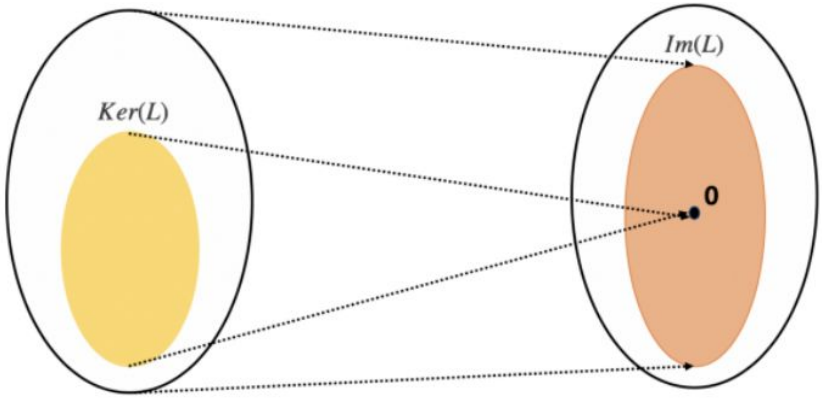
Linear Mapping
Kernel and Image space of a linear map
Matrix associated with linear map

Mappings

- **Let S, S' be two sets. A mapping from S to S' is an association which to every element of S associates an element of S'.**

$$F\colon S \rightarrow S'$$

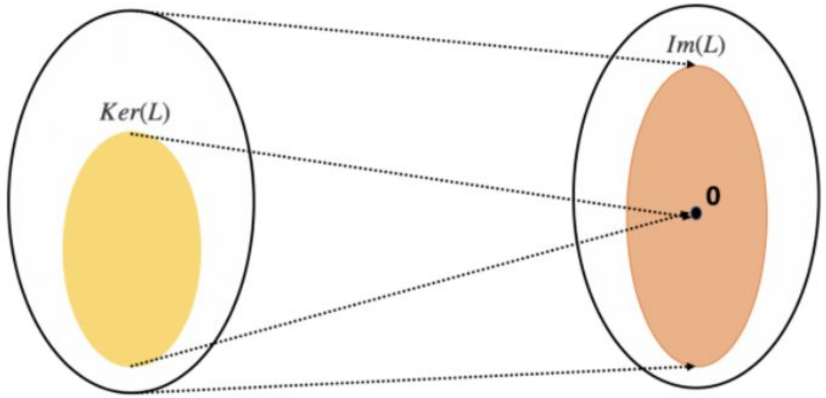
- **For instance, if $\boldsymbol{T}\colon S \rightarrow S'$ is a mapping, and if u is an element of S, then we denote by $T(u)$, or Tu, the element of S' associated to $\boldsymbol{u}$ by T.**
- **$T(u)$ is called the value of T at u, or also the image of u under T, i.e., $\boldsymbol{u} \mapsto \boldsymbol{T(u)}$.**
- **The set of all elements $T(u)$, when u ranges over all elements of S, is called the image of T.**



Mappings

- For any set S we have the identity mapping $I: S \to S$. It is defined by $I(x) = x$ for all x.

- Let S be the set $\mathbb{R}^3$, i.e., the set of *3-tuples*. Let $A = (2,3,-1)$ and $L: \mathbb{R}^3 \to \mathbb{R}$ be the mapping whose value at a vector $\boldsymbol{x} = (x, y, z)$ is $A \cdot \boldsymbol{x}$.
  - Then $L(\boldsymbol{x}) = A \cdot \boldsymbol{x}$.
  - If $\boldsymbol{x} = (1, 1, -1)$, then the value of L at X is 6.

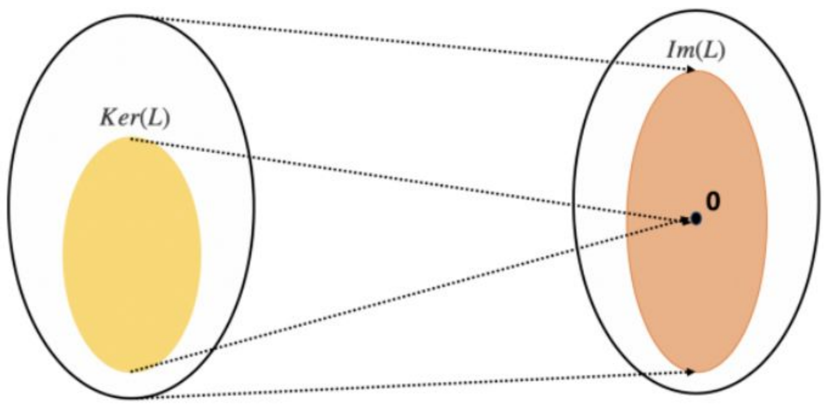


Mappings

- In the m × n linear system $Ax = 0$, we can regard A as transforming elements of $\mathbb{R}^n$(as column vectors) into elements of $\mathbb{R}^m$ via the rule

$$T(x) = Ax$$

- Then solving the system amounts to finding all of the vectors $x \in \mathbb{R}^n$ such that $T(x) = 0$.

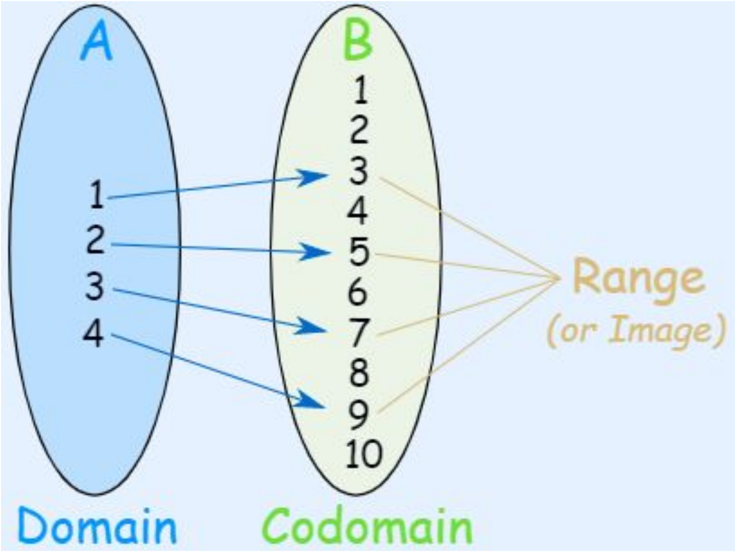


Linear Transformation

- Let V and W be vector spaces. A function $T : V \rightarrow W$ is called a **linear transformation** if for any vectors u, v in V and scalar c
  1. $\mathbf{T(u + v) = T(u) + T(v)}$,
  2. $\mathbf{T(cu) = cT(u)}$.
- V is called the **domain** and W the **codomain** of T.
- Example
  - Let A be a given $m\ x\ n$ matrix. Define

$$\mathbf{T_A}:\ \mathbb{R}^n \rightarrow \mathbb{R}^m$$

by the formula $T_A(x) = Ax$ Then T_A is linear



$T:\ x \rightarrow 2x + 1$

Image Source: https://www.mathsisfun.com/sets/domain-range-codomain.html

Linear Transformation

- Let V, W be any vector spaces. The mapping which associates the element 0 in W to any element u of V is called the zero mapping and is obviously linear.
- Let $L: V \to W$ be a linear map. Then $L(O) = O$.
  - Proof
    - We have $L(\mathbf{0}) = L(\mathbf{0} + \mathbf{0}) = L(\mathbf{0}) + L(\mathbf{0})$. Subtracting L(O) from both sides yields $0 = L(\mathbf{0})$, as desired.
- Let $L: V \to W$ be a linear map. Then $L(-v) = -L(v)$.
  - Proof
    - We have $0 = L(O) = L(v - v) = L(v) + L(-v)$. Add $-L(v)$ to both sides to get the desired assertion

Linear Transformation

- **Let $L: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map. Suppose that $L(1,1) = (1,4)$ and $L(2,-1) = (-2,3)$. Find L(3, - 1).**

- **We write (3, -1) as a linear combination of (1,1) and (2, -1). Thus, we have to solve**

$$(3,-1) = x(\mathbf{1},\mathbf{1}) + y(2,-1)$$

- **This amounts to solving**

$$x + 2y = 3,$$
$$x - y = -1$$

The solution is $x = \frac{1}{3}$, $y = \frac{4}{3}$. Hence

$$L(3,-1) = xL(1,1) + yL(2,-1) = \frac{\mathbf{1}}{\mathbf{3}}(1,4) + \frac{\mathbf{4}}{\mathbf{3}}(-2,3) = \left(\frac{-\mathbf{7}}{\mathbf{3}}, \frac{\mathbf{16}}{\mathbf{3}}\right).$$

The coordinates of a linear map

- Let $\boldsymbol{F}: \boldsymbol{V} \rightarrow \mathbb{R}^{\boldsymbol{n}}$, be any mapping. Then each value $F(v)$ is an element of $\mathbb{R}^{\boldsymbol{n}}$, and so has coordinates. Thus, we can write

$$\boldsymbol{F}(\boldsymbol{v}) = \left(\boldsymbol{F}_{\mathbf{1}}(\boldsymbol{v}), \boldsymbol{F}_{\mathbf{2}}(\boldsymbol{v}), \dots, \boldsymbol{F}_{\boldsymbol{n}}(\boldsymbol{v})\right) \textbf{ or } \mathbf{F} = (\mathrm{F}_1, \dots \mathrm{F}_\mathrm{n})$$

- Each F_i is a function of V into $\mathbb{R}$, which we write

$$\boldsymbol{F}_{\boldsymbol{i}}: \boldsymbol{V} \rightarrow \mathbb{R}$$

- Example: Let $F: \mathbb{R}^{\mathbf{2}} \rightarrow \mathbb{R}^{\mathbf{3}}$ be the mapping

$$F(x, y) = (2x - y, 3x + 4y, x - 5y).$$

- Then,

$$F_1(x, y) = 2x - y, F_2(x, y) = 3x + 4y, F_3(x, y) = x - 5y$$

The coordinates of a linear map

- $F_1(x, y) = 2x - y, F_2(x, y) = 3x + 4y, F_3(x, y) = x - 5y$
- **Each coordinate function can be expressed in terms of a dot product. For instance, let**

$$A_1 = (2, -1), A_2 = (3,4), A_3 = (1, -5)$$

- **Then,**

$$\boldsymbol{F_i(x, y) = A_i.[x, y]} \text{ for } \boldsymbol{i = 1, 2, 3}$$

- **Each function $\boldsymbol{x \mapsto A_i x}$, is linear**

The coordinates of a linear map

- **Proposition.** Let $F: V \to \mathbb{R}^n$ be a mapping of a vector space V into $\mathbb{R}^n$. Then F is linear if and only if each coordinate function $F_i: V \to \mathbb{R}$ is linear, for $i = 1, \ldots, n$.
- **Proof.** For $v, w \in V$ we have

$$\boldsymbol{F}(\boldsymbol{w}) = \left(\boldsymbol{F}_1(\boldsymbol{w}), \ldots, \boldsymbol{F}_n(\boldsymbol{w})\right)$$
$$\boldsymbol{F}(\boldsymbol{v}) = \left(\boldsymbol{F}_1(\boldsymbol{v}), \ldots, \boldsymbol{F}_n(\boldsymbol{v})\right)$$
$$\boldsymbol{F}(\boldsymbol{v} + \boldsymbol{w}) = (\boldsymbol{F}_1(\boldsymbol{v} + \boldsymbol{w}), \ldots, \boldsymbol{F}_n(\boldsymbol{v} + \boldsymbol{w})),$$

Thus $\boldsymbol{F}(\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{F}(\boldsymbol{v}) + \boldsymbol{F}(\boldsymbol{w})$ if and only $\boldsymbol{if}$ $\boldsymbol{F}_i(\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{F}_i(\boldsymbol{v}) + \boldsymbol{F}_i(\boldsymbol{w})$ for all $\boldsymbol{i} = \boldsymbol{1}, \ldots, \boldsymbol{n}$ by the definition of addition of n-tuples.

- The same argument shows that if $c \in \mathbb{R}$, then $F(cv) = cF(v)$ if and only if $\boldsymbol{f}_i(\boldsymbol{cv}) = \boldsymbol{cf}_i, \forall \boldsymbol{i} = \boldsymbol{1}, \boldsymbol{2}, \ldots, \boldsymbol{n}$

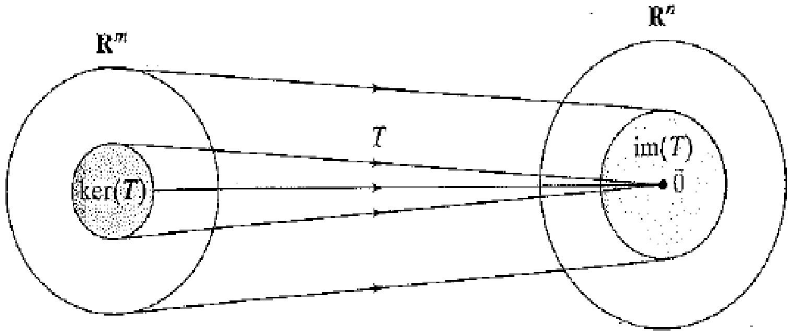
Linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$

- **Let A be an $m \times n$ matrix with real entries and define $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ by $T(x) = Ax$. Verify that T is a linear transformation**
  - **If x is an n × 1 column vector, then Ax is an m × 1 column vector.**
  - $T(x + y) = A(x + y) = Ax + Ay = T(x) + T(y)$
  - $T(cx) = A(cx) = cAx = c\,T(x)$
- **Such a transformation is called a matrix transformation.**

The Kernel and Image of a Linear Map

- Let $F: V \to W$ be a linear map. The **image** of F is the set of elements w in W such that there exists an element v of V such that $F(v) = w$.
- The image of F is a subspace of W
- Proof.
  - $F(0) = 0$, and hence 0 is in the image.
  - Suppose that w_1, w_2 are in the image. Then there exist elements v_1, v_2 of V such that $F(v_1) = w_1$ and $F(v_2) = w_2$. Hence

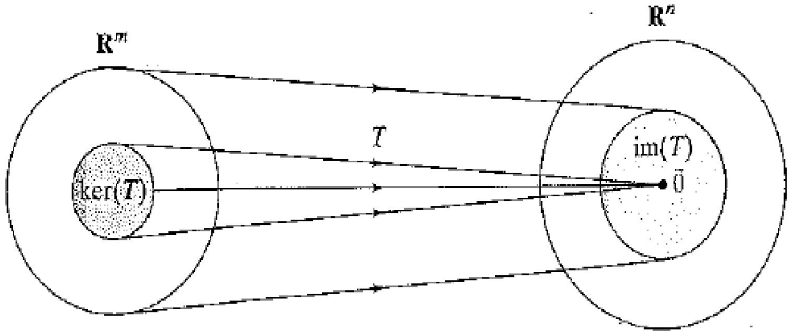
    $\boldsymbol{F(v_1 + v_2) = F(v_1) + F(v_2) = w_1 + w_2}$ is in the image
  - If c is a number, then $F(cv_1) = cF(v_1) = cw_1$. Hence cw_1 is in the image



- Let V be a vector space, and let H be a subset of V. Assume that H satisfies the following conditions.:
  1. The zero vector of V is in H.
  2. If u, v are elements of H, then their sum $u + v$ is also an element of H.
  3. If v is an element of H and c a number, the vector cu is in H.
- Properties (1), (2), and (3) guarantee that a subspace H of V is itself a vector space, under the vector space operations already defined in V.
- Every subspace is a vector space.

The Kernel and Image of a Linear Map

- Let V, W be vector spaces, and let $F: V \to W$ be a linear map. The set of elements $v \in V$ such that $F(v) = 0$ is called the **kernel** of F.
- kernel is a subspace
- Proof.
  - $F(0) = 0$, 0 is in the kernel.
  - Then $F(v + w) = F(v) + F(w) = 0 + 0 = 0$, so that $v + w$ is in the kernel.
  - If c is a number, then $F(cv) = cF(v) = 0$ so that cv is also in the kernel.
  - Hence the kernel is a subspace



- Let V be a vector space, and let H be a subset of V. Assume that H satisfies the following conditions.:
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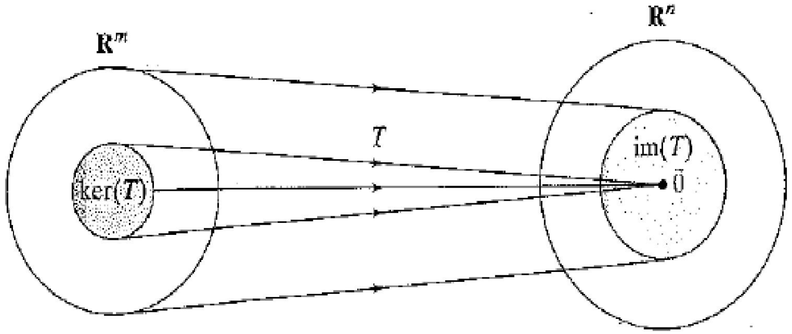
- Every subspace is a vector space.

The Kernel and Image of a Linear Map

- Let V, W be vector spaces, and let $F: V \to W$ be a linear map. The set of elements $v \in V$ such that $F(v) = 0$ is called the **kernel** of F.
- Example
  - Let $L: \mathbb{R}^3 \to \mathbb{R}$ be the map such that
    $$L(x, y, z) = 3x - 2y + z$$
  - Thus, if $A = (3, -2, 1)$, then we can write $L(X) = X \cdot A = A \cdot X$.
  - Then the kernel of L is the set of solutions of the equation.
    $$3x - 2y + z = 0$$



- Let V be a vector space, and let H be a subset of V. Assume that H satisfies the following conditions.:
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The Kernel and Image of a Linear Map

- Let V, W be vector spaces, and let $F: V \to W$ be a linear map. The set of elements $v \in V$ such that $F(v) = 0$ is called the kernel of F.
- Example
  - Let $\boldsymbol{P}: \mathbb{R}^{\mathbf{3}} \to \mathbb{R}^{\mathbf{2}}$ be the map such that
    $$\boldsymbol{P}(x, y, z) = (\boldsymbol{x}, \boldsymbol{y})$$
  - Then P is a linear map whose kernel consists of all vectors in $\mathbb{R}^3$ whose first two coordinates are equal to 0, i.e., all vectors $(0, 0, z)$ with arbitrary component z.

Matrix transformations

- **Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is described by the matrix transformation $T(x) = Ax$.**

$$\boldsymbol{A} = [\boldsymbol{T}(\boldsymbol{e_1}), \boldsymbol{T}(\boldsymbol{e_2}), \dots, \boldsymbol{T}(\boldsymbol{e_n})]$$

  and $\boldsymbol{e_1}, \boldsymbol{e_2}, \dots, \boldsymbol{e_n}$ denote the standard basis vectors for $\mathbb{R}^n$. This A is called the matrix of T

- **Given a linear map $\boldsymbol{L}: \mathbb{R}^n \to \mathbb{R}^{\boldsymbol{m}}$, there is some associated matrix A such that $L = L_A$**

The Matrix Associated with a Linear Map

- **Given a linear map $L: \mathbb{R}^n \to \mathbb{R}^m$, there is some associated matrix A such that $L = L_A$**
- **Proof.**
  - **Let $E^1, \dots, E^n$ be the unit column vectors of $\mathbb{R}^n$. For each $j = 1, \dots, n$, let $L(E^j) = A^j$, where A^j is a column vector in $\mathbb{R}^m$. Thus**

$$L(E^1) = \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix} = A^1, \dots, \qquad L(E^n) = \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix} = A^n.$$

- **Then for every element $\boldsymbol{x}$ in $\mathbb{R}^n$, we can write**

$$\boldsymbol{x} = x_1 E^1 + x_2 E^2 + \cdots + x_n E^n = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

The Matrix Associated with a Linear Map

- Then for every element $\boldsymbol{x}$ in $\mathbb{R}^n$, we can write

$$\boldsymbol{x} = x_1E^1 + x_2E^2 + \cdots + x_nE^n = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

- Therefore

$$\begin{aligned} L(x) &= x_1L(E^1) + x_2L(E^2) + \cdots + x_nL(E^n) \\ &= x_1A^1 + x_2A^2 + \cdots + x_nA^n \\ &= Ax \end{aligned}$$

  where A is the matrix whose column vectors are $A^1, \ldots, A^n$.

- The matrix A is then called the matrix associated with the linear map L.

The Matrix Associated with a Linear Map

- **Example: Determine the matrix of the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^2$ defined by**

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- **Then the matrix associated with T is**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Reference

- Introduction to Linear Algebra, Serge Lang, Second Edition
- Linear Transformations, Math 240 , Calculus III, Summer 2013, Session II