Continuous Random Variables

Continuous Random Variables

- Random variables whose set of possible values is uncountable.
- For a continuous random variable (r.v.) X, a probability $p(X = x)$ is meaningless and zero.
- Instead, we use $f(x)$ to denote the probability density and is often expressed in terms of an integral between two points, $p(a \leq X \leq b)$.

Continuous Random Variables

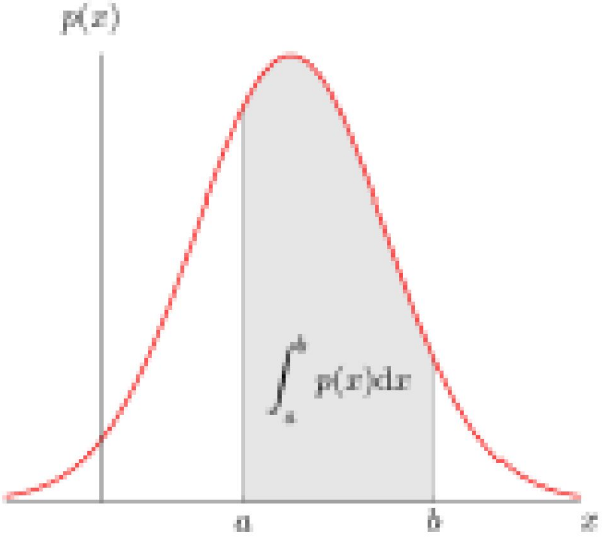
- A real valued function $f(x)$ is called a probability density function of the continuous random variable X iff

$$P(a \leq X \leq b) = \int_a^b f(x)dx$$

for any real constants a and b with a ≤ b

- If we let $a = b$ in the preceding, then

$$P[X = a] = \int_a^a f(x)dx = 0$$



this equation states that the probability that a continuous random variable will assume any *particular* value is zero.

Continuous Random Variables

- A real valued function $f(x)$ is called a probability density function of the continuous random variable X iff

$$\boldsymbol{P(a \leq X \leq b) = \int_{a}^{b} f(x)dx}$$

 for any real constants a and b with a ≤ b

- The relationship between the cumulative distribution $F(\cdot)$ and the probability density $f(\cdot)$ is expressed by

$$\boldsymbol{F(a) = P\{X \in (-\infty, a]\} = \int_{-\infty}^{a} f(x)dx}$$

Continuous Random Variables

- **Theorem:** ***A function can serve as a probability density function of a continuous random variable X if its values f(x) satisfy the conditions***
  1. $f(x) \geq 0 \; for \; -\infty < X < \infty$
  2. $\int_{-\infty}^{\infty} f(x)dx = 1$

Continuous Random Variables

- **Example: Metal Cylinder Production**
  - **Suppose that the random variable *x* is the diameter of a randomly chosen cylinder manufactured by the company. Since this random variable can take any value between 49.5 and 50.5, it is a continuous random variable.**
  - **Suppose that the diameter of a metal cylinder has a p.d.f**

$$f(x) = 1.5 - 6(x - 50.2)^2 \quad \text{for} \quad 49.5 \le x \le 50.5$$

$$f(x) = 0, \quad \text{elsewhere}$$

Is this valid p.d.f.?

Continuous Random Variables

$$\int_{49.5}^{50.5} (1.5 - 6(x - 50.0)^2)dx = [1.5x - 2(x - 50.0)^3]_{49.5}^{50.5}$$

$$= [1.5 \times 50.5 - 2(50.5 - 50.0)^3]$$

$$-[1.5 \times 49.5 - 2(49.5 - 50.0)^3]$$

$$= 75.5 - 74.5 = 1.0$$

Continuous Random Variables

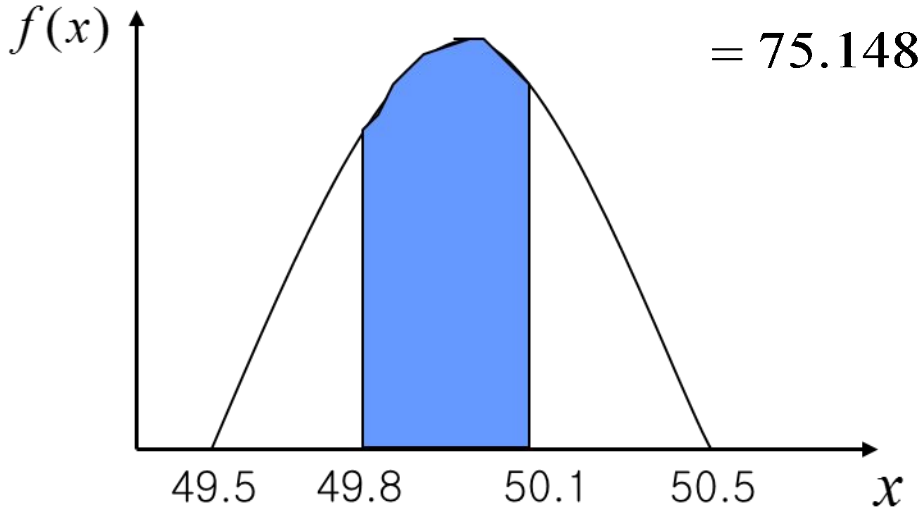
- **The probability that a metal cylinder has a diameter between 49.8 and 50.1 mm can be calculated to be**

$$\int_{49.8}^{50.1} (1.5 - 6(x-50.0)^2)dx = [1.5x - 2(x-50.0)^3]_{49.8}^{50.1}$$

$$= [1.5 \times 50.1 - 2(50.1 - 50.0)^3]$$

$$-[1.5 \times 49.8 - 2(49.8 - 50.0)^3]$$

$$= 75.148 - 74.716 = 0.432$$



Continuous Random Variables

- **Find the constant c such that the function**

$$f(x) = \begin{cases} cx^2 & 0 < x < 3 \\ 0 & otherwise \end{cases}$$

  is a density function

- **Solution:**

$$\int_{-\infty}^{\infty} cx^2 dx = \int_0^3 cx^2 dx = \left[\frac{cx^3}{3}\right]_0^3 = 1 \Rightarrow c = \frac{1}{9}$$

Continuous Random Variables

- **Theorem:** *If X is a continuous random variable and a and b are real constants with a ≤ b, then*

  $$P(a \leq X \leq b) = P(a < X \leq b) = P(a \leq X < b) = P(a < X < b)$$

  - **In case f(x) is continuous, the probability that X is equal to any particular value is zero.**

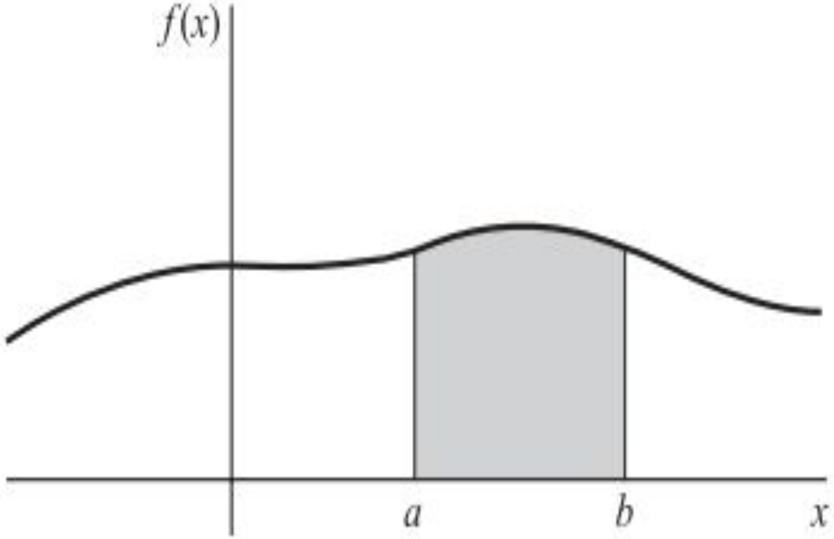
Cumulative Distribution Function

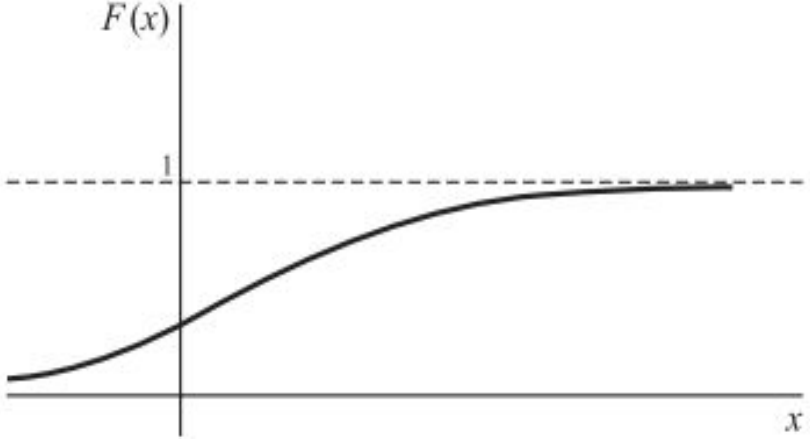
- **Definition:** *If X is a continuous random variable with pdf $f(x)$, then the distribution function of X is given by*

$$F(a) = P(X \leq a) = \int_{-\infty}^{\boldsymbol{a}} \boldsymbol{f(x)dx}$$

Graphical Interpretations

- If $f(x)$ is the density function for a random variable X, then we can represent $y = f(x)$ graphically by a curve.
  - Since $f(x) \geq 0$, the curve cannot fall below the x axis
  - Since $\int_{-\infty}^{\infty} f(x) = 1$, the entire area bounded by the curve and the x axis must be 1
- Geometrically the probability that X is between a and b, i.e., $P(a < X < b)$, is then represented by the area shown shaded
- Distribution function $F(x) = P(X \leq x)$ is a monotonically increasing function





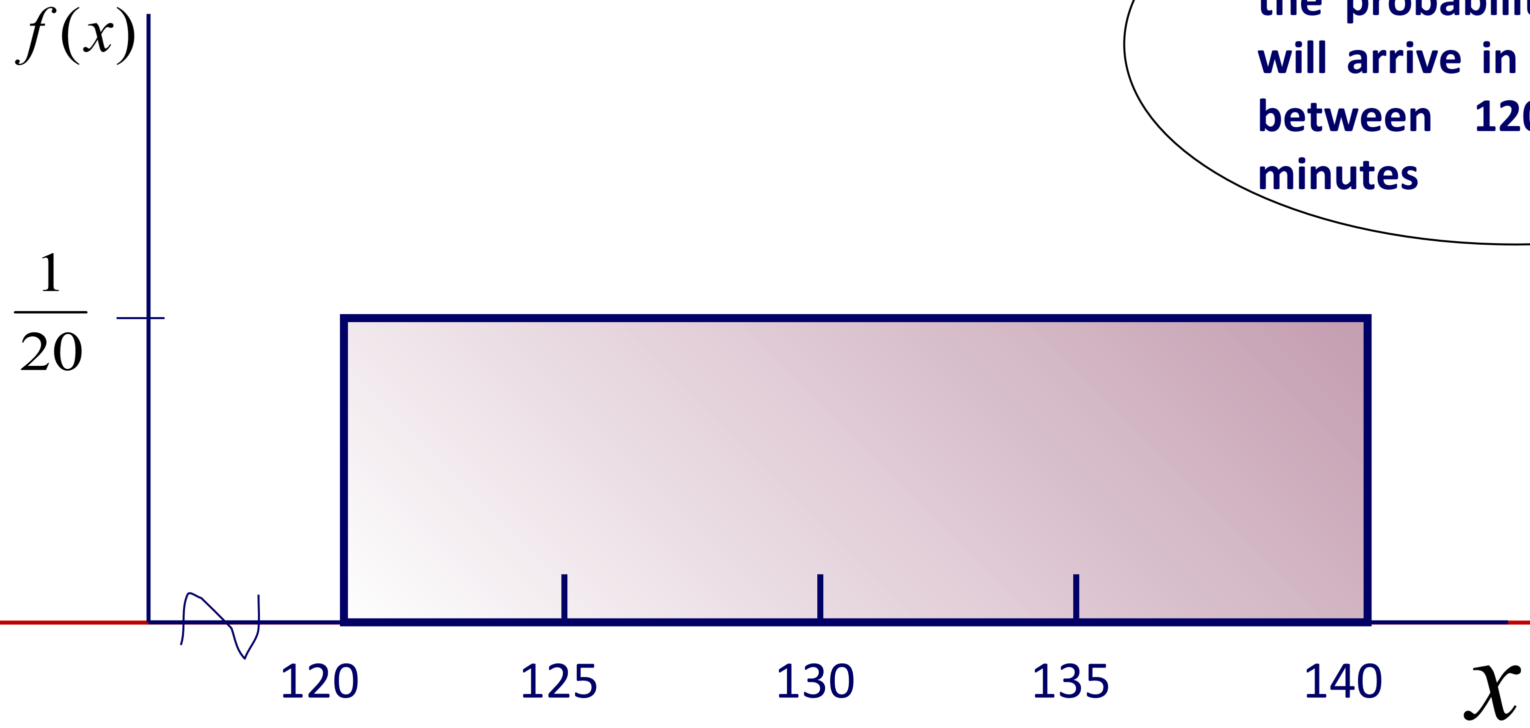
The Uniform Random Variable

- **Example**
  - Consider the random variable x representing the flight time of an airplane traveling from Delhi to Chennai.
  - Under normal conditions, flight time is between 120 and 140 minutes.
  - Because flight time can be any value between 120 and 140 minutes, x is a continuous variable.
  - PMF
    - $f(x) = \begin{cases} \frac{1}{20} & for\ 120 \leq x \leq 140 \\ 0 & elsewhere \end{cases}$

With every one-minute interval being equally likely, the random variable x is said to have a uniform probability distribution

The Uniform Random Variable

- $$f(x) = \begin{cases} \dfrac{1}{b-a} & for\ a \le x \le b \\ 0 & elsewhere \end{cases}$$
- **Uniform Probability Density Function for Flight time**

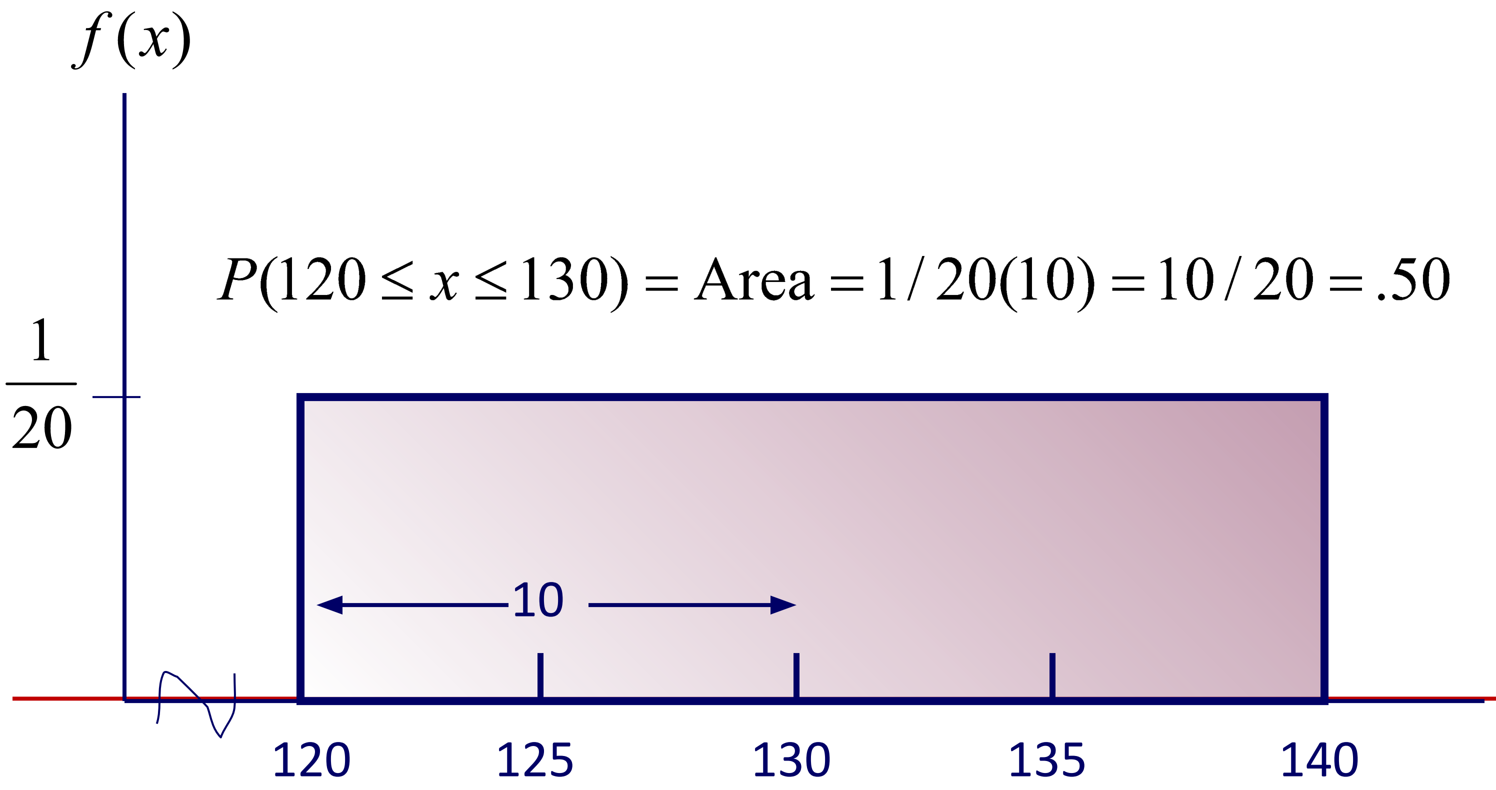


The shaded area indicates the probability the flight will arrive in the interval between 120 and 140 minutes

Probability as an Area

- **Question: What is the probability that arrival time will be between 120 and 130 minutes—that is:**

$$P(120 \le x \le 130)$$



$$P(120 \le x \le 130) = \text{Area} = 1/20(10) = 10/20 = .50$$

The Uniform Random Variable

- **Calculate the cumulative distribution function of a random variable uniformly distributed over (α, β).**
- **Solution**
  - Since, $F(a) = P(X \leq a) = \int_{-\infty}^{\boldsymbol{a}} \boldsymbol{f(x)dx}$
  - $F(a) = \begin{cases} 0 & a \leq \alpha \\ \frac{a-\alpha}{\beta-\alpha} & \alpha < a < \beta \\ 1 & elsewhere \end{cases}$

Expectation of a Uniform Random Variable

- Expectation of a random variable uniformly distributed over (α, β)

$$E[X] = \int_{\alpha}^{\beta} \frac{x}{\beta - \alpha} dx$$

$$= \left[\frac{x^2}{2(\beta - \alpha)}\right]_{\alpha}^{\beta}$$

$$= \frac{\beta^2 - \alpha^2}{2(\beta - \alpha)}$$

$$= \frac{\beta + \alpha}{2}$$

Exponential Random Variables

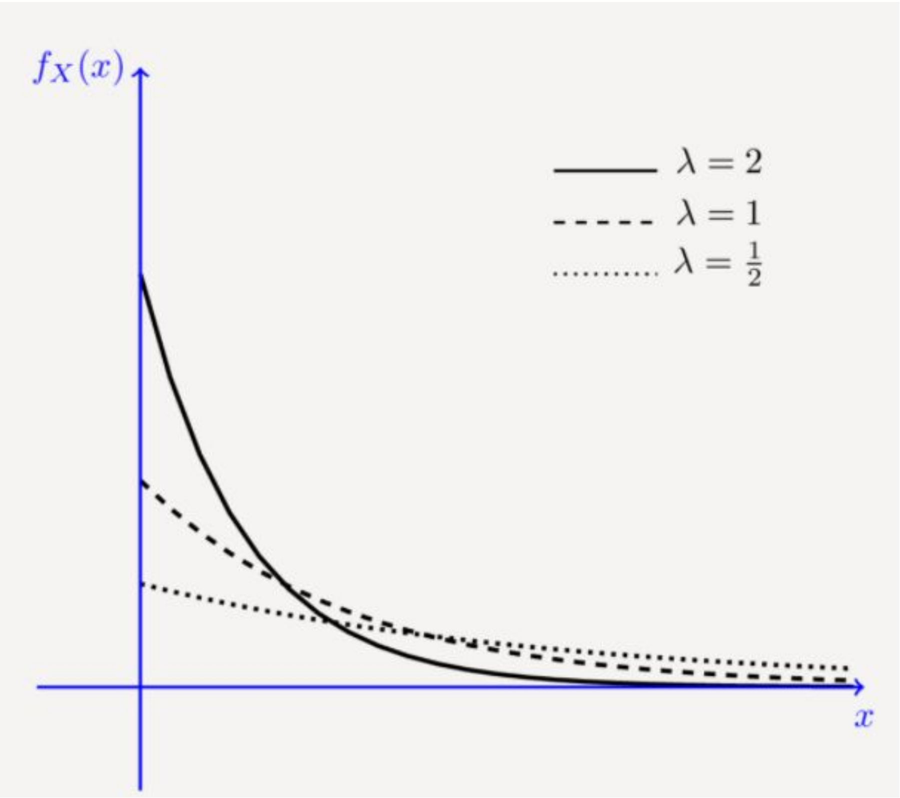
- A continuous random variable whose probability density function is given, for some $\lambda > 0$, by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & if\ x \geq 0 \\ 0 & if\ x < 0 \end{cases}$$

  - is said to be an exponential random variable with parameter λ.

- Question: Cumulative distribution function F

$$F(a) = \int_0^a \lambda e^{-\lambda x} dx = 1 - e^{-\lambda a}, a \geq 0$$



Expectation of Exponential Random Variable

- Let X be exponentially distributed with parameter λ.

$$E[X] = \int_0^{\infty} x\lambda e^{-\lambda x} dx$$

- Integrating by parts $(u = x, dv = \lambda e^{-\lambda x})$ yields

$$E[X] = \left[-xe^{-\lambda x}\right]_0^{\infty} + \int_0^{\infty} e^{-\lambda x} dx$$

$$E[X] = 0 - \left[\frac{e^{-\lambda x}}{\lambda}\right]_0^{\infty} = \frac{1}{\lambda}$$

Gamma Random Variables

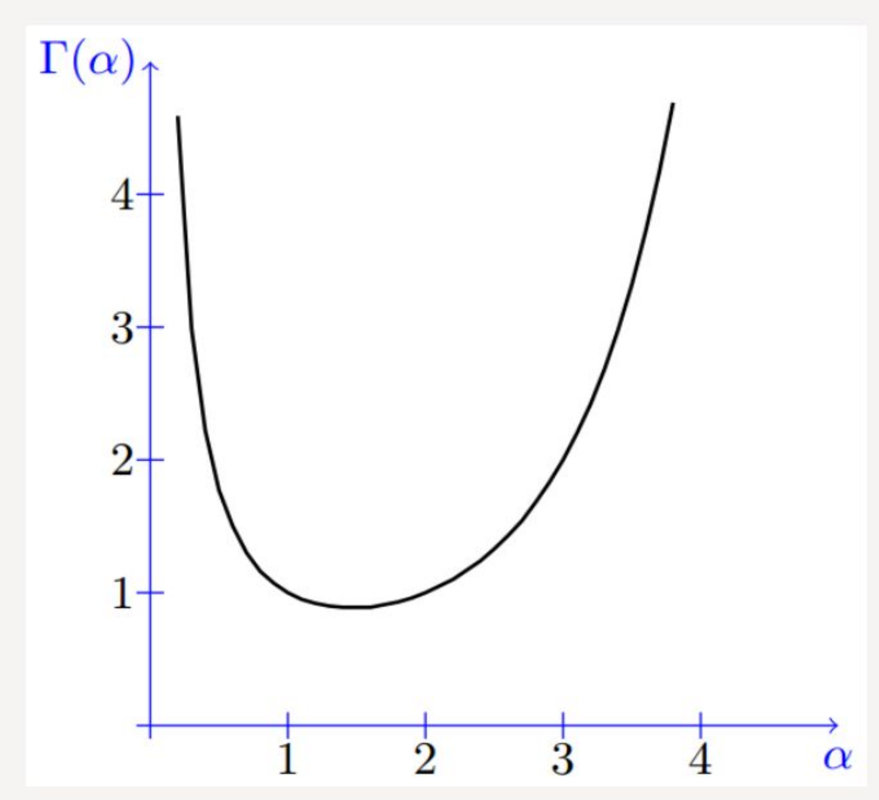
- A continuous random variable whose density is given by

$$f(x) = \begin{cases} \dfrac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} & if\ x \geq 0 \\ 0 & x < 0 \end{cases}$$

  For some $\lambda > 0, \alpha > 0$ is said to be a gamma random variable with parameters λ, α.

- The quantity $\Gamma(\alpha)$ is called the gamma function and is defined by

$$\Gamma(\alpha) = \int_0^{\infty} e^{-x} x^{\alpha-1} dx$$

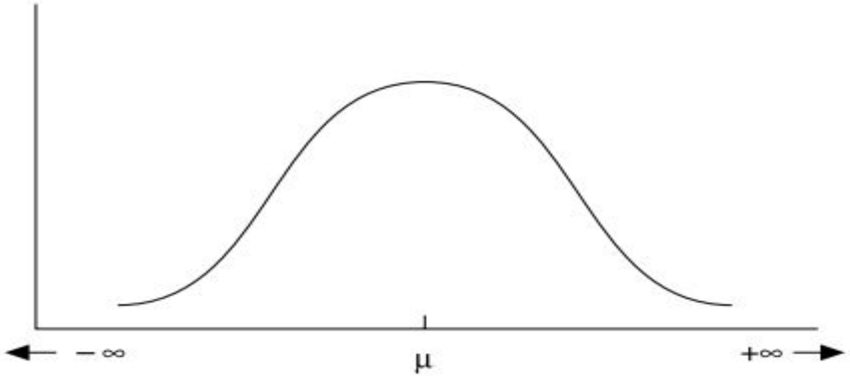


Normal Random Variables

- We say that X is a normal random variable with parameters μ and σ^2 if the density of X is given by

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, -\infty < x < \infty$$

- This density function is a bell-shaped curve that is symmetric around μ



https://www.probabilitycourse.com/chapter4/4_2_3_normal.php

Joint Distributions

- **CONTINUOUS CASE.**
  - **If X and Y are two continuous random variables, we define the joint probability function (or, joint density function) of X and Y by**

$$p(a \leq X \leq b, c \leq Y \leq d) = \int_c^d \int_a^b f(x, y)\,dx\,dy$$

where,

1. $f(x, y) \geq 0$
2. $\int_a^b \int_c^d f(x, y)\,dx\,dy = 1$, i.e.

More generally, if A represents any event, there will be a region $\mathbb{R}_A$ $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\,dx\,dy$

Joint Distributions

- **The joint density function of X and Y is given by**

$$f(x,y) = \begin{cases} 2e^{-x}e^{-2y}, & 0 < x < \infty, 0 < y < \infty \\ 0, & otherwise \end{cases}$$

- **Compute** $P\{X > 1, Y < 1\}$

$$\begin{aligned} P\{X > 1, Y < 1\} &= \int_0^1 \int_1^\infty 2e^{-x}e^{-2y}\,dx\,dy \\ &= \int_0^1 2e^{-2y}\left(-e^{-x}\big|_1^\infty\right) dy \\ &= e^{-1}\int_0^1 2e^{-2y}\,dy \\ &= e^{-1}(1 - e^{-2}) \end{aligned}$$

Joint Distributions

- **DISCRETE CASE**
  - If X and Y are two discrete random variables, we define the joint probability function of X and Y by

$$P(X = x, Y = y) = f(x, y)$$

where,

1. $f(x, y) \geq 0$
2. $\sum_x \sum_y f(x, y) = 1$, i.e., the sum over all values of x and y is 1.

- Can be represented by a joint probability table

Joint probability		Throwing a Die						Marginal probability
		1	2	3	4	5	6	
Tossing a Coin	Head							
	Tail							

Joint Distributions

- **Suppose that 3 balls are randomly selected from an urn containing 3 red, 4 white, and 5 blue balls. If we let X and Y denote, respectively, the number of red and white balls chosen, then the joint probability mass function of X and Y, $p(i,j) = P\{X=i, Y=j\}$, is given by**

$$p(0,0) = \binom{5}{3}\Big/\binom{12}{3} = \frac{10}{220}$$

$$p(0,1) = \binom{4}{1}\binom{5}{2}\Big/\binom{12}{3} = \frac{40}{220}$$

$$p(0,2) = \binom{4}{2}\binom{5}{1}\Big/\binom{12}{3} = \frac{30}{220}$$

$$p(0,3) = \binom{4}{3}\Big/\binom{12}{3} = \frac{4}{220}$$

$$p(1,0) = \binom{3}{1}\binom{5}{2}\Big/\binom{12}{3} = \frac{30}{220}$$

$$p(1,1) = \binom{3}{1}\binom{4}{1}\binom{5}{1}\Big/\binom{12}{3} = \frac{60}{220}$$

$$p(1,2) = \binom{3}{1}\binom{4}{2}\Big/\binom{12}{3} = \frac{18}{220}$$

$$p(2,0) = \binom{3}{2}\binom{5}{1}\Big/\binom{12}{3} = \frac{15}{220}$$

$$p(2,1) = \binom{3}{2}\binom{4}{1}\Big/\binom{12}{3} = \frac{12}{220}$$

$$p(3,0) = \binom{3}{3}\Big/\binom{12}{3} = \frac{1}{220}$$

Reference

- Lecture notes on Probability Theory by Phanuel Mariano
- **Introduction to Probability Models, Sheldon M. Ross, Tenth Edition**