Review of Probability Theory

Introduction to Probability

- Experiments
- Counting Rules
- Events and Their Probability
- Some Basic Relationships of Probability
- Conditional Probability
- Bayes' Theorem

Why to study Probability?

- Do you know what happen tomorrow?
- Nothing in life is certain. In everything we do, we gauge the chances of successful outcomes.
- Uncertainties in the World
  - Weather and natural events
  - Getting hit by a disease or an epidemic
  - Political party winning an election
  - Transport

- **In all these cases, we understand that some repetitive process generates outcomes that are not deterministic but nevertheless exhibit a rational pattern.**

Why to study Probability?

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- In all these cases, we understand that some repetitive process generates outcomes that are not deterministic but nevertheless exhibit a rational pattern.

- For example, each rotation of the planet brings a new day, on which rain may or may not occur.
  - We interpret a 40% chance of rain as meaning that over an extended run of days, 40% of them will be rainy.

Probability as a Numerical Measure of the Likelihood of Occurrence

- Probability theory provides a rigorous mathematical framework for investigating phenomena that exhibit repeatable patterns, even though individual outcomes may appear random.

- The probability of an event is a numerical value that measures the likelihood that the event can occur

  - Probability values are always assigned on a scale from 0 to 1.

  - A probability near zero indicates an event is quite unlikely to occur

  - A probability near one indicates an event is almost certain to occur.

Probability Experiments

- A probability experiment is an action through which specific results (counts, measurements or responses) are obtained.
  - Example: Toss a die and observing the number that is rolled is a probability experiment.

- The result of a single trial in a probability experiment is the outcome.

- The set of all possible outcomes for an experiment is the sample space.

  - Example: The sample space when tossing a die has six outcomes. {1, 2, 3, 4, 5, 6}

Event

- An event consists of one or more outcomes and is a subset of the sample space.

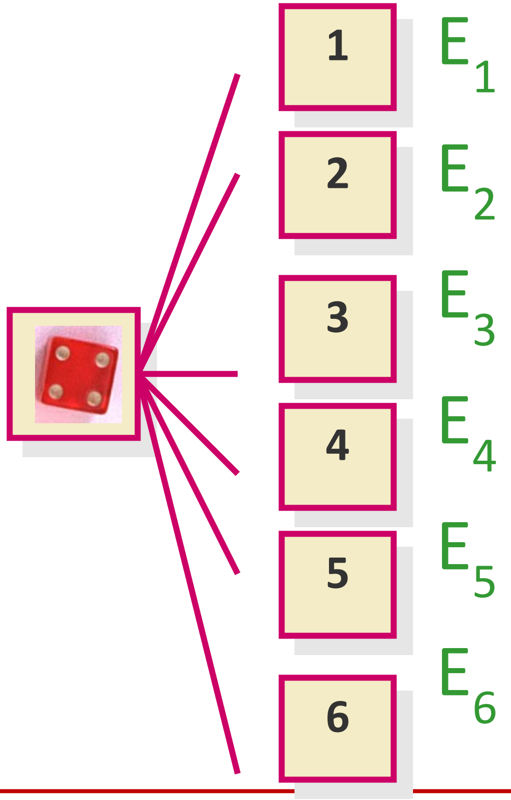
  - Example: A die is tossed. Event A is observing an even number.
    - A = {2, 4, 6}.

- A simple event is an event that consists of a single outcome.

Event

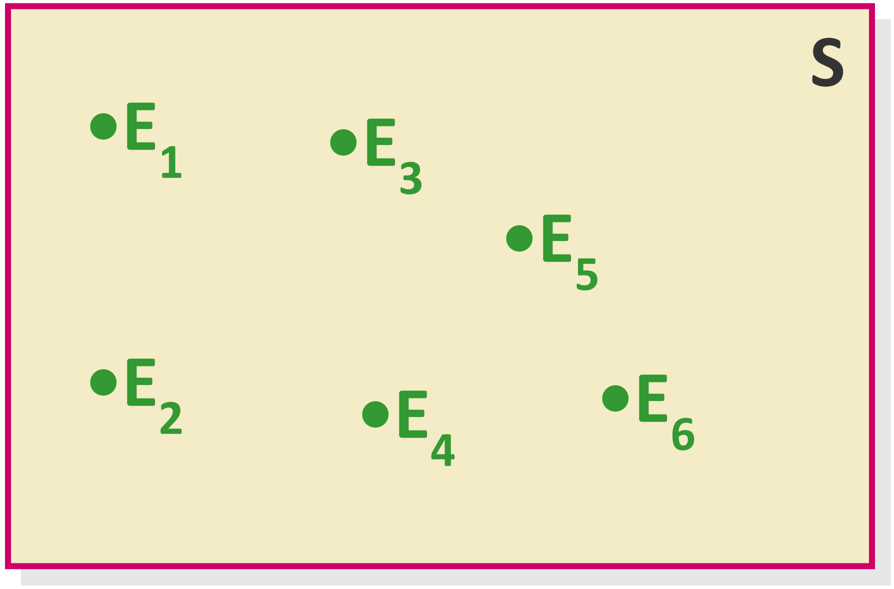
- The die toss:

- Simple events:



Sample space:

$S = \{E_1, E_2, E_3, E_4, E_5, E_6\}$



Event

- **Two events are mutually exclusive if, when one event occurs, the other cannot, and vice versa.**
  - **Example: Toss a die.**
    - **A: Observe an odd number**
    - **B: Observe a number greater than 2**
    - **C: Observe a 6**
    - **D: observe a 3**

A and B?	Not Mutually Exclusive
C and D?	Mutually Exclusive
B and C?	Not Mutually Exclusive
B and D?	Not Mutually Exclusive

The Probability of an Event

- **The probability of an event A measures "how often" A will occur. We write P(A).**
- **Suppose that an experiment is performed n times. The relative frequency for an event A is**

$$\mathrm{P}(A) = \frac{\text{Number of } \textit{times A occurs}}{\text{Total number of outcomes in sample space}} = \frac{f}{n}.$$

- **Example: A die is Tossed. Find the probability of Event A: rolling a 5.**

There is one outcome in Event *A*: {5}

$$P(A) = \frac{1}{6} \approx 0.167$$

"Probability of Event *A*."

The Probability of an Event

- P(A) must be between 0 and 1.
- The sum of the probabilities for all simple events in S equals 1.
- Probabilities can be found using
  - Estimates from empirical studies
  - Common sense estimates based on equally likely events
- Examples:
  - Toss a fair coin.

P(Head) = 1/2

  - Suppose that 10% of the U.S. population has red hair. Then for a person selected at random

P(Red hair) = .10

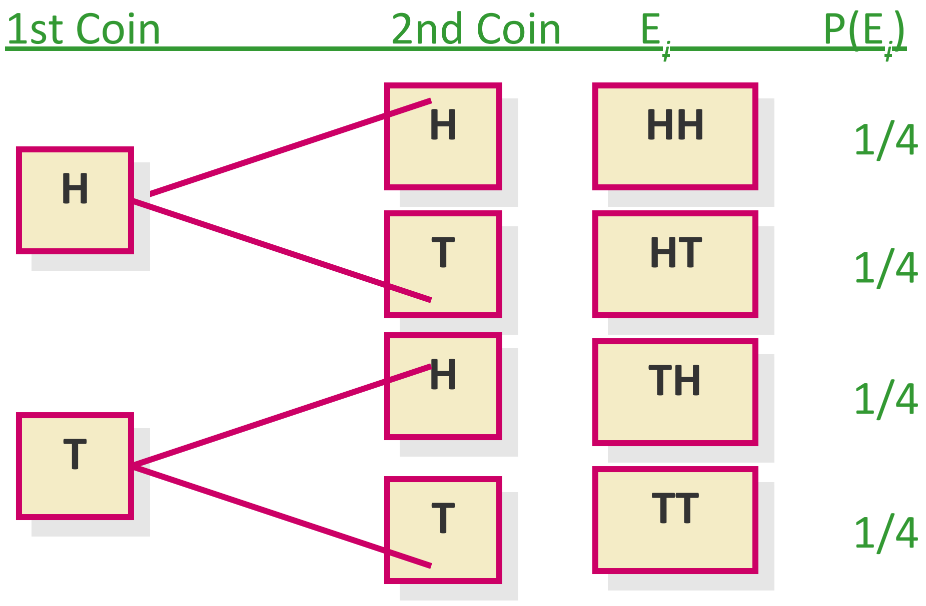
Using Simple Events

- The **probability of an event A** is equal to the sum of the probabilities of the simple events contained in A
- If the simple events in an experiment are equally likely, then

$$P(A) = \frac{n_A}{N} = \frac{\text{number of simple events in A}}{\text{total number of simple events}}$$

Using Simple Events: Example 1

- Toss a fair coin twice. What is the probability of observing at least one head?



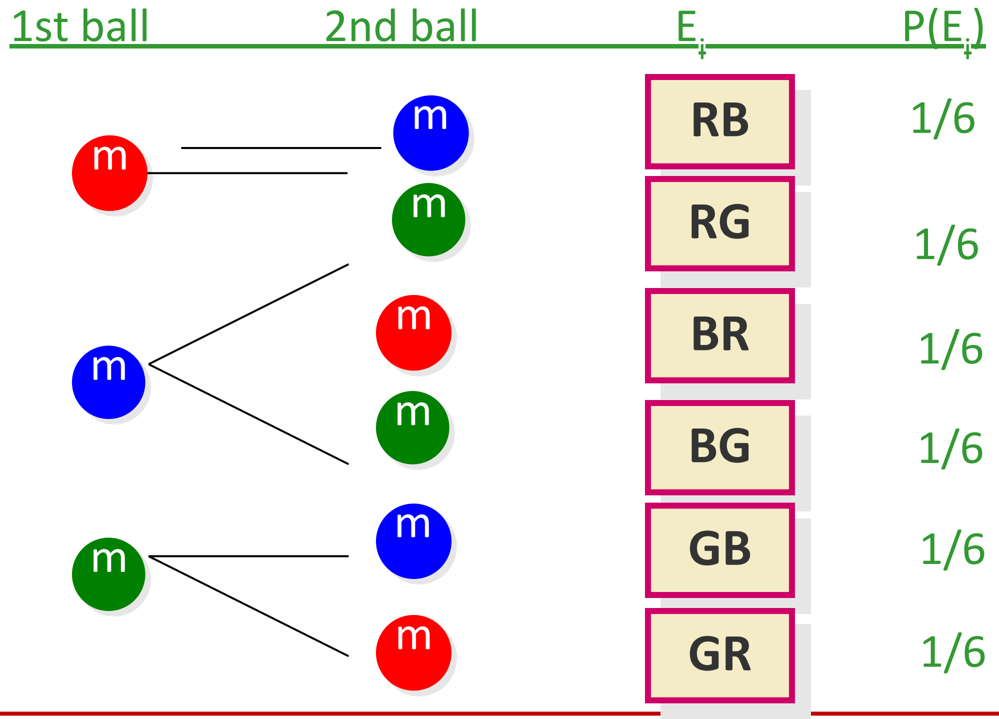
P(at least 1 head)

$= P(E_1) + P(E_2) + P(E_3)$

$= 1/4 + 1/4 + 1/4 = 3/4$

Using Simple Events: Example 2

- A bowl contains three balls, one red, one blue and one green. A child selects two ball at random. What is the probability that at least one is red?



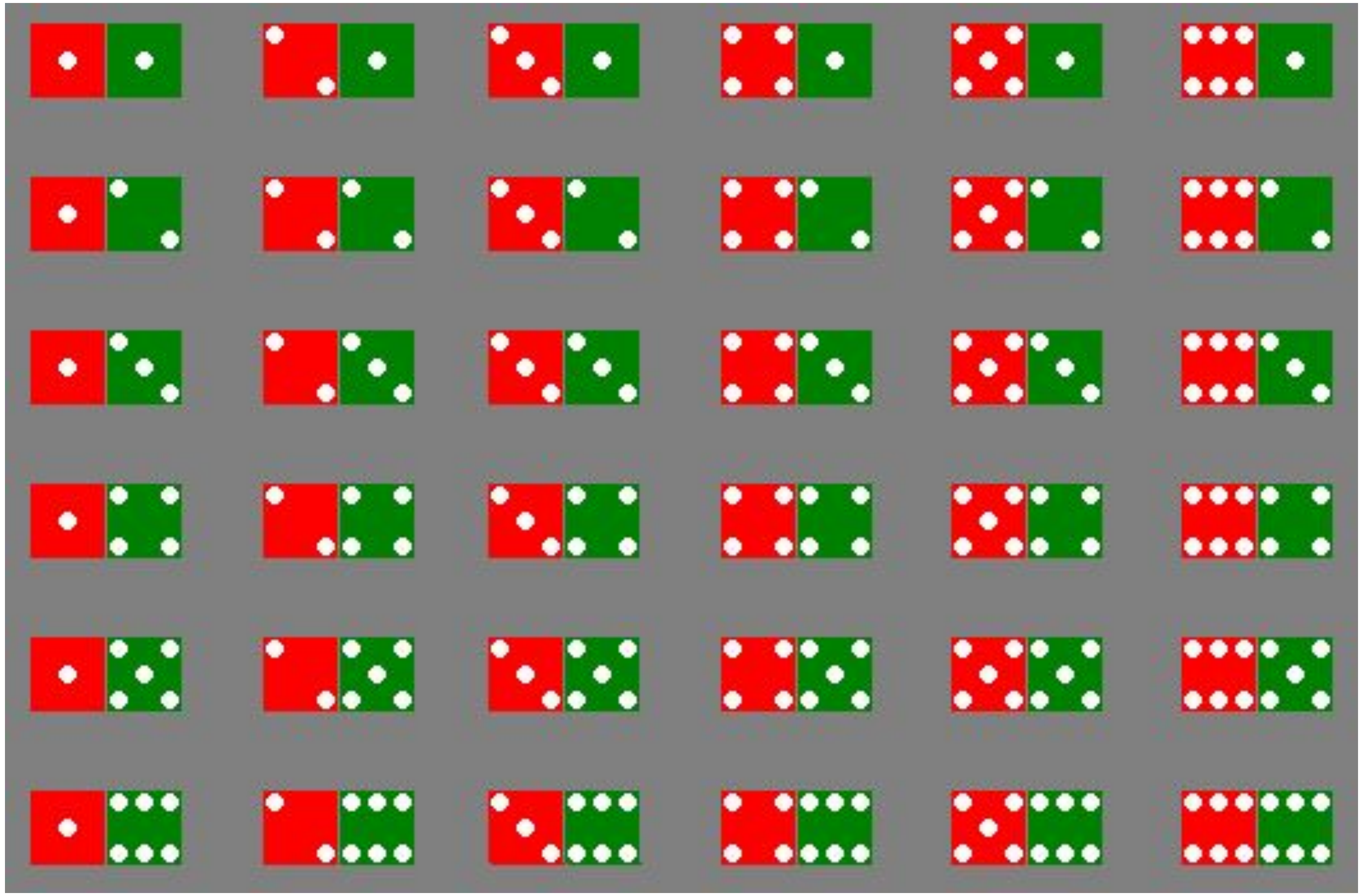
P(at least 1 red)

= P(RB) + P(BR)+ P(RG) + P(GR)

= 4/6 = 2/3

Using Simple Events: Example 3

- The sample space of throwing a pair of dice is



Using Simple Events: Example 3

Event	Simple events	Probability
Dice add to 3	(1,2),(2,1)	2/36
Dice add to 6	(1,5),(2,4),(3,3), (4,2),(5,1)	5/36

Counting Rules

- Sample space of throwing 3 dice has 216 entries, sample space of throwing 4 dice has 1296 entries, …
- At some point, we have to stop listing and start thinking …
- We need some counting rules

The *mn* Rule

- If an experiment is performed in two stages, with m ways to accomplish the first stage and n ways to accomplish the second stage, then there are mn ways to accomplish the experiment.
- This rule is easily extended to k stages, with the number of ways equal to

$$n_1 n_2 n_3 \dots n_k$$

Example: Toss two coins. The total number of simple events is: $\boxed{2 \times 2 = 4}$

The *mn* Rule

Example: Toss three coins. The total number of simple events is:

$$2 \times 2 \times 2 = 8$$

Example: Toss two dice. The total number of simple events is:

$$6 \times 6 = 36$$

Example: Toss three dice. The total number of simple events is:

$$6 \times 6 \times 6 = 216$$

Combinatorial reasoning

- Permutations of n items - number of sequences of n items $n!$

- The number of ways you can arrange n distinct objects, taking them r at a time is $\frac{n!}{(n-r)!}$

- The number of distinct combinations of n distinct objects that can be formed, taking them r at a time is $\frac{n!}{r!\,(n-r)!}$

Example

- **How many 3-digit lock combinations (without repetition) can we make from the numbers 1, 2, 3, and 4?**

The order of the choice is important! →

$$\frac{4!}{(4-3)!} = 24$$

- **Three members of a 5-person committee must be chosen to form a subcommittee. How many different subcommittees could be formed?**

The order of the choice is not important! →

$$\frac{5!}{3!\,(5-3)!} = 10$$

Example

- **A box contains six balls four red and two green. A child selects two balls at random. What is the probability that exactly one is red?**

The order of the choice is not important!

$$C_2^6 = \frac{6!}{2!4!} = \frac{6(5)}{2(1)} = 15$$

ways to choose 2 balls.

$$C_1^2 = \frac{2!}{1!1!} = 2$$

ways to choose 1 green ball.

$$C_1^4 = \frac{4!}{1!3!} = 4$$

ways to choose 1 red ball.

4 × 2 =8 ways to choose 1 red and 1 green balls.

P(exactly one red) = 8/15

Some Basic Relationships of Probability

- **There are some basic probability relationships that can be used to compute the probability of an event without knowledge of all the sample point probabilities.**
    - **Complement of an Event**
    - **Union of Two Events**
    - **Intersection of Two Events**
    - **Mutually Exclusive Events**

Complement of an Event

- The **complement** of event A is defined to be the event consisting of all sample points that are not in A.
- The complement of A is denoted by A^c.

Sample Space S

Example: Select a student from the classroom and record his/her hair color and gender.

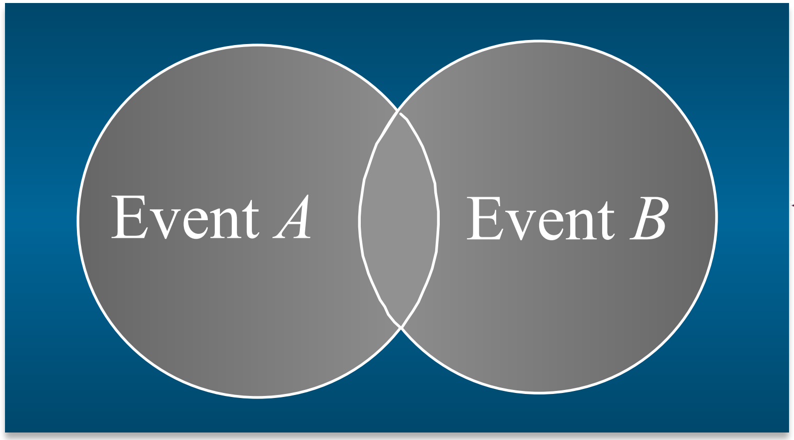
- A: student has brown hair
- B: student is female
- C: student is male

$C^{C?}$

B

Union of Two Events

- The union of events A and B is the event containing all sample points that are in A or B or both.
- The union of events A and B is denoted by $A \cup B$



Sample Space S

Example: Select a student from the classroom and record his/her hair color and gender.

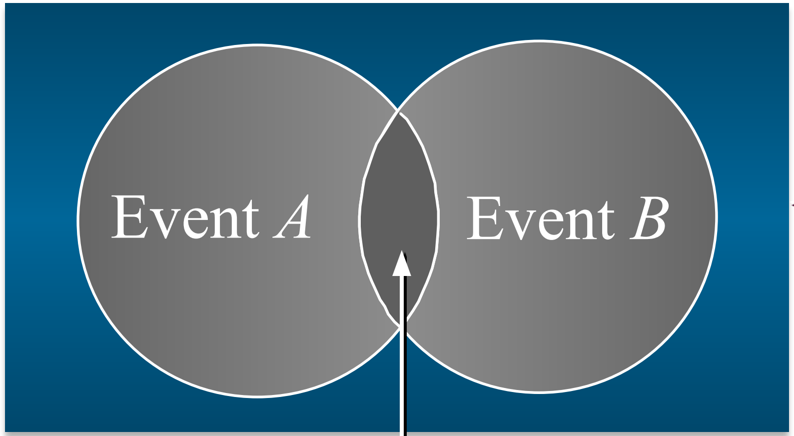
- A: student has brown hair
- B: student is female
- C: student is male

$B \cup C$?

All students

Intersection of Two Events

- The intersection of events A and B is the set of all sample points that are in both A and B.
- The intersection of events A and B is denoted by $A \cap B$



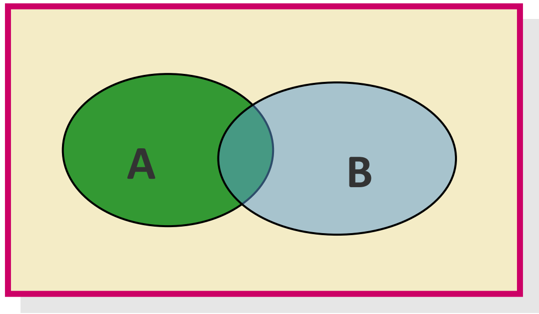
Sample Space S

Intersection of A and B

The Additive Rule for Unions

- For any two events, A and B, the probability of their union, P($A \cup B$), is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



Example: Additive Rule

- Example: Suppose that there were 120 students in the classroom, and that they could be classified as follows:
  - A: brown hair
    - P(A) = 50/120
  - B: female
    - P(B) = 60/120

	Brown	Not Brown
Male	20	40
Female	30	30

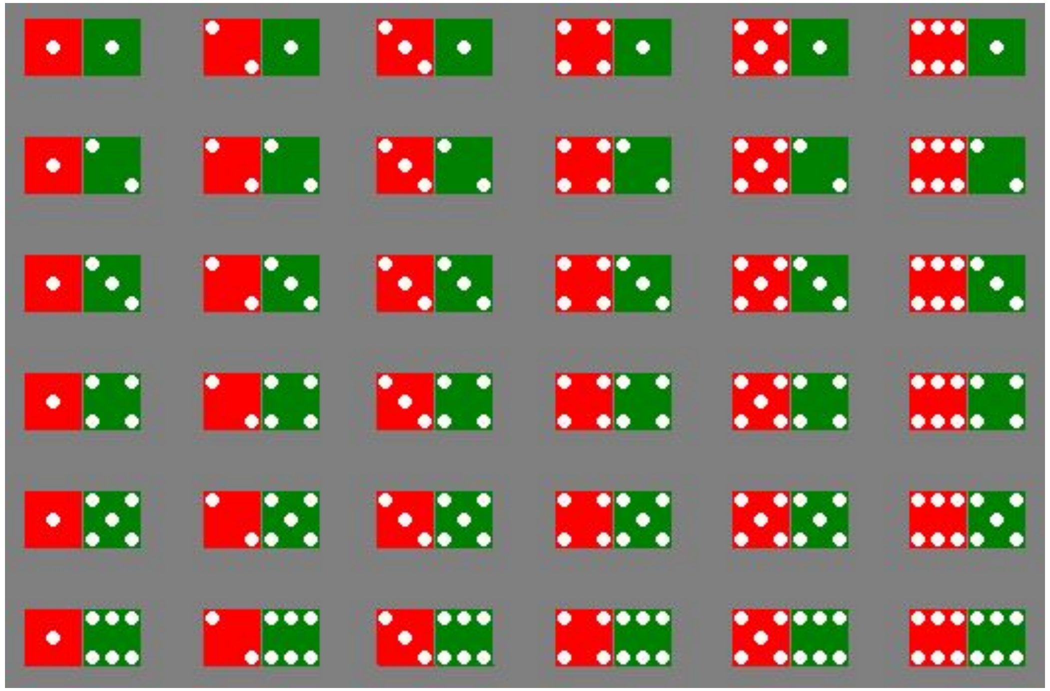
$P(A \cup B)?$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$= 50/120 + 60/120 - 30/120$
$= 80/120 = 2/3$

Example: Two Dice

A: red die show 1

B: green die show 1



$P(A \cup B)$?

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$= 6/36 + 6/36 - 1/36$
$= 11/36$

A Special Case

- **When two events A and B are mutually exclusive,** $P(A \cap B) = 0$ **and** $P(A \cup B) = P(A) + P(B)$.

A: male with brown hair

$P(A) = 20/120$

B: female with brown hair

$P(B) = 30/120$

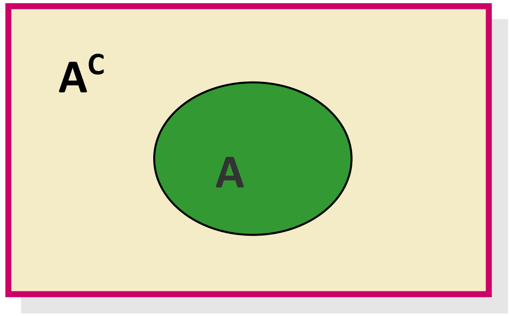
	Brown	Not Brown
Male	20	40
Female	30	30

A and B are mutually exclusive, so that

$P(A \cup B) = P(A) + P(B)$
$= 20/120 + 30/120$
$= 50/120$

Calculating Probabilities for Complements

- $P(A \cap A^C) = 0$
- $P(A \cup A^C) = 1$
  - Since either A or A^C must occur,
- $P(A) + P(A^C) = 1$
    - $P(A^C) = 1 - P(A)$



Example

- **Select a student at random from the classroom. Define:**

A: male

$P(A) = 60/120$

B: female

$P(B) = ?$

	Brown	Not Brown
Male	20	40
Female	30	30

A and B are complementary, so that

$P(B) = 1 - P(A)$

$= 1 - 60/120 = 60/120$

Calculating Probabilities for Intersections

- In the previous example, we found $P(A \cap B)$ directly from the table.

- Sometimes this is impractical or impossible. The rule for calculating $P(A \cap B)$ depends on the idea of independent and dependent events.
  - Two events, A and B, are said to be independent if the occurrence or nonoccurrence of one of the events does not change the probability of the occurrence of the other event.

Practice Exercises

- Two cards are randomly drawn from a deck of 52 playing cards. find the probability that both cards will be greater than 3 and less than 8.
- 4 candidates are seeking a vacancy on a school board. If A is twice as likely to be elected as B, and B and C are given about the same chance of being elected, while C is twice as likely to be elected as D, then what are the probabilities that C will win? A will not win?

Topic for the next class

- **Conditional Probabilities**

Reference

- Statistics with Economics and Business Applications, Chapter 3 Probability and Discrete Probability Distributions
- Modern Business Statistics, Slides by John Loucks