PROBABILITY

Qus1) An urn contains 9 balls, 2 of which are Red, 3 are blue and 4 black. 3 balls are drawn from the urn at random. Find the probability that

(i) Three balls are of same color.

The total no. of ways in which 3 balls can be drawn = ${}^9C_3$.

No. of ways of drawing 3 balls of same color = ${}^4C_3 + {}^3C_3$.

$$P = \frac{{}^4C_3 + {}^3C_3}{{}^9C_3} = \boxed{\frac{5}{84}}$$

(ii) Three balls are of different color.

No. of ways of drawing 3 balls of different color = ${}^2C_1 \times {}^4C_1 \times {}^3C_1$.

$$P = \frac{{}^2C_1 \times {}^4C_1 \times {}^3C_1}{{}^9C_3} = \frac{24}{84} = \frac{12}{42} = \boxed{\frac{2}{7}}.$$

(iii) Two are of same color and third is of different color.

2 blue balls & (1 Red or 1 black)
+ 2 Red & (1 blue or 1 black)
+ 2 black & (1 Red or 1 blue)

$$P \Rightarrow \frac{{}^3C_2({}^2C_1+{}^4C_1)}{{}^9C_3} + \frac{{}^2C_2({}^3C_1+{}^4C_1)}{{}^9C_3} + \frac{{}^4C_2 \times ({}^2C_1+{}^3C_1)}{{}^9C_3}$$

$$P = \frac{3(6)}{84} + \frac{1(7)}{84} + \frac{6\times(5)}{84}$$

$$\boxed{P = \frac{55}{84}}$$

Qus 2) A bag contain 4 Red and 3 blue balls. Two drawing of 2 balls are made. Find the probability that the first drawing gain 2 Red balls and 2nd drawing gain 2 blue balls when:

(i) If the balls are returned after 1st drawing.

(ii) if balls are not returned.

(i) The no. of ways in which two balls are drawn is ${}^7C_2$.

Probability of drawing 2 Red balls $= {}^4C_2/{}^7C_2 = \frac{2}{7}$.

Probability of drawing two blue balls $= {}^3C_2 / {}^7C_2 = \frac{1}{7}$.

$\therefore$ Chance of compound event is

$= \frac{1}{7} \times \frac{2}{7} = \frac{2}{49}$

(ii) Chance of drawing two red balls $= \frac{2}{7}$.

but now balls are not returned
so chance of drawing two blue balls $= \frac{{}^3C_2}{{}^5C_2} = \frac{3}{10}$.

$\therefore$ Chance of compound event is $= \frac{2}{7} \times \frac{3}{10}$

$= \frac{3}{35}$

Qus3) The probability that A can solve the problem is 30% and for B it is 40%. What is the probability that a given problem will be solved by A or B?

Probability A can solve problem
$= 0.3$

P

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Probability B can solve ~~A~~ problem
$= 0.4$.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

as Both are independent
$$P(A \cap B) = P(A) \cdot P(B)$$

$$P(A \cup B) = 0.4 + 0.3 - 0.4 \times 0.3$$
$$= \underline{\underline{0.58}}$$

Ques 5)

Ques 4) The probability that a teacher will given an unannounced test during a class is $\frac{1}{5}$, if a student is absent twice. Find the probability he/she will miss atleast one test?

$$P(\text{unannounced test}) = \frac{1}{5}$$

P(miss atleast one test) = P(miss one test) + P(miss both test)

$$P(\text{miss one test}) = \frac{1}{5} \times \frac{4}{5} + \frac{4}{5} \times \frac{1}{5}$$

(out of two day; one day test will happen and other day not happen)

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$P(\text{miss both test}) = \frac{1}{5} \times \frac{1}{5}$.

Both the days unannounced test will happen

$$P = \frac{1}{5} \times \frac{4}{5} \times 2 + \frac{1}{25} = \boxed{\frac{9}{25}}.$$

Qus 5) A and B take turns in throwing two dice. A starts first. The first person who throws 9 (i.e. sum of number appeared is 9) wins a prize. Prove that the chances of their winning in the ratio 9 : 8.

$P(\text{sum} = 9) = \frac{4}{36}$ $\quad ((4,5), (5,4), (6,3), (3,6))$

A = A got sum = 9 $\quad$ B = B got sum = 9.

$P(\text{A wins}) = P(\text{sum} = 9) + P(\bar{A}\bar{B}A) + P(\bar{A}\bar{B}\bar{A}\bar{B}A) + \cdots$

$\bar{A}$ = A not get sum = 9

$\bar{B}$ = B not get sum 9.

$$P(\text{A wins}) = \frac{4}{36} + \frac{32}{36} \cdot \frac{32}{36} \cdot \frac{4}{36} + \left(\frac{32}{36}\right)^4 \cdot \frac{4}{36} + \cdots$$

$$= \frac{4}{36} + \left(\frac{32}{36}\right)^2 \cdot \frac{4}{36} + \left(\frac{32}{36}\right)^4 \cdot \frac{4}{36} + \cdots$$

$$= \frac{4}{36}\left(1 + \left(\frac{32}{36}\right)^2 + \cdots\right)$$

$$= \frac{4}{36}\left(\frac{1}{1-(32/36)^2}\right)$$

$$= \frac{4}{36} \cdot \frac{36^2}{\{(36)^2-(32)^2\}} = \frac{4 \times 36}{4 \times 68}$$

$$= \frac{36}{68}$$

$$P(B \text{ wins}) = 1 - P(A \text{ wins})$$

$$= 1 - \frac{36}{68} = \frac{32}{68}.$$

$$\frac{P(A \text{ wins})}{P(B \text{ wins})} = \frac{36/68}{32/68} = \frac{9}{8}.$$

Qus 6) Bag A contains 2 white and 3 Red balls and Bag B contains 4 white and 5 Red balls. One ball is drawn at random from one of the bag and is found to be red. Find the probability that it was drawn from Bag B.

Let E_1 = event that the ball drawn from bag A.

E_2 is event ball drawn from bag B.

E event that ball drawn is red.

$\therefore \quad P(E_1) = \frac{1}{2} = P(E_2)$

since the ball drawn is red

~~$P(E_2/E) = \frac{3}{5}$~~

$P(E/E_1) = \frac{3}{5} \qquad P(E/E_2) = \frac{5}{9}$

Baye's Theorem

$$P(E_2/E) = \frac{P(E_2)\,P(E|E_2)}{P(E_1)\,P(E|E_1) + P(E_1)\,P(E|E_1)}$$

$$= \frac{\frac{1}{2} \times \frac{5}{9}}{\frac{1}{2} \times \frac{3}{5} + \frac{1}{2} \times \frac{5}{9}} = \frac{25}{52}$$

Qus 7) A factory has 3 machines A, B, C producing 1000, 2000 and 3000 bolts per day respectively. A produces 1% defective, B produces 1.5% and C produces 2% defective. A bolt is checked at random at the end of

a day and is found to be defective Qus8)
Find the probability that it
was produce by machine A ?

Let E_1, E_2, E_3 be events that the bolt was chosen from machine A, B, C respectively. Let E be the event That it is defective.

$$P(E_1) = \frac{1000}{1000+2000+3000} = \frac{1}{6}.$$

$$P(E_2) = \frac{2}{6}.$$

$$P(E_3) = \frac{3000}{6000} = \frac{3}{6}.$$

$$P(E/E_1) = \frac{1}{100} = 0.01$$

$$P(E|E_2) = 0.015 \qquad P(E|E_3) = 0.02.$$

Hence by Bayés Theorem.

$$P(E_1|E) = \frac{P(E_1) \cdot P(E|E_1)}{P(E_1) \cdot P(E|E_1) + P(E_2)\, P(E|E_2) + P(E_3) \times P(E|E_3)}$$

$$= \frac{1/6 \times 0.01}{1/6 \times 0.01 + 2/6 \times 0.015 + 3/6 \times 0.02}$$

$$= \frac{0.01}{0.10} = \frac{1}{10} = 0.1$$

time

Qus8) Calculate the mean and variance of the distribution.

a) x: 1 2 3 4 5 6

p: $1/6$ $1/6$ $1/6$ $1/6$ $1/6$ $1/6$.

olt

$$\text{Mean} = \sum x_i p_i = 1\cdot\frac{1}{6} + 2\times\frac{1}{6} + 3\times\frac{1}{6} + 4\times\frac{1}{6} + 5\times\frac{1}{6} + 6\times\frac{1}{6}.$$

$$= \frac{21}{6}$$

$$\text{Variance} = \sum x_i^2 p_i - \bar{x}^2$$

$$= \frac{1}{6}\left(1^2+2^2+3^2+4^2+5^2+6^2\right) - \left(\frac{21}{6}\right)^2.$$

$$= \frac{91}{6} - \frac{441}{36} = \frac{15}{36}$$

b) x: 0 1 2 x

p: p pq q^2p . . . . $q^x p$.

$E_3) \times$

$E/E_3)$

Qus 9) Six perfect dice are thrown 729 times. How many times do you expect at least 3 dice to show a five or six?

No. of trials $(n) = 6$.

No. of experiments $(N) = 729$.

$P(\text{success}) = p = \frac{2}{6} = \frac{1}{3}$.

$q = 1 - p = \frac{2}{3}$.

The required no. $= NP(3) + NP(4) + NP(5) + NP(6)$

$= N\{1 - P(0) - P(1) - P(2)\}$.

$= 729\left\{1 - {}^6C_0 \left(\frac{2}{3}\right)^6 - {}^6C_1 \cdot \frac{1}{3}\left(\frac{2}{3}\right)^5 - {}^6C_2 \left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^4\right\}$

$= 729\left[1 - \left\{\frac{64}{729} + \frac{6 \times 32}{729} + \frac{15 \times 2^4}{729}\right\}\right]$

$= 729\left\{1 - \frac{496}{729}\right\}$

$= \frac{233 \times 729}{729} = \boxed{233}$.

$1-p = \frac{2}{3}$

... no. $= NP(3) + NP(4) + NP(5) + NP(6)$

$= N\{1 - P(0) - P(1) - P(2)\}$

$- 6C_1 \cdot \frac{1}{3}\left(\frac{2}{3}\right)^5 \ldots \frac{15 \times 16}{729} \}]$

$\{ \frac{64}{729} + \frac{6 \times 32}{729} + \frac{15 \times 16}{729} \}$

$1 - \frac{496}{729} \}$

$\frac{233}{729} \times 729 = \boxed{233}$

Qus 10) A perfect cubic die is thrown a large no. of time in set of 8. Occurrence of five or six is called a success. In what proportion of sets would you expect to get 3 success ?

$P(\text{success}) = p = \frac{1}{3}$.

$q = \frac{2}{3}$.

n = no. of trials in one experiment

$\underline{n = 8}$. $\underline{r = 3}$

Expected Proportion $= \frac{N \cdot P(3)}{N} = P(3)$

$= {}^8C_r \, p^r q^{n-r}$

$= {}^8C_3 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^5$

$= \frac{8 \times 7 \times 6}{6} \times \frac{2^5}{3^8} = \frac{1792}{6561}$

$= \boxed{0.273}$.

$\boxed{27.3\%}$

Qus 11) It is given that 20% of the bolts produced by a machine are defective. Five bolts are selected at random. Find the probability that atmost 2 bolts will be defective.

$P(\text{defective}) = p = 0.2$

$q = 0.8 = P(\text{Non defective})$.

P(atmost 2 are defective)

$= P(0 \text{ defective}) + P(1 \text{ defective}) + P(2 \text{ defective})$

$= {}^5C_0 (0.2)^0 (0.8)^5 + {}^5C_1 (0.2)^1 (0.8)^4 + {}^5C_2 (0.2)^2 (0.8)^3$

$= (0.8)^5 + (0.8)^4 + 0.4 \times (0.8)^3$

$= (0.8)^3 \{0.64 + 0.8 + 0.4\}$.

$= \boxed{0.94}$

Ques 12) Let x and y be two independent binomial variable $b(n,p) = (5, 1/2)$ and $b(7, 1/3)$ respectively. Find the probability that $x+y = 3$.

$x+y=3$ 4 cases $(0,3)$ $(3,0)$ $(1,2)$ $(2,1)$.

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Qus 13) Prove that if x_1 and x_2 are independent random variables then

$$M_0^{x_1+x_2}(t) = M_0^{x_1}(t) \times M_0^{x_2}(t)$$

where $M_0^{x_1}(t)$, $M_0^{x_2}(t)$ & $M_0^{x_1+x_2}(t)$ denotes the moment generating function of x_1, x_2 & (x_1+x_2) about the origin.

$$M_0(t) = E\{e^{tx}\}.$$

$$M_0^{x_1}(t) = \sum e^{tx_1} P(x_1) \quad — (1)$$

$$M_0^{x_2}(t) = \sum e^{tx_2} P(x_2) \quad — (2)$$

$$M_0^{x_1+x_2}(t) = \sum e^{t(x_1+x_2)}\, P(x_1+x_2) \quad -(3)$$

As given in question x_1 & x_2 are independent random variables then

$$P(x_1+x_2) = P(x_1) \cdot P(x_2)$$

$$M_0^{x_1+x_2}(t) = \sum e^{tx_1} \cdot e^{tx_2} \cdot P(x_1) \cdot P(x_2)$$
$$= \sum e^{tx_1} P(x_1)\; e^{tx_2} \cdot P(x_2)$$

From eqn (1) & (2)

$$\boxed{M_0^{x_1+x_2}(t) = M_0^{x_1}(t) \times M_0^{x_2}(t)}$$

Qus 14) Prove that if x_1 and x_2 are two independent poisson variable with parameters m_1 & m_2 respectively. Then (x_1+x_2) is also a poisson variate which has parameter (m_1+m_2).

$$M_1(t) = e^{m_1(e^t-1)}$$ → Moment generating function with parameter m_1

Moment generating function with parameter m_2

$$M_2(t) = e^{m_2(e^t-1)}$$

The m.g.f. of the sum (x_1+x_2) is given by result of above question

$$M(t) = M_1(t) \times M_2(t) = e^{m_1(e^t-1)} \times e^{m_2(e^t-1)}$$

$$\boxed{M(t) = e^{(m_1+m_2)(e^t-1)}}$$

Parameter $\boxed{m_1+m_2}$

Q16)

Ques 15) Find the probability that atmost 5 defective fuses will be ~~formed~~ found in a box of 200 fuses if experience shows that 2% of such fuses are defective.

$p = 0.02$

$n = 200$ $\quad$ $n >> p$.

Hence $m = np = 200 \times 0.02$

$= 4$.

Hence using poissons distribution

$$P(x) = \frac{e^{-m} \cdot m^x}{\lfloor x}.$$

$$P(x \le 5) = \sum_{x=0}^{5} \frac{e^{-m} \cdot m^x}{\lfloor x}.$$

$$= e^{-4} + e^{-4}\cdot 4 + e^{-4}\cdot\frac{4^2}{\lfloor 2} + e^{-4}\frac{4^3}{\lfloor 3} + e^{-4}\cdot\frac{4^4}{\lfloor 4}$$

$$+ e^{-4}\frac{4^5}{\lfloor 5}$$

$$= e^{-4}\left(1+4+\frac{4^2}{2}+\frac{4^3}{6}+\frac{4^4}{24}+\frac{4^5}{120}\right)$$

$$P = \boxed{0.7852}$$

Q16) Take 100 sets of 10 tosses of a coin. In how many cases do you expect to get atleast 7 Heads.

No. of experiments $(N) = 100$

No. of trials in a experiment $= 10$.

$$p(\text{success}) = \frac{1}{2}$$

$$q = \frac{1}{2}.$$

The required No $= NP(7) + NP(8) + NP(9) + NP(10)$

$$= 100\left[{}^{10}C_7\cdot\left(\tfrac{1}{2}\right)^{10} + {}^{10}C_8\left(\tfrac{1}{2}\right)^{10} + {}^{10}C_9\left(\tfrac{1}{2}\right)^{10} + {}^{10}C_{10}\left(\tfrac{1}{2}\right)^{10}\right]$$

$$= \frac{100}{2^{10}}\left[120+45+11\right].$$

$$= \frac{100\times 176}{1024} \cong 17.1875 \simeq \underline{\underline{17}}$$

The required no. is $\underline{\underline{17}}$

Qus 17) Suppose that the no. of telephone calls received by an operator b/w 9.00 am to 9.05 am follows a poisson distribution with mean 3. Find the probability that.

(i) no call is received in that time interval.
(ii) a total of 1 call is received in that time interval in 3 consecutive days.

(i) Mean = 3
$\Rightarrow$ $m = 3$

No call is received = $P(x=0)$

$$P(x) = \frac{e^{-m} \cdot m^x}{\lfloor x}$$

$$P(x=0) = \frac{e^{-3} \cdot 3^0}{\lfloor 0} = e^{-3}$$

$$\boxed{P(x=0) = 0.049787}$$

(ii) Given in 3 consecutive days
so $m = 3 \times 3 = 9$

$P(x=1)$ with $m = 9$

$$P(x=1) = \frac{e^{-9} \cdot 9^1}{\lfloor 1} = 9e^{-9} = 1.1 \times 10^{-3}$$

Qus 18)
(i
(ii

Qus 18) A car hire firm has two cars which it hires out day by day. The no. of demands for car has a poisson distribution with mean 1.5. Find the proportion of days on which.

(i) neither car is used.

(ii) proportion of days on which some demand is refused.

(i) Mean of poissons distribution $= m = 1.5$

$\underline{m = 1.5}$

$$P(x) = \frac{e^{-m} \cdot m^x}{\lfloor x}$$

$$P(x=0) = e^{-1.5} = \underline{0.2231}$$

(ii) P(some demands are refused)

= 1 - P(some demands are entertained)

$$P = 1 - P(x \leq 2)$$

$$= 1 - \sum_{x=0}^{2} \frac{e^{-m} \cdot m^x}{\lfloor x}$$

$$= 1 - \left\{ e^{-1.5} \left(\frac{1.5^0}{\lfloor 0} + \frac{1.5^1}{\lfloor 1} + \frac{1.5^2}{\lfloor 2} \right) \right\}$$

$$= 1 - e^{-1.5}(3.625)$$

$$= 1 - 0.8098 = \underline{0.19}.$$

Ques 19) A manufacturer of pins knows that 5% of his product is defective. If he sells the pin in boxes of 100 and gurantees that not more than 10 pins will be defective. What is the probability that a box will fail to meet gurantee quality.

$p = 0.05$

$n = 100$ $\quad \underline{n > p}$ $\quad x =$ No. of defective pins.

$\underline{m = np = 5}$

P(box will fail to meet gurantee)

$= P(x > 10)$

$P(x>10) = 1 - P(x \leq 10)$

$= 1 - \sum_{x=0}^{10} \frac{e^{-m} \cdot m^x}{\lfloor x}$

$= 1 - \sum_{x=0}^{10} \frac{e^{-5} \cdot 5^x}{\lfloor x}$

$P(x>10) = 1 - \left\{ e^{-5} \left(\frac{5^0}{\lfloor 0} + \frac{5^1}{\lfloor 1} + \frac{5^2}{\lfloor 2} + \cdots \frac{5^{10}}{\lfloor 10} \right) \right\}$

$= \underline{\underline{0.013695}}$

Qus 20) A source of liquid is known to contain bacteria with mean number of bacteria per c.c. = 3. If 10 1.cc test tubes are filled with liquid then assuming Poisson distribution for no. of bacteria per c.c. Find the probability that.

(a) all test tubes will show growth
(b) exactly 7 test tubes will show growth

mean = $\underline{m = 3}$.

Ques 21) Let x, y be independent poisson variates with means 1 & 2 respectively. Show that $P(x+y<4) = 13\,e^{-3}$

As we know if x & y are independent variable with parameters m_1 & m_2 then $x+y$ is a poisson variable with parameter m_1+m_2.

$$m_1+m_2 = 1+2 = 3.$$

$$P(x+y<4) = \sum_{(y+x)=0}^{9} \frac{e^{-3} \cdot 3^{(x+y)}}{\lfloor(x+y)}$$

$$= e^{-3}\left\{1+3+\frac{3^2}{2!}+\frac{3^3}{3!}\right\}.$$

$$= \underline{\underline{13e^{-3}}}$$

Ques 22) If x is poisson variable such that $P(x=1) = 2P(x=2)$ find the mean and variance.

$$P(x=1) = \frac{e^{-m} \cdot m}{\lfloor 1} = me^{-m}$$

$$P(x=2) = \frac{e^{-m}\cdot m^2}{\lfloor 2} = \frac{e^{-m}\cdot m^2}{2}$$

$$P(x=1) = 2P(x=2)$$

$$me^{-m} = \cancel{2}\cdot\frac{e^{-m}\cdot m^2}{\cancel{2}}$$

$$m^2 - m = 0$$

$$m = 0 \quad \times$$

$$m = 1$$

$$\boxed{m=1}$$

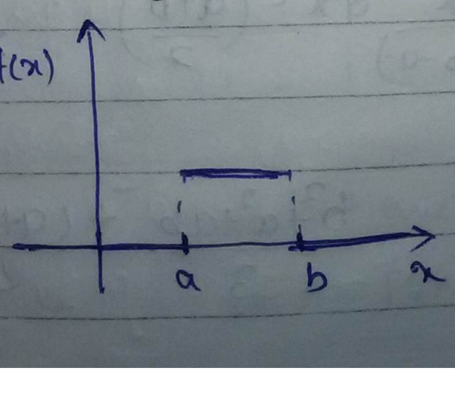
$$\boxed{\text{mean} = \text{variance} = 1}$$

Qus 23) Find the value of k so that the following function becomes a density function

$$f(x) = \frac{1}{k} \qquad a \le x \le b$$

$$= 0 \qquad \text{otherwise}$$

Find the mean and variance of the distribution. Also find cumulative distributive function.



By definition $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\Rightarrow \int_a^b f(x)\,dx = 1$$

$$\int_a^b \frac{1}{k}\,dx = 1.$$

$$\frac{1}{k}(b-a) = 1$$

$$\boxed{k = (b-a)}$$

Qus 2

$$\text{Mean}(x) = \int_{-\infty}^{\infty} x f(x)\,dx \quad \text{(By definition)}$$

$$= \int_a^b x(b-a)^{-1}\,dx$$

$$= \int_a^b \frac{x}{(b-a)}\,dx = \frac{1}{(b-a)} \frac{x^2}{2}\Big|_a^b$$

$$= \frac{1}{2(b-a)}(b^2-a^2) = \frac{b+a}{2}$$

$$\text{Var}(x) = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \bar{x}^2 \quad \text{(By definition)}$$

$$= \int_a^b \frac{x^2}{(b-a)}\,dx - \left(\frac{a+b}{2}\right)^2 = \frac{1}{3(b-a)}(b^3-a^3) - \frac{(a+b)^2}{4}$$

$$= \frac{b^2+a^2+ab}{3} - \frac{(a+b)^2}{4}$$

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$$= \frac{4(b^2+a^2+ab) - 3(a^2+b^2+2ab)}{12}$$

$$= \frac{a^2+b^2-2ab}{12} = \frac{(a-b)^2}{12} = \boxed{\frac{(b-a)^2}{12}}.$$

Qus 24) Verify whether the following is cummulative distribution function

(i) $$F(x) = \begin{cases} 0 & x < -a \\ \frac{1}{2}(x/a + 1) & -a \le x \le a \\ 1 & x > a \end{cases}$$

As we know

density function $f(x) = \frac{dF(x)}{dx}$

$$f(x) = \begin{cases} \frac{1}{2a} & -a \le x \le a \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{-a} f(x)\,dx + \int_{-a}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx$$

$$\int 0 \cdot dx + \int_{-a}^{a} \frac{1}{2a}\,dx + \int 0 \cdot dx.$$

$$\frac{1}{2a} \cdot 2a = 1$$

~~test~~ Hence $f(x)$ is probability density function and $F(x)$ is distribution function.

(b) Show that

$$F(x) = 0 \qquad -\infty < x < 0$$
$$= 1 - e^{-x} \qquad 0 \le x < \infty$$

is a possible distribution function and find the density function.

As we know

density function $\left(f(x)\right) = \dfrac{dF(x)}{dx}$

$$f(x) = \begin{cases} 0 & -\infty < x < 0 \\ e^{-x} & 0 \le x < \infty. \end{cases}$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{0}^{\infty} e^{-x}\,dx = -e^{-x}\Big|_0^{\infty}$$
$$= \underline{\underline{1}}$$

Hence the $f(x)$ is probability density function and $F(x)$ is cummulative distribution function.

Qus 25) If $f(x) = 3x^2 \quad 0 \le x \le 1$ then find a number 'a' such that x is equally likely to be greater than or less than 'a'.

$$\left.\begin{aligned} P(x > a) &= \int_{a}^{\infty} 3x^2\,dx. \\ P(x < a) &= \int_{-\infty}^{a} 3x^2\,dx. \end{aligned}\right\} \text{By definition}$$

Given they are equally likely

2) $\int_a^{\infty} 3x^2\, dx = \int_{-\infty}^{a} 3x^2\, dx$.

$\int_a^{1} 3x^2 dx = \int_0^{a} 3x^2 dx$ (By given limits).

$x^3 \Big|_a^1 = x^3 \Big|_0^a$

$1 - a^3 = a^3$

$2a^3 = 1$

$\boxed{a = \left(\frac{1}{2}\right)^{1/3}}$

Q26) Suppose life in hours of certain kind of radio tube has density function

$$f(x) = 100/x^2 \qquad x \geq 100$$
$$= 0 \qquad x < 100$$

Find the probability that none of three such tubes will have to be replaced during the first 150 hrs of operation.

Qus 27) For the probability distribution given by the following density function find the moment generating function about the origin and hence find the mean and variance

$$f(x) = \alpha e^{-\alpha x} \quad x > 0$$
$$= 0 \quad \text{otherwise}$$

$\underline{\alpha > 0}$

$$M_0(t) = \int_{-\infty}^{\infty} e^{tx} \cdot \alpha e^{-\alpha x} \, dx .$$

$$= \alpha \int_0^\infty e^{(t-\alpha)x}\,dx = \alpha \int_0^\infty e^{-(\alpha-t)x}\,dx$$

$$= \frac{-\alpha}{\alpha-t} e^{-(\alpha-t)x} \Big|_0^\infty = \frac{\alpha}{\alpha-t}$$

$$\text{Mean}(x) = \frac{d M_0(t)}{dt}\Big|_{t=0} = \frac{\alpha}{(\alpha-t)^2} = \frac{1}{\alpha}$$

$$\text{Var}(x) = \frac{d^2 M_{\bar{x}}(x)}{dt^2}\Big|_{t=0}.$$

$$M_{\bar{x}} = e^{-\bar{x}t} \cdot \frac{\alpha}{\alpha-t}$$

$$\text{Var}(x) = \frac{\alpha}{\alpha-t}(-\bar{x})\, e^{-\bar{x}t} = \frac{\alpha}{\alpha-t}(\bar{x})^2 e^{-\bar{x}t}\Big|_{t=0}.$$

$$= \frac{\alpha}{\alpha^2(\alpha-t)} = \frac{1}{\alpha(\alpha-t)}$$

Qus 28) (a) In a normal distribution whose mean is 12 and s.d. 2 find $P(9.6 < x < 13.8)$.

(b) If $x \sim N(2,3)$. Find a value of the variate, such that probability from mean to that value is 0.4115.

(a) $$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$$

$$f(x) = \frac{1}{2\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-12}{2}\right)^2}$$

$$P(9.6 < x < 13.8) = \int_{9.6}^{13.8} f(x)\,dx = \int_{9.6}^{13.8} \frac{1}{2\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-12}{2}\right)^2} dx$$

substitute $\frac{x-12}{2} = z$.

$$\int_{-1.2}^{0.9} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz = \phi(0.9) - \phi(-1.2)$$

$$= \phi(0.9) + \phi(1.2)$$

$$= 0.3159 + 0.3849$$

$$= \boxed{0.7008.}$$

(b) $$f(x) = \frac{1}{3\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-2}{3}\right)^2}.$$

$$P(2 < x < x_0) = 0.4115$$

$$\int_{2}^{x_0} \frac{1}{3\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-2}{3}\right)^2} dx = 0.4115.$$

Replace $\frac{x-2}{3} = z$

$$\int_{0}^{\left(\frac{x_0-2}{3}\right)} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz = 0.4115$$

$$\phi\left(\frac{x_0-2}{3}\right) = 0.4115$$

$$\phi(1.35) = 0.4115$$

$$\frac{x_0-2}{3} = 1.35$$

$$\boxed{x_0 = 6.05}$$

Qus 29) If $x \sim N(11, 1.5)$ Find x_0 such that

a) $P(x > x_0) = 0.3$

b) $P(x < x_0) = 0.09$

(a) $$f(x) = \frac{1}{1.5\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-11}{1.5}\right)^2}$$

$$P(x>x_0) = P(x_0 < x < \infty) = \int_{x_0}^{\infty} \frac{1}{1.5\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-11}{1.5}\right)^2} dx$$

$$= \int_{x_0}^{m} \frac{1}{1.5\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-11}{1.5}\right)^2} dx + \underbrace{\int_{m}^{\infty} \frac{1}{1.5\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-11}{1.5}\right)^2} dx}_{1/2}$$

$$0.3 = \int_{\left(\frac{x_0-11}{1.5}\right)}^{0} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz = 0.5$$

or $-\phi\left(\frac{x_0-11}{1.5}\right) = -0.2$

$$\phi\left(\frac{x_0-11}{1.5}\right) = 0.2.$$

$$\phi(0.53) = 0.2019.$$

$$\frac{x_0-11}{1.5} = 0.53$$

$$\boxed{x_0 = 11.795}$$

(b) $P(x<x_0) = 0.09.$

$$f(x) = \frac{1}{1.5\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-11}{1.5}\right)^2}.$$

$$P(x<x_0) = P(-\infty<x<x_0) = \int_{-\infty}^{x_0} f(x)\,dx.$$

$$= \int_{-\infty}^{m} f(x)\,dx + \int_{m}^{x_0} f(x)\,dx.$$

$$= 0.5 + \int_{m}^{x_0} \frac{1}{1.5\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-11}{1.5}\right)^2}.$$

$$0.09 = 0.5 + \int_{0}^{\left(\frac{x_0-11}{1.5}\right)} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz.$$

$$\phi\left(\frac{x_0-11}{1.5}\right) = 0.09 - 0.5$$

$$\phi\left(\frac{x_0-11}{1.5}\right) = -0.41$$

$$\phi\left(\frac{11-x_0}{1.5}\right) = 0.41$$

$$\phi(1.34) = 0.4099$$

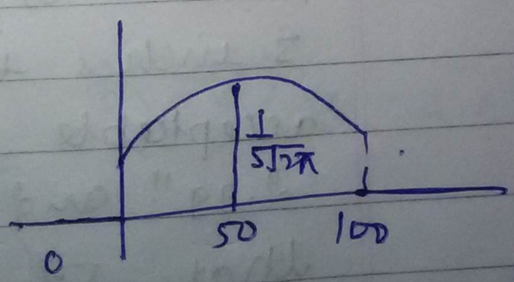
$$\frac{11-x_0}{1.5} = 1.34$$

$$\boxed{x_0 = 8.99}$$

Qus 30) In a certain experiment 2000 student appeared in statistics. The average marks obtained were 50% and the s.d. was 5%. How many students do you expect to get more than 60% marks, assuming the marks to be distributed normally.

$$f(x) = \frac{1}{5\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-50}{5}\right)^2} \qquad 0 \le x \le 100.$$

$2000\, P(x>60)$



$$P(60<x<100) = \int_{60}^{100} \frac{1}{5\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-50}{5}\right)^2} dx$$

$$= \int_{2}^{10} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$

$$= \phi(10) - \phi(2).$$

as $\phi(10)$ is not given in table.
so we have to process like

$$= \int_{60}^{m} \frac{1}{5\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-50}{5}\right)^2} dx + \int_{m}^{100} \frac{1}{5\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-50}{5}\right)^2} dx$$

$$= 0.5 - \phi(2)$$
$$= 0.5 - 0.4772$$
$$= 0.0228.$$

$$200\,P(x>60) = 2000 \times 0.0228.$$
$$= 4.56$$
$$\simeq 4.6$$

Qus 31) steel rods are manufactured to be 3 inches in dia but they are acceptable if the diameters b/w 2.99" and 3.01". It is observed that 5% are rejected oversize and

5% rejected undersize. Find the s.d. of the distribution. Find the proportion of rejected rods if the permissible limits are widened to 2.985" and 3.015"

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$$

$$P(x > 3.01") = 0.05$$
$$P(x < 2.99") = 0.05$$

$$P(x > 3.01") = 0.05$$

$$\int_{3.01}^{m} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} dx + 0.5 = 0.05$$

$$\int_{\left(\frac{3.01-m}{\sigma}\right)}^{0} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz = 0.05 - 0.5$$

$$\phi\left(\frac{3.01-m}{\sigma}\right) = 0.45 \quad \text{--- (1)}$$

Now $P(x < 2.99") = 0.05$

$$0.5 + \int_{m}^{2.99} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} dx = 0.05$$

$$\phi\left(\frac{2.99-m}{\sigma}\right) = -0.45 \quad \text{--- (1)}$$

$$\phi\left(\frac{m-2.99}{\sigma}\right) = 0.45 \quad \text{--- (2)}$$

$$\phi(1.65) = 0.4505$$

~~for~~ from eq^n 1.

$$\frac{3.01-m}{\sigma} = 1.65 \quad \text{--- (3)}$$

$$\frac{m-2.99}{\sigma} = 1.65 \quad \text{--- (4)}$$

From eq^n 3 and 4.

$$\boxed{m=3}$$

$$\sigma = \frac{0.01}{1.65} = \frac{1}{165}$$

(b)

$$P(2.985 < x < 3.015)$$

$$= \int_{2.985}^{3.015} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} dx$$

$$= \phi\left(\frac{0.015}{\sigma}\right) - \phi\left(\frac{-0.015}{\sigma}\right)$$

$$= 2\phi(0.015 \times 165)$$

$$= 2\phi(2.475)$$

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$= 2 \times 0.4932$

$= \boxed{0.9864}$

Qus32) In a normal distribution, 31% of the items are under 45 and 8% are over 64. Find m, σ.

$P(x<45) = 0.31$

$P(x>64) = 0.08$.

$$P(-\infty < x < 45) = 0.5 + \int_m^{45} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} dx$$

$$0.31 = 0.5 + \phi\left(\frac{45-m}{\sigma}\right)$$

$$\phi\left(\frac{45-m}{\sigma}\right) = -0.19$$

$$\phi\left(\frac{m-45}{\sigma}\right) = 0.19 \quad \text{—} (1)$$

Now.

$$P(64 < x < \infty) = 0.5 + \int_{64}^{m} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} = 0.08.$$

$$\Rightarrow 0.5 \cancel{\phi} - \phi\left(\frac{64-m}{\sigma}\right) = 0.08$$

$$\phi\left(\frac{64-m}{\sigma}\right) = 0.42 \quad - (2)$$

$$\phi(0.5) = 0.1915$$

$$\Rightarrow \frac{m-45}{\sigma} = 0.5 \quad - (3)$$

$$\phi(1.4) = 0.4192$$

$$\frac{64-m}{\sigma} = 1.4 \quad - (4)$$

$$m-45 = 0.5 \times \sigma$$
$$64-m = 1.4 \times \sigma$$

$$19 = 1.9 \times \sigma$$

$$\boxed{\cancel{\sigma = \frac{10}{10} = 0.01}} \qquad \boxed{\sigma = 10}$$

$$\cancel{m = 45 + \overset{10}{0.1} \times 0.5} \qquad m = 45 + 5$$
$$= \cancel{45 + 0.05} \qquad \boxed{m = 50}$$
$$= \cancel{45.05}$$
$$\boxed{\cancel{m = 45.05}}$$

Q33) Prove that in a normal distribution

$$M_0(t) = e^{mt + \frac{1}{2}t^2\sigma^2}.$$

$$M_0(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} dx.$$

substitute $\frac{x-m}{\sigma} = z$.

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{t(\sigma z+m)} e^{-\frac{1}{2}z^2} dz.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2\sigma t z)} e^{mt} dz.$$

$$= \frac{e^{mt}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2\sigma t z + \sigma^2 t^2 - \sigma^2 t^2)} dz.$$

$$= e^{(\frac{1}{2}\sigma^2 t^2 + mt)} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-\sigma t)^2} dz.$$

$$\boxed{M_0(t) = e^{\frac{1}{2}(\sigma^2 t^2 + mt)}}$$

as $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-\sigma t)^2} dz = 1$

Total Probability = 1.

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Q.34) If $x_1, x_2, \ldots x_n$ are independent variates with normal distribution respectively mean $m_1, m_2 \ldots m_n$ and variance $\sigma_1^2, \sigma_2^2, \ldots \sigma_n^2$. Then prove that $\sum_{i=1}^{n} c_i x_i$ is also a normal variate with mean $\Sigma c_i m_i$ and variance $\sum_{i=1}^{n} c_i^2 \sigma_i^2$.

$$M_0^x(t) = E(e^{tx})$$

$$M_0^{cx}(t) = E(e^{tcx}) = M_0^x(ct)$$

$$M_0^{x_1} = e^{m_1 t + \frac{1}{2} t^2 \sigma_1^2}.$$

$$M_0^{c_1 x_1} = M_0^{x_1}(c_1 t) = e^{c_1 m_1 t + \frac{1}{2} c_1^2 t^2 \sigma_1^2}.$$

$$M_0^{\Sigma c_i x_i}(t) = e^{(\Sigma c_i m_i) t + \frac{1}{2} t^2 (\Sigma c_i^2 \sigma_i^2)}$$

$$\Rightarrow \boxed{\begin{array}{l} \text{Mean} = \Sigma c_i m_i \\ \text{Variance} = \Sigma c_i^2 \sigma_i^2 \end{array}}$$

Qus 35) If $x \sim N(m, \sigma)$ then $c_1 x + c_2$ is also a normal variate which has mean $c_1 m + c_2$ and variance $c_1^2 \sigma^2$

$$M_0^x(t) = E(e^{tx})$$

$$M_0^{x+K}(t) = E(e^{(t+K)})$$
$$= e^{tK}\, E(e^{tx})$$
$$= e^{tK}\, M_0(t)$$

$$M_0^{c_1 x}(t) = M_0^{x}(c_1 t) = e^{m_1 c_1 t + \frac{1}{2}t^2 c_1^2 \sigma^2}.$$

$$M_0^{c_1 x + c_2}(t) = e^{c_2 t} \cdot e^{c_1 m_1 t + \frac{1}{2}t^2 c_1^2 \sigma^2}.$$
$$= e^{(c_1 m + c_2)t + \frac{1}{2}c_1^2\sigma^2 t^2}.$$

equating with $M_0(t) = e^{mt + \frac{1}{2}\sigma^2 t}$.

$$\boxed{\begin{aligned} \text{mean} &= m = c_1 m + c_2 \\ \text{Var} &= c_1^2 \sigma^2 \end{aligned}}$$