Conditional Probability

slides are based on lecture notes on Probability Theory by Phanuel Mariano

Practice Exercises

- Two cards are randomly drawn from a deck of 52 playing cards. Find the probability that both cards will be greater than 3 and less than 8.
  - {4, 5, 6, 7}, total 16 card. $\frac{16}{52}\frac{15}{51}$
- 4 candidates are seeking a vacancy on a school board. If A is twice as likely to be elected as B, and B and C are given about the same chance of being elected, while C is twice as likely to be elected as D, then what are the probabilities that C will win? A will not win?
  - Let x be the probability that D will win, then
  - $B = C = 2x, A = 4x$
  - $4x + 2x + 2x + x = 1 \Longrightarrow x = \frac{1}{9}$

Recall

- **Sample Space**
  - **The set of all possible outcomes for an experiment is the sample space.**
    - **Tossing a coin: {H, T}**
    - **Tossing two coin: {HH,HT, TH, TT}**
- **Event**
  - **Consists of one or more outcomes and is a subset of the sample space.**
    - **A die is tossed. Event A is observing an even number.**
      - A = {2, 4, 6}
  - **An event that consists of a single outcome is called simple event**

Recall

- Two events are mutually exclusive if, when one event occurs, the other cannot, and vice versa.
  - Toss a die.
    - A: Observe a 6
    - B: Observe a 3
- Equally Likely Outcomes
  - Probability of an event A

$$p(A) = \frac{\text{Favorable Outcomes}}{\text{Total Outcomes}}$$

  - This method for calculating probabilities is only appropriate when the outcomes of the sample space are equally likely.

Recall

- The probability of an event A is equal to the sum of the probabilities of the simple events contained in A
  - Toss a fair coin twice. What is the probability of observing at least one head?
  - $P(HH) + P(HT) + P(TH) = \frac{3}{4}$
- Basic Rules of Probability
  - Rule I: For any event A, $0 \leq P(A) \leq 1$
  - The sum of the probabilities for all simple events in S equals 1.
    - Toss a fair coin twice: $P(HH) + P(HT) + P(TH) + P(TT) = 1$

Recall

- **Counting Principle**
  - **Sample space of throwing 3 dice has 216 entries, sample space of throwing 4 dice has 1296 entries, …**
  - **We need a way to help us count faster rather than counting by hand one by one.**

(Basic Counting Principle) Suppose 2 experiments are to be performed.

If one experiment can result in m possibilities
Second experiment can result in n possibilities
Then together there are mn possibilities

Recall

- **(Basic Counting Principle)** Suppose 2 experiments are to be performed.

  If one experiment can result in m possibilities
  Second experiment can result in n possibilities
  Then together there are mn possibilities

- **A college planning committee consists of 3 freshmen, 4 second year's, 5 juniors, and 2 seniors. A subcommittee of 4 consists 1 person from each class. How many such subcommittee?**
  - $\mathbf{3 \times 4 \times 5 \times 2 = 120}$

Recall

- Permutations
  - How many different ordered arrangements of the letters a, b, c are possible? $3!$
  - With n objects. There are $n!$ different permutations of the n objects.
- ORDER matters when it comes to Permutations
- Question: How many ways can one arrange 4 math books, 3 chemistry books, 2 physics books, and 1 biology book on a bookshelf so that all the math books are together, all the chemistry books are together, and all the physics books are together?
  - $4!\,(4!\,.\,3!\,.\,2!\,.\,1!)$

Recall

- Permutations
  - How many different ordered arrangements of the letters a, b, c are possible? $3!$
  - With n objects. There are $n!$ different permutations of the n objects.
- ORDER matters when it comes to Permutations
- Question: Repetitions: How many ways can one arrange the letters a, a, b, c?
- There are $\frac{n!}{n_1!n_2!\ldots n_r!}$ different permutations of n objects of which n_1 are alike, n_2 are alike, n_r are alike.

Recall

- Combinations
  - We are often interested in selecting r objects from a total of n objects.
  - If r ≤ n, then $\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$ represents the number of possible combinations of objects taken r at a time.
  - Order DOES NOT Matter here
- Question: A person has 8 friends, of whom 5 will be invited to a party. How many choices are there if 2 of the friends are feuding and will not attend together?

$$\binom{6}{5} + \binom{2}{1}\binom{6}{4}$$

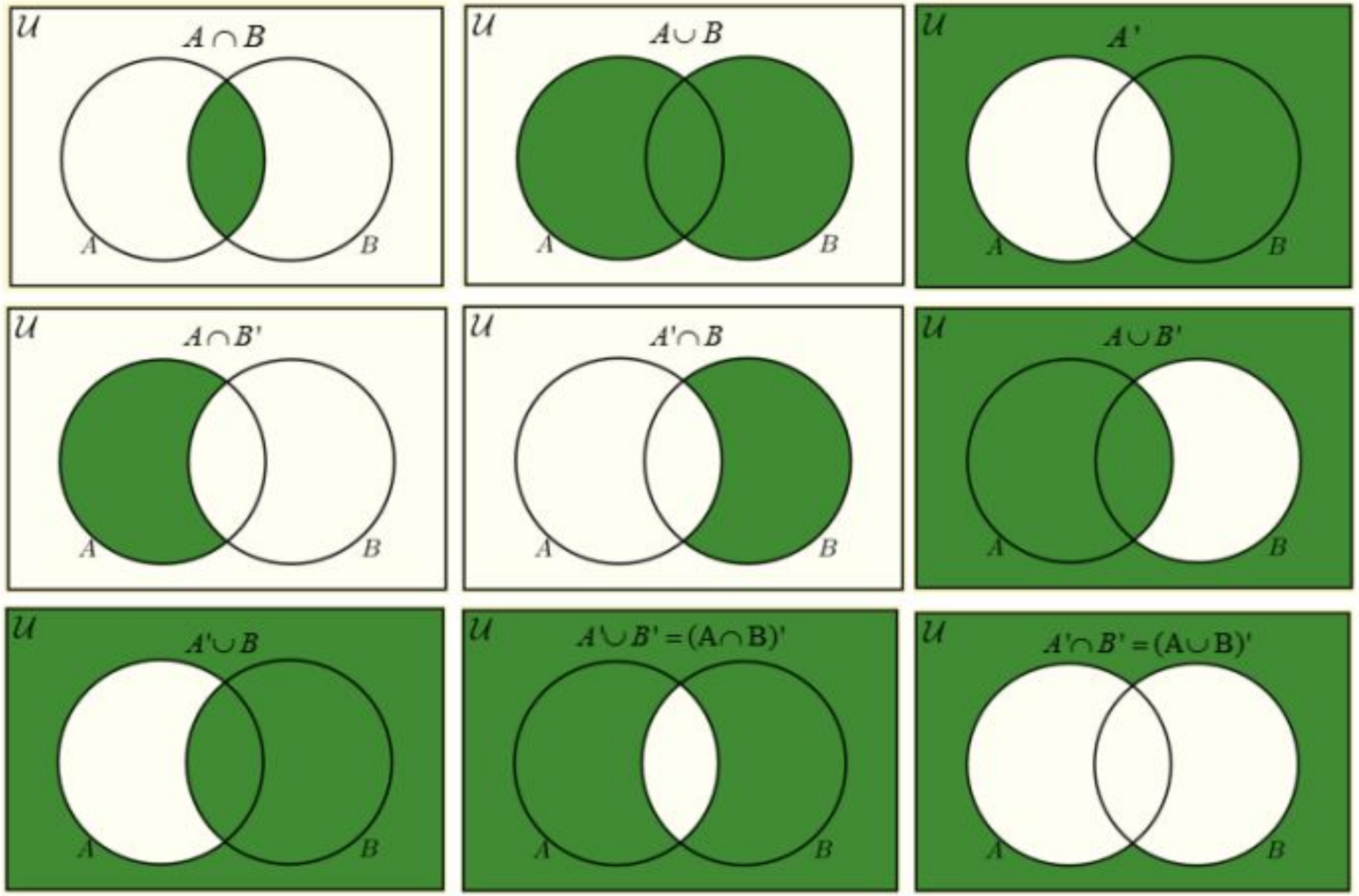
Recall

- Question: Suppose one has 9 people and one wants to divide them into one committee of 3, one of 4, and a last of 2. How many different ways are there?

$$\binom{9}{3}\binom{6}{4}\binom{2}{2} = \frac{9!}{3!\,4!\,2!}$$

- Divide n objects into one group of n_1, one group of n_2, … and a kth group of n_k, where $n = n_1 + \cdots + n_k$, then there are $\frac{n!}{n_1!n_2!\ldots n_k!}$ ways

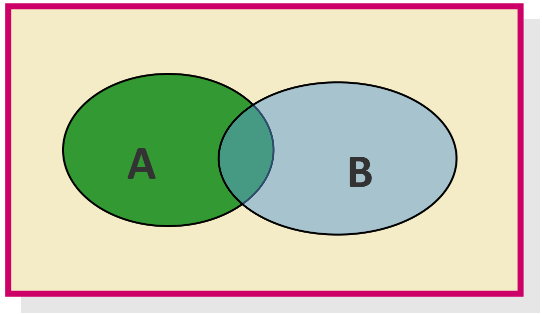
Recall



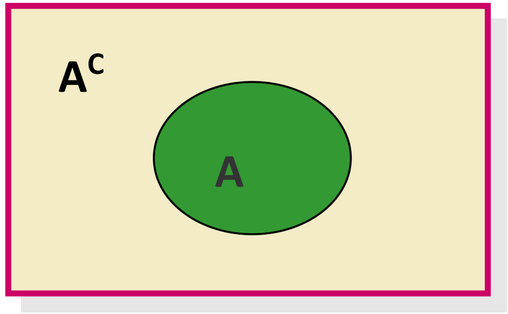
Recall

-

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



- **When two events A and B are mutually exclusive,** $P(A \cap B) = 0$ **and**
$$P(A \cup B) = P(A) + P(B).$$
- $P(A \cap A^C) = 0$
- $P(A \cup A^C) = 1$
- $P(A) + P(A^C) = 1$



Will You Play?

- Suppose that a “friend” offers you to play a game. He has three cards, one red on both sides, one black on both sides, and one red on one side and black on the other.
- He mixes the three cards in a hat, picks one at random, and places it flat on the table with only one side showing.
- Suppose that one side is red. He then offers to bet his $4 against your $3 that the other side of the card is also red.

Independent Events

- **E and F are independent events if**

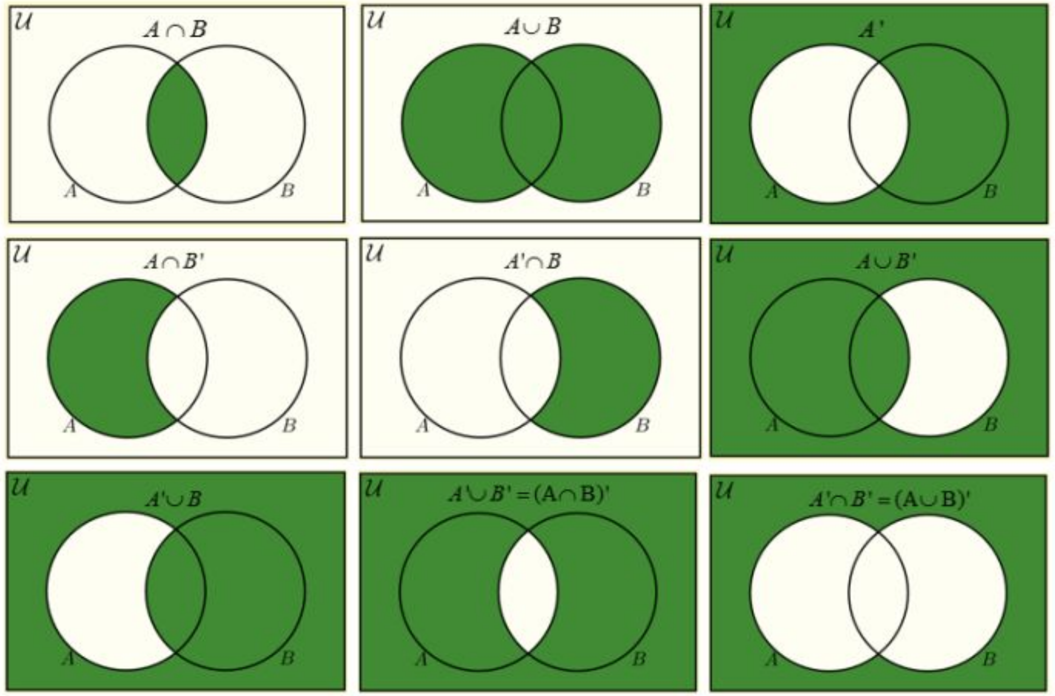
$$\boldsymbol{P(E \cap F) = P(E)P(F)}$$

  - **Toss two fair coin**
    - **The event that you get heads on the second coin is independent of the event that you get tails on the first.**
      - A_t: event of getting is tails for the first coin
      - B_h: event of getting heads for the second coin

$$\boldsymbol{P(A_t \cap B_h) = P(A_t)P(B_h) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}}$$

- **Independence and mutually exclusive, are two different things!**

Independent Events



- **If E and F are independent, then**

$$P(E \cap F^c) = ?$$
$$= P(E) - P(E \cap F)$$
$$= P(E) - P(E)P(F)$$
$$= P(E)\big(1 - P(F)\big)$$
$$= P(E)P(F^c)$$

- **Events $A_1, \ldots, A_n$ are independent if for all subcollections $i_1, \ldots, i_r \in \{1, \ldots, n\}$ we have**

$$P\left(\cap_{j=1}^{r} A_{i_j}\right) = \prod_{j=1}^{r} P(A_{i_r})$$

Independent Events

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- Question:
  - An urn contains 10 balls: 4 red and 6 blue.
  - A second urn contains 16 red balls and an unknown number of blue balls.
  - A single ball is drawn from each urn. The probability that both balls are the same color is 0.44
  - Calculate the number of blue balls in the second urn.

R_i: Event that a red ball drawn from urn i
B_i: Event that a blue ball drawn from urn i

Let x be the number of blue balls in urn 2

$$P\big((R_1 \cap R_2) \cup (B_1 \cap B_2)\big) = 0.44$$
$$P(R_1 \cap R_2) + P(B_1 \cap B_2) = 0.44$$

$$P(R_1)\, P(R_2) + P(B_1) P(B_2) = 0.44$$

$$\frac{4}{10} \times \frac{16}{16+x} + \frac{6}{10} \times \frac{x}{16+x} = 0.44$$

$$x = 4$$

Conditional Probability and Independence

- **Flip three different fair coins, then**
  $$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$
- **Question: What is the probability that the first coin comes up heads?**
  - $P(\{HHH, HHT, HTH, HTT\}) = \frac{4}{8}$
  - **Additional Information: Exactly two of the three coins came up heads.**
    - **Now, what is the probability that the first coin was heads?**

“conditional on,” or “given that

$$P(\text{first coin heads} \mid \text{two coins heads}) = \frac{2}{3}$$

More precisely, we have computed a conditional probability. That is, we have determined that, conditional on knowing that exactly two coins came up heads, the conditional probability of the first coin being head is 2/3.

Conditional Probability

- **Given two events A and B with $P(B) > 0$, the conditional probability of A given B, written**
$$P(A \mid B)$$

Fraction of the time that *A* occurs once we *know* that *B* occurs.

- **The conditional probability of *A* given *B* is equal to**

$$\boldsymbol{P(A \mid B) = \frac{P(A \cap B)}{P(B)}}$$

Conditional Probability

- The conditional probability of *A* given *B* is equal to

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

- Question: Suppose $P\,(A \mid B) = P(A)$, What this implies?
    - A and B are independent of each other.

Conditional Probability

- The conditional probability of *A* given *B* is equal to

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

- Suppose a box has 3 red marbles and 2 black ones. We select 2 marbles. Question: What is the probability that second marble is red given that the first one is red? Apply conditional probability

R_1: First marble is is red

R_2: Second marble is red

$$P(R_2|R_1) = \frac{P(R_1 \cap R_2)}{P(R_2)}$$

$$= \frac{(2\ red)(0\ black)/\binom{5}{2}}{\frac{3}{5}}$$

$$= \frac{\binom{3}{2}\binom{2}{0}/\binom{5}{2}}{\frac{3}{5}} = \frac{1}{2}$$

Conditional Probability

- Suppose that person X and Y each draw 13 cards from a standard deck of 52. Given that X has exactly two King, what is the probability that Y has exactly one King?
- Solution:
  - A: Event that X has two King
  - B: Event that Y has one King

$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$

$$P(A) = \frac{\binom{4}{2}\binom{48}{11}}{\binom{52}{13}}$$

$$P(B \cap A) = \frac{\binom{4}{2}\binom{48}{11}\binom{2}{1}\binom{37}{12}}{\binom{52}{13}\binom{39}{13}}$$

$$P(B|A) = \frac{\frac{\binom{4}{2}\binom{48}{11}\binom{2}{1}\binom{37}{12}}{\binom{52}{13}\binom{39}{13}}}{\frac{\binom{4}{2}\binom{48}{11}}{\binom{52}{13}}}$$

Conditional Probability

- $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$ then
- $P(A \cap B) = P(A \mid B)P(B)$

- **If** $E_1, \dots, E_n$ **are events then**

$$P(E_1 \cap E_2 \cap \cdots \cap E_n) = P(E_1)P(E_2|E_1)P(E_3|E_1 \cap E_2), \dots, P(E_n|E_1 \cap E_2 \cap \cdots \cap E_{n-1})$$

Practice Exercises

- Suppose an urn has 5 White balls and 7 Black balls. Each ball that is selected is returned to the urn along with an additional ball of the same color. Suppose draw 3 balls. Question: What is the probability that you get 3 white balls.
- Phan wants to take a Biology course or a Chemistry course. Given that the students take Biology, the probability that they get an A is is $4/5$. While the probability of getting an A given that the student took Chemistry is $1/7$. If Phan makes a decision on the course to take randomly, what's probability of getting an A in Chem?

Next Class

- **Bayes's Formula**

Reference

- **Statistics with Economics and Business Applications, Chapter 3 Probability and Discrete Probability Distributions**
- **Modern Business Statistics, Slides by John Loucks**
- lecture notes on Probability Theory by Phanuel Mariano