Master of Computer Applications

MCAC103: Mathematical Techniques for Computer Science Applications
Unique Paper Code: 223401103
Semester I
March-2021
Year of admission: 2020

Time: Three Hours **Maximum marks: 70**

Instructions:

- Answer any 4 questions. All questions carry equal marks.
- Show complete solution with proper notation for full credit.

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1. (a) Let X and Y be two events such that $P(X) = 0.5$, $P(Y) = 0.70$ and $P(X \cup Y) = 0.85$. Are the events, X and Y, independent? Justify your answer. [2.5]

   (b) In mathematics course, majority of the students enrolled are from computer science (CS) background and some students don't major in CS. In the EoSE exam, only 50% of the CS students and 30% of the non-CS students pass. Unfortunately, 60% of the entire class are non-CS students. What is the percentage of CS students among those that actually pass the exam? [4]

   (c) Let $x = (1, 1, 0, 1)$, $y = (1, -2, 0, 0)$, and $z = (1, 0, -1, 2)$ be three vectors in $\mathbb{R}^4$. Find an orthonormal basis for the above given vectors. [5]

   (d) You are performing a random experiment that results in one of three possible outcomes with outcome i occurring with probability p_i. You performed n such independent identical experiments. Let X_i, $i = 1, 2, 3$ denote the number of times outcome i appears. Determine the conditional expectation of X_1 given that $X_2 = m$. [6]

2. (a) A box contains 10 red balls and 15 blue balls. You are asked to select the balls randomly one at a time, until a blue one is obtained. If we assume that each ball selected is replaced before the next one is drawn, what is the probability of exactly 6 draws are needed? [2.5]

   (b) Let $X \backsim Binomial(n, p)$ be a random variable. Using Chebyshev's inequality, find an upper bound on $P(X \geq \alpha n)$, where $p < \alpha < 1$. Find the bound for $p = 1/2$ and $\alpha = 3/5$. [4]

   (c) Let $X = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 6 & 5 \\ 3 & 9 & 1 \\ -2 & -6 & 0 \end{pmatrix}$ define a linear transformation $F : \mathbb{R}^3 \to \mathbb{R}^4$ by $y \mapsto Ay$. Find the kernel of the given linear map. [5]

   (d) Let $\{x_1, x_2, \ldots, x_m\}$ be a set of linearly independent vectors in X and y is another vector in X such that $\{x_1 + y, x_2 + y, \ldots, x_m + y\}$ are linearly dependent. Prove that $y \in Span\{x_1, x_2, \ldots, x_m\}$. [6]

3. (a) Suppose X has the following probability mass function: $p(1) = 0.3$, $p(2) = 0.6$, $p(3) = 0.1$. Calculate $E[X^3]$ and $Var(X^3)$. [2.5]

(b) Five friends, along with their wives, went to a restaurant for dinner. Luckily, they got a round table with ten chairs. Let X be the number of wives who sit next to their husbands. What is $E[X]$? [4]

(c) Let $A = [a_{ij}]$ be a symmetric matrix of size $m \times m$, where $a_{ij} \in \{0, 1, 2\}$. Determine the number of different ways to construct the matrix A. [5]

(d) Find the eigenvalues and a basis for the eigenspaces of the $Y = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$. [6]

4. (a) It is known that any balloon sold by a shopkeeper will be defective with probability 0.1, independently of any other balloon. What is the probability that in a sample of three balloons, at most one will be defective? [2.5]

(b) Let a, b, c be three numbers. Show that $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b-a)(c-a)(c-b)$ [4]

(c) Solve the system of linear equations: $3a + 2b + 4c = 1$, $2a - b + c = 0$,& $a + 2b + 3c = 1$ [5]

(d) The classes for MCAC-103 are scheduled regularly in a 6 seater classroom. The number of students enrolled into the course is 9. Students turn up for the class independently of other students with probability p.

(i) What is the probability that on a particular day, the number of seats in the classroom will not be sufficient. [3]

(ii) What is the probability that on a particular day, the class completes with empty seats. [3]

5. (a) Let X be a continuous random variable with the following density function.

$$f(x) = \begin{cases} c + dx^3, 0 \le x \le 1 \\ 0, \text{otherwise} \end{cases}$$

If $E[X] = 3/5$, find c and d. [2.5]

(b) A box contains 12 black, 16 yellow, and 18 white balls. Your friend asked you to withdraw seven balls randomly from the box. Find probability that at least 2 black balls are withdrawn. [4]

(c) There are n urns numbered from $1, 2 \ldots, n$. The ith urn contains i number of balls. In each urn, a ball is damaged with probability $1/2$. Assuming that each urn is equally likely to be chosen, you have selected an urn and found that none of the balls are damaged. Find the probability that you have selected the 1st urn. [5]

(d) Find the real matrix X of size 3×3, which satisfies the following.
$X \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ -3 \\ -3 \end{pmatrix}$, $X \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and $X \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -4 \end{pmatrix}$. [6]

6. (a) Let $X \backsim Uniform(0, 1)$ be a random variable. Calculate $E[X^n]$ and $Var(X^n)$. [2.5]

(b) Let m be the number of files and n be the number of folders. You have to place every file in a folder, where a folder can hold an arbitrarily large number of files. What is the expected number of empty folders?

(c) The following are marks secured by 15 students in MCAC-103.
$14, 36, 39, 41, 25, 21, 17, 22, 18, 33, 31, 18, 19, 16, 21$

(i) Draw the box and whisker plots for the above data. [3]

(ii) Draw the cumulative frequency graph by partitioning the set of students into 9 groups based on the marks. [2]

(d) Let A and B be two events.

(i) Given $P(A^c \cap B^c) = 2/3$ and find $P(A \cup B)$. [2]

(ii) Given $P(A) = 0.25, P(B) = 0.45$, and $P(A \cap B) = 0.1$, find $P(A^c \cap B)$. [2]

(iii) Given A and B are independent, then E and F^c will also be independent. True or False? Justify your answer. [2]