Expectation and Variance of a Random Variable

Recall: The Discrete Case

- If X is a discrete random variable having a probability mass function $p(x)$, then the expected value of X is defined by

$$E[X] = \sum_{x:p(x)>0} xp(x)$$

The expected value of X is a weighted average of the possible values that X can take on; each value being weighted by the probability that X assumes that value.

- Example: Toss a die

$$p(1) = p(2) = p(3) = p(4) = p(5) = p(6) = \frac{1}{6}$$

$$E[X] = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 6 \times \frac{1}{6}$$
$$= 3.5$$

Recall: The Discrete Case

- If X is a discrete random variable having a probability mass function $p(x)$, then the variance of X is defined by

$$Var[X] = E\left[(X-\mu)^2\right], \mu = E[X]$$

The variance of X measures the expected square of the deviation of X from its expected value.

$$\begin{aligned} Var(X) = \sigma^2 &= E\left[(X-\mu)^2\right] \\ &= E\left[X^2 - 2.X.\mu + \mu^2\right] \\ &= E\left[X^2\right] - 2\mu E[X] + (E[X])^2 \\ &= E\left[X^2\right] - 2E[X]E[X] + (E[X])^2 \\ &= E\left[X^2\right] - (E[X])^2 \end{aligned}$$

Recall: The Discrete Case

- If X is a discrete random variable having a probability mass function $p(x)$, then the variance of X is defined by

$$Var[X] = E\left[(X - E[X])^2\right] = E\left[X^2\right] - (E[X])^2$$

- Example: Toss a die

$$E\left[X^2\right] = 1 \times \frac{1}{6} + 2^2 \times \frac{1}{6} + 3^2 \times \frac{1}{6} + 4^2 \times \frac{1}{6} + 5^2 \times \frac{1}{6} + 6^2 \times \frac{1}{6}$$

$$E\left[X^2\right] = \frac{91}{6}$$

$$Var[X] = \frac{91}{6} - \frac{49}{4} = \frac{182 - 147}{12} = \frac{35}{12} \approx 2.9167$$

Recall: Bernoulli Random Variable

- If X is a Bernoulli random variable with parameter p, i.e., $\boldsymbol{p}(\mathbf{0}) = \mathbf{1} - p, \boldsymbol{p}(\mathbf{1}) = p$

$$\boldsymbol{E}[\boldsymbol{X}] = \mathbf{0}(\mathbf{1} - p) + \mathbf{1}(p) = p$$

$$\boldsymbol{E}[\boldsymbol{X}^2] = \mathbf{0}^2(\mathbf{1} - p) + \mathbf{1}^2(p) = \boldsymbol{p}$$

$$Var[X] = p - p^2 = p(1 - p)$$

Recall: Binomial Random Variable

- If X is binomially distributed with parameters n and p

$$
\begin{aligned}
E[X] &= \sum_{i=0}^{n} i p(i) \\
&= \sum_{i=0}^{n} i \binom{n}{i} p^i (1-p)^{n-i} \\
&= \sum_{i=1}^{n} \frac{i n!}{(n-i)!\, i!} p^i (1-p)^{n-i} \\
&= \sum_{i=1}^{n} \frac{n!}{(n-i)!(i-1)!} p^i (1-p)^{n-i} \\
&= np \sum_{i=1}^{n} \frac{(n-1)!}{(n-i)!(i-1)!} p^{i-1} (1-p)^{n-i} \\
&= np \sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{n-1-k} \\
&= np[p + (1-p)]^{n-1} \\
&= np
\end{aligned}
$$

Refer slides on Discrete random variables posted earlier

- $Var(X) = np(1-p)$ [Proof Next Slide]

Recall: Variance of a Binomial Random Variable

$$\begin{aligned}
\mathsf{E}\left(X^2\right) &= \sum_{k \geq 0}^{n} k^2 \binom{n}{k} p^k q^{n-k} && \text{Definition of Binomial Distribution: } p + q = 1 \\
&= \sum_{k=0}^{n} kn \binom{n-1}{k-1} p^k q^{n-k} && \text{Factors of Binomial Coefficient: } k\binom{n}{k} = n\binom{n-1}{k-1} \\
&= np \sum_{k=1}^{n} k \binom{n-1}{k-1} p^{k-1} q^{(n-1)-(k-1)} && \text{Change of limit: term is zero when } k-1=0 \\
&= np \sum_{j=0}^{m} (j+1) \binom{m}{j} p^j q^{m-j} && \text{putting } j = k-1, m = n-1 \\
&= np \left(\sum_{j=0}^{m} j \binom{m}{j} p^j q^{m-j} + \sum_{j=0}^{m} \binom{m}{j} p^j q^{m-j} \right) && \text{splitting sum up into two} \\
&= np \left(\sum_{j=0}^{m} m \binom{m-1}{j-1} p^j q^{m-j} + \sum_{j=0}^{m} \binom{m}{j} p^j q^{m-j} \right) && \text{Factors of Binomial Coefficient: } j\binom{m}{j} = m\binom{m-1}{j-1} \\
&= np \left((n-1)\, p \sum_{j=1}^{m} \binom{m-1}{j-1} p^{j-1} q^{(m-1)-(j-1)} + \sum_{j=0}^{m} \binom{m}{j} p^j q^{m-j} \right) && \text{Change of limit: term is zero when } j-1=0 \\
&= np \left((n-1)\, p(p+q)^{m-1} + (p+q)^m \right) && \text{Binomial Theorem} \\
&= np \left((n-1)\, p + 1 \right) && \text{as } p+q=1 \\
&= n^2p^2 + np\,(1-p) && \text{by algebra}
\end{aligned}$$

$$\begin{aligned}
\mathrm{var}\,(X) &= \mathsf{E}\left(X^2\right) - \left(\mathsf{E}\,(X)\right)^2 \\
&= np\,(1-p) + n^2p^2 - (np)^2 \\
&= np\,(1-p)
\end{aligned}$$

Recall: Poisson Distribution

- If X is Poisson random variable with parameters λ

$$
\begin{aligned}
E[X] &= \sum_{i=0}^{\infty} \frac{i e^{-\lambda} \lambda^i}{i!} \\
&= \sum_{i=1}^{\infty} \frac{e^{-\lambda} \lambda^i}{(i-1)!} \\
&= \lambda e^{-\lambda} \sum_{i=1}^{\infty} \frac{\lambda^{i-1}}{(i-1)!} \\
&= \lambda e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \\
&= \lambda e^{-\lambda} e^{\lambda} \\
&= \lambda
\end{aligned}
$$

Refer slides on Discrete random variables posted earlier

Recall: Poisson Distribution

- $Var(X) = \lambda$

$$E[X^2] = E[X(X-1) + X] = E[X(X-1)] + \lambda$$

$$E[X(X-1)] = \sum_{x=0}^{\infty} (x)(x-1)e^{-\lambda}\frac{\lambda^x}{x!}$$

Recall: Poisson Distribution

- $$E[X(X-1)] = \sum_{x=0}^{\infty} (x)(x-1)e^{-\lambda}\frac{\lambda^x}{x!}$$

$$E[X(X-1)] = \sum_{x=2}^{\infty} (x)(x-1)e^{-\lambda}\frac{\lambda^x}{x!}$$

$$E[X(X-1)] = \sum_{x=2}^{\infty} e^{-\lambda}\frac{\lambda^x}{(x-2)!}$$

Recall: Poisson Distribution

- $$E[X(X-1)] = \sum_{x=2}^{\infty} e^{-\lambda} \frac{\lambda^x}{(x-2)!}$$

$$E[X(X-1)] = \lambda^2 e^{-\lambda} \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!}$$

$$E[X(X-1)] = \lambda^2 e^{-\lambda} e^{\lambda}$$

$$E[X^2] = \lambda^2 + \lambda$$

$$Var[X] = \lambda^2 + \lambda - \lambda^2 = \lambda$$

Recall: Negative Binomial Random Variable

- $p(X = x|p, r) = \binom{x-1}{r-1} p^r (1-p)^{x-r}$, x = r, r + 1, …

$$E[X] = \sum_{x=r}^{\infty} x \binom{x-1}{r-1} p^r (1-p)^{x-r}$$

$$E[X] = \sum_{x=r}^{\infty} x \frac{(x-1)!}{(x-r)!\,(r-1)!} p^r (1-p)^{x-r}$$

Let, $k = x - r$

Recall: Negative Binomial Random Variable

- $$E[X] = \sum_{x=r}^{\infty} \frac{x!}{(x-r)!\,(r-1)!} p^r (1-p)^{x-r}$$

Let, $k = x - r$

$$E[X] = \sum_{k=0}^{\infty} \frac{(k+r)!}{k!\,(r-1)!} p^r (1-p)^k$$

Recall: Negative Binomial Random Variable

- $$E[X] = r\sum_{k=0}^{\infty}\frac{(k+r)!}{k!\,r!}p^r(1-p)^k$$

$$E[X] = rp^r\sum_{k=0}^{\infty}\binom{k+r}{k}(1-p)^k$$

$$\binom{-\beta-1}{k} = (-1)^k\binom{k+\beta}{k}.$$

$$E[X] = rp^r\sum_{k=0}^{\infty}\binom{-(r+1)}{k}(-1)^k(1-p)^k$$

Recall: Negative Binomial Random Variable

- $$E[X] = rp^r \sum_{k=0}^{\infty} \binom{-(r+1)}{k} (-1)^k (1-p)^k$$

$$E[X] = rp^r \left(1 - (1-p)\right)^{-(r+1)}$$

$$E[X] = rp^r (p)^{-(r+1)} = \frac{r}{p}$$

$$Var[X] = \frac{r(1-p)}{p^2}$$

Recall: Geometric Random Variable

- **If X a geometric random variable having parameter p.**

$$\boldsymbol{E[X] = \sum_{n=1}^{\infty} np(1-p)^{n-1}}$$

- Setting, q=p-1

$$\boldsymbol{E[X] = p\sum_{n=1}^{\infty} nq^{n-1}}$$

Recall: Geometric Random Variable

- $$E[X] = p \sum_{n=1}^{\infty} nq^{n-1}$$

$$E[X] = p \sum_{n=1}^{\infty} \frac{d}{dq} q^n$$

$$E[X] = p \frac{d}{dq} \left(\sum_{n=1}^{\infty} q^n \right)$$

Recall: Geometric Random Variable

- $$E[X] = p\frac{d}{dq}\left(\sum_{n=1}^{\infty} q^n\right)$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{u}{v}\right)$$

$$E[X] = p\frac{d}{dq}\left(\frac{q}{1-q}\right)$$

$$\frac{dy}{dx} = \frac{v\frac{d}{dx}(u) - u\frac{d}{dx}(v)}{v^2}$$

$$E[X] = p\left(\frac{(1-q)(1) - q(-1)}{(1-q)^2}\right) = p\left(\frac{1}{(1-q)^2}\right)$$

$$E[X] = \frac{1}{p}$$

$$Var[X] = \frac{1-p}{p^2}$$

Recall: Expectation of a Hypergeometric Random Variable

- PMF
  - $P(X = q) = \frac{\binom{K}{q}\binom{N-K}{t-q}}{\binom{N}{t}}$

$$E[X] = \sum_{q=0}^{K} q \frac{\binom{K}{q}\binom{N-K}{t-q}}{\binom{N}{t}}$$

$$E[X] = \sum_{q=1}^{K} q \frac{\binom{K}{q}\binom{N-K}{t-q}}{\binom{N}{t}}$$

Recall: Expectation of a Hypergeometric Random Variable

- $$E[X] = \sum_{q=1}^{K} q \frac{\binom{K}{q}\binom{N-K}{t-q}}{\binom{N}{t}}$$

$$E[X] = \sum_{q=1}^{K} q \frac{\binom{K}{q}\binom{N-K}{t-q}}{\frac{N}{t}\binom{N-1}{t-1}}$$

$$E[X] = \sum_{q=1}^{K} \frac{K\binom{K-1}{q-1}\binom{N-K}{t-q}}{\frac{N}{t}\binom{N-1}{t-1}}$$

Recall: Expectation of a Hypergeometric Random Variable

- $$E[X] = \sum_{q=1}^{K} \frac{K\binom{K-1}{q-1}\binom{N-K}{t-q}}{\frac{N}{t}\binom{N-1}{t-1}}$$

$$E[X] = \frac{Kt}{N} \sum_{q=1}^{K} \frac{\binom{K-1}{q-1}\binom{N-K}{t-q}}{\binom{N-1}{t-1}}$$

Let, $j = q - 1$

Recall: Expectation of a Hypergeometric Random Variable

- $$E[X] = \frac{Kt}{N}\sum_{q=1}^{K}\frac{\binom{K-1}{q-1}\binom{N-K}{t-q}}{\binom{N-1}{t-1}}$$

Let, $j = q - 1$

$$E[X] = \frac{Kt}{N}\sum_{j=0}^{K-1}\frac{\binom{K-1}{j}\binom{N-K}{t-j-1}}{\binom{N-1}{t-1}}$$

Recall: Expectation of a Hypergeometric Random Variable

- $$E[X] = \frac{Kt}{N} \boxed{\sum_{j=0}^{K-1} \frac{\binom{K-1}{j}\binom{N-K}{t-j-1}}{\binom{N-1}{t-1}}}$$

Probability that when you draw t−1 balls from an urn that contains K−1 red and N-k white, you will get exactly j red balls.

1

$$E[X] = \frac{Kt}{N}$$

$$Var[X] = \frac{tK}{N}\left(\frac{N-K}{N}\right)\left(\frac{N-t}{N-1}\right)$$

Continuous Random Variable

Continuous Random

- If X is a continuous random variable having a probability density function $f(x)$, then the expected value of X is defined by

$$\boldsymbol{E[X] = \int_{-\infty}^{\infty} xf(x)dx}$$

- Variance

$$\boldsymbol{Var[X] = \int_{-\infty}^{\infty} (x - E[X])^2 f(x)dx = E\left[X^2\right] - E[X]}$$

Recall: Uniform Random Variable

- **Expectation of a random variable uniformly distributed over (α, β)**

$$E[X] = \int_{\alpha}^{\beta} \frac{x}{\beta - \alpha} dx$$

$$= \left[\frac{x^2}{2(\beta - \alpha)}\right]_{\alpha}^{\beta}$$

$$= \frac{\beta^2 - \alpha^2}{2(\beta - \alpha)}$$

$$= \frac{\beta + \alpha}{2}$$

$$f(x) = \begin{cases} \frac{1}{\beta - \alpha} & for\ \beta \leq x \leq \alpha \\ 0 & elsewhere \end{cases}$$

Recall: Uniform Random Variable

- Expectation of a random variable uniformly distributed over (α, β)

$$Var[X] = \int_{\alpha}^{\beta} \frac{x^2}{\beta - \alpha} dx - \left(\frac{\beta + \alpha}{2}\right)^2$$

$$Var[X] = \left[\frac{x^3}{3(\beta - \alpha)}\right]_{\alpha}^{\beta} - \left(\frac{\beta + \alpha}{2}\right)^2$$

$$Var[X] = \frac{\beta^2 - \alpha\beta + \alpha^2}{3} - \left(\frac{\beta + \alpha}{2}\right)^2$$

Recall: Uniform Random Variable

- $$Var[X] = \frac{\beta^2 - \alpha\beta + \alpha^2}{3} - \left(\frac{\beta + \alpha}{2}\right)^2$$

$$Var[X] = \frac{\beta^2 - \alpha\beta + \alpha^2}{3} - \frac{\beta^2 + \alpha^2 + 2\alpha\beta}{4}$$

$$Var[X] = \frac{4\beta^2 - 4\alpha\beta + 4\alpha^2 - 3\beta^2 - 3\alpha^2 - 6\alpha\beta}{12}$$

$$Var[X] = \frac{\beta^2 - 2\alpha\beta + \alpha^2}{12} = \frac{(\beta - \alpha)^2}{12}$$

Recall: Exponential Random Variable

- Let X be exponentially distributed with parameter λ.

$$E[X] = \int_0^\infty x\lambda e^{-\lambda x} dx$$

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & if\ x \geq 0 \\ 0 & if\ x < 0 \end{cases}$$

- Integrating by parts $(u = x, dv = \lambda e^{-\lambda x})$ yields

$$E[X] = \left[-xe^{-\lambda x}\right]_0^\infty + \int_0^\infty e^{-\lambda x} dx$$

$$E[X] = 0 - \left[\frac{e^{-\lambda x}}{\lambda}\right]_0^\infty = \frac{1}{\lambda}$$

$$Var[X] = \frac{1}{\lambda^2}$$

Recall: Normal Random Variable

- $$E[X] = \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} x e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx$$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, -\infty < x < \infty$$

- Writing x as (x − μ) + μ yields

$$E[X] = \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} (x-\mu) e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx + \mu \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx$$

- Letting y = x − μ

$$E[X] = \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} y e^{-\frac{y^2}{2\sigma^2}} \, dy + \mu \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} f(x) \, dx$$

Recall: Normal Random Variable

- $$E[X] = \frac{1}{\sqrt{2\pi\sigma}}\int_{-\infty}^{\infty} y e^{-\frac{y^2}{2\sigma^2}}\, dy + \mu \frac{1}{\sqrt{2\pi\sigma}}\int_{-\infty}^{\infty} f(x)\, dx$$

$$E[X] = 0 + \mu \frac{1}{\sqrt{2\pi\sigma}}\int_{-\infty}^{\infty} f(x)\, dx = \mu$$

$$Var[X] = \sigma^2$$

Recall: Gamma Random Variable

- $E[X] = \frac{\alpha}{\lambda}$

- $Var[X] = \frac{\alpha}{\lambda^2}$

$$f(x) = \begin{cases} \frac{\lambda e^{-\lambda x} (\lambda x)^{\alpha - 1}}{\Gamma(\alpha)} & if\ x \geq 0 \\ 0 & x < 0 \end{cases}$$

Reference

- Lecture notes on Probability Theory by Phanuel Mariano
- **Introduction to Probability Models, Sheldon M. Ross, Tenth Edition**