Eigenvectors & Eigenvalues

Transformation Matrices

- Consider the following:

$$\begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \times \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 12 \\ 8 \end{bmatrix} = 4 \times \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

- Now assume we take a multiple of (3,2)

$$2 \times \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \times \begin{bmatrix} 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 24 \\ 16 \end{bmatrix} = 4 \times \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

Transformation Matrices

- Scale vector (3,2) by a value 2 to get (6,4)
- Multiply by the square transformation matrix
- And we see that the result is still scaled by 4.
- WHY?

A vector consists of both length and direction. Scaling a vector only changes its length and not its direction. This is an important observation in the transformation of matrices leading to formation of eigenvectors and eigenvalues. Irrespective of how much we scale (3,2) by, the solution (under the given transformation matrix) is always a multiple of 4.

Eigenvalue Problem

- The eigenvalue problem is any problem having the following form:

$$A\,.\,v\;=\lambda\,.\,v$$

A: $m\times\; m$ matrix

v: $m\times\; 1$ non-zero vector

λ: scalar

Any value of λ for which this equation has a solution is called the eigenvalue of A and the vector v which corresponds to this value is called the eigenvector of A.

Eigenvalue Problem

- **Going back to our example:**

$$\begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \times \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 12 \\ 8 \end{bmatrix} = 4 \times \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

A × v λ × v

- **Therefore, (3,2) is an eigenvector of the square matrix A and 4 is an eigenvalue of A**
- **The question is:**
  - **Given matrix A, how can we calculate the eigenvector and eigenvalues for A?**

Calculating Eigenvectors & Eigenvalues

- **Simple matrix algebra shows that:**

$$A\,.\,v = \lambda\,.\,v$$

$$A\,.\,v - \lambda\,.\,I\,.\,v = 0$$

$$(A - \lambda\,.\,I\,).\,v = 0$$

- **When does this homogeneous system have a solution other than v=0?**
- **Must have that $A - \lambda\,.\,I$ is not invertible, which means that $|A - \lambda\,.\,I\,|$ = 0.**
- **Finding the roots of |A - λ . I| will give the eigenvalues and for each of these eigenvalues there will be an eigenvector**
- **Example …**

Calculating Eigenvectors & Eigenvalues

- **Let** $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$
- **Then:**

$$|A - \lambda I| = \left| \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right|$$

$$= \left| \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right|$$

$$= \left| \begin{bmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{bmatrix} \right| = \big(\lambda \times (-3-\lambda)\big) - (-2 \times 1)$$

$$= \lambda^2 + 3\lambda + 2$$

- **And setting the determinant to 0, we obtain 2 eigenvalues:** $\lambda_1 = -1 \; and \; \lambda_2 = -2$

Calculating Eigenvectors & Eigenvalues

- **For λ_1 the eigenvector is:**

$$(A - \lambda.I).v = 0$$

$$\begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0$$

$v_{11} + v_{12} = 0$ **and** $-2v_{11} - 2v_{12} = 0$

$$v_{11} = -v_{12}$$

Calculating Eigenvectors & Eigenvalues

- Therefore, eigenvector v_1 is

$$v_1 = k_1 \begin{bmatrix} +1 \\ -1 \end{bmatrix}$$

where k_1 is some constant.

- Similarly, we find that eigenvector v_2

$$v_2 = k_2 \begin{bmatrix} +1 \\ -2 \end{bmatrix}$$

where k_2 is some constant.

Properties of Eigenvectors and Eigenvalues

- **Eigenvectors can only be found for square matrices and not every square matrix has eigenvectors.**
- **Given an *m* x *m* matrix (with eigenvectors), we can find *m* eigenvectors.**
- **All eigenvectors of a symmetric matrix are perpendicular to each other, no matter how many dimensions we have.**
- **In practice eigenvectors are normalized to have unit length.**

Reference

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