System of linear equations-II

Recall: Gauss-Jordan Elimination

- The Gauss-Jordan elimination method is a technique for solving systems of linear equations of any size.
- The operations of the Gauss-Jordan method are
  - Interchange any two equations.
  - Replace an equation by a nonzero constant multiple of itself.
  - Replace an equation by the sum of that equation and a constant multiple of any other equation.

Recall: Row-Reduced Form of a Matrix

- Each row consisting entirely of zeros lies below all rows having nonzero entries.
- The first nonzero entry in each nonzero row is 1 (called a leading/pivot 1).
- In any two successive (nonzero) rows, the leading 1 in the lower row lies to the right of the leading 1 in the upper row.
- If a column contains a leading 1, then the other entries in that column are zeros.

Row Operations

1. Interchange any two rows.
2. Replace any row by a nonzero constant multiple of itself.
3. Replace any row by the sum of that row and a constant multiple of any other row.

Recall: The Gauss-Jordan Elimination Method

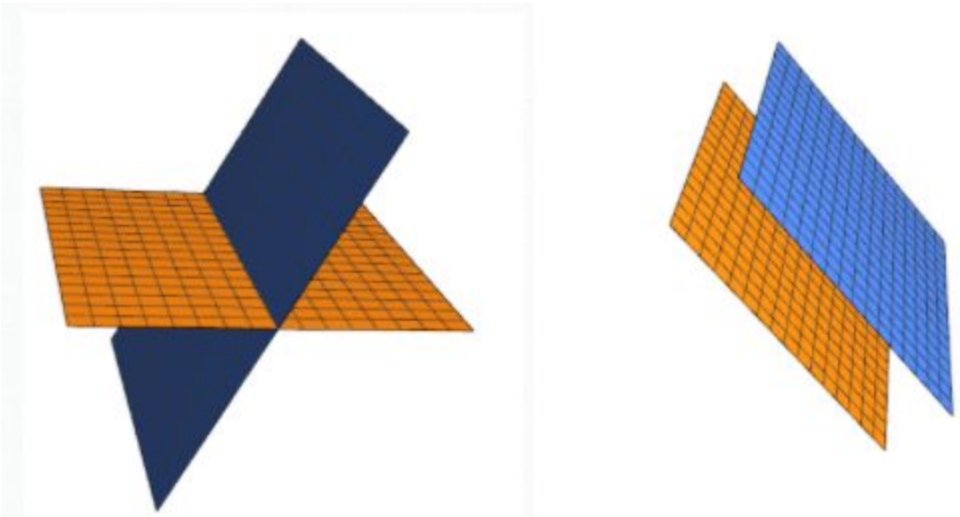
1. Write the augmented matrix corresponding to the linear system.
2. Interchange rows, if necessary, to obtain an augmented matrix in which the first entry in the first row is nonzero. Then pivot the matrix about this entry.
3. Interchange the second row with any row below it, if necessary, to obtain an augmented matrix in which the second entry in the second row is nonzero. Pivot the matrix about this entry.
4. Continue until the final matrix is in row-reduced form.

Recall: System of linear equations

- Consider a linear system $Ax = b$. Then
  1. a solution of Ax = b is a vector y such that the matrix product Ay indeed equals b.
  2. the set of all solutions is called the solution set of the system.
  3. this linear system is called consistent if it admits a solution and is called inconsistent if it admits no solution

Recall: System of linear equations

- **Underdetermined if n>m: “more unknowns than equations”**
  - cannot have a unique solution.
  - In this case, there are either infinitely many or no solutions.
  - For an example of this, refer to what can happen with only two planes in three dimensions:



- **Overdetermined if m>n: “more equations than unknowns”**
  - An overdetermined system may also have infinitely many or no solutions or may have a unique solution

System of linear equations

- **Theorem.** **If there is a row in an augmented matrix containing all zeros to the left of the vertical line and a nonzero entry to the right of the line, then the corresponding system of equations has no solution.**

$$
\begin{aligned}
x_1 + x_2 + x_3 &= 1 \\
3x_1 - x_2 - x_3 &= 4 \\
x_1 + 5x_2 + 5x_3 &= -1
\end{aligned}
$$

$$
\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 3 & -1 & -1 & 4 \\ 1 & 5 & 5 & -1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -4 & -4 & 1 \\ 0 & 0 & 0 & -1 \end{array}\right]
$$

inconsistent

Homogenous system of linear equations

- **The homogeneous system of linear equations always has a solution, namely the solution obtained by letting all $x_i = 0$. This solution will be called the trivial solution.**
- **A solution $(x_1, \ldots, x_n)$ such that some x_i $is \neq 0$ is called non-trivial.**

Homogenous system of linear equations

- **Theorem.**

Let

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ \vdots \qquad\qquad\qquad\qquad & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

where for $1 \leq i \leq m$ and $1 \leq j \leq n$; $a_{ij} \in R$, be a system of m linear equations in n variables $x_1, x_2, \ldots, x_n$, and assume that $n > m$. Then the system has a non-trivial solution.

Homogenous system of linear equations

- Case I: one equation in n unknowns, $n > 1$:
- If all coefficients $a_1, \dots, a_n$ are equal to 0, then any value of the variables will be a solution, and a non-trivial solution certainly exists.
- Suppose that some coefficient a_i $is \neq 0$. After renumbering the variables and the coefficients, we may assume that it is a_1.
- Then we give $x_2, \dots, x_n$ arbitrary values, for instance we let $x_2 = \dots = x_n = 1$, and solve for x_1, letting

$$x_1 = -\frac{1}{a_1}(a_2 + \cdots + a_n)$$

- In that manner, we obtain a non-trivial solution for our system of equations.

Homogenous system of linear equations

- Let us now assume that our theorem is true for a system of m – 1 equations in more than m - 1 unknowns. We shall prove that it is true for m equations in n unknowns when n > m.
- If all coefficients a_{ij} are equal to 0, we can give any non-zero value to our variables to get a solution.
- Suppose that some coefficient a_{ij} $is \neq 0$. After renumbering the variables and the coefficients, we may assume that it is a_{11} .
- We shall subtract a multiple of the first equation from the others to eliminate x_1.

Homogenous system of linear equations

- Let us now assume that our theorem is true for a system of m – 1 equations in more than m - 1 unknowns. We shall prove that it is true for m equations in n unknowns when n > m.
- Namely, we consider the system of equations

$$\left(A_2 - \frac{a_{21}}{a_{11}} A_1\right) X = 0$$
$$\left(A_3 - \frac{a_{31}}{a_{11}} A_1\right) X = 0$$
$$\vdots$$
$$\left(A_m - \frac{a_{m1}}{a_{11}} A_1\right) X = 0$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

A_i refers to the ith row of A, and

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Homogenous system of linear equations

- Let us now assume that our theorem is true for a system of m – 1 equations in more than m - 1 unknowns. We shall prove that it is true for m equations in n unknowns when n > m.
- which can also be written in the form

$$A_2X - \frac{a_{21}}{a_{11}}A_1X = 0$$

$$A_3X - \frac{a_{31}}{a_{11}}A_1X = 0$$

$$\vdots$$

$$A_mX - \frac{a_{m1}}{a_{11}}A_1X = 0$$

In this system, the coefficient of x_1 is equal to 0. Hence, we may view the above equation as a system of $m - 1$ equations in $n - 1$ unknowns, and we have $n - 1 > m - 1$.

- According to our assumption, we can find a non-trivial solution $(x_2, \dots, x_n)$ for this system.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

A_i refers to the ith row of A, and

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Homogenous system of linear equations

- **Case II:** m **equation in** n **unknowns,** $\boldsymbol{n} > \boldsymbol{m}$**:**
- **We can then solve for** x_1 **in the first equation, namely**

$$\boldsymbol{x_1 = -\frac{1}{a_{11}}(a_{12}x_2 + \cdots + a_{1n}x_n)}$$

- **In that way, we find a solution of** $A_1\ X = O$**. But previously we have**

$$\boldsymbol{A_iX = \frac{a_{i1}}{a_{11}}A_1X, for\ i = 2, \ldots m.}$$

- **Hence** $A_i \cdot X = 0$ **for** $\boldsymbol{i = 2, \ldots m}$**, and we have found the non-trivial solution for the same previously, therefore we have found a non-trivial solution to our original system**

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

A_i refers to the ith row of A, and

$$\boldsymbol{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Homogenous system of linear equations

- Use Gaussian elimination to solve the following homogeneous system of equations.

$$x_1 - x_2 - x_3 + 3x_4 = 0$$
$$x_1 + x_2 - 2x_3 + x_4 = 0$$
$$4x_1 - 2x_2 + 4x_3 + x_4 = 0$$

Source: https://portal.tpu.ru/SHARED/k/KONVAL/Sites/English_sites/M/s_HomoE.htm

Homogenous system of linear equations: non-trivial Solution

$$x_1 - x_2 - x_3 + 3x_4 = 0$$
$$x_1 + x_2 - 2x_3 + x_4 = 0$$
$$4x_1 - 2x_2 + 4x_3 + x_4 = 0$$

- $\begin{bmatrix} 1 & -1 & -1 & 3 \\ 1 & 1 & -2 & 1 \\ 4 & -2 & 4 & 1 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - R_1} \begin{bmatrix} 1 & -1 & -1 & 3 \\ 0 & 2 & -1 & -2 \\ 4 & -2 & 4 & 1 \end{bmatrix}$

$$\xrightarrow{R_3 \leftarrow R_3 - 4R_1} \begin{bmatrix} 1 & -1 & -1 & 3 \\ 0 & 2 & -1 & -2 \\ 0 & 2 & 8 & -11 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - R_2} \begin{bmatrix} 1 & -1 & -1 & 3 \\ 0 & 2 & -1 & -2 \\ 0 & 0 & 9 & -9 \end{bmatrix}$$

Homogenous system of linear equations: non-trivial Solution

$$\begin{aligned} x_1 - x_2 - x_3 + 3x_4 &= 0 \\ x_1 + x_2 - 2x_3 + x_4 &= 0 \\ 4x_1 - 2x_2 + 4x_3 + x_4 &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & -1 & -1 & 3 \\ 0 & 2 & -1 & -2 \\ 0 & 0 & 9 & -9 \end{bmatrix}$$

- The rank of this matrix equals 3, and so the system with four unknowns has an infinite number of solutions, depending on one free variable.

Homogenous system of linear equations: non-trivial Solution

$$\begin{aligned} x_1 - x_2 - x_3 + 3x_4 &= 0 \\ x_1 + x_2 - 2x_3 + x_4 &= 0 \\ 4x_1 - 2x_2 + 4x_3 + x_4 &= 0 \end{aligned} \longrightarrow \begin{bmatrix} 1 & -1 & -1 & 3 \\ 0 & 2 & -1 & -2 \\ 0 & 0 & 9 & -9 \end{bmatrix}$$

- **The rank of this matrix equals 3, and so the system with four unknowns has an infinite number of solutions, depending on one free variable.**
- **If we choose x_4 as the free variable and set x_4 =c, then the leading unknowns have to be expressed through the parameter c.**

Homogenous system of linear equations: non-trivial Solution

$$\begin{aligned} x_1 - x_2 - x_3 + 3x_4 &= 0 \\ x_1 + x_2 - 2x_3 + x_4 &= 0 \\ 4x_1 - 2x_2 + 4x_3 + x_4 &= 0 \end{aligned} \longrightarrow \begin{bmatrix} 1 & -1 & -1 & 3 \\ 0 & 2 & -1 & -2 \\ 0 & 0 & 9 & -9 \end{bmatrix}$$

- If we choose x_4 as the free variable and set x_4 =c, then the leading unknowns have to be expressed through the parameter c.
- The above matrix corresponds to the following homogeneous system

$$\begin{aligned} x_1 - x_2 - x_3 + 3c &= 0 \\ 2x_2 - x_3 - 2c &= 0 \\ 9x_3 - 9c &= 0 \end{aligned}$$

Homogenous system of linear equations: non-trivial Solution

$$\begin{aligned} x_1 - x_2 - x_3 + 3x_4 &= 0 \\ x_1 + x_2 - 2x_3 + x_4 &= 0 \\ 4x_1 - 2x_2 + 4x_3 + x_4 &= 0 \end{aligned} \rightarrow \begin{bmatrix} 1 & -1 & -1 & 3 \\ 0 & 2 & -1 & -2 \\ 0 & 0 & 9 & -9 \end{bmatrix} \rightarrow \begin{aligned} x_1 - x_2 - x_3 + 3c &= 0 \\ 2x_2 - x_3 - 2c &= 0 \\ 9x_3 - 9c &= 0 \end{aligned}$$

- The last equation implies

$$x_3 = c$$

- Using the method of back substitution, we obtain

$$2x_2 = x_3 + 2c = 3c \Rightarrow x_2 = \frac{3}{2}c$$

$$x_1 = x_2 + x_3 - 3c = \frac{3}{2}c + c - 3c \Rightarrow x_1 = -\frac{1}{2}c$$

Homogenous system of linear equations: non-trivial Solution

$$\begin{aligned} x_1 - x_2 - x_3 + 3x_4 &= 0 \\ x_1 + x_2 - 2x_3 + x_4 &= 0 \\ 4x_1 - 2x_2 + 4x_3 + x_4 &= 0 \end{aligned} \rightarrow \begin{bmatrix} 1 & -1 & -1 & 3 \\ 0 & 2 & -1 & -2 \\ 0 & 0 & 9 & -9 \end{bmatrix} \rightarrow \begin{aligned} x_1 - x_2 - x_3 + 3c &= 0 \\ 2x_2 - x_3 - 2c &= 0 \\ 9x_3 - 9c &= 0 \end{aligned}$$

- $$X = \begin{bmatrix} -\frac{1}{2}c \\ \frac{3}{2}c \\ c \\ c \end{bmatrix} = c \begin{bmatrix} -\frac{1}{2} \\ \frac{3}{2} \\ 1 \\ 1 \end{bmatrix}$$

Homogenous system of linear equations: Trivial Solution

$$\begin{aligned} -x_1 + x_2 - x_3 &= 0 \\ 3x_1 - x_2 - x_3 &= 0 \\ 2x_1 + x_2 - 3x_3 &= 0 \end{aligned} \longrightarrow \begin{bmatrix} -1 & 1 & -1 \\ 3 & -1 & -1 \\ 2 & 1 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 & 1 & -1 \\ 0 & 2 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

- The rank of this matrix equals 3. Therefore, there are no free variables and the system

$$\begin{aligned} -x_1 + x_2 - x_3 &= 0 \\ 2x_2 - 3x_3 &= 0 \\ x_3 &= 0 \end{aligned}$$

has only the trivial solution:

$$x_1 = x_2 = x_3 = 0$$

Tips for efficient implementation

- **The row operations which we used to solve linear equations can be represented by matrix operations.**
- **Let $1 \leq r \leq m$ and $1 \leq s \leq m$. Let I_{rs} be the square $m \times m$ matrix which has component 1 in the *rs* place, and 0 elsewhere:**

$$I_{rs} = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & 1_{rs} \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

Tips for efficient implementation

- Let $A = (a_{ij})$ be any $m \times n$ matrix. What is the effect of multiplying $I_{rs}A$?

$$
r\left\{ \underbrace{\begin{pmatrix} 0 & \cdots\cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots 1_{rs} \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots\cdots & 0 \end{pmatrix}}_{s} \right.
\left.\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{s1} & \cdots & a_{sn} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}\right\} s
=
\left.\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ a_{s1} & \cdots & a_{sn} \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}\right\} r .
$$

- $I_{rs}A$ is the matrix obtained by putting the s-th row of A in the r-th row, and zeros elsewhere.
- If r = s then I_{rr} has a component 1 on the diagonal place, and 0 elsewhere. Multiplication by I_{rr} then leaves the r-th row fixed and replaces all the other rows by zeros.

Tips for efficient implementation

- If $r \neq s$ let

$$J_{rs} = I_{rs} + I_{sr}$$

- Then,

$$J_{rs}A = I_{rs}A + I_{sr}A$$

- Thus, $J_{rs}A$ interchanges the r-th row and the s-th row and replaces all other rows by zero.
- Example

$$J = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 3 & 2 & -1 \\ 1 & 4 & 2 \\ -2 & 5 & 1 \end{pmatrix}.$$

Tips for efficient implementation

- let
- $E = I_{12} + I_{21} + I_{33}$ and $\mathrm{A} = \begin{bmatrix} -1 & 1 & -1 \\ 3 & -1 & -1 \\ 2 & 1 & -3 \end{bmatrix}$
- Question: What will be result of EA?

EA = $\begin{bmatrix} 0 & \mathbf{1} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -3 & \mathbf{1} & -\mathbf{1} \\ 1 & -\mathbf{1} & -\mathbf{1} \\ \mathbf{2} & \mathbf{1} & -\mathbf{3} \end{bmatrix}$

First two rows of A will be interchanged

Tips for efficient implementation

- Let E be the matrix obtained from the unit $n \times n$ matrix by interchanging two rows. Let A be an $n \times n$ matrix. Then EA is the matrix obtained from A by interchanging these two rows

Interchange any two rows.

- Let $\boldsymbol{cI_{rr}}$ is a matrix with entry c at rr-th position and zero elsewhere. Let A be an $n \times n$ matrix. Then, $\mathrm{cI_{rr}A}$ is obtained from A by multiplying the *r-th* row of A by *c*.

Replace any row by a nonzero constant multiple of itself.

- Let E be the matrix obtained from the unit $n \times n$ matrix by multiplying the *r-th* row with a number *c* and adding it to the *s-th* row, $r \neq s$. Let A be an $n \times n$ matrix. Then EA is obtained from A by multiplying the *r-th* row of A by *c* and adding it to the *s-th* row of A

Replace any row by the sum of that row and a constant multiple of any other row.

Matrix Inverse using Elementary Row Operations

- **Let *A* be the matrix**

$$A = \begin{bmatrix} 1 & -2 & -4 \\ 2 & -3 & -6 \\ -3 & 6 & 15 \end{bmatrix}$$

Inverse of a 3 x 3 Matrix

- We begin with the 3 x 6 matrix whose left half is *A* and whose right half is the identity matrix.

$$\left[\begin{array}{rrr|rrr} 1 & -2 & -4 & 1 & 0 & 0 \\ 2 & -3 & -6 & 0 & 1 & 0 \\ -3 & 6 & 15 & 0 & 0 & 1 \end{array}\right]$$

- We then transform the left half of this new matrix into the identity matrix— using sequence of elementary row operations on the entire new matrix.

Inverse of a 3 x 3 Matrix

$$\left[\begin{array}{ccc|ccc} 1 & -2 & -4 & 1 & 0 & 0 \\ 2 & -3 & -6 & 0 & 1 & 0 \\ -3 & 6 & 15 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3+3R_1 \to R_3]{R_2-2R_1 \to R_2} \left[\begin{array}{ccc|ccc} 1 & -2 & -4 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 3 & 3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{3}R_3} \left[\begin{array}{ccc|ccc} 1 & -2 & -4 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & \frac{1}{3} \end{array}\right]$$

Inverse of a 3 x 3 Matrix

$$\xrightarrow{\frac{1}{3}R_3} \left[\begin{array}{ccc|ccc} 1 & -2 & -4 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & \frac{1}{3} \end{array}\right]$$

$$\xrightarrow{R_1+2R_2 \rightarrow R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 2 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & \frac{1}{3} \end{array}\right]$$

$$\xrightarrow{R_2-2R_3 \rightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 2 & 0 \\ 0 & 1 & 0 & -4 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 & 1 & 0 & \frac{1}{3} \end{array}\right]$$

Inverse of a 3 x 3 Matrix

- We have now transformed the left half of this matrix into an identity matrix.
  - This means we've put the entire matrix in reduced row-echelon form.
  - The right half is now A^{-1}.

$$A^{-1} = \begin{bmatrix} -3 & 2 & 0 \\ -4 & 1 & -\frac{2}{3} \\ 1 & 0 & \frac{1}{3} \end{bmatrix}$$

Matrix Inverse using Row Elementary Operation

- **A square matrix A is invertible if and only if A is row equivalent to the unit matrix. Any upper triangular matrix with nonzero diagonal elements is invertible.**

A matrix B is row equivalent to a matrix A if B result from A via elementary row operations.

$$A = \begin{bmatrix} 1 & -2 & -4 \\ 2 & -3 & -6 \\ -3 & 6 & 15 \end{bmatrix} \qquad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_1 = R_1 - 4R_3]{R_1 = R_1 - 2R_2} \begin{bmatrix} 1 & -2 & -4 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 = R_2 + 2R_1} \begin{bmatrix} 1 & -2 & - \\ 2 & -3 & -1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 = R_2 + 8R_3} \begin{bmatrix} 1 & -2 & -4 \\ 2 & -3 & -6 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 = 3R_3} \begin{bmatrix} 1 & -2 & -4 \\ 2 & -3 & -6 \\ 0 & 0 & 3 \end{bmatrix} \xrightarrow{R_3 = R_3 - 3R_1} \begin{bmatrix} 1 & -2 & -4 \\ 2 & -3 & -6 \\ -3 & 6 & 15 \end{bmatrix}$$

System of linear equations

- **Let $AX = B$ be a system of n linear equations in n unknowns. Assume that the matrix of coefficients A is invertible. Then there is a unique solution X to the system, and**

$$\boldsymbol{X = A^{-1}B}$$

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