Discrete Mathematics Study Center

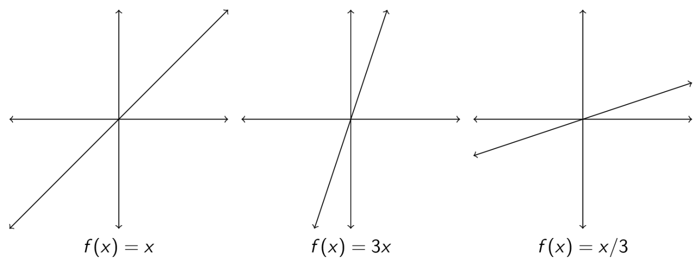
Growth of Functions

The growth of a function is determined by the highest order term: if you add a bunch of terms, the function grows about as fast as the largest term (for large enough input values).

For example, $f(x) = x^2 + 1$ grows as fast as $g(x) = x^2 + 2$ and $h(x) = x^2 + x + 1$, because for large x, x^2 is *much* bigger than 1, 2, or $x + 1$.

Similarly, constant multiples don't matter that much: $f(x) = x^2$ grows as fast as $g(x) = 2x^2$ and $h(x) = 100x^2$, because for large x, multiplying x^2 by a constant does not change it "too much" (at least not as much as increasing x).

Essentially, we are concerned with the shape of the curve:



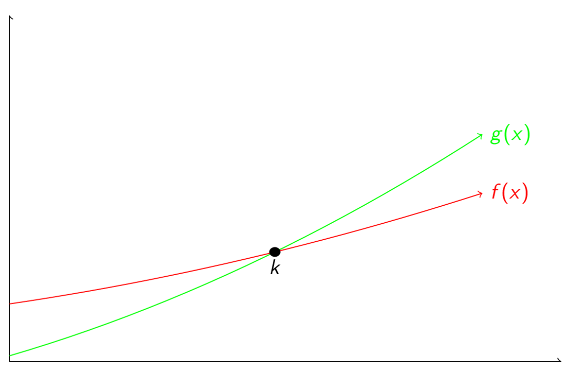
All three of these functions are lines; their exact slope/y-intercept does not matter.

Only caring about the highest order term (without constant multiples) corresponds to ignoring differences in hardware/operating system/etc. If the CPU is twice as fast, for example, the algorithm still *behaves* the same way, even if it executes faster.

Big-Oh Notation

Let f and g be functions from $\mathbb{Z} \to \mathbb{R}$ or $\mathbb{R} \to \mathbb{R}$. We say $f(x)$ is $O(g(x))$ if there are **constants** $c > 0$ and $k > 0$ such that $0 \leq f(n) \leq c \times g(n)$ for all $x \geq k$. The constants c and k are called **witnesses**. We read $f(x)$ is $O(g(x))$ as "$f(x)$ is big-Oh of $g(x)$". We write $f(x) \in O(g(x))$ or $f(x) = O(g(x))$ (though the former is more technically correct).

Basically, $f(x)$ is $O(g(x))$ means that, after a certain value of x, f is always smaller than some constant multiple of g:



Here are some examples that use big-Oh notation:

- To show that $5x^2 + 10x + 3$ is $O(x^2)$:

  $$5x^2 + 10x + 3 \leq 5x^2 + 10x^2 + 3x^2 = 18x^2$$

  Each of the above steps is true for all $x \geq 1$, so take $c = 18$, $k = 1$ as witnesses.
- To show that $5x^2 - 10x + 3$ is $O(x^2)$:

  $$5x^2 - 10x + 3 \leq 5x^2 + 3 \leq 5x^2 + 3x^2 = 8x^2$$

The first step is true as long as $10x > 0$ (which is the same as $x > 0$) and the second step is true as long as $x \geq 1$, so take $c = 8$, $k = 1$ as witnesses.

- Is it true that x^3 is $O(x^2)$?

  Suppose it is true. Then $x^3 \leq cx^2$ for $x > k$. Dividing through by x^2, we get that $x \leq c$. This says that "x is always less than a constant", but this is not true: a line with positive slope is not bounded from above by any constant! Therefore, x^3 is **not** $O(x^2)$.

Typically, we want the function inside the Oh to be as small and simple as possible. Even though it is true, for example, that $5x^2 + 10x + 3$ is $O(x^3)$, this is not terribly informative. Similarly, $5x^2 + 10x + 3$ is $O(2x^2 + 11x + 3)$, but this is not particularly useful.

Here are some important big-Oh results:

- If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ where $a_n > 0$, then $f(x)$ is $O(x^n)$.

  Proof: If $x > 1$, then:

$$\begin{array}{rcl} f(x) & = & a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \\ & \leq & |a_n|x^n + |a_{n-1}|x^{n-1} + \cdots + |a_1|x + |a_0| \\ & \leq & |a_n|x^n + |a_{n-1}|x^n + \cdots + |a_1|x^n + |a_0|x^n \\ & = & x^n(|a_n| + |a_{n-1}| + \cdots + |a_1| + |a_0|) \end{array}$$

  Therefore, take $c = |a_n| + |a_{n-1}| + \cdots + |a_1| + |a_0| > 0$ and $k = 1$.

- What is the sum of the first n integers?

$$1 + 2 + 3 + \cdots + n \leq n + n + n + \cdots + n = n \times n = n^2$$

  Take $c = 1, k = 1$ to see that sum is $O(n^2)$. Notice that this agrees with the formula we derived earlier: $\sum_{i=1}^{n} i = n(n+1)/2$, which is $O(n^2)$.

- What is the growth of $n!$?

$$\begin{array}{rcl} n! & = & n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1 \\ & = & n \times n \times n \times \cdots n \\ & = & n^n \end{array}$$

  Therefore, $n!$ is $O(n^n)$ with $c = k = 1$

- What is the growth of $\log n!$?

  Take the logarithm of both sides of the previous equation to get $\log n! \leq \log n^n$, so $\log n! \leq n \log n$. Therefore, $\log n!$ is $O(n \log n)$ with $c = k = 1$.

- How does $\log_2 n$ compare to n?

  We know that $n < 2^n$ (we will prove this later). Taking the logarithm of both sides, we have that $\log_2 n < \log_2 2^n = n$. So $\log_2 n$ is $O(n)$ with $c = k = 1$.

When using logarithms inside big-Oh notation, the base does not matter. Recall the change-of-base formula: $\log_b n = \frac{\log n}{\log b}$. Therefore, as long as the base b is a constant, it differs from $\log n$ by a constant factor.

Here are some common functions, listed from slowest to fastest growth:

$$O(1), O(\log n), O(n), O(n \log n), O(n^2), O(2^n), O(n!)$$

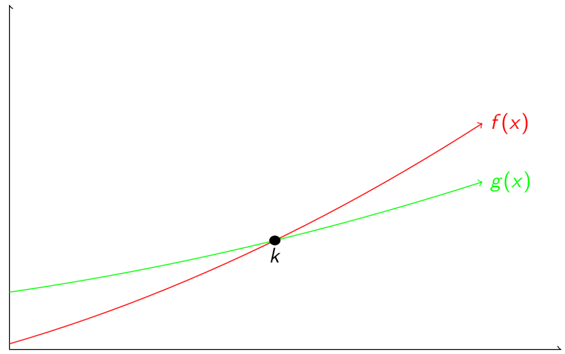
Caution: there are infinitely many functions between each element of this list!

Big-Omega Notation

As we saw above, big-Oh provides an *upper* bound for a function. To specify a *lower* bound, we use big-Omega notation.

Let f and g be functions from $\mathbb{Z} \to \mathbb{R}$ or $\mathbb{R} \to \mathbb{R}$. We say $f(x)$ is $\Omega(g(x))$ if there are **constants** $c > 0$ and $k > 0$ such that $0 \leq c \times g(n) \leq f(n)$ for all $x \geq k$. The constants c and k are called **witnesses**. We read $f(x)$ is $\Omega(g(x))$ as "$f(x)$ is big-Oh of $g(x)$". We write $f(x) \in \Omega(g(x))$ or $f(x) = \Omega(g(x))$ (though the former is more technically correct).

Basically, $f(x)$ is $\Omega(g(x))$ means that, after a certain value of x, f is always bigger than some constant multiple of g:



Here are some examples that use big-Omega notation:

- To show that $5x^3 + 3x^2 + 2$ is $\Omega(x^3)$:

$$5x^3 + 3x^2 + 2 \geq 5x^3 \geq x^3$$

  Therefore, take $c = k = 1$.
- To show that $x^2 - 3x + 4$ is $\Omega(x^2)$:

$$\begin{aligned} x^2 - 3x + 4 &= \tfrac{1}{2}x^2 + \tfrac{1}{2}x^2 - 3x + 4 \\ &= \tfrac{1}{2}x^2 + \left(\tfrac{1}{2}x^2 - 3x + 4\right) \\ &\geq \tfrac{1}{2}x^2 \end{aligned}$$

  The last step is true as long as $\frac{1}{2}x^2 - 3x + 4 \geq 0$, which is true when $x > 6$. Therefore, take $c = 1/2, k = 6$.
- Is it true that $3x + 1$ is $\Omega(x^2)$?

  Suppose it is true. Then $3x + 1 \geq cx^2$ for $x > k$. Dividing through by x^2, we get that $3/x + 1/x^2 \geq c$. Notice that as x gets bigger, the left hand side gets smaller, so this cannot be true. Therefore, $3x + 1$ is **not** $\Omega(x^2)$.

- What is the sum of the first n integers?

$$\begin{aligned} & 1 + 2 + 3 + \cdots + n \\ \geq\ & \left\lceil \tfrac{n}{2} \right\rceil + \left(\left\lceil \tfrac{n}{2} \right\rceil + 1\right) + \left(\left\lceil \tfrac{n}{2} \right\rceil + 2\right) + \cdots + n \\ \geq\ & \left\lceil \tfrac{n}{2} \right\rceil + \left\lceil \tfrac{n}{2} \right\rceil + \cdots + \left\lceil \tfrac{n}{2} \right\rceil \\ =\ & \left(n - \left\lceil \tfrac{n}{2} \right\rceil + 1\right) \left\lceil \tfrac{n}{2} \right\rceil \\ \geq\ & \left(\tfrac{n}{2}\right)\left(\tfrac{n}{2}\right) \\ =\ & \tfrac{n^2}{4} \end{aligned}$$

  Therefore, take $c = 1/4, k = 1$ to show that the sum is $\Omega(n^2)$ (which matches with our formula for this sum).

Big-Theta Notation

In the previous example, we showed that $\sum_{i=1}^{n} i = \Omega(n^2)$. Earlier, we also showed that this sum is $O(n^2)$. We have special notation for such situations:

Let f and g be functions from $\mathbb{Z} \to \mathbb{R}$ or $\mathbb{R} \to \mathbb{R}$. We say $f(x)$ is $\Theta(g(x))$ if $f(x)$ is $O(g(x))$ **and** $f(x)$ is $\Omega(g(x))$. We read $f(x)$ is $\Theta(g(x))$ as "$f(x)$ is big-Theta of $g(x)$". We write $f(x) \in \Theta(g(x))$ or $f(x) = \Theta(g(x))$ (though the former is more technically correct).

It might be helpful to think of big-Oh/Omega/Theta as follows:

- $\leq$ is to numbers as big-Oh is to functions
- $\geq$ is to numbers as big-Omega is to functions
- $=$ is to numbers as big-Theta is to functions