Random Variable

Random Variable

- **Probability measure assigns probability $p(o_i)$ to each outcome o_i, such that $\forall o_i \in S$; $0 \leq p(o_i) \leq 1$ and $\sum \boldsymbol{p(o_i)} = 1$**

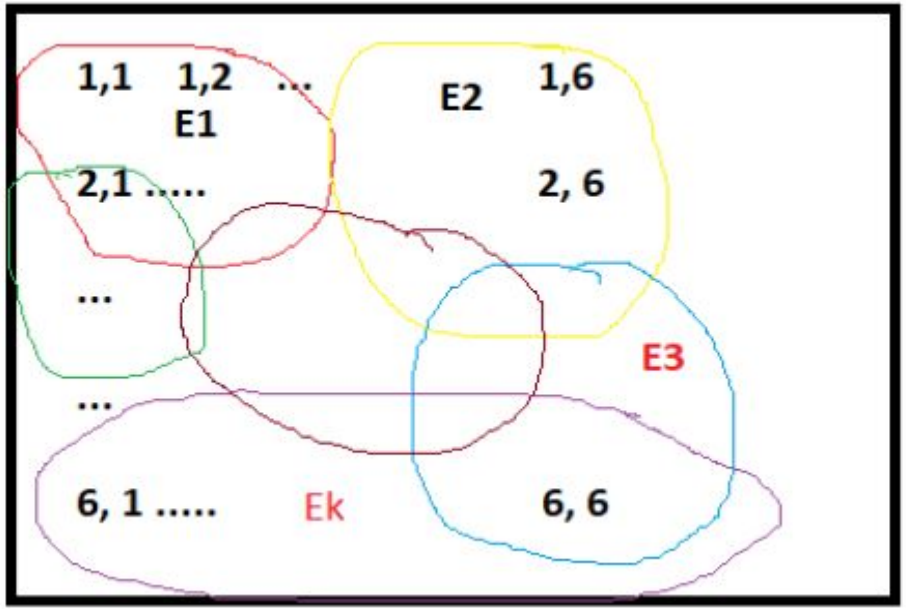
- **If all outcomes are mutually exclusive and equally likely then $\forall \boldsymbol{i, j}\ \boldsymbol{p(o_i) = (p_j) = \frac{1}{n}}$, where n is the size of S**

1,1	1,2	…			1,6
2,1 …..					2, 6
…		4,3	4,4	4,5	4,6
…					
6, 1 …..					6, 6

Sample space for 2-dice rolling experiment, $|S| = 36$

Random Variable

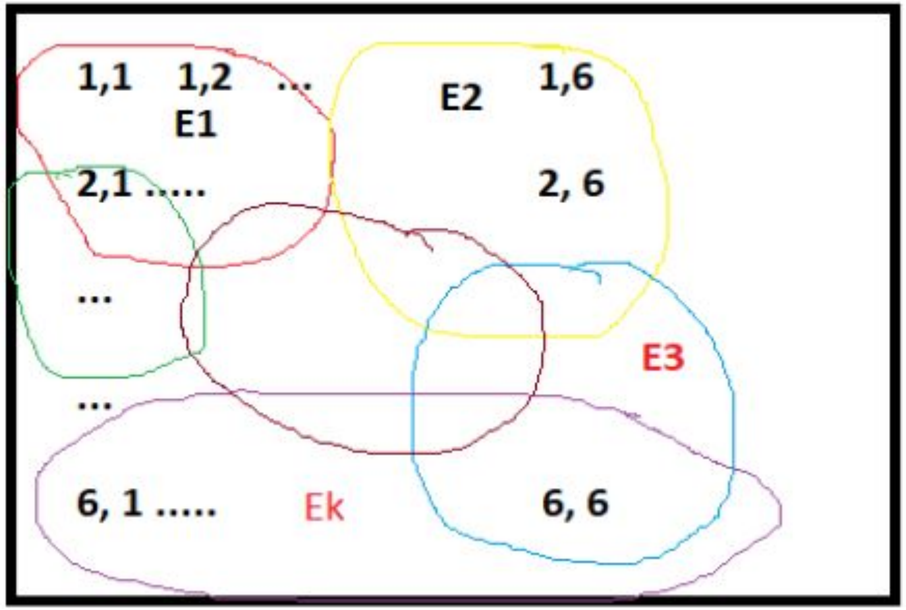
- **Multiple events can be mapped to S**
  - **Event E_1: Sum of two numbers greater than 8**
    - E_1: $\{(4,5);(5,4);(3,6);\dots;(6,6)\}$
  - **Event E_2: First number bigger than second**
    - E_2: $\{(2,1);(3,1);\ \dots;(6,5)\}$
  - **Event E_3: Max of the two number**
    - E_3: $\{2;3;\ \dots;6\}$
- **Suppose we are interested in more general questions about the outcomes of the experiment.**
  - **What is the sum of the two numbers? {2, ... 12}**



Sample space for 2-dice rolling experiment, $|S| = 36$

Random Variable

- Suppose we are interested in more general questions about the outcomes of the experiment.
  - What is the sum of the two numbers? $\{2, \ldots . 12\}$
  - Which is the bigger number? $\{2, 3, \ldots 6\}$
  - Sum of the numbers, when first number is divisible by second (not = 1). $\{ ??? \}$



Sample space for 2-dice rolling experiment, $|S| = 36$

Define a function that maps all outcomes in S to a set of values

$$f : S \to R$$

Random Variable

- **Definition 1:** *A random variable is a function from a sample space* $S(or, \Omega)$ *to the real numbers. Conventionally, random variables are denoted with capital letters, e.g.,* $X : S \to R \coloneqq (-\infty, +\infty)$
- **If the outcome of the random experiment is** ω**, then the value of the random variable is** $X(\boldsymbol{\omega}) \in \mathbb{R}$
- **Examples**
  - **Toss a coin 10 times and let X be the number of Heads**
  - **Choose a random person in a class and let X be the height of the person, in inches.**

Random Variable

- **Several random variables can be defined for a set of outcomes.**
  - **Experiment of tossing four coins**
    - X_1 **counts # of Heads,**
      - Example: $X(\{H,H,T,H\}) \rightarrow 3$
    - X_2 **count# of Tails**
      - Example: $X(\{H,H,T,H\}) \rightarrow 1$
    - X_3 **denotes the case when all faces are same**
      - Example: $X(\{H,H,T,H\}) \rightarrow 0$

- **Can be discrete (i.e., finite many possible outcomes) or continuous**

Discrete Random Variable

- Random variables that can assume a countable number (finite or infinite) of values are called discrete.
- Examples

Experiment	Random Variable	Possible Values
Make 100 Sales Calls	# Sales	0, 1, 2, ..., 100
Inspect 70 Radios	# Defective	0, 1, 2, ..., 70
Answer 33 Questions	# Correct	0, 1, 2, ..., 33

Continuous Random Variable

- **A continuous random variable is a random variable with infinitely many possible values (in an interval of real numbers).**

Experiment	Random Variable	Possible Values
Weigh 100 People	Weight	45.1, 78, ...
Measure time taken	Hours	900, 875.9, ...
Amount spent on food	$ amount	54.12, 42, ...
Measure Time Between Arrivals	Inter-Arrival Time	0, 1.3, 2.78, ...

Types of Probability

- Can either be **marginal**, **joint** or **conditional**.
- **Marginal Probability**: If A is an event, then the marginal probability is the probability of that event occurring, P(A).
  - Example: Let we toss a coin (first event) and throw a dice (second event).

		Throwing a Die						
		1	2	3	4	5	6	
Tossing a Coin	Head							
	Tail							

Marginal probability

Types of Probability

- Can either be **marginal**, **joint** or **conditional**.
- **Joint Probability:** The probability of the intersection of two or more events.
- If A and B are two events then the joint probability of the two events is written as $P(A \cap B)$ or, $P(X = x, Y = y)$, X and Y are random variables.

Joint probability

		Throwing a Die						
		1	2	3	4	5	6	
Tossing a Coin	Head							
	Tail							

Marginal probability

Types of Probability

- Can either be marginal, joint or conditional.
- Conditional Probability: The conditional probability is the probability that some event(s) occur given that we know other events have already occurred. If A and B are two events, then the conditional probability of A occurring given that B has occurred is written as $P(A|B)$ or $P(X=x|Y=y)$.
  - Example: the probability that a card is a four given that we have drawn a red card is $P(4|\text{red}) = 2/26 = 1/13$.

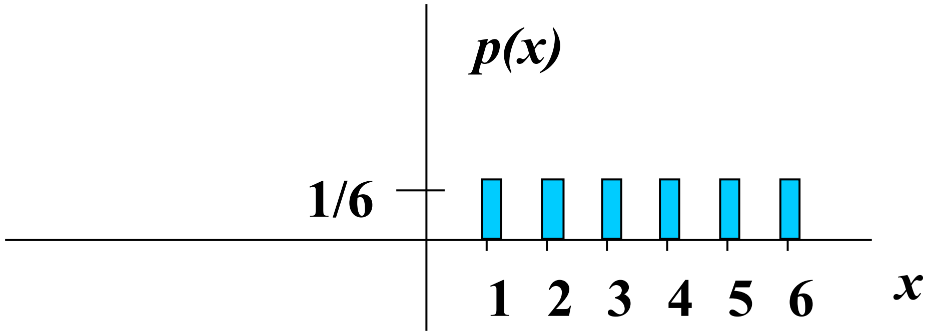
Probability Functions

- We are often interested in knowing the probability of a random variable taking on a certain value.
- We may assign probabilities to the different values of the random variable:
  - Counts # of Heads, when tossing three coins
    - $P(X = 0) = P((T,T,T)) = 1/\mathbf{2^3} = 1/8$
    - $P(X = 1) = P((T,T,H);\ (T,H,T);\ (H,T,T)) = 3/8$
    - $P(X = 2) = P((T,H,H);\ (H,H,T);\ (H,T,H)) = 3/8$
    - $P(X = 3) = P((H,H,H)) = 1/8$
  - Note that since X must take the values of 0 through 3 then
    - $\mathbf{1 = P\left(\cup_{i=0}^{3}\{X = i\}\right) = \sum_{i=0}^{3} P(X = i)}$

Probabilities for random variables can be computed from the probability measure defined on underlying sample space.

Probability Functions

- **A probability function maps the possible values of *x* against their respective probabilities of occurrence, *p(x)***
  - **p(x) is a number from 0 to 1.0.**
  - **The area under a probability function is always 1.**
- **Discrete example: roll of a die**



$$\sum_{\text{all } x} P(x) = 1$$

Which of the following are probability functions?

a. *f(x)=.25* for x=9,10,11,12

b. *f(x)= (3-x)/2* for x=1,2,3,4

c. *f(x)=* $(x^2+x+1)/25$ for x=0,1,2,3

Probability Distributions and Probability Mass Functions

- **Definition: Probability Distribution**
  - *A probability distribution of a random variable X is a description of the probabilities associated with the possible values of X*
- **Example**
  - **Let X = # of heads observed when a coin is flipped twice**

Number of Heads	0	1	2
Probability	1/4	2/4	1/4

Probability Distributions and Probability Mass Functions

- **The probability distribution of a discrete random variable is a graph, table, or formula that specifies the probability associated with each possible value the random variable can assume.**
- **Requirements for the Probability Distribution of a Discrete Random Variable X**
  1. $p(x) \geq 0$ for all values of x
  2. $\Sigma\, p(x) = 1$

Example: (Probability defined by function p(x))

x	1	2	3	4
P(X = x) = f(x)	0.1	0.2	0.3	0.4

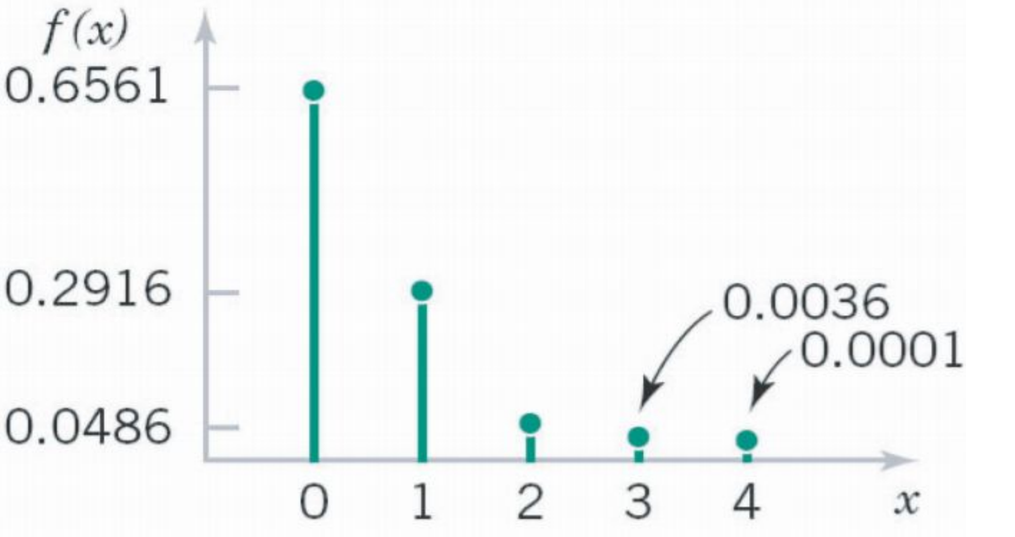
Function of X: $p(x) = \frac{1}{10}x$ for $x \in \{1, 2, 3, 4\}$

Probability Distributions and Probability Mass Functions

- **Example (Bits Transmission)**
  - There is a chance that a bit transmitted through a digital transmission channel is received in error.
  - Let X equal the number of bits in error in the next four bits transmitted. The possible values for X are $\{0, 1, 2, 3, 4\}$.
  - Suppose that the probabilities are...

0	0.6561
1	0.2916
2	0.0486
3	0.0036
4	0.0001

The probability distribution shown graphically



Probability Mass Function (PMF)

- **For a discrete random variable X with possible values $x_1, x_2, x_3, \ldots, x_n$, a probability mass function $\boldsymbol{p}(x_i)$ is a function such**
  1. $\boldsymbol{p}(x_i) \geq 0$ **for all values of** x_i
  2. $\sum p(x_i) = 1$
  3. $\boldsymbol{p}(\boldsymbol{x}_i) = \boldsymbol{P}(\boldsymbol{X} = \boldsymbol{x}_i)$

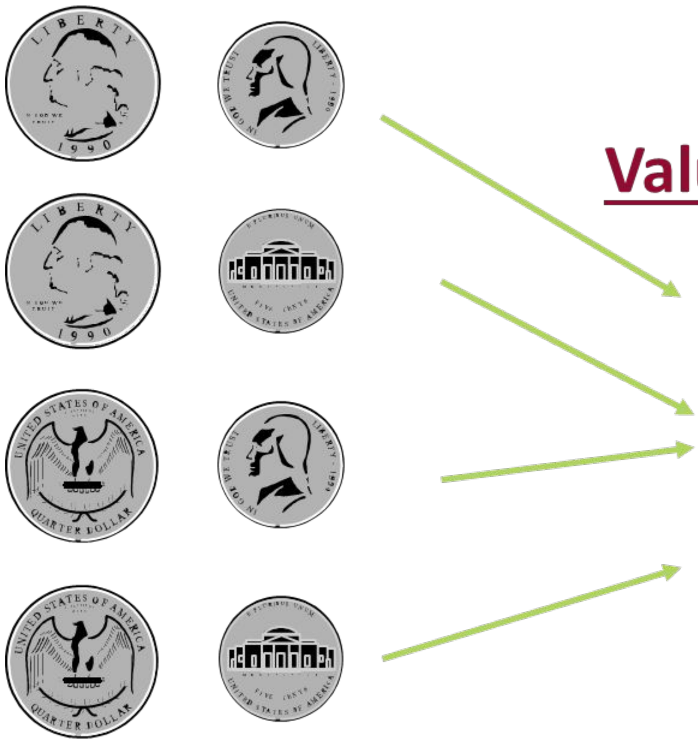
The probability distribution for a discrete random variable is described with a probability mass function (probability distributions for continuous random variables will use different terminology).

Example (Probability Mass Function (PMF)): Tossing a die

x	*p(x)*
1	*p(x=1)*=1/6
2	*p(x=2)*=1/6
3	*p(x=3)*=1/6
4	*p(x=4)*=1/6
5	*p(x=5)*=1/6
6	*p(x=6)*=1/6
	1.0

Discrete Probability Distribution Example

- **Experiment:** Toss 2 coins. Count number of tails.



Probability Distribution

Values, x	Probabilities, $p(x)$
0	1/4 = .25
1	2/4 = .50
2	1/4 = .25

Visualizing Discrete Probability Distributions

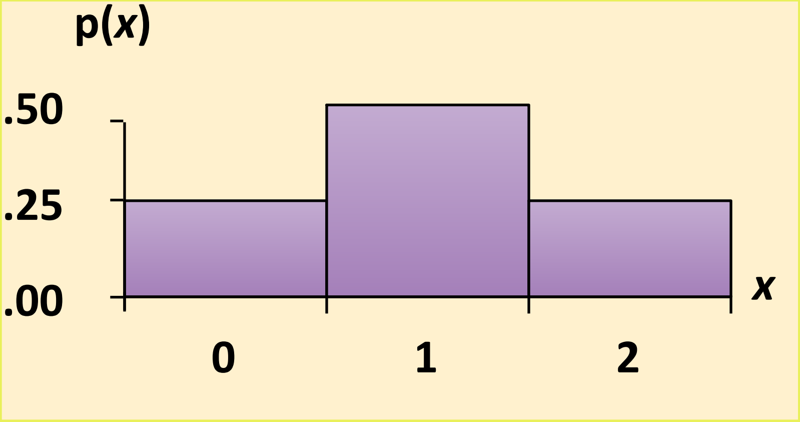
Listing

{ (0, .25), (1, .50), (2, .25) }

Table

# Tails	f(x) Count	p(x)
0	1	.25
1	2	.50
2	1	.25

Graph



Formula

$$p(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

Cumulative Distribution Function (CDF)

- **Sometimes it's useful to quickly calculate a cumulative probability, or $P(X \leq x)$, denoted as $F(x)$, which is the probability that X is less than or equal to some specific x.**
- **Example: Toss a die, then the probability mass function for X**

x	*p(x)*
1	*p(x=1)*=1/6
2	*p(x=2)*=1/6
3	*p(x=3)*=1/6
4	*p(x=4)*=1/6
5	*p(x=5)*=1/6
6	*p(x=6)*=1/6

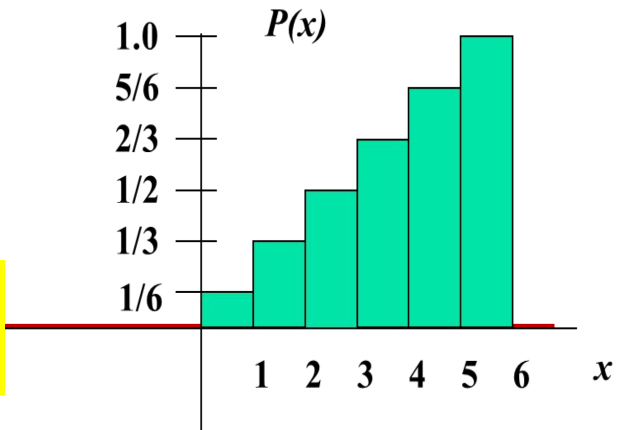
Cumulative Distribution Function (CDF)

x	$p(x)$
1	$p(x=1)=1/6$
2	$p(x=2)=1/6$
3	$p(x=3)=1/6$
4	$p(x=4)=1/6$
5	$p(x=5)=1/6$
6	$p(x=6)=1/6$

Cumulative Probabilities

1
2
3
4
5
6

- **Suppose we're interested in the probability of getting 3 or less**
  - $P(X \leq 3) = P(X = 1) + P(X = 2) + P(X = 3) = 3/6 = 1/2$



As x increases across the possible values for x, the cumulative probability increases, eventually getting 1, as you accumulate all the probability.

Cumulative Distribution Function (CDF)

- The cumulative distribution function of a discrete random variable X, denoted as F(x), is

$$F(x) = P(X \leq x) = \sum_{x_i \leq x} p(x_i)$$

- Properties
  1. $F(x) = P(X \leq x) = \sum_{x_i \leq x} p(x_i)$
  2. $0 \leq F(x) \leq 1$
  3. If $x \leq y$, then $F(x) \leq F(y)$
- The CDF is defined on the real number line.
- The CDF is a non-decreasing function of X (i.e., increases or stays constant as x → ∞).

Summary Measures

- **Expected Value (Mean of probability distribution)**
  - **Weighted average of all possible values**
  - $\mu = E(x) = \Sigma x\, p(x)$
- **Variance**
  - **Weighted average of squared deviation about mean**
  - $\sigma^2 = E[(x - \mu)^2] = \Sigma\, (x - \mu)^2\, p(x)$
- **Standard Deviation**
  - $\sigma = \sqrt{\sigma^2}$

Summary Measures Calculation Table

- **Experiment:** Toss 2 coins. Count number of tails.

x	$p(x)$	$x\,p(x)$	$x - \mu$	$(x - \mu)^2$	$(x - \mu)^2 p(x)$
Total		$\Sigma x\,p(x)$			$\Sigma(x - \mu)^2\,p(x)$

Summary Measures Calculation Table

- **Experiment:** Toss 2 coins. Count number of tails.

x	$p(x)$	$x\,p(x)$	$x - \mu$	$(x - \mu)^2$	$(x - \mu)^2 p(x)$
0	.25	0	–1.00	1.00	.25
1	.50	.50	0	0	0
2	.25	.50	1.00	1.00	.25
		$\mu = 1.0$			$\sigma^2 = .50$
					$\sigma = .71$

Practice Problem

- The number of ships to arrive at a harbor on any given day is a random variable represented by *x*. The probability distribution for *x* is:

x	10	11	12	13	14
$P(x)$	.4	.2	.2	.1	.1

- Question: Find the probability that on a given day:

a. exactly 14 ships arrive $\quad p(x=14) = .1$

b. At least 12 ships arrive $\quad p(x \geq 12) = (.2 + .1 + .1) = .4$

c. At most 11 ships arrive $\quad p(x \leq 11) = (.4 + .2) = .6$

Practice Problem

- You are lecturing to a group of 1000 students. You ask them to each randomly pick an integer between 1 and 10. Assuming, their picks are truly random:

  - What percentage of the students would you expect picked a number less than or equal to 6?

    Since $p(x \leq 6) = 1/10 + 1/10 + 1/10 + 1/10 + 1/10 + 1/10 = .6$

    = 60%

Reference

- Lecture notes on Probability Theory by Phanuel Mariano