Vector Space

Recall: Vector space

- A vector space is a nonempty set V of objects, called vectors, which can be added and multiplied by numbers, in such a way that the sum of two elements of V is again an element of V, the product of an element of V by a number is an element of V, and the following properties are satisfied:
- Given the elements u, v, w of V, we have
  1. The sum of u and v, denoted by $\mathrm{u} + \mathrm{v}$, is in V.
  2. $\mathrm{u} + \mathrm{v} = \mathrm{v} + \mathrm{u}$
  3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
  4. There is a zero vector 0 in V such that, $\mathbf{u} + \mathbf{0} = \mathbf{u}, \mathbf{0} + \mathbf{u} = \mathbf{u}$
  5. for each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

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6. The scalar multiple of u by c, denoted by $c\boldsymbol{u}$, is in V
7. If c is a number, $c(\boldsymbol{u}+\boldsymbol{v}) = c\boldsymbol{u} + c\boldsymbol{v}$
8. $(\boldsymbol{u}+\boldsymbol{v})c = c\boldsymbol{u} + c\boldsymbol{v}$
9. If c, d are two number, then $(cd)v = c(d\boldsymbol{v})$
10. For all elements u of V, we have $1 \cdot u = u$ (1 here is the number one).

Recall: Subspaces

- Let V be a vector space, and let H be a subset of V. Assume that H satisfies the following conditions.:
  1. The zero vector of V is in H.
  2. If u, v are elements of H, then their sum $u + v$ is also an element of H.
  3. If v is an element of H and c a number, the vector cu is in H.
- Properties (1), (2), and (3) guarantee that a subspace H of V is itself a vector space, under the vector space operations already defined in V.
- Every subspace is a vector space.

Linear Combination

- Let V be a vector space, and $v_1, v_2, \ldots, v_n$ be elements of V. Let n real number $c_1, c_2, \ldots, c_n$, then the vector w obtained by

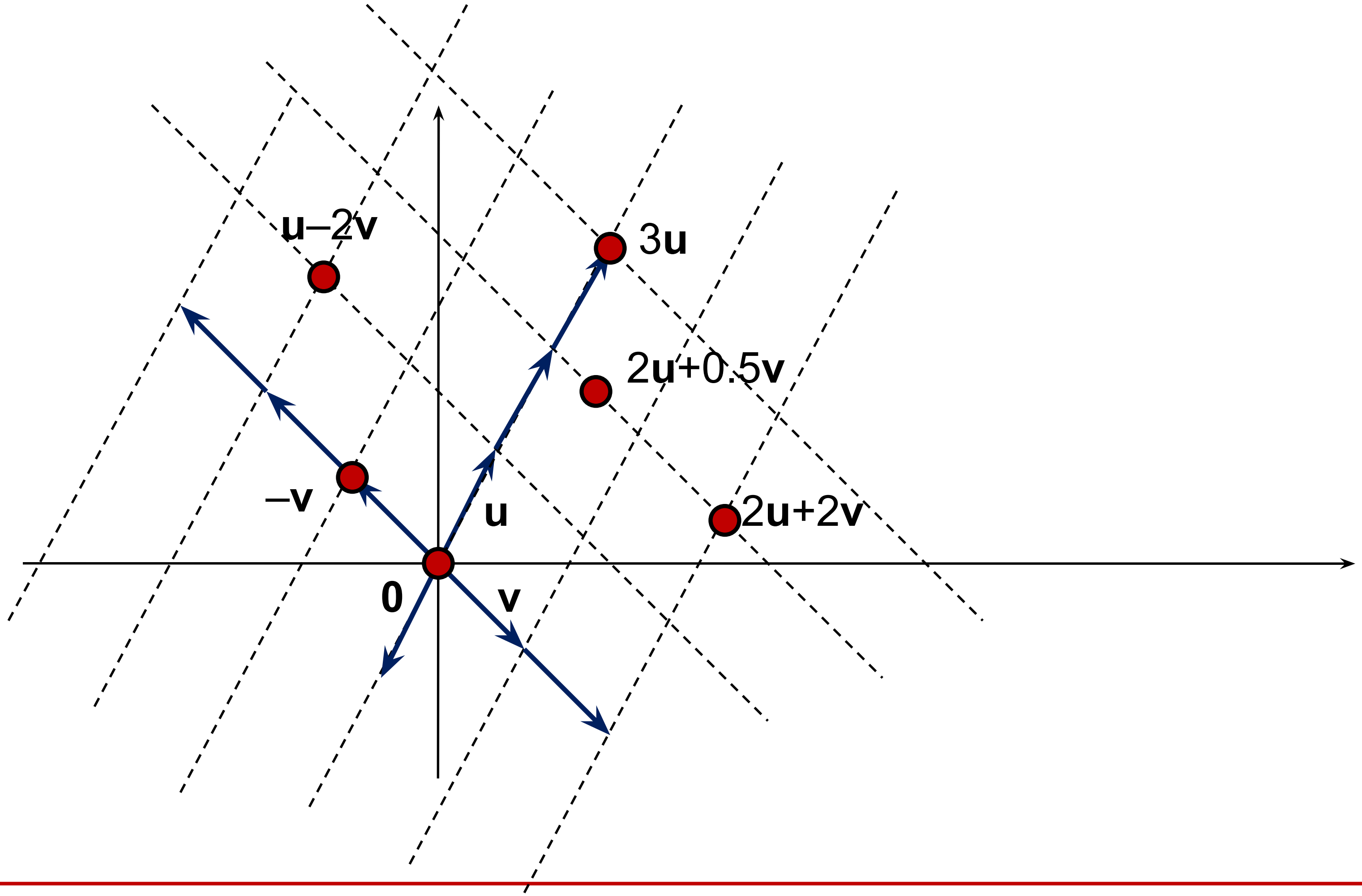
$$w = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n$$

  is called a linear combination of $v_1, v_2, \ldots, v_n$, and $c_1, c_2, \ldots, c_n$ are called coefficients of the linear combination.

- The set of all linear combinations of $v_1, v_2, \ldots, v_n$ is a subspace of V

Picture

- **Linear combinations of two vectors u and v.**



A Subspace Spanned by a Set

- **The set consisting of only the zero vector in a vector space *V* is a subspace of *V*, called the zero subspace and written as {0}.**
- **Span $\{v_1,\ldots,v_p\}$ denotes the set of all vectors that can be written as linear combinations of $v_1,\ldots,v_p$.**

A Subspace Spanned by a Set

- Example 2: Given v_1 and v_2 in a vector space V, let $H = span\{v_1, v_2\}$. Show that H is a subspace of V.

- Solution: The zero vector is in H, since $\mathbf{0} = 0v_1 + 0v_2$.
  - To show that H is closed under vector addition, take two arbitrary vectors in H, say,

$$u = s_1v_1 + s_2v_2 \text{ and } w = t_1v_1 + t_2v_2$$

  - By Axioms 2, 3, and 7 for the vector space V, (Commutative law, Associative law and Distributive law)

$$u + w = (s_1v_1 + s_2v_2) + (t_1v_1 + t_2v_2)$$

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- Theorem 1: If $v_1,\dots,v_p$ are in a vector space *V*, then Span $\{v_1,\dots,v_p\}$ is a subspace of *V*.
- We call Span $\{v_1,\dots,v_p\}$ the subspace spanned (or generated) by $\{v_1,\dots,v_p\}$.

Convex Sets

- Let S be a subset of a vector space V. We shall say that S is convex if given points P, Q in S then the line segment between P and Q is contained in S.

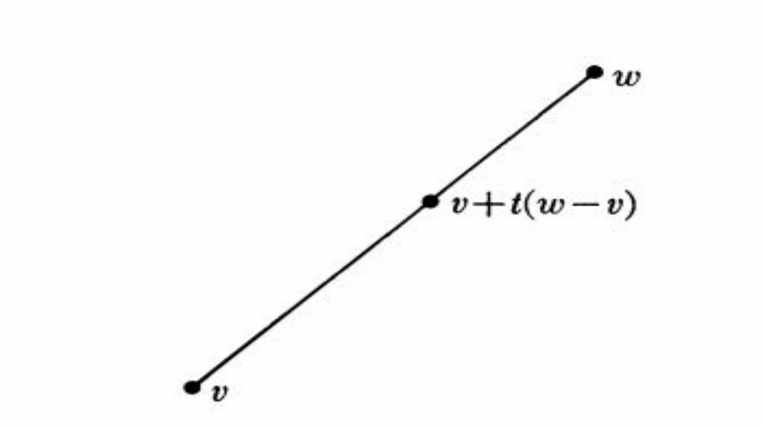
Convex set   *Not* convex

- The line segment between P and Q consists of all points

$$(1-t)P + tQ \ with \ , 0 \leq t \leq 1$$

- Line Segment: If v, w are elements of V, let u = w - v. Then the line segment between v and w is the set of all points v + tu, or

$$v + t(w - v), 0 \leq t \leq 1$$



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