ is $\parallel$ to $\vec{b}$ then $\sin\theta = 0$
& $\vec{a}\times\vec{b} = 0$

4) Distributive law: $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c}$
& $(\vec{b}+\vec{c})\vec{a} = \vec{b}\times\vec{a} + \vec{c}\times\vec{a}$

5) $\vec{a}\times\vec{a} = \vec{0}$

6) $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

7) Area of $\parallel$gm ABCD $\frac{1}{2}|\vec{AC}\times\vec{BD}|$

8) Area of $\parallel$gm ABC = $\frac{1}{2}|\vec{AB}\times\vec{AC}|$

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Scalar Triple Product

1) $[\vec{a}\vec{b}\vec{c}] = (\vec{a}\times\vec{b})\cdot\vec{c}$ $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$\downarrow$

volume of parallelopiped.

2) $[abc] = [bca] = [cab]$ but

$[abc] = -[acb]$

3) If any two vectors of $\vec{a}, \vec{b}, \vec{c}$ are equal, then $[abc] = 0$.

4) $(\vec{a}\times\vec{b})\cdot\vec{c} = \vec{a}\cdot(\vec{b}\times\vec{c})$

5) If two vectors in $[\vec{a}\vec{b}\vec{c}]$ are $\parallel$ then $[\vec{a}\vec{b}\vec{c}] = 0$.

6) $\vec{a}, \vec{b}, \vec{c}$ are coplanar if $[\vec{a}\vec{b}\vec{c}] = 0$.

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Matrices

$$A = \begin{array}{c} \xrightarrow{\text{rows}} \\ \begin{bmatrix} a_{11} & a_{12} & --- & a_{1n} \\ a_{21} & a_{22} & --- & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & --- & a_{mn} \end{bmatrix} \end{array} \Big\downarrow \text{columns}$$

size of matrix $= \underbrace{m}_{\text{no. of rows}} \times \underbrace{n}_{\text{no. of columns}}$

Types:-

1) Row Matrix :→ A matrix having only one row is called a row matrix or a row-vector.
Ex:- $[2, 1, 3]$

2) Column Matrix :→ A matrix having only one column is called column matrix or a column-vector.
Ex:- $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

3) Square Matrix :→ A matrix in which the number of rows is equal to the number of columns. say n Matrix called of n-order square matrix.

$$\begin{bmatrix} 2 & 4 & 5 \\ 1 & 2 & 3 \\ 3 & 6 & 2 \end{bmatrix}$$

4) Diagonal Matrix :→ A square matrix is called diagonal matrix if its all elements $a_{ij} = 0$ for $i \neq j$.
a_{ij} = leading diagonal element for $i = j$

Scalar Matrix :→ A square matrix $A = [a_{ij}]_{n \times n}$ is called scalar matrix if

(i) $a_{ij} = 0 \quad \forall\ i \neq j$

(ii) $a_{ii} = c \quad \forall i, c \neq 0$.

Note :→ Diagonal matrix in which all diagonal element are equal is called scalar matrix.

Identity Matrix :→ A square matrix $A = [a_{ij}]_{n \times n}$ is called an identity or a unit matrix, if $a_{ij} = 0 \quad \forall\ i \neq j$

$a_{ii} = 1 \quad \forall\ i$

It is denoted by I_n.

Null Matrix :→ A matrix whose all elements are zero is called null or zero matrix.

Upper Triangular Matrix :→ A square matrix $A = [a_{ij}]$ is called an upper triangular matrix if $a_{ij} = 0 \quad \forall\ i > j$.

Note: All element below diagonal element are zero.

Lower Triangular Matrix :→ A square matrix $A = [a_{ij}]$ is called a lower triangular matrix if $a_{ij} = 0 \quad \forall\ i < j$.

Note :- All element above diagonal are zero.

Strictly triangular if diagonal elements also zero.

Trace of a Matrix: → Sum of diagonal elements.

$$tr(A) = \sum_{i=1}^{n} a_{ii}.$$

Equality of Matrices:-

let $A_{m\times n}$ & $B_{p\times q}$ are two matrix then we say $A = B$ if $m = p$, $n = q$

& $a_{ij} = b_{ij}$ $\forall$ $i, j = 1, 2, \ldots n$

Algebra of Matrices.

1) **Addition:-** $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & \cdots & & a_{mn} \end{bmatrix}$, $B = \begin{bmatrix} b_{11}, & b_{12} & \cdots & b_{1n} \\ \vdots & & & \\ b_{m1} & \cdots & & b_{mn} \end{bmatrix}$,

$$A + B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ \vdots & & & \\ a_{m1}+b_{m1} & \cdots & & a_{mn}+b_{mn} \end{bmatrix}$$

two matrices can be added when both are of same order.

Property-

1) Commutative : $A+B = B+A$

2) Associative : $(A+B)+C = A+(B+C)$.

3) Identity : $A + O = A = O + A$

4) Inverse : $A + (-A) = O = (-A) + A$

5) Cancellation Law Ex- $A+B = A+C$ $\Rightarrow B = C$

Scalar Multiplication :->

let k be any number & A be any matrix.

$$kA = [ka_{ij}]_{m\times n}.$$

Properties :-> If $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{m\times n}$ & k, l are scalars, then.

#.
(i) $k(A+B) = kA + kB.$
(ii) $(k+l)A = kA + lA$
(iii) $(kl)A = k(lA) = l(kA).$
(iv) $(-k)A = -(kA) = k(-A)$
(v) $1A = A$
(vi) $(-1)A = -A$

Multiplication :-> Easily understandable through

ex:-

$$A = \begin{bmatrix} 2 & 4 \\ 1 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 2 \\ 1 & 9 \end{bmatrix}$$

i th row; j th column

$$AB = \begin{bmatrix} 6+4 & 4+36 \\ 3+6 & 2+54 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 40 \\ 9 & 56 \end{bmatrix}$$

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Properties:→

1) Multiplication is not commutative $AB \neq BA$.

2) Multiplication is associative.
$$(AB)C = A(BC)$$
whenever both sides are defined.

3) Distributive:-
① $A(B+C) = AB + AC$.
② $(A+B)C = AC + BC$

4) $$I_m A = A = A I_n$$

5) Product of two matrices can be null while neither of them is null matrix.

6) Product of null matrix & any matrix is null.

7) If $AB = 0$ it does not $\Rightarrow$ BA is also 0.

Idempotent Matrix:→ When $A^2 = A$

Nilpotent Matrix:→ When $A^n = 0$ for $n =$ smallest positive int

Transpose of a Matrix:→ A^T is obtained from A by changing its rows into columns and columns into rows.

Ex: $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ $A^T = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$

Properties :→

(i) $(A^T)^T = A$

(ii) $(A+B)^T = A^T + B^T$

(iii) $(KA)^T = K(A^T)$.

(iv) $(AB)^T = B^T A^T$ {Reversal law}.

Symmetric Matrix

If $A^T = A$

Skew Symmetric Matrix

If $A^T = -A$.

Singular Matrix :→ A square matrix is a singular matrix if its determinant is zero. Otherwise it is Non-singular.

Inverse of a Matrix :→ A square matrix of order n is invertible if $\exists$ a square matrix B of same order such that

$$AB = I_n = BA$$
$$\Rightarrow A^{-1} = B.$$

Orthogonal Matrix :→ A square matrix A is called an orthogonal matrix if
$AA^T = A^T A = I$.

Rank of a Matrix :→ A number r is said to be the rank of $m \times n$ matrix A if

(a) every submatrix of order $(r+1)$ or more is singular

(b) $\exists$ atleast one square submatrix of order r which is non-singular

OR:-

Number of non-zero rows formed in Echelon row matrix of a given matrix.

Echelon form of a Matrix:→

A matrix A is said to be in Echelon form if either A is the null matrix or A satisfies the following conditions:

(i). Every non-zero row in A precedes every zero row.

(ii) The number of zeros before the first non-zero element in a row is less than the number of such zeroes in the next row.

leading coefficient

Ex:- $\begin{bmatrix} 0 & \underline{3} & 2 & 1 \\ 0 & 0 & \underline{2} & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is echelon form & rank of matrix = 2.

$\begin{bmatrix} 0 & 2 & 3 \\ 0 & 0 & 5 \\ 0 & 0 & 3 \end{bmatrix}$ is not an echelon form as no. of zeroes in R_2 is not less than the no. of zeroes in R_3.

Note:- When we multiply a matrix by a row vector we get a row vector.

When we multiply a matrix by column vector we get a column vector.

System of Simultaneous Linear Equation

$$a_{11}x_1 + a_{12}x_2 \cdots a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 \cdots a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 \cdots a_{mn}x_n = b_m$$

This system can be written as.
$AX = B$ where.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$$

Coefficient matrix

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}$$

Solution:- A set of values of the variables $x_1, x_2, \ldots x_n$ which simultaneously satisfy all the equation is called a solution of the system of equation.

If the system of equations has one or more solutions, then it is said to be a consistent system of equation otherwise it is an inconsistent system of equations.

Homogenous System:-> A system of eq^n $AX=B$ is called a homogenous system, if $B=0$
Otherwise

It is non-Homogenous System.

Solution of Non-Homogeneous System :-

Matrix method :-> let $AX = B$ be a system of n linear eqn & A^{-1} exists.

$$AX = B$$
$$\Rightarrow A^{-1}(AX) = A^{-1}B$$
$$X = A^{-1}B.$$

If A is non-singular $\Rightarrow |A| \neq 0$ then unique solⁿ.

If A is singular $\Rightarrow |A| = 0$ then infinite many solⁿ OR No solution.

Rank Method :-> Consider a system of m simultaneous linear equations in n unknowns $x_1, x_2, -- x_n$ given by

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \quad --- \quad a_{1n}x = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \quad --- \quad a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \quad --- \quad a_{mn}x_n = b_m.$$

$$AX = B$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & -- & a_{1n} \\ a_{21} & a_{22} & - & - & a_{2n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & -- & a_{mn} \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

first of all we created augmented matrix

$$[A:B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & : b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & : b_2 \\ \vdots & & & & & \vdots \\ a_{m1} & a_{m2} & a_{m2} & \cdots & a_{mn} & : b_m \end{bmatrix}$$

Matrix Method

let $AX=0$ be a homogenous system of n-linear equations with n-unknowns. If A is a non-singular matrix, then the system of equation has a unique solution $X=0$ or $x_1=x_2=\cdots=x_n$. This solution is known as a trivial solution

$AX=0$ of n homogenous linear equations in n unknowns has non-trivial solution iff the coefficient matrix A is singular.

Rank Method: In case of a homogenous system of linear eqn the rank of the augmeted matrix is always same as that of the coefficient matrix. So, a homogenous system of linear eqn is always consistent.

If $r(A)=n=$ no. of variables, then $AX=0$ has a unique solun. $X=0$ i.e, $x_1=x_2=\cdots x_n=0$

If $r(A)=r<n$ (= Number of variables), then the system of equations has infinitely many solutions.

Row operations and Gauss Eliminations.

let $Ax = B$ be a system of m linear eqn n unknowns then to find the solution through augmented matrix we apply some Row Operation. that are -

(1) Multiply one row by a non-zero number.

(2) Add one row to another.

(3) Interchange rows.

(4) Multiply by scalar then add two rows.

* Two matrices are row equivalent if one can be obtained from other by successively applying ERO's.

Reduced Row Matrix

In the form of
$$\begin{bmatrix} a & 0 & 0 & - & - \\ 0 & b & 0 & & - \\ 0, & 0 & c & & - \end{bmatrix}.$$
where leading coefficient should be 1 & follow these rules.

~~This~~ ~~th~~

1. All non-zero rows precced zero rows when both types are contained in matrix.

Convert into reduced row form :-

Que 1

$$\begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1$, $R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to R_1 - 2R_2$

$$\begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Ques 2

$$\begin{bmatrix} 0 & 1 & 3 & -2 \\ 2 & 1 & -4 & 3 \\ 2 & 3 & 2 & 1 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 2 & 1 & -4 & 3 \\ 0 & 1 & 3 & -2 \\ 2 & 3 & 2 & 1 \end{bmatrix}$$

2. The first non-zero element of each non-zero row is unity.

3. When the first non-zero element of row appears in column c then all other elements in column c are zero.

4. The first non-zero element of any non-zero row appears in a later column then the first non-zero element of any preceding row.

Some Questions :-

Ques 1:- If $A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ then find A^2, A^3.

$$A^2 = A \cdot A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Que2:- Solve $2x+3y-z=0$ —— ①

$x+y+z=0$ —— ②

Since number of variable > no. of eqn.

$\therefore$ there is one independent & two dependent variables.

add eq ① & ② we get

$$3x+4y=0$$

$$4y=-3x$$

$$y=\frac{-3x}{4}$$

put $x=4$, $y=-3$

the put in eq ①

$$2\times 4-9-z=0$$

$$8-9-z=0$$

$$-1-z=0$$

$$z=-1$$

Que find Echolean form of Matrix:-

$$\begin{bmatrix} 3 & -2 & 1 & 2 \\ 1 & 1 & -1 & -1 \\ 2 & -1 & 3 & 0 \end{bmatrix}$$

$R_1 \to R_1 - R_3$, $R_2 \to R_3 - 2R_2$

$$\begin{bmatrix} 1 & -1 & -2 & 2 \\ 0 & -3 & 5 & 2 \\ 2 & -1 & 3 & 0 \end{bmatrix}$$

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$R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & -1 & -2 & 2 \\ 0 & -3 & 5 & 2 \\ 0 & 1 & 7 & -4 \end{bmatrix}$$

$R_2 \to \frac{1}{-3} R_2$

$$\begin{bmatrix} 1 & -1 & -2 & 2 \\ 0 & 1 & -5/3 & -2/3 \\ 0 & 1 & 7 & -4 \end{bmatrix}$$

$R_3 \to R_3 - R_2$ $\quad -4+\frac{2}{3}$

$$\begin{bmatrix} 1 & -1 & -2 & 2 \\ 0 & 1 & -5/3 & -2/3 \\ 0 & 0 & \frac{26}{7} & \frac{-10}{3} \end{bmatrix}$$

Ques Find row equivalent matrix in row echolean form.

(i) $$\begin{bmatrix} 6 & 3 & -4 \\ -4 & 1 & -6 \\ 1 & 2 & -5 \end{bmatrix}$$

$R_1 \to \frac{R_1}{6}$

$$\begin{bmatrix} 1 & \frac{1}{2} & -\frac{2}{3} \\ -4 & 1 & -6 \\ 1 & 2 & -5 \end{bmatrix}$$ $\quad -6 - \frac{8}{3}$

$-5 + \frac{2}{3}$

$R_3 \to R_3 - R_1$, $R_2 \to R_2 + 4R_1$

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$$\begin{bmatrix} 1 & 1/2 & -2/3 \\ 0 & 3 & -\frac{26}{3} \\ 0 & \frac{3}{2} & \frac{-13}{3} \end{bmatrix}$$

$R_2 \to \frac{R_2}{3}$

$$\begin{bmatrix} 1 & 1/2 & -2/3 \\ 0 & 1 & -26/9 \\ 0 & 3/2 & \frac{-13}{3} \end{bmatrix}$$

$\frac{-13}{3} + \frac{\cancel{3}}{\cancel{2}} \times \frac{26}{\cancel{9}3}$ 13

$R_3 \to R_3 - 3R_2$

$$\begin{bmatrix} 1 & 1/2 & -2/3 \\ 0 & 1 & -26/9 \\ 0 & 0 & 0 \end{bmatrix}$$

(ii) $$\begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 2 & 4 & 1 & -2 & 3 \\ 3 & 6 & 2 & -6 & 5 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 3 & 6 & 1 \\ 0 & 0 & 5 & -12 & 2 \end{bmatrix}$$

$R_3 \to R_3 + 5R_1$

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 3 & 6 & 1 \\ 0 & 0 & 0 & -2 & 7 \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 2 & 1 & -4 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 2 & 6 & -2 \end{bmatrix}$$

$R_1 \to \frac{R_1}{2}$

$$\begin{bmatrix} 1 & 1/2 & -2 & 3/2 \\ 0 & 1 & 3 & -2 \\ 0 & 2 & 6 & -2 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} 1 & 1/2 & -2 & 3/2 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$R_1 \to R_1 - \frac{R_2}{2}$ $\quad -2-\frac{3}{2} \quad \frac{3}{2}$

$$\begin{bmatrix} 1 & 0 & -7/2 & 1/2 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$R_1 \to R_1 - \frac{R_3}{4}$, $R_2 \to R_2 + R_3$

$$\begin{bmatrix} 1 & 0 & -7/2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$R_3 \to \frac{R_3}{2}$

$$\begin{bmatrix} 1 & 0 & -7/2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Que3:-

$$\begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 2 & -5 & 3 & 1 \\ 4 & 1 & 1 & 5 \end{bmatrix}$$

$R_3 \to R_3 - 2R_1 \quad , R_4 \to R_4 - 4R_1$

$$\begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 0 & -11 & 5 & -3 \\ 0 & -11 & 5 & -3 \end{bmatrix}$$

$R_4 \to R_4 \mp R_3$

$$\begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 0 & -11 & 5 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to \frac{R_2}{11}$

$$\begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & \frac{-5}{11} & \frac{3}{11} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$-1 + \frac{15}{11}$

$2 - \frac{3}{11}$

$R_1 \to R_1 - 3R_2$

$$\begin{bmatrix} 1 & 0 & \frac{4}{11} & \frac{19}{11} \\ 0 & 1 & -\frac{5}{11} & \frac{3}{11} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Ques find Rank.

(i) $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

Rank → 2

(ii) $\begin{bmatrix} 2 & 0 & 0 \\ -5 & 1 & 2 \\ 3 & 8 & -7 \end{bmatrix}$

$R_2 \to \frac{R_2}{2}$

$$\begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 2 \\ 3 & 8 & -7 \end{bmatrix}$$

$R_2 \to R_2 + 5R_1$

$R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 8 & -7 \end{bmatrix}$$

Rank - 3

Vector Space:

let $F(+,\cdot)$ is a field. The elements of F will be called scalars. Let V be a non-empty set whose elements will be called vectors.

Here V is a vector space over the field F, if it holds.

<1> $(V,+)$ is an abelian group.

(i) $(V,+)$ is closed
$\forall \alpha, \beta \in V \quad \alpha+\beta \in V$.

(ii) $(V,+)$ holds associative law
i.e. $(\alpha+\beta)+\gamma = \alpha+(\beta+\gamma) \quad \forall \alpha, \beta, \gamma \in V$

(iii) $(V,+)$ holds additive Identity.
i.e. $\alpha+0 = \alpha+0+\alpha$ here 0 is addi. Identity

(iv) $(V,+)$ holds Inverse
i.e. $\alpha+(-\alpha) = 0 = (-\alpha)+\alpha$
where $-\alpha$ is inverse of α. $\forall \alpha, -\alpha \in V$

(v) $(V,+)$ holds commutative law
i.e. $\alpha+\beta = \beta+\alpha \quad \forall \alpha, \beta \in V$.

<2> V is closed under scalar Multiplication
if $\alpha \in V$, $a \in F \Rightarrow a\alpha \in V$.

<3> $a,b \in F$ & $\alpha, \beta \in V$
(i) $a(\alpha+\beta) = a\alpha + a\beta$
(ii) $(a+b)\alpha = a\alpha + b\alpha$
(iii) $a(b\alpha) = (a\alpha)b = (ab)\alpha$
(iv) $1\cdot\alpha = \alpha$. $(1 \in F)$

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Example:- Set C of all complex Numbers over R
$C(R)$
Set of Real numbers over R
$R(R)$
$R(C)$ is not a vector space.

Subspace :- let V be a vector space over the field F and let $W \subseteq V$ then W is called a subspace of V if W itself is a vector space over F w.r.t. operation of vector addition & scalar multiplication in V.

Some properties:-

(i) for $\alpha, \beta \in V$ & $a \in F$
if $\alpha \pm \beta \in W$
$a\alpha \in W$ then W is called subspace of V.

(ii) for $\alpha, \beta \in V$ & $a, b \in F$
if $a\alpha + b\beta \in W$ then W is called subspace of V.

(iii) Intersection of two subspace is also a subspace.

iv) Union of two subspaces is a subspace iff one is contained in other.

Linear Dependence:- let $V(F)$ be a vector space. A finite set $\{\alpha_1, \alpha_2 \dots \alpha_n\}$ of vectors of V is said to be lineary dependent if $\exists$ $a_1, a_2 \dots a_n \in F$ not all of them are 0. for

$$a_1\alpha_1 + a_2\alpha_2 \dots a_n\alpha_n = 0$$

Linear Independence:- let $V(F)$ be a vector space. A finite set $\{\alpha_1, \alpha_2, \dots \alpha_n\}$ of vectors is said to be linearly independent if

$$a_1\alpha_1 + a_2\alpha_2 \dots a_n\alpha_n = 0$$

$(a_i \in F, \quad 1 \le i \le n)$

$\Rightarrow$ all $a_i = 0 \quad (1 \le i \le n)$.

Note:-

Any Infinite set of vectors of V is said to be linearly independent if its every finite subset is linearly independent, otherwise it is linearly dependent.

In terms of matrices. for a matrix A of some elements is called linearly dependent if $|A| = 0$

Called linearly independent if $|A| \neq 0$.

for ex:- ~~let~~ $(1,2,3), (4,1,5), (-4,6,2)$

let a, b, c are scalars

$$a(1,2,3) + b(4,1,5) + c(-4,6,2) = (0,0,0).$$

$$(a+4b-4c,\ 2a+b+6c,\ 3a+5b+2c) = (0,0,0)$$

$$a+4b-4c = 0$$
$$2a+b+6c = 0$$
$$3a+5b+2c = 0.$$

$$\begin{bmatrix} 1 & 4 & -4 \\ 2 & 1 & 6 \\ 3 & 5 & 2 \end{bmatrix}$$

$$|A| = 1[2-30] + 4[18-4] - 4[10-3]$$
$$= -28 + 56 - 28 \quad = 56-56 = 0$$

$\Rightarrow$ $|A| = 0$

$\therefore$ These vectors are lineary dependent.

for ex:- Prove set $\{(1,0,0), (0,1,0), (1,1,1)\}$ is linearly independent.

let $a, b, c \in F$.

$$a(1,0,0) + b(0,1,0) + c(1,1,1) = (0,0,0).$$
$$(a+c, b+c, c) = (0,0,0).$$

$c = 0$
$b+c = 0$
$\Rightarrow b = 0$
$a+c = 0$
$\Rightarrow a = 0$

all scalars are zero
$\therefore$ set is linearly Independent.

OR

$a+c=0$
$b+c=0$
$c=0$.

coefficient matrix
$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

$|A| = 1[1-0] = 1 \neq 0$
$\Rightarrow$ L.I.

Basis of Vector Space :→

A set of linearly independent vectors that spans all of V is called basis.

Standard Basis for $\mathbb{R}^3$ $\begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}$

In general $\left\{\begin{bmatrix}1\\0\\0\\\vdots\end{bmatrix}\begin{bmatrix}0\\1\\0\\\vdots\end{bmatrix}\begin{bmatrix}0\\0\\1\\\vdots\end{bmatrix}\cdots\right\}$ is called S.B of $\mathbb{R}^n$

Ques:→ Verify if $\vec{v_1}, \vec{v_2}, \vec{v_3}$ forms basis for $\mathbb{R}^3$ or not

$$\vec{v_1} = \begin{bmatrix}1\\1\\0\end{bmatrix}, \quad \vec{v_2} = \begin{bmatrix}1\\0\\1\end{bmatrix}, \quad \vec{v_3} = \begin{bmatrix}0\\1\\1\end{bmatrix}$$

$$a\vec{v_1} + b\vec{v_2} + c\vec{v_3} = 0$$

$$a\begin{bmatrix}1\\1\\0\end{bmatrix} + b\begin{bmatrix}1\\0\\1\end{bmatrix} + c\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}a+b\\a+c\\b+c\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$\Rightarrow b - c = 0$

$b + c = 0$

$2b = 0 \Rightarrow b = 0$

$c = 0, \ a = 0, \ b = 0.$

$\therefore \vec{v_1}, \vec{v_2}, \vec{v_3}$ are ~~the~~ Linearly Independent.

$\Rightarrow$ It forms a Basis.

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Dimension of Finitely Generated Vector Space →

The number of elements in any basis of a vector space V(F) is called dimension of the vector space V(F) & denoted by dim V.

Ex Dimension of set of all matrices of order 2x2 is 4.

Other Questions

(1) Prove that set $S = \{(1,2,1), (2,1,0), (1,-1,2)\}$ forms a basis for $V_3(F)$.

first of all we have to prove it is L.I set.

So, $a(1,2,1) + b(2,1,0) + c(1,-1,2) = (0,0,0)$

$(a+2b+c,\ 2a+b-c,\ a+2c) = (0,0,0)$.

~~$a+2c=0$~~
~~$2c=-a$~~

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix}$$

$|A| = 1[2+0] + 2[-1-4] + 1[-1]$

$= 2 - 10 - 1$

$= 2 - 11$

$= -9$

$\Rightarrow |A| \neq 0$.

system has trivial solution

$\Rightarrow c = 0$

$\Rightarrow a = 0$ also $b = 0$.

$\therefore$ S forms basis for $V_3(F)$

Null Space of a Matrix :→

Null of space of $n \times n$ matrix is written as Nul A is set of all solution of homogenous equation.

$$Ax = 0$$

In Set Notation

$$\text{Nul } A = \{x : x \text{ is in } \mathbb{R}^n \ \& \ Ax = 0\}$$

Ques:- Let A be a matrix

$$A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$$ Check $u = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$ belongs to Nul A or Not ;

$$Au = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 5-9+4 \\ -25+27-2 \end{bmatrix} = \begin{bmatrix} 9-9 \\ 27-27 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Yes u belongs to Nul A.

Theorem :→ Prove that Null space of $m \times n$ matrix A is a subspace of $\mathbb{R}^n$, equivalently set of all solutions to $Ax = 0$ of m homogeneous linear equation in n-unknowns is a subspace of $\mathbb{R}^n$.

Proof:- Nul A is a subset of $\mathbb{R}^n$.

$\because$ A has n-colums. We show that Nul A satisfies 3 properties of a subspace of course 0 is in Null A.

Let u, v be any two vectors in Null A. Then
$Au = 0$ & $Av = 0$

To show that $u+v$ is in Null A,
$A(u+v) = 0$
$Au + Av = 0 + 0 = 0$
$\Rightarrow A(u+v) = 0$
$\Rightarrow u+v$ is in Null A.

let c be any scalar
then $A(cu) = c(Au) = c(0) = 0$
$\Rightarrow cu \in$ Null A
$\Rightarrow$ Nul A is a subspace of $\mathbb{R}^n$.

Column Space :->

Column of an $m \times n$ matrix A written as col A is set of all linear combinations of column of A.
If $A = [a_1, a_2 \dots a_n]$ then Col A =

$$\text{span}\{a_1, a_2 \dots a_n\}$$

$$\text{Dim}(\text{Col} A) = m.$$

Theorem:- Column Space of $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

i.e. $\text{Col} A = \{b : b = Ax \text{ for some } x \text{ is in } \mathbb{R}^n\}$

If $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & 5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$ is a column space

(i) of suspace of $\mathbb{R}^k$ then what is k?

k = Number of Rows $\to 3$.

(ii) If Null space is supspace of $\mathbb{R}^k$ then what is k?

k = Number of column $\to 4$

Ques Find non-zero vector in col (A) & non-zero vector in null A.

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$$

let u be the vector in Null A

$$A \cdot u = 0$$

$$\begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2x + 4y - 2z + w \\ -2x - 5y + 7z + 3w \\ 3x + 7y - 8z + 6w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 2 & 4 & -2 & 1 & 0 \\ -2 & -5 & 7 & 3 & 0 \\ 3 & 7 & -8 & 6 & 0 \end{array}\right]$$

$R_2 \rightarrow R_1 + R_2$

$$\left[\begin{array}{cccc|c} 2 & 4 & -2 & 1 & 0 \\ 0 & -1 & 5 & 4 & 0 \\ 3 & 7 & -8 & 6 & 0 \end{array}\right]$$

$R_1 \rightarrow R_1/2$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1/2 & 0 \\ 0 & -1 & 5 & 4 & 0 \\ 3 & 7 & -8 & 6 & 0 \end{array}\right]$$

$R_3 \longrightarrow R_3 - 3R_1$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1/2 & 0 \\ 0 & -1 & 5 & 4 & 0 \\ 0 & 1 & -5 & 9/2 & 0 \end{array}\right]$$

$R_3 \longrightarrow R_2 + R_3$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1/2 & 0 \\ 0 & -1 & 5 & 4 & 0 \\ 0 & 0 & 0 & 17/2 & 0 \end{array}\right]$$

$$\frac{17}{2} w = 0$$

$$\boxed{w = 0}$$

$$-y + 5z + 4w = 0$$

$$-y + 5z = 0$$

$$5z = y$$

$$\boxed{z = 1}, \boxed{y = 5}$$

$x + 2y - z - \tfrac{1}{2}w \Rightarrow 0$

$x + 10 - 1 \Rightarrow 0$

$x = 9$

Vector in Null A

$$\begin{bmatrix} 9 \\ 5 \\ 1 \\ 0 \end{bmatrix}$$

let $v = \begin{bmatrix} a \\ b \\ c \\ \cancel{d} \end{bmatrix}$ in Column A.

Now we know that $Ax = b \quad \forall x \in R$

$$\begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ \cancel{d} \end{bmatrix}$$

$$\begin{bmatrix} 2-4-4-2 \\ -2+5+14-6 \\ 3-7-16-12 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ \cancel{d} \end{bmatrix} \Rightarrow \begin{bmatrix} -8 \\ 11 \\ -32 \end{bmatrix}$$

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__Ques:-__ Determine if u is in Nul A could u be in Col A. $u = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}$

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$$

If ~~A~~ u is in Null A then $Au = 0$.

$$\begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix} =$$

$$\begin{bmatrix} 6-8+2+0 \\ -6+10-7+0 \\ 9-14+8+0 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix}$$

It is not in Nul A

Since ~~A·u~~ = $\cancel{\begin{bmatrix} 0 \\ -3 \\ -3 \end{bmatrix}}$

~~& u∈~~

$$\begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} =$$

It can't belong to Col(A) as its dimension is 4 & dim. of vector in Col(A) ~~mo~~ should be zero.

Ques:→ Determine if $\vec{w} = \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix}$ is in Null A where

$$A = \begin{bmatrix} 3 & -5 & -3 \\ 6 & -2 & 0 \\ -8 & 4 & 1 \end{bmatrix}$$

$$A \cdot w = \begin{bmatrix} 3 & -5 & -3 \\ 6 & -2 & 0 \\ -8 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 - 15 + 12 \\ 6 - 6 + 0 \\ -8 + 12 - 4 \end{bmatrix} = \begin{bmatrix} 15 - 15 \\ 6 - 6 \\ 12 - 12 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Yes it is belong to Null A.

Ques:→ Determine if w is in Col A

$$A = \begin{bmatrix} -6 & 12 \\ -3 & 6 \end{bmatrix}, \quad w = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$A = \begin{bmatrix} -6 & 12 \\ -3 & 6 \end{bmatrix}$, $w = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

$x = \begin{bmatrix} x \\ y \end{bmatrix}$ then x should be belong to Real Num

$A \cdot x = w$

$$\begin{bmatrix} -6 & 12 \\ -3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -6x + 12y \\ -3x + 6y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$-6x + 12y = 2$$

$$-x + 2y = 1 \quad — (1) \times 3$$

$$-3x + 6y = 1 \quad — (2)$$

There is no real values for ~~w~~ x & y.

$\therefore$ w does not belong to A.

<u>for $m \times n$ Matrix.</u>

Null A	Col A
1. Null Space is subspace of $\mathbb{R}^n$	1. Col. Space is subspace of $\mathbb{R}^m$.
2. It takes time to find vectors of Null A. Row operations $[A\,0]$ req	2. Easy to find vectors in Col A
3. No relations between Nul A & entries o/A	3. There is a relation b/w Col A & entries o/A

4. Vector v in Nul A has property $Av = 0$	4. Vector v in Col A has property that $Ax = v$ is consistent
5. Easy to tell if vector is in Null A	5. May take time to tell if v is in Col A.
6. Nul A = {0} if $Ax = 0$ has only the trivial solution	6. Col A = $\mathbb{R}^m$ if $Ax = b$ has solⁿ for every b in $\mathbb{R}^n$.

Inverse Matrix:- A Matrix A is called invertible if ∃ matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$

$$(A^{-1})^{-1} = A$$

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ If $ad - bc \neq 0$ then A is invertible & $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Example:- Check C is the inverse of A or not

$$A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}, \quad C = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}.$$

$$AC = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}\begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -14+15 & -10+10 \\ 21 \cancel{+} -21 & 15-14 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Ques:- Find inverse of $A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$

$|A| = 18 - 20 = -2.$

Cofactors $A_{11} = 6$ Adj $A = \begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix}$

$A_{12} = -5$

$A_{21} = -4$

$A_{22} = 3$

$$A^{-1} = \frac{\text{Adj } A}{|A|} = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix}$$

Theorem:- If A is an invertible $n \times n$ matrix then for each b in $\mathbb{R}^n$, $a = A^{-1}b$

Proof:- Take any b in $\mathbb{R}^n$. A solution exists because of $A^{-1}b$ is substituted for x.

Then $Ax = A(A^{-1}b) = Ib = b$ so $A^{-1}b$ is a solution.

To prove that solution is unique, show that if u is any solution then u be $A^{-1}b$.

Indeed if $Au = b$

$$A^{-1}Au = A^{-1}b.$$
$$Iu = A^{-1}b$$
$$\boxed{u = A^{-1}b}$$

Ques:-> Use inverse of matrix to solve the system of equation.

$$3x_1 + 6x_2 = 3 \quad \text{——} (1)$$
$$5x_1 + 6x_2 = 7 \quad \text{——} (2).$$

$A = \begin{bmatrix} 3 & 6 \\ 5 & 6 \end{bmatrix} \Rightarrow B = \begin{bmatrix} 3 \\ 7 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ \ldots \end{bmatrix}$

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$AX = B$ $\quad A_{11} = 6$

$X = A^{-1}B.$ $\quad A_{12} = -5$

$A_{21} = -6$

$A_{22} = 3.$

$$\text{Adj } A = \begin{bmatrix} 6 & -6 \\ -5 & 3 \end{bmatrix} \qquad |A| = 18 - 30 = -12.$$

$$A^{-1} = \begin{bmatrix} 6/-12 & -6/-12 \\ -5/-12 & 3/-12 \end{bmatrix} = \begin{bmatrix} -1/2 & 1/2 \\ 5/12 & -1/4 \end{bmatrix}$$

$$X = A^{-1}B = \begin{bmatrix} -1/2 & 1/2 \\ 5/12 & -1/4 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} -3/2 + 7/2 \\ 15/12 - 21/12 \end{bmatrix} = \begin{bmatrix} 4/2 \\ -6/12 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1/2 \end{bmatrix}$$

$x_1 = 2, \quad x_2 = -1/2$

Some Important Points

(a) If A is an invertible matrix then A^{-1} is
$(A^{-1})^{-1} = A$

(b) If A & B are $n \times n$ invertible matrices then so is AB & $(AB)^{-1} = B^{-1}A^{-1}$

(c) If A is invertible matrix then so is A^T & inverse of A^T is the transpose of A^{-1}.

$$(A^T)^{-1} = (A^{-1})^T$$

Elementary Matrices:-

An elementary matrix is one that is obtained by performing a single elementary row operation on an identity matrix.

Let $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

$$E_1 \cdot A = \begin{bmatrix} a & b & c \\ d & e & f \\ -4a+g & -4b+h & -4c+i \end{bmatrix}$$

$$E_2 \cdot A = \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix}$$

$$E_3 \cdot A = \begin{bmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{bmatrix}$$

The row operations which are used to solve linear equation can be represented by matrix operations.

let I_{rs} $\begin{bmatrix} 0 & - & - & 0 \\ \vdots & I_{rs} & & \vdots \\ 0 & & & 0 \\ \vdots & & & \vdots \\ 0 & - & - & 0 \end{bmatrix}$ where $I_{rs} = m \times n$ matrix let $A = a_{ij}$ be $m \times n$ matrix. $I_{rs} A$?

$$\begin{bmatrix} 0 & - & - & 0 \\ | & I_{rs} & & | \\ 0 & - & - & 0 \end{bmatrix} \begin{bmatrix} a_{11} & - & - & a_{1n} \\ | & & & | \\ a_{m1} & - & - & a_{mn} \end{bmatrix}$$

$I_{rs} A$ is a matrix obtained by putting sth row of A in r-th row and zero elsewhere. If $r = s$ then I_{rs} has a component on diagonal place and zero elsewhere.

Multiplication by I_{rs} then leaves r-th row fixed & replace all other rows by zeroes.

If $r \neq s$ $\quad I_{rs} = I_{rs} + I_{sr}$

$$I_{rs} \cdot A = I_{rs} A + I_{sr} A$$

Inverse of Scalar Matrix :-

Matrix = $\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$.

Inverse = $\begin{bmatrix} 1/d_1 & 0 & 0 \\ 0 & 1/d_2 & 0 \\ 0 & 0 & 1/d_3 \end{bmatrix}$

Ex:- $\quad A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} 1/5 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$

* If an Elementary row operation is performed on an $m \times n$ matrix A the resultant matrix can be written as EA where $m \times m$ matrix E is created by performing the same row operation on I_m.

* Each elementary matrix E is invertible. The inverse of E is invertible matrix of the same type that transforms E_{rev} to I.

for example:- find inverse of $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$

To find inverse we first convert it into Identity matrix then again apply same operation to get inverse.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$$

$R_3 \rightarrow R_3 + 4R_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \rightarrow R_3 + 4R_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} = \text{Invese}$$

Ques find inverse of $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Inverse can be obtained by dividing 3rd Row of Identity Matrix by 2.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}.$$

Theorem:- A $m \times n$ matrix is invertible iff A is row equivalent to I_n and in this case any sequence of elementary row operations that reduces A to I_n also transforms I_n to A.

If we apply elementary row operations on some matrix A this is equivalent to premultiplying ~~A~~ A by one of three special types of matrices, which I generally denoted with E.

The result of a series of n row operations on M has the form

$$E_n E_{n-1} \text{---} E_3 E_2 E_1 A.$$

Each elementary matrix E_i is invertible, so if M is row equivalent to the indentity matrix I,

$$I = E_n E_{n-1} \text{---} E_3 E_2 E_1 A$$

then the inverse of M has the form,

$$A^{-1} = E_n E_{n-1} \cdots E_3 E_2 E_1$$

So matrix being invertible and a matrix being row-equivalent to the identity are same thing.

Gauss-Jordan Method :-

$$AA^{-1} = I$$

$$\cancel{A} \, I A^{-1} = \cancel{A}^{-1}$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 \to R_1 + R_2$.

$$\begin{bmatrix} 1 & 1 & 5 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - 4R_1$

$$\begin{bmatrix} 1 & 1 & 5 \\ 0 & -1 & -2 \\ 0 & -7 & -12 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ -4 & -4 & 1 \end{bmatrix}$$

$R_1 \to R_1 + R_2$. $R_2 \to -R_2$

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & -7 & -12 \end{bmatrix} A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -4 & -4 & 1 \end{bmatrix}$$

$R_3 \to R_3 + 7R_2$

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 3 & -4 & 1 \end{bmatrix}$$

$R_3 \to \dfrac{R_3}{2}$

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 3/2 & -2 & 1/2 \end{bmatrix}$$

$R_2 \to R_2 - 2R_3, \quad R_1 \to R_1 - 3R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}$$

find inverse of

$$A = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 3 & 0 \\ 0 & 4 & 0 & 0 \\ 5 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ 0 & 0 & 6 & 5 \\ 0 & 0 & 7 & 6 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1/2 \\ 0 & 0 & 1/3 & 0 \\ 0 & 1/4 & 0 & 0 \\ 1/5 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ 0 & 0 & 6 & 5 \\ 0 & 0 & 7 & 6 \end{bmatrix} \quad B^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_1 \to R_2 - R_1$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ 0 & 0 & 6 & 5 \\ 0 & 0 & 7 & 6 \end{bmatrix} B^{-1} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 4R_1$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 6 & 5 \\ 0 & 0 & 7 & 6 \end{bmatrix} B^{-1} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 4 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_1 \to R_1 + R_2$

$R_2 \to -R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 6 & 5 \\ 0 & 0 & 7 & 6 \end{bmatrix} B^{-1} = \begin{bmatrix} 3 & 2 & 0 & 0 \\ -4 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_4 \to R_4 - R_3.$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 6 & 5 \\ 0 & 0 & 1 & 1 \end{bmatrix} B^{-1} = \begin{bmatrix} 3 & 2 & 0 & 0 \\ -4 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

$R_3 \to R_3 - 5R_4.$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} B^{-1} = \begin{bmatrix} 3 & 2 & 0 & 0 \\ -4 & 3 & 0 & 0 \\ 0 & 0 & 6 & -5 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$R_4 \to R_4 - R_3.$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} B^{-1} = \begin{bmatrix} 3 & 2 & 0 & 0 \\ -4 & 3 & 0 & 0 \\ 0 & 0 & 6 & -5 \\ 0 & 0 & -7 & 6 \end{bmatrix}$$

Distance between u and v $\forall u, v \in \mathbb{R}^n$ is length of vector $\vec{u} - \vec{v}$.

i.e. $dist(u, v) = |(u-v)|$

Orthogonal Vectors :→ Two vectors in $\mathbb{R}^n$ are orthogonal to each other if $u \cdot v = 0$.

Note:- Zero vector is orthogonal to every vector in $\mathbb{R}^n$.

Theorem:- Two vectors u and v are orthogonal iff $|u+v|^2 = |u|^2 + |v|^2$.

let $\quad |u+v|^2 = |u|^2 + |v|^2$

Now, $\quad |u-v|^2 = |u|^2 + |v|^2 - 2u \cdot v$

But $|u+v|^2 = |u-v|^2$

& norms are +ve

$\therefore \quad |u+v|^2 = |u-\cancel{v}|^2$

$|u|^2 + |v|^2 + 2u \cdot v = |u|^2 + |v|^2 - 2u \cdot v$

By equality $\quad 2u \cdot v = -2u \cdot v$

$\Rightarrow \quad 4u \cdot v = 0$

$u \cdot v = 0$

$\Rightarrow$ u & v are orthogonal.

Orthogonal Compliments:→ If a vector z is orthogonal to every vector in a subspace W of $\mathbb{R}^n$ then z is said to be orthogonal complement to W. and is denoted by $W^\perp$ (W, perpendicular)

Ex1→ A plane W to the $\mathbb{R}^3$ and a line z be the line ⊥ to plane then z is called orthogonal compliment to W.

Ques Determine which pairs of vectors are orthogonal.

(i) $a = \begin{bmatrix} 12 \\ 3 \\ -5 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$

$a \cdot b = 24 - 9 - 15 = 24 - 24 = 0.$

Yes they are orthogonal to each other.

(ii) $a = \begin{bmatrix} -3 \\ 7 \\ 4 \\ 0 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ -8 \\ 15 \\ -7 \end{bmatrix}$

$a \cdot b = [-3 - 56 + 60 + 0] = 60 - 59 = 1$

No, a & b are not orthogonal.

Que find distance b/w $x = \begin{bmatrix} 10 \\ -3 \end{bmatrix}$, $y = \begin{bmatrix} -1 \\ -5 \end{bmatrix}$

$x - y = \begin{bmatrix} 11 \\ 2 \end{bmatrix}$

$|x - y| = \sqrt{11^2 + 2^2} = \sqrt{121 + 4} = \sqrt{125} = 5\sqrt{5}$

Orthogonal Set :→ A set of vectors $u_1, u_2 \ldots$ -- u_p in $\mathbb{R}^n$ is said to be orthogonal set if each pair of distinct vectors from the set is orthogonal i.e.

$$u_i \cdot u_j = 0 \quad \text{if } i \neq j$$

Ques:- Show that u_1, u_2, u_3 is orthogonal set.

$$u_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \quad u_2 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \quad u_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}$$

$$u_1 \cdot u_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = -3+2+1 = -3+3 = 0$$

$$u_2 \cdot u_3 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix} = \frac{1}{2} - 4 + \frac{7}{2}$$

$$\cancel{\frac{8}{2}}^{4} \cancel{\frac{7}{2}} - 4 = 0.$$

$$u_3 \cdot u_1 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = -3/2 - 2 + 7/2$$

$$= -7/2 + 7/2 = 0$$

$\Rightarrow \{u_1, u_2, u_3\}$ forms orthogonal set.

Theorem:- $S = \{\vec{u_1}, \vec{u_2} \ldots \vec{u_p}\}$ is orthogonal set of non-zero vectors in $\mathbb{R}^n$ the S is linearly independent and Hence is the basis for subspace spans by S.

Proof- If $0 = c_1 u_1 + c_2 u_2 \ldots c_p u_p$ for some scalars $c_1, c_2 \ldots c_p$

$0 = 0 \cdot u_1 = (c_1u_1 + c_2u_2 \cdots c_pu_p) \cdot u_1$

$0 = c_1(u_1 \cdot u_1) + c_2(u_2 \cdot u_1) \cdots$

$0 = c_1(u_1 \cdot u_1)$

because u_1 is orthogonal to $u_1 \cdots u_p$.
Since u_1 is non-zero $\therefore u_1 \cdot u_1$ is not zero

$\Rightarrow c_1 = 0$ Similarly $c_2 \cdots c_p$ must be 0.
Thus S is linearly Independent.

<u>Theorem</u> :- let $u_1, u_2 \cdots u_p$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$ for each y in W can be written in linear combination $y = c_1u_1 + c_2u_2 \cdots c_pu_p$ are

given by $c_j = \dfrac{y \cdot u_j}{u_j \cdot u_j}$ $[j = 1 \cdots p]$

<u>Proof</u>:-
As in preceding proof the orthogonility of $\{u_1, u_2 \cdots u_p\}$ shows that

$y \cdot u_1 = (c_1u_1 + c_2u_2 + \cdots c_pu_p) \cdot u_1$

$y \cdot u_1 = c_1(u_1 \cdot u_1)$

$\Rightarrow c_1 = \dfrac{y \cdot u_1}{u_1 \cdot u_1}$ $[\because u_1 \cdot u_1$ is non-zero.

In general way

$$c_j = \frac{y \cdot u_j}{u_j \cdot u_j}$$

Hence Proved.

Ques!- for $u_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, $u_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}$

The set $S = \{u_1, u_2, u_3\}$ is an orthogonal basis in R^3.

Express vector $y = \begin{bmatrix} 6 \\ 1 \\ -8 \end{bmatrix}$ as linear combination of vectors in S.

We have to show that $y = c_1 u_1 + c_2 u_2 + c_3 u_3$ for which we have to find values of c_1, c_2 and c_3.

By using the last theorem.

$$c_1 = \frac{y \cdot u_1}{u_1 \cdot u_1} = \frac{18+1-8}{9+1+1} = \frac{11}{11} = 1.$$

$$c_2 = \frac{y \cdot u_2}{u_2 \cdot u_2} = \frac{-6+2-8}{1+4+1} = \frac{-12}{6} = -2$$

$$c_3 = \frac{y \cdot u_3}{u_3 \cdot u_3} = \frac{-3-2-28}{\frac{1}{4}+\frac{16}{4}+\frac{49}{4}} = \frac{-\cancel{33} \times \cancel{4}^{2}}{\cancel{66}_{2}} = -2.$$

$$\boxed{y = u_1 - 2u_2 - 2u_3}$$

Ques!- Determine which set of vectors are orthogonal

(a) $a_1 = \begin{bmatrix} 2 \\ -5 \\ -3 \end{bmatrix}$, $a_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $a_3 = \begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix}$

$a_1 \cdot a_2 = 0$, $a_2 \cdot a_3 = 0$, $a_3 \cdot a_1 = \begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -5 \\ -3 \end{bmatrix}$

$8+10-18 = 18-18 = 0$

Yes, it is

(b) $a_1 = \begin{bmatrix} 5 \\ -4 \\ 0 \\ 3 \end{bmatrix}$, $a_2 = \begin{bmatrix} -4 \\ 1 \\ -3 \\ 8 \end{bmatrix}$, $a_3 = \begin{bmatrix} 3 \\ 3 \\ 5 \\ -1 \end{bmatrix}$

$a_1 \cdot a_2 = -20 - 4 + 0 + 24 = -24 + 24 = 0$
$a_2 \cdot a_3 = -12 + 3 - 15 - 8 = -32$
It is not an orthogonal set

Ques:- Show that $\{u_1, u_2, u_3\}$ or $\{u_1, u_2\}$ orthogonal basis. Then express x as linear combination of u's.

(a) $u_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$ $x = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$.

(b) $u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$, $u_3 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$, $x = \begin{bmatrix} 8 \\ -4 \\ -3 \end{bmatrix}$

$u_1 \cdot u_2 = -6 + 6 = 0$
$\Rightarrow$ u_1 & u_2 are orthogonal.
so x can be written as-

$$x = c_1 u_1 + c_2 u_2$$

$$c_1 = \frac{x \cdot u_1}{u_1 \cdot u_1} = \frac{-18+3}{9+1} = \frac{-15}{10} = \frac{-3}{2}$$

$$c_2 = \frac{x \cdot u_2}{u_2 \cdot u_2} = \frac{12+18}{4+36} = \frac{30}{40} = \frac{3}{4}$$

$$\boxed{x = -\frac{3}{2} u_1 + \frac{3}{4} u_2 .}$$

(b) $u_1 \cdot u_2 = -1+0+1 = 0$, $u_2 \cdot u_3 = -2+4-2 = 4-4 = 0$

$u_3 \cdot u_1 = 2+0-2 = 0$

$\therefore$ u_1, u_2 & u_3 are orthogonals.

Now $c_1 = \dfrac{x \cdot u_1}{u_1 \cdot u_1} = \dfrac{8+0-3}{1+0+1} = \dfrac{5}{2}$

$$c_2 = \frac{x \cdot u_2}{u_2 \cdot u_2} = \frac{-8-16-3}{1+16+1} = \frac{-\cancel{27}\,3}{\cancel{18}\,2} = \frac{-3}{2}$$

$$c_3 = \frac{x \cdot u_3}{u_3 \cdot u_3} = \frac{16-4+6}{4+1+4} = \frac{\cancel{18}\,2}{\cancel{9}} = 2.$$

$$\boxed{x = \frac{5}{2}u_1 - \frac{3}{2}u_2 + 2u_3}$$

Orthonormal Set :→ A set $\{u_1, u_2 \ldots u_p\}$ is an orthonormal set if is an orthogonal set of unit vectors.

If w is a subspace span by such a set then $\{u_1, u_2 \ldots u_p\}$ is an orthonormal basis for w.

Since the set is automatically linearly independent.

Simplest Example:- Standard basis $\{e_1, e_2 \ldots e_n\}$ for $\mathbb{R}^n$

$$u_i \cdot u_j = \left.\begin{matrix} 1 \\ 0 \end{matrix}\right\} \begin{matrix} \text{if } i = j \\ i \neq j \end{matrix}$$

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Ques:- Show that $\{v_1, v_2, v_3\}$ is orthonormal basis.

$$v_1 = \begin{bmatrix} 3/\sqrt{11} \\ 1/\sqrt{11} \\ 1/\sqrt{11} \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}, \quad v_3 = \begin{bmatrix} -1/\sqrt{66} \\ -4/\sqrt{66} \\ 7/\sqrt{66} \end{bmatrix}$$

$$v_1 \cdot v_2 = \frac{-3}{\sqrt{66}} + \frac{2}{\sqrt{66}} + \frac{1}{\sqrt{66}} = 0$$

$$v_2 \cdot v_3 = \frac{1}{\sqrt{396}} - \frac{8}{\sqrt{396}} + \frac{7}{\sqrt{396}} = 0$$

$$v_3 \cdot v_1 = \frac{-3}{11\sqrt{6}} - \frac{4}{11\sqrt{6}} + \frac{7}{11\sqrt{6}} = 0$$

$$v_1 \cdot v_1 = \frac{9}{11} + \frac{1}{11} + \frac{1}{11} = \frac{11}{11} = 1$$

$$v_2 \cdot v_2 = \frac{1}{6} + \frac{4}{6} + \frac{1}{6} = \frac{6}{6} = 1$$

$$v_3 \cdot v_3 = \frac{1}{66} + \frac{16}{66} + \frac{49}{66} = \frac{66}{66} = 1$$

Since $v_1 \cdot v_2 = v_2 \cdot v_3 = v_3 \cdot v_1 = 0$

& $v_1 \cdot v_1 = v_2 \cdot v_2 = v_3 \cdot v_3 = 1$

$\Rightarrow$ Set $\{v_1, v_2, v_3\}$ is an orthonormal set

Theorem:- An $m \times n$ matrix U has orthonormal columns if and only if $U^T U = I$

Proof:- Let U has only 3 column each a vector in $\mathbb{R}^m$ let $U = \{u_1, u_2, u_3\}$

$$U^T U = \begin{bmatrix} u_1^T \\ u_2^T \\ u_3^T \end{bmatrix} \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} = \begin{bmatrix} u_1^T u_1 & u_1^T u_2 & u_1^T u_3 \\ u_2^T u_1 & u_2^T u_2 & u_2^T u_3 \\ u_3^T u_1 & u_3^T u_2 & u_3^T u_3 \end{bmatrix}$$

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Entries in matrix are dot product. The columns of U are orthogonal if

$$U_1^T U_2 = U_2^T U_1 = 0$$
$$U_1^T U_3 = U_3^T U_1 = 0$$
$$U_2^T U_3 = U_3^T U_2 = 0$$

Columns of U have unit length if.

$$U_1^T U_1 = 1, \quad U_2^T U_2 = 1, \quad U_3^T U_3 = 1$$

Hence $\quad U^T U = I$

Ques:- $U_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \quad U_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \quad U_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}$

Make Set $\{U_1, U_2, U_3\}$ Orthonormal.

$$|U_1| = \sqrt{9+1+1} = \sqrt{11}$$

$$|U_2| = \sqrt{1+4+1} = \sqrt{6}$$

$$|U_3| = \sqrt{\frac{1}{4} + \frac{16}{4} + \frac{49}{4}} = \sqrt{\frac{\cancel{66}\,33}{\cancel{4}\,2}} = \sqrt{\frac{33}{2}}$$

$$U_1 = \begin{bmatrix} 3/\sqrt{11} \\ 1/\sqrt{11} \\ 1/\sqrt{11} \end{bmatrix} \quad U_2 = \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}, \quad U_3 = \begin{bmatrix} -\frac{\sqrt{33}}{2\sqrt{2}} \\ -\sqrt{66} \\ \frac{7}{2}\sqrt{\frac{33}{2}} \end{bmatrix}$$

Decomposition of Vector:-

Given a non-zero vector is in $\mathbb{R}^n$ consider the problem of decomposing a vector u in $\mathbb{R}^n$ into sum of two vectors in which one is multiple of u and other is orthogonal to u.

$$\Rightarrow y = \hat{y} + z$$

where $\hat{y} = \alpha u$ for some scalar α.
z = some vector orthogonal to u.

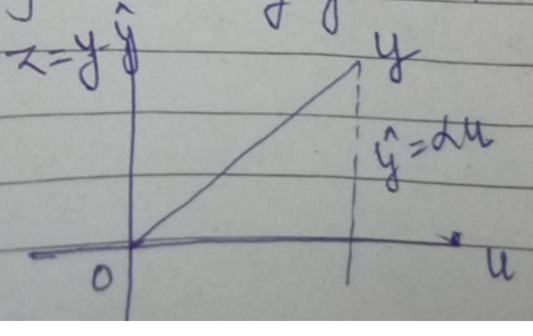
Given any scalar α, let $z = y - \alpha u$
Then $z = y - \hat{y}$ is orthogonal to u iff

$$\Rightarrow (y - \hat{y})\cdot u = 0$$
$$\Rightarrow y\cdot u - (\alpha u)u = 0 \qquad (\text{as } \hat{y} = \alpha u)$$
$$\Rightarrow y\cdot u - \alpha(u\cdot u) = 0$$

$$\alpha = \frac{y\cdot u}{u\cdot u}$$

& $$\hat{y} = \frac{y\cdot u}{u\cdot u}\cdot u$$

Here $\hat{y}$ is called the orthogonal projection of y onto u & z is called component of y orthogonal to u. $\hat{y}$ is sometimes denoted by proj y & is called projection of y onto L.



$$\text{Proj } y = \frac{y\cdot u}{u\cdot u}\cdot u$$

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Ques:- Find projection of y on u. Then write y as a sum of two orthogonal vectors one is span of u & other one is orthogonal to u.

(i) $y = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$, $u = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

$$\hat{y} = \frac{y \cdot u}{u \cdot u} u = \left[\frac{28+12}{16+4}\right] u$$

$$= \frac{\cancel{40}\,2}{\cancel{20}} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

$$z = y - \hat{y} = -\begin{bmatrix} 8 \\ 4 \end{bmatrix} + \begin{bmatrix} 7 \\ 6 \end{bmatrix} = \begin{bmatrix} -1 \\ +2 \end{bmatrix}$$

(ii) $y = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $u = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$

$$\hat{y} = \frac{y \cdot u}{u \cdot u} u = \frac{8-21}{16+49} \begin{bmatrix} 4 \\ -7 \end{bmatrix}$$

$$= -\frac{\cancel{13}}{\cancel{65}\,5} \begin{bmatrix} 4 \\ -7 \end{bmatrix} = \begin{bmatrix} -4/5 \\ 7/5 \end{bmatrix}$$

$$z = y - \hat{y} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} - \begin{bmatrix} -4/5 \\ 7/5 \end{bmatrix} = \begin{bmatrix} 14/5 \\ 8/5 \end{bmatrix}$$

Orthogonal Decompostion Theorem:-

If u_1, u_2 & u_3 is an ~~orthogonal~~ orthonormal basis for a subspace W of $\mathbb{R}^n$ then Proj. of y on w

$$\text{Proj}_w\, y = (y \cdot u_1)u_1 + (y \cdot u_2)u_2 + \cdots (y \cdot u_p)u_p$$

If $U = [u_1, u_2 \cdots u_p]$ then $\text{proj}_w\, y = UU^T y$ $\forall$ y in $\mathbb{R}^n$.

Proof:- let W be the subspace of $\mathbb{R}^n$ then each y in $\mathbb{R}^n$ can be written uniquely in the form $y = \hat{y} + z$ where $\hat{y}$ is in W & z is in $W_\perp$
infact if $\{u_1, u_2, \cdots u_p\}$ is any orthogonal basis of W then

$$\hat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2 \cdots$$

$$\frac{y \cdot u_p}{u_p \cdot u_p} u_p.$$

Then $\{v_1, v_2 \cdots v_p\}$ is an orthogonal basis of W and span $\{x_1, x_2 \cdots x_k\}$ =
Span $\{v_1, v_2 \cdots v_k\}$ for $1 \le k \le p$

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Gram Schmidt Process :→ It is a simple algorithm for producing a orthogonal or orthonormal basis for any non-zero subspace of $\mathbb{R}^n$.

$$W = \{\vec{v_1}, \vec{v_2} -- \vec{v_n}\}$$

Span $(v_1) = u_1$
Span $(v_1, v_2) = u_1, u_2$

We have to find u_2 such that v_1 and u_2 are $\perp$r to each other.

basically $u_2 = \frac{v_2 \cdot u_1}{u_1 \cdot u_1} u_1$

$$Z = V_2 - \frac{V_2 \cdot u_1}{u_1 \cdot u_1} V_1$$

Given a Basis $\{x_1, --- x_p\}$ for a non-zero subspace W of $\mathbb{R}^n$ define

$$V_1 = x_1$$

$$V_2 = x_2 - \frac{x_2 \cdot V_1}{V_1 \cdot V_1} V_1$$

$$V_3 = x_3 - \frac{x_3 \cdot V_1}{V_1 \cdot V_1} V_1 - \frac{x_3 \cdot V_2}{V_2 \cdot V_2} \cdot V_2$$

$$V_p = x_p - \frac{x_p V_1}{V_1 \cdot V_1} V_1 - \frac{x_p \cdot V_2}{V_2 \cdot V_2} V_2 - \cdots$$

$$\frac{x_p \cdot V_{p-1}}{x_{p-1} V_{p-1}} V_{p-1}$$

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Transformation :→ A transformation or function or mapping T from $\mathbb{R}^n$ to $\mathbb{R}^n$ is a rule that assigns to each vector x in $\mathbb{R}^n$ to a vector $T(x)$ in $\mathbb{R}^n$.

$$T : \underset{\text{domain}}{\mathbb{R}^n} \rightarrow \underset{\text{codomain}}{\mathbb{R}^n}$$

for all x $T(x)$ is called image & set of all images $T(x)$ is called Range of T.

Ex :→ $A \cdot x = b$, So mapping is from $\mathbb{R}^4$ to $\mathbb{R}^2$

as $A = \begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$

Matrix Transformation :→ for x in $\mathbb{R}^n$ $T(x)$ is computed as Ax where A is $m \times n$ matrix we denote it as as $x \mapsto Ax$.

If $x \in \mathbb{R}^n$ {Then $\mathbb{R}^n$ is called domain}

$Ax \in \mathbb{R}^n$ {Then $\mathbb{R}^n$ is called codomain}

Range of T is set of all linear combination of columns of A.

Linear Transformation :- A transformation T is linear if (i) $T(u+v) = T(u) + T(v)$ $\forall\ u, v \in dom(T)$

(ii) $T(cu) = cT(u)$ $\forall$ scalars c & $u \in dom(T)$

Theorem :→ If T is linear transformation Then $T(0) = 0$ & $T(cu + dv) = cT(u) + dT(v)$ $\forall\, u, v \in \text{dom}(T)$ & scalar c, d.

Proof :- (i) $T(0) = T(0.u) = 0T(u) = 0$

(ii) $T(cu + dv) = T(cu) + T(dv)$
$= cT(u) + dT(v)$

In General
$T(c_1v_1 + c_2v_2 \cdots\cdots c_pv_p) = c_1T(v_1) + c_2T(v_2) \cdots$

Ques :- Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, $U = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $b = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$,
$C = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$

Transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ by $T(x) = Ax$

a) Find T(u) the image of u under transformation.
b) Find an x in $\mathbb{R}^2$ whose image under T is b.
c) Is there more than one x whose image under T is b?
d) Determine if C is image of T?

a) $T(u) = Au = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2+3 \\ 6-5 \\ -2-7 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$

b) $A \cdot x = b \Rightarrow \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$

$$\begin{bmatrix} x-3y \\ 3x+5y \\ -x+7y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$$

On Comparing
By

$$x-3y = 3 \quad \text{—— (1)}$$
$$3x+5y = 2 \quad \text{—— (2)}$$
$$-x+7y = -5 \quad \text{—— (3)}$$

from (1) & (3)

$$\cancel{x}-3y = 3$$
$$-\cancel{x}+7y = -5$$
$$4y = -2$$

$$y = \frac{-\cancel{2}^{1}}{\cancel{4}_{2}} = \frac{-1}{2}$$

$$x + \frac{3}{2} = 3 \Rightarrow x = 3 - \frac{3}{2} = \frac{3}{2}$$

from (2) & (3)

$$3\cancel{x}+5y = 2$$
$$-3x+21y = -15$$
$$26y = -13 \qquad y = -1/2.$$

$$3x - \frac{5}{2} = 2$$

$$3x = 2 + \frac{5}{2} = \frac{9}{2}$$

$$\boxed{x = 3/2}.$$

c) We get same answer from every equation
∴ Unique result.

d)

$$\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$$

$$x_1 - 3x_2 = 3 \quad \text{--- (1)}$$
$$3x_1 + 5x_2 = 2 \quad \text{--- (2)}$$
$$-x_1 + 7x_2 = 5 \quad \text{--- (3)}$$

Using (1) & (3)

$$\cancel{x_1} - 3x_2 = 3$$
$$-\cancel{x_1} + 7x_2 = 5$$

$$4x_2 = 8$$

$$x_2 = 2 \quad \text{--- (4)}$$

Using (1) & (2)

$$\cancel{3x_1} - 9x_2 = 9$$
$$\cancel{3x_1} + 5x_2 = 2$$

$$-14x_2 = 7$$

$$x_2 = -\frac{1}{2} \quad \text{--- (5)}$$

eqn ~~Using~~ (4) & (5) contradicts.

$\therefore$ No Solutions.

Ques:- Given a scalar r, define $T: \mathbb{R}^2 \to \mathbb{R}^3$ by $T(x) = r(x)$. T is called contraction if $u \le r < 1$ & delation if $r > 1$. let $r = 3$ & show that T is a Linear Transformation.

let $r = 3$. let u & v be the vectors $\in \mathbb{R}^2$ & a, b are scalars.

$$T(au + bv) = 3(au + bv) = 3au + 3bv$$
$$= a(3u) + b(3v)$$
$$= aT(u) + bT(v).$$

Hence T is linear combination

Ques:- T is defined as $T(x) = Ax$ find vector whose image under T is b and determine if x is unique.

(a) $A = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 6 \\ 3 & -2 & -5 \end{bmatrix}$, $b = \begin{bmatrix} -1 \\ 7 \\ -3 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & -5 & -7 \\ 3 & 7 & 5 \end{bmatrix}$, $b = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$

let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

(a) $Ax = b$

$$\begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 6 \\ 3 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 - 2x_3 \\ -2x_1 + x_2 + 6x_3 \\ 3x_1 - 2x_2 - 5x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \\ -3 \end{bmatrix}$$

$x_1 - 2x_3 = -1$
$-2x_1 + x_2 + 6x_3 = 7$
$3x_1 - 2x_2 - 5x_3 = -3$

$-4x_1 + 2x_2 + 12x_3 = 14$
$3x_1 - 2x_2 - 5x_3 = -3$

$-x_1 + 7x_3 = 11$
$x_1 - 2x_3 = -1$

$5x_3 = 10$ $\boxed{x_3 = 2}$

$x_1 - 4 = -1$
$\boxed{x_1 = 3}$

$-6 + x_2 + 12 = 7$
$x_2 + 6 = 7$
$\boxed{x_2 = 1}$

$\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$

Unique Solution

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(b) $$\begin{bmatrix} 1 & -5 & -7 \\ 3 & 7 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

Solution is not possible as ~~x~~ ~~dimension~~ no. of eq < no. of variables

$$\begin{bmatrix} x_1 - 5x_2 - 7x_3 \\ 3x_1 + 7x_2 + 5x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

Not ~~a~~ Unique solution we have to take

$\boxed{x_3 = 0}$

$x_1 - 5x_2 = -2$

$3x_1 + 7x_2 = -2$

$3x_1 - 15x_2 = -6$

$22x_2 = 4$

$\boxed{x_2 = \frac{2}{11}}$

$x_1 - \frac{10}{11} = -2$

$x_1 = -2 + \frac{10}{11} = \frac{-22+10}{11}$

$\boxed{x_1 = -\frac{12}{11}}$

Ques:- Let A be 6x5 matrix. What must be a & b in order to define $T: \mathbb{R}^a \to \mathbb{R}^b$ by
$T(x) = A \cdot x$

Ans $a = 5, \ b = 6$

Que 3:→ How many rows and columns matrix A have in order to define mapping from $\mathbb{R}^4$ to $\mathbb{R}^5$ by rule $T(x) = Ax$?

Ans:- 5×4

Ques 4:→ Find x in $\mathbb{R}^4$ that are mapped to zero vector by transformation $x \to Ax$ for given Matrix A.

$$A = \begin{bmatrix} 1 & -4 & 7 & -5 \\ 0 & 1 & -4 & 3 \\ 2 & -6 & 6 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -4 & 7 & -5 \\ 0 & 1 & -4 & 3 \\ 2 & -6 & 6 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\cancel{x_1 - 4x_2 + 7x_3}$

$R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & -4 & 7 & -5 \\ 0 & 1 & -4 & 3 \\ 0 & 2 & -8 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} 1 & -4 & 7 & -5 \\ 0 & 1 & -4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 - 4x_2 + 7x_3 - 5x_4 \\ x_2 - 4x_3 + 3x_4 \end{bmatrix} \begin{matrix} = 0 \\ = 0 \end{matrix}$$

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let $x_3 = 1$
$x_4 = 2$

$x_2 - 4 + 6 = 0$

$x_2 + 2 = 0$

$\boxed{x_2 = -2}$

$x_1 + 8 + 7 - 10 = 0$

$x_1 + 15 - 10 = 0$

$x_1 + 5 = 0$

$\boxed{x_1 = -5}$

$$\begin{bmatrix} -5 \\ -2 \\ 1 \\ 2 \end{bmatrix}$$

Linear Transformation as a Matrix vector product

Let we have $n \times n$ Identity matrix

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & & & 1 \end{bmatrix}$$

If we multiply I_n by a vector $\vec{x}$ we get.

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

let $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \end{bmatrix}$ $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}$ $\cdots$ $e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$

Here $\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots x_n\vec{e}_n$

$$T(\vec{x}) = T(x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots x_n\vec{e}_n)$$

$$= T(x_1\vec{e}_1) + T(x_2\vec{e}_2) + \cdots T(x_n\vec{e}_n)$$

where T is a L.T.

$$T(\vec{x}) = \begin{bmatrix} T(e_1) & T(e_2) & \cdots & T(e_n) \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Theorem:- Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation then $\exists$ a unique matrix A such that $T(x) = Ax \quad \forall \; x \in \mathbb{R}^n$.

Here A is $m \times n$ matrix whose j^{th} column is a vector $T(e_j)$ where e_j is the j^{th} column of Identity matrix in $\mathbb{R}^n$. So,

$$A = [T(e_1)\; T(e_2) \; --- \; T(e_n)]$$

Ques:- $T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 3x_2 \\ 5x_2 - x_1 \\ 4x_1 + x_2 \end{bmatrix}$, find matrix A.

let $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

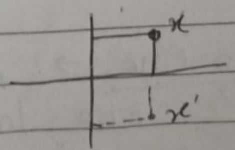
$$T(e_1) = T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

$$T(e_2) = T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}$$

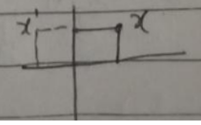
$$A = [T(e_1)\; T(e_2)] = \begin{bmatrix} 1 & 3 \\ -1 & 5 \\ 4 & 1 \end{bmatrix}$$

If we want to find reflection of any vector then standard values of A if

(i) with x-axis $\quad A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$



(ii) with y-axis $\quad A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$



Linear Transformation as a Matrix vector Product

Let we have $n \times n$ Identity matrix

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots \\ 0 & \cdots & \cdots & \cdots & 1 \end{bmatrix}$$

If we multiply I_n by a vector $\vec{x}$ we get.

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots \\ 0 & \cdots & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

let $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \end{bmatrix}$ $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}$ $\cdots$ $e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$

Here $\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots x_n\vec{e}_n$

$$T(\vec{x}) = T(x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots x_n\vec{e}_n)$$

$$= T(x_1\vec{e}_1) + T(x_2\vec{e}_2) + \cdots T(x_n\vec{e}_n)$$

where T is a L.T.

$$T(\vec{x}) = \begin{bmatrix} T(e_1) & T(e_2) & \cdots & T(e_n) \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

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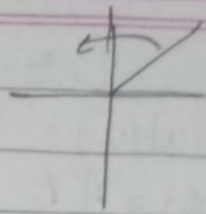
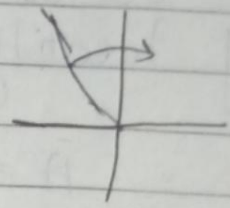
3) Along a line towards left:-

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



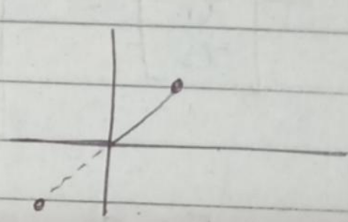
4) Along a line towards right:-

$$A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$



5) Along origin

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$



6) Contraction:- horizontal $A = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$

Verifical $A = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$

Onto:→ A mapping T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is said to be On-to $\mathbb{R}^m$. If each $b \in \mathbb{R}^m$ is the image of atleast one x in $\mathbb{R}^n$.

OR, We can say that $\exists$ atleast one solution of $T(x) = b$

One-to-One:→ A mapping $T: \mathbb{R}^n \to \mathbb{R}^m$ is said to be one-to-one if each b in $\mathbb{R}^m$ is the image of atmost one x in $\mathbb{R}^n$.

Eqn $T(x) = b$ has either a unique soluti[on] or non solution at all.

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Ques:→ Let T be linear transformation whose Standard matrix is

$$A = \begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Does T map $\mathbb{R}^4$ onto $\mathbb{R}^3$?
Is T a one-to-one mapping.

If $x \in \mathbb{R}^4$, let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

$$T(x) = Ax$$

$$= \begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

four variables & three rows ⇒ infinitely many solutions

∴ T is onto. It is not one-to-one.

Ques:- Let T be the linear transformation whose Standard matrix is given. Decide if T is one-to-one Mapping.

$$A = \begin{bmatrix} -5 & 10 & -5 & 4 \\ 8 & 3 & -4 & 7 \\ 4 & -9 & 5 & -3 \\ -3 & -2 & 5 & 4 \end{bmatrix}$$

let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, $T(x) = Ax$

$$\begin{bmatrix} -5 & 10 & -5 & 4 \\ 8 & 3 & -4 & 7 \\ 4 & -9 & 5 & -3 \\ -3 & -2 & 5 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

let us covert the matrix into echolean form so that we can find rank and decide it w/as which type of solution exists.

$R_1 \to R_1 + R_3$

$$\begin{bmatrix} -1 & 1 & 0 & 1 \\ 8 & 3 & -4 & 7 \\ 4 & -9 & 5 & -3 \\ -3 & -2 & 5 & 4 \end{bmatrix}$$

$R_2 \to R_2 + 8R_1$, $R_3 \to R_3 + 4R_1$, $R_4 \to R_4 - 3R_1$

$$\begin{bmatrix} -1 & 1 & 0 & 1 \\ 0 & 11 & -4 & 15 \\ 0 & -5 & 5 & 1 \\ 0 & -5 & 5 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_4$

$$\begin{bmatrix} -1 & 1 & 0 & 1 \\ 0 & 11 & -4 & 15 \\ 0 & 0 & 0 & 0 \\ 0 & -5 & 5 & 1 \end{bmatrix}$$

$\Rightarrow$ ~~the~~ no. of eqn < no. of variables $\Rightarrow$ many soln.

$\Rightarrow$ On-to mapping

Not one-to-one

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Ques:- Find Standard Matrix A for $T(x) = \begin{bmatrix} 3x_1 - 2x_3 \\ 4x_1 \\ x_1 - x_2 + x_3 \end{bmatrix}$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$T(e_1) = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}, \quad T(e_2) = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \quad T(e_3) = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} T(e_1) & T(e_2) & T(e_3) \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 0 & -2 \\ 4 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

Ques:- Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that $T(x_1, x_2) = (x_1 + x_2, 4x_1 + 5(x_2))$ find x such that $T(x) = (3, 8)$

$$T(x_1, x_2) = (3, 8)$$

$$\Rightarrow (x_1 + x_2, 4x_1 + 5x_2) = (3, 8)$$

$$4x_1 + 4x_2 = 12$$

$$-4x_1 \pm 5x_2 = \mp 8$$

$$x_2 = -4$$

$$x_1 - 4 = 3$$

$$\boxed{x_1 = 7}$$

$$x = [7, -4]. = \begin{bmatrix} 7 \\ -4 \end{bmatrix}$$

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Theorem:- Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a L.T Then T is one to One iff the eqn $T(x)=0$ has only the trivial solution.

Proof:- Since T is linear so $T(0)=0$ If T is one-to-one the eqn $T(x)=0$ has almost one solution and hence only the trivial solution. If T is not one-to-one then there is a b $\exists$ image of atleast two different vectors in $\mathbb{R}^n$ say u & v that is $T(u)=b$ & $T(v)=b$.

Since T is linear.

$$\therefore T(u-v) = T(u)-T(v)$$
$$= b-b$$
~~The vec~~
$$= 0$$

The vector $(u-v)$ is not zero since $u \neq v$ hence the equation $T(x)=0$ has more than one solution. So either the two conditions in the theorem are both true or ~~are~~ both false.

Theorem:- Let $T: R^n \to R^m$ be a linear transformation and let A be the standard matrix for T then:

a) T maps R^n onto R^m iff the columns of A spans R^m.

b) T is one to one iff columns of A are linearly independent.

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Eigen Values and Eigen Vectors :→

Let $A = [a_{ij}]_{n\times n}$ be a matrix if
$x = [x_1, x_2 \;---\; x_n]$ then
$AX = \lambda X$
$AX - \lambda X = 0$
$(A - \lambda I)X = 0$
where I is unit matrix of order n.

Eigen Matrix :→ If $A = [a_{ij}]_{n\times n}$ then the matrix $(A-\lambda I)$ is called the characteristic matrix of A.

Eigen Polynomial :→ $\det[A-\lambda I]$ is called eigen polynomial of A & denoted by $\phi(\lambda)$.

Eigen Equation :→

$(A - \lambda I) = 0$ is called characterstic eqn of A.

Characteristic / Eigen Root / Eigen Value (λ) :-
The root of eigen eqn $|A - \lambda I| = 0$ of a square matrix A are called eigen roots.

Eigen Vector :- Corresponding to eigen root λ of a square matrix A, if there exists a non-zero vector X such that
$(A - \lambda I)X = 0$ where
X is called characteristic vector corresponding to the characteristic vector.

Ques Find eigen value and eigen vectors of the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

$$|A - \lambda I| = 0$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = 0$$

$$\left| \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 1 = 0$$

$$\lambda^2 = 1$$

$$\lambda = \pm 1$$

eigen values 1 & -1.

for $\lambda = 1$

$$[A - I] X = 0$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} X = 0$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} -x_1 + x_2 \\ x_1 - x_2 \end{bmatrix} = 0$$

$$\Rightarrow x_1 = x_2$$

Vector $\begin{bmatrix} x_1 \\ x_1 \end{bmatrix} = K \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \forall K \in R$

for $\lambda = -1$ $\quad [A + I] X = 0$.

$$\left[\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right] X = 0.$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} x_1 + x_2 \\ x_1 + x_2 \end{bmatrix} = 0$$

$x_1 + x_2 = 0$

$\Rightarrow \quad x_1 = -x_2$

$\Rightarrow$ vector $\quad k\begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} \Rightarrow k\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

OR

for a $n \times n$ matrix A, for some ~~x~~ vectors if $Ax = \lambda x$ for some scalar λ. then λ is called eigen value and ~~x is called eigen vector~~. if $\exists$ non-trivial solution x of $Ax = \lambda x$ such ~~as~~ x is called eigen vector corresponding to eigen value λ.

Ques let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. An eigen value of A is 2. Find basis of corresponding eigen space.

Note, $\quad (A - \lambda I)\vec{x} = 0.$

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$$\left(\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\right) \cdot \vec{x} = 0$$

$$\begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix} \cdot \vec{x} = 0$$

$R_3 \to R_3 - R_2$, $R_2 \to R_2 - R_1$.

$$\begin{bmatrix} 2 & -1 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Hence $(A - \lambda I)$ is dependent.

$$2x_1 - x_2 + 6x_3 = 0$$

let $x_2 = x_2$, $x_3 = x_3$

$$2x_1 = x_2 - 6x_3$$

$$x_1 = \frac{x_2}{2} - 3x_3$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

The basis $= \left\{ \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

Theorem:- Eigen value of triangular matrix are entries on its main diagonals.

Proof:-

$$A - \lambda I = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

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$$= \begin{bmatrix} a_{11}-\lambda & 0 & 0 \\ 0 & a_{22}-\lambda & a_{23} \\ 0 & 0 & a_{33}-\lambda \end{bmatrix}$$

Scalar λ is an eigen value of A iff equation $(A-\lambda I)x = 0$ has non trivial solution i.e. iff eq has free variables. So, $\lambda = a_{11}, a_{22}, a_{33}$.