Linear Algebra

Suggested Books

- Serge Lang, Introduction to Linear Algebra, 2nd Edition, Springer, 1986.
- Gilbert Strang, Introduction to Linear Algebra, 4th Edition, Wellesley-Cambridge Press, 2009.

LINEAR ALGEBRA

Outline

- Vector arithmetic
- Matrix arithmetic
- Matrix properties
- Linear Independence
- Bases & Orthonormal Bases
- Determinant of a matrix
- Rank of a Matrix
- Eigenvectors & eigenvalues
- System of linear equations
- Principal Component Analysis
- Singular Value Decomposition
- Applications

What is a Vector ?

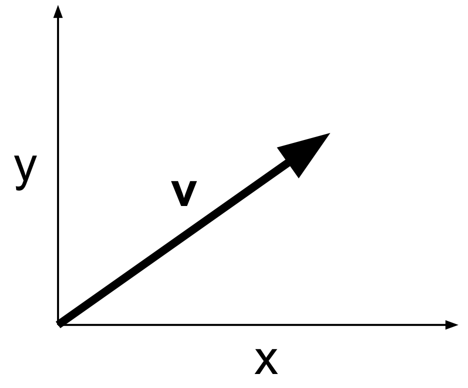
- Vector in R^n is an ordered set of n real numbers.
  - e.g. v = (1,6,3,4) is in R^4
- Think of a vector as a ***directed line segment in N-dimensions***! (has "length" and "direction")
- Basic idea: convert geometry in higher dimensions into algebra!
  - Once you define a "nice" ***basis*** along each dimension: x-, y-, z-axis …
  - Vector becomes a 1 x N matrix!
  - v = [a b c]^T

$$\vec{v} = [a \quad b \quad c]$$

Row vector

$$\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Column vector



What is a Matrix?

- A matrix is a set of elements, organized into rows and columns

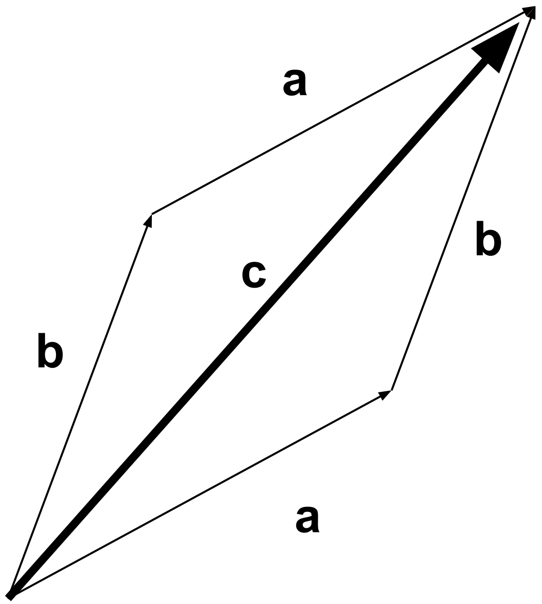
rows

columns

$$\begin{bmatrix} a_{11} & a_{21} & a_{31} & a_{41} \\ a_{12} & a_{22} & a_{32} & a_{42} \\ a_{13} & a_{23} & a_{33} & a_{43} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{bmatrix}$$

Vector Addition

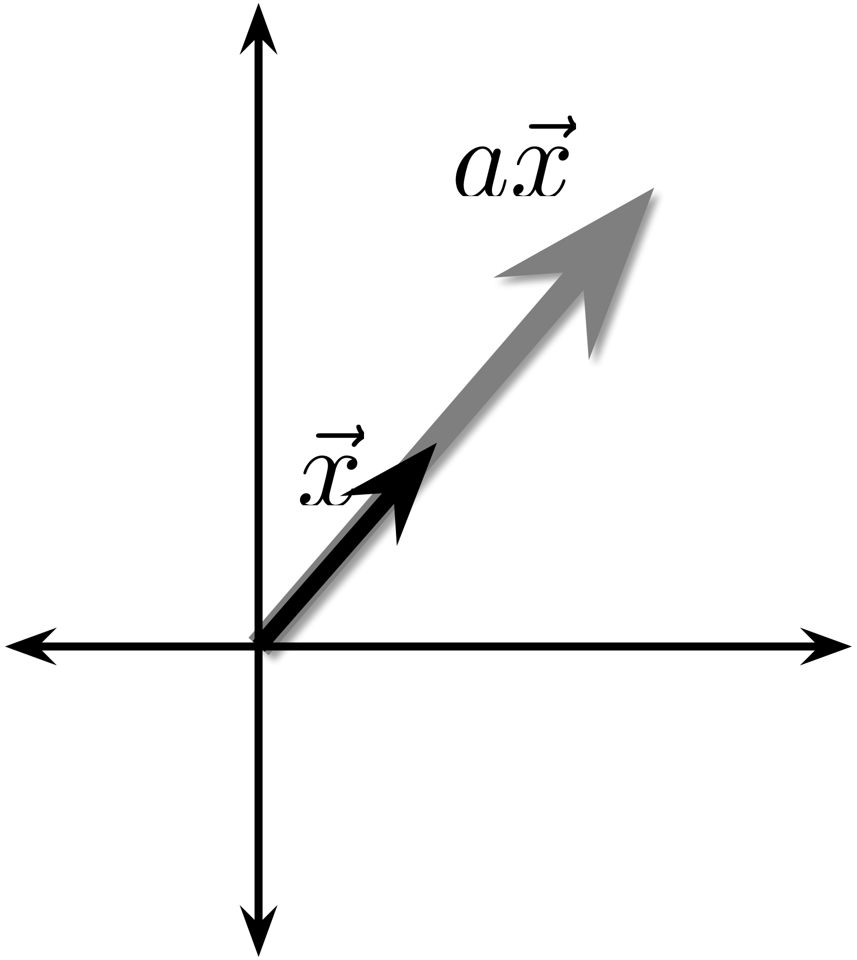
a+b $\mathbf{v} + \mathbf{w} = (x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$



a+b = c

Scalar times Vector

Change only the length ("scaling"), but keep *direction fixed*.



$$a\vec{x} = a \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} ax_1 \\ ax_2 \\ \vdots \\ ax_N \end{pmatrix}$$

Product of 2 Vectors

- Three ways to multiply
  - Element-by-element
  - Inner product
  - Outer product

Element-by-element product (Hadamard product)

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} .* \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 b_1 \\ a_2 b_2 \end{pmatrix}$$

- **Element-wise multiplication (.* in MATLAB)**

Vectors: Dot Product (inner product)

- **Think of the dot product as a matrix multiplication**

$$\vec{x} \cdot \vec{y} =$$

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_N \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix} = x_1 y_1 + x_2 y_2 + \cdots + x_N y_N$$

$$= \sum_{i=1}^{N} x_i y_i$$

1 X N N X 1 1 X 1

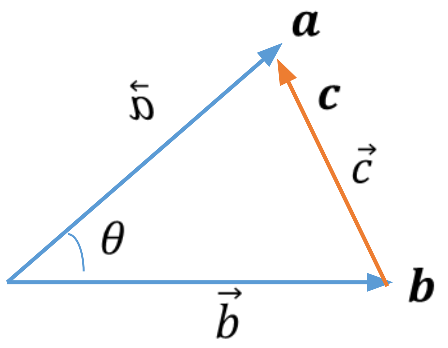
- MATLAB: 'inner matrix dimensions must agree'

Outer dimensions give size of resulting matrix

Vectors: Dot Product (inner product)

$\boldsymbol{a} \cdot \boldsymbol{b} \; = ||a|| \; ||b|| \cos \theta$

The dot product is also related to the angle between the two vectors



Law of cosines

$$||\boldsymbol{c}||^2 = ||\boldsymbol{a}||^2 + ||\boldsymbol{b}||^2 - 2||\boldsymbol{a}|| \cdot ||\boldsymbol{b}||\boldsymbol{cos\theta} \;\; (\boldsymbol{1})$$

$|| \cdot ||$ The magnitude/length of a vector. For example, let $\boldsymbol{a} = (a_1, a_2)$ be a two-dimensional vector, then the formula for its magnitude is $\sqrt{a_1^2 + a_2^2}$

$$\boldsymbol{c} = \boldsymbol{a} - \boldsymbol{b}$$
$$\boldsymbol{c} \cdot \boldsymbol{c} = (\boldsymbol{a} - \boldsymbol{b}) \cdot (\boldsymbol{a} - \boldsymbol{b})$$
$$||\boldsymbol{c}||^2 = \boldsymbol{a} \cdot \boldsymbol{a} - \boldsymbol{a} \cdot \boldsymbol{b} - \boldsymbol{b} \cdot \boldsymbol{a} + \boldsymbol{b} \cdot \boldsymbol{b}$$
$$||\boldsymbol{c}||^2 = ||\boldsymbol{a}||^2 + ||\boldsymbol{b}||^2 - 2\boldsymbol{a} \cdot \boldsymbol{b} \;\; (\boldsymbol{2})$$

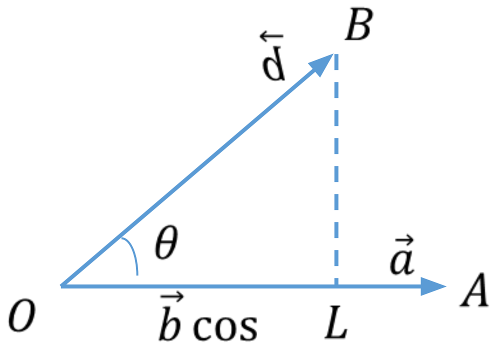
Vectors: Dot Product Geometrical Interpretation

$$\boldsymbol{a} \cdot \boldsymbol{b} = ||a||\ ||b||\cos\theta$$

The dot product is also related to the angle between the two vectors

- **In ΔOLB**

$$\cos\theta = \frac{OL}{OB}$$



$$OL = OB\cos\theta$$
$$OL = ||\boldsymbol{b}||\cos\theta$$

$$A.B = ||\boldsymbol{a}||OL$$

Magnitude of vector A

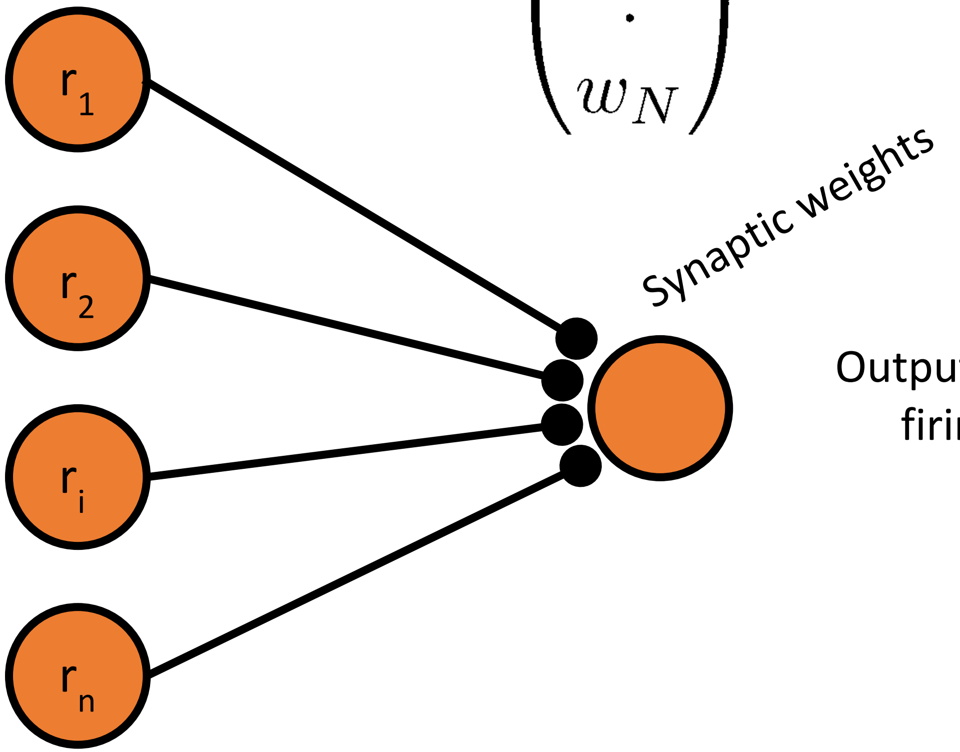
Projection of vector B on A

$$A.B = 0 \implies A \perp B$$

Dot Product Example: linear feed-forward network

$$\begin{pmatrix} r_1 & r_2 & \cdots & r_N \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{pmatrix} = r_1 w_1 + r_2 w_2 + \cdots + r_N w_N$$

Input neurons'
Firing rates



Output neuron's
firing rate

- **<u>Insight</u>: for a given input (L2) magnitude, the response is maximized when the input is parallel to the weight vector**

Multiplication: Outer product

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} \begin{pmatrix} y_1 & y_2 & \cdots & y_M \end{pmatrix} = \begin{pmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_M \\ x_2 y_1 & x_2 y_2 & \cdots & x_2 y_M \\ \vdots & \vdots & \ddots & \ldots \\ x_N y_1 & x_N y_2 & \cdots & x_N y_M \end{pmatrix}$$

N X 1 1 X M N X M

Multiplication: Outer product

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$$

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Multiplication: Outer product

$$
\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}
\begin{pmatrix} y_1 & y_2 & \cdots & y_M \end{pmatrix}
=
\begin{pmatrix}
x_1 y_1 & x_1 y_2 & \cdots & x_1 y_M \\
x_2 y_1 & x_2 y_2 & \cdots & x_2 y_M \\
\vdots & \vdots & \ddots & \vdots \\
x_N y_1 & x_N y_2 & \cdots & x_N y_M
\end{pmatrix}
$$

Matrix times a vector

$$\overrightarrow{y} = \overleftrightarrow{W}\,\overrightarrow{x}$$

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_M \end{pmatrix} = \begin{pmatrix} W_{11} & W_{12} & \cdots & W_{1N} \\ W_{21} & W_{22} & \cdots & W_{2N} \\ \vdots & \vdots & & \vdots \\ W_{M1} & W_{M2} & \cdots & W_{MN} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}$$

M X 1 M X N N X 1

Matrix times a vector: inner product interpretation

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_i \\ \vdots \\ y_M \end{pmatrix} = \begin{pmatrix} W_{11} & W_{12} & \cdots & W_{1N} \\ W_{21} & W_{22} & \cdots & W_{2N} \\ \vdots & \vdots & & \vdots \\ W_{i1} & W_{i2} & \cdots & W_{iN} \\ \vdots & \vdots & \ddots & \vdots \\ W_{M1} & W_{M2} & \cdots & W_{MN} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}$$

- **Rule: the ith element of y is the dot product of the ith row of W with x**

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- **Rule: the i^{th} element of y is the dot product of the i^{th} row of W with x**

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Example of the outer product method

$$\overleftrightarrow{M}$$

$$\begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} =$$

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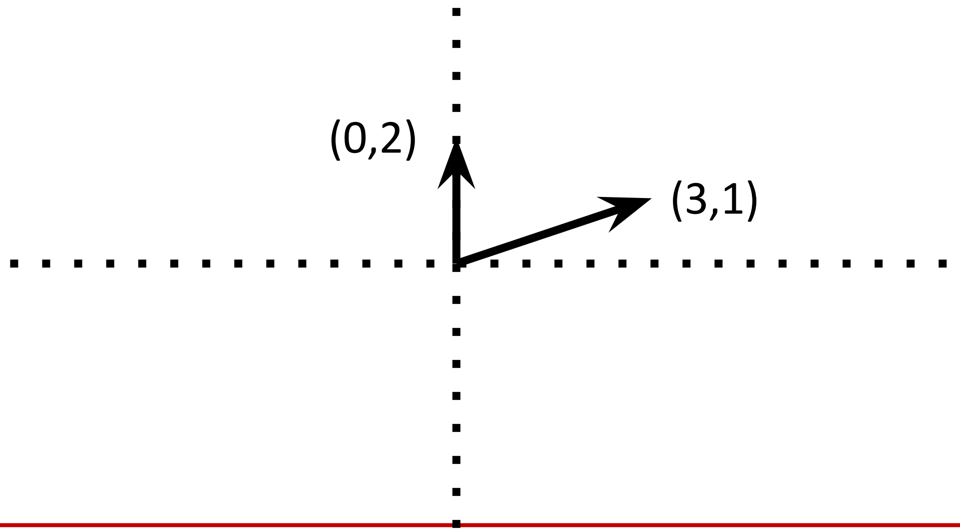
$$\overleftrightarrow{M}$$

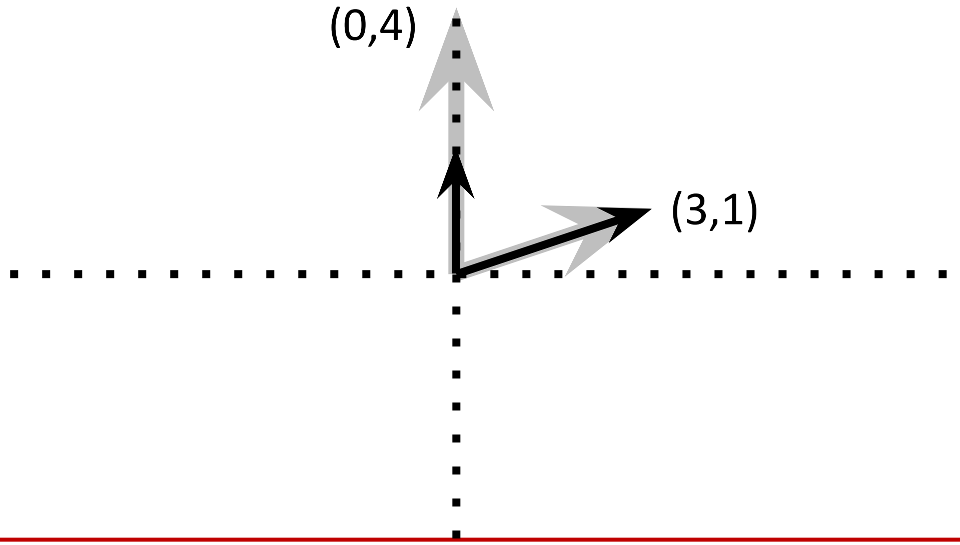
$$\begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 0 \\ 2 \end{pmatrix} + 1 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$



Example of the outer product method

$$\overleftrightarrow{M}$$

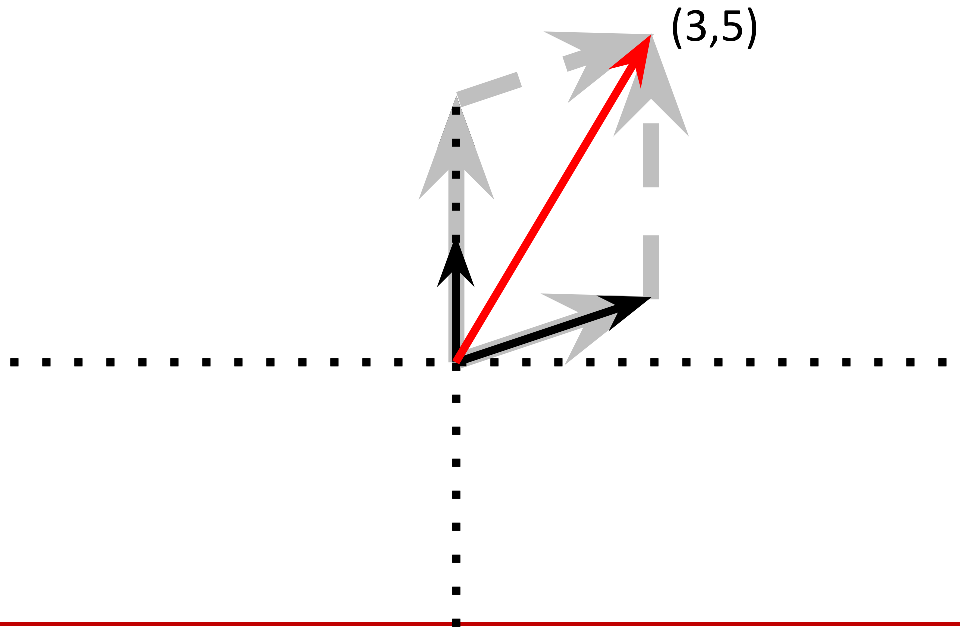
$$\begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 0 \\ 2 \end{pmatrix} + 1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$



Example of the outer product method

$$\overleftrightarrow{M}$$

$$\begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 0 \\ 2 \end{pmatrix} + 1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$



Basic Matrix Operations

- **Addition, Subtraction, Multiplication: creating new matrices**

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

Just add elements

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a-e & b-f \\ c-g & d-h \end{bmatrix}$$

Just subtract elements

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

Multiply each row by each column

Product of 2 Matrices

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1P} \\ A_{21} & A_{22} & \cdots & A_{2P} \\ \vdots & \vdots & & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{NP} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1M} \\ B_{21} & B_{22} & \cdots & B_{2M} \\ \vdots & \vdots & & \vdots \\ B_{P1} & B_{P2} & \cdots & B_{PM} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1M} \\ C_{21} & C_{22} & \cdots & C_{2M} \\ \vdots & \vdots & & \vdots \\ C_{N1} & C_{N2} & \cdots & C_{NM} \end{pmatrix}$$

N X P   P X M   N X M

- **MATLAB: ‘inner matrix dimensions must agree’**
- **Note: Matrix multiplication doesn’t (generally) commute, AB ≠ BA**

Matrix times Matrix: by inner products

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1P} \\ A_{21} & A_{22} & \cdots & A_{2P} \\ \vdots & \vdots & & \vdots \\ A_{i1} & A_{i2} & \cdots & A_{iP} \\ \vdots & \vdots & & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{NP} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1j} & \cdots & B_{1M} \\ B_{21} & B_{22} & \cdots & B_{2j} & \cdots & B_{2M} \\ \vdots & \vdots & & \vdots & & \vdots \\ B_{P1} & B_{P2} & \cdots & B_{Pj} & \cdots & B_{PM} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1M} \\ C_{21} & C_{22} & \cdots & C_{2M} \\ \vdots & \vdots & C_{ij} & \vdots \\ C_{N1} & C_{N2} & \cdots & C_{NM} \end{pmatrix}$$

- C_{ij} is the inner product of the i^{th} row with the j^{th} column

Matrix times Matrix: by inner products

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1P} \\ A_{21} & A_{22} & \cdots & A_{2P} \\ \vdots & \vdots & & \vdots \\ A_{i1} & A_{i2} & \cdots & A_{iP} \\ \vdots & \vdots & & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{NP} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1j} & \cdots & B_{1M} \\ B_{21} & B_{22} & \cdots & B_{2j} & \cdots & B_{2M} \\ \vdots & \vdots & & \vdots & & \vdots \\ B_{P1} & B_{P2} & \cdots & B_{Pj} & \cdots & B_{PM} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1M} \\ C_{21} & C_{22} & \cdots & C_{2M} \\ \vdots & \vdots & C_{ij} & \vdots \\ C_{N1} & C_{N2} & \cdots & C_{NM} \end{pmatrix}$$

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$$C_{ij} = \sum_{k=1}^{P} A_{ik} B_{kj}$$

- C_{ij} is the inner product of the i^{th} row with the j^{th} column

Matrix times Matrix: by outer products

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1P} \\ A_{21} & A_{22} & \cdots & A_{2P} \\ \vdots & \vdots & & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{NP} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1M} \\ B_{21} & B_{22} & \cdots & B_{2M} \\ \vdots & \vdots & & \vdots \\ B_{P1} & B_{P2} & \cdots & B_{PM} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1M} \\ C_{21} & C_{22} & \cdots & C_{2M} \\ \vdots & \vdots & & \vdots \\ C_{N1} & C_{N2} & \cdots & C_{NM} \end{pmatrix}$$

$$\overleftrightarrow{C} =$$

Matrix times Matrix: by outer products

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1P} \\ A_{21} & A_{22} & \cdots & A_{2P} \\ \vdots & \vdots & & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{NP} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1M} \\ B_{21} & B_{22} & \cdots & B_{2M} \\ \vdots & \vdots & & \vdots \\ B_{P1} & B_{P2} & \cdots & B_{PM} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1M} \\ C_{21} & C_{22} & \cdots & C_{2M} \\ \vdots & \vdots & & \vdots \\ C_{N1} & C_{N2} & \cdots & C_{NM} \end{pmatrix}$$

$$\overleftrightarrow{C} = \begin{pmatrix} \\ A^{\mathrm{c}1} \\ \\ \end{pmatrix} \begin{pmatrix} & B^{\mathrm{r}1} & \end{pmatrix} +$$

Matrix times Matrix: by outer products

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1P} \\ A_{21} & A_{22} & \cdots & A_{2P} \\ \vdots & \vdots & & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{NP} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1M} \\ B_{21} & B_{22} & \cdots & B_{2M} \\ \vdots & \vdots & & \vdots \\ B_{P1} & B_{P2} & \cdots & B_{PM} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1M} \\ C_{21} & C_{22} & \cdots & C_{2M} \\ \vdots & \vdots & & \vdots \\ C_{N1} & C_{N2} & \cdots & C_{NM} \end{pmatrix}$$

$$\overleftrightarrow{C} = \begin{pmatrix} \\ A^{c1} \\ \\ \end{pmatrix} \begin{pmatrix} \quad B^{r1} \quad \end{pmatrix} + \begin{pmatrix} \\ A^{c2} \\ \\ \end{pmatrix} \begin{pmatrix} \quad B^{r2} \quad \end{pmatrix} + $$

Matrix times Matrix: by outer products

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1P} \\ A_{21} & A_{22} & \cdots & A_{2P} \\ \vdots & \vdots & & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{NP} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1M} \\ B_{21} & B_{22} & \cdots & B_{2M} \\ \vdots & \vdots & & \vdots \\ B_{P1} & B_{P2} & \cdots & B_{PM} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1M} \\ C_{21} & C_{22} & \cdots & C_{2M} \\ \vdots & \vdots & & \vdots \\ C_{N1} & C_{N2} & \cdots & C_{NM} \end{pmatrix}$$

$$\overleftrightarrow{C} = \left(A^{\mathrm{c1}} \right) \left(\quad B^{\mathrm{r1}} \quad \right) + \left(A^{\mathrm{c2}} \right) \left(\quad B^{\mathrm{r2}} \quad \right) + \cdots + \left(A^{\mathrm{cP}} \right) \left(\quad B^{\mathrm{rP}} \quad \right)$$

- C is a sum of outer products of the columns of A with the rows of B

Rules for Matrix Addition and Scalar Multiplication

- $A + B = B + A$
- $(A + B) + C = A + (B + C) \quad (\text{written } A + B + C)$
- $A + 0 = A$
- $A + (-A) = 0$, here 0 denotes the zero matrix (of size $m \times n$), that is, the $m \times n$ matrix with all entries zero.
- $c(A + B) = cA + cB$
- $(c + k)A = cA + kA$
- $c(kA) = (ck)A \quad (\text{written } ckA)$
- $1A = A$

Rules for Matrix Multiplication

- **Matrix Multiplication Is Not Commutative, AB ≠ BA in General**
  - $(\mathbf{kA})\mathbf{B} = \mathbf{k}(\mathbf{AB}) = \mathbf{A}(\mathbf{kB})$ **written kAB or Ak**
  - $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$ **written ABC** (associative law)
  - $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$ (distributive law)
  - $\mathbf{C}(\mathbf{A} + \mathbf{B}) = \mathbf{CA} + \mathbf{CB}$ (distributive law)

Transpose of a Matrix

- We obtain the transpose of a matrix by writing its rows as columns (or equivalently its columns as rows).
- Also note that, if A is the given matrix, then we denote its transpose by A^T.
- Rules for transposition are
  - $(A^T)^T = A$
  - $(A + B)^T = A^T + B^T$
  - $c(A)^T = cA^T$
  - $(AB)^T = B^T A^T$

$$A = \begin{bmatrix} 2 & 13 \\ -9 & 11 \\ 3 & 17 \end{bmatrix}_{3\times 2}$$

$$B = \begin{bmatrix} 2 & -9 & 3 \\ 13 & 11 & 17 \end{bmatrix}_{2\times 3}$$

Equality of Matrices

- **Definition**
  - **Two matrices A = $[a_{jk}]$ and B = $[b_{jk}]$ are equal, written A = B, if and only if (1) they have the same size and (2) the corresponding entries are equal, that is, $a_{11} = b_{11}$, $a_{12} = b_{12}$, and so on.**

  - **Matrices that are not equal are called different. Thus, matrices of different sizes are always different.**

Matrix Properties

- (A few) special matrices
- Matrix transformations & the determinant
- Matrices & systems of algebraic equations

Special matrices: diagonal matrix

$$\overleftrightarrow{D} = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{pmatrix}$$

$$\overleftrightarrow{D}\,\overrightarrow{x} = \begin{pmatrix} d_1 x_1 \\ d_2 x_2 \\ \vdots \\ d_n x_n \end{pmatrix}$$

- **This acts like scalar multiplication**

Special matrices: identity matrix

$$\overleftrightarrow{1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

$$\text{for all } \overleftrightarrow{A}, \quad \overleftrightarrow{1}\overleftrightarrow{A} = \overleftrightarrow{A}\overleftrightarrow{1} = \overleftrightarrow{A}$$

Special matrices: Inverse Matrix

- **Some matrices have an inverse, such that: $AA^{-1} = I$**
- **Inverse exists only for <u>square matrices</u> that are <u>non-singular</u>**
- **Inversion is tricky: $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$**
  - **Derived from non-commutativity property**

$$\overleftrightarrow{A}\,\overleftrightarrow{A}^{-1} = \overleftrightarrow{A}^{-1}\,\overleftrightarrow{A} = \overleftrightarrow{1}$$

Symmetric and Skew-Symmetric Matrices.

- **Symmetric matrices are square matrices whose transpose equals the matrix itself.**
  - $\mathbf{A}^{\mathsf{T}} = \mathbf{A}$ (thus $a_{kj} = a_{jk}$),
- **Skew-symmetric matrices are square matrices whose transpose equals minus the matrix.**
  - $\mathbf{A}^{\mathsf{T}} = -\mathbf{A}$ (thus $a_{kj} = -a_{jk}$), hence $a_{jj} = 0$).

Triangular Matrices.

- Upper triangular matrices are square matrices that can have nonzero entries only on and above the main diagonal, whereas any entry below the diagonal must be zero.

$$\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 6 \end{bmatrix}$$

- Similarly, lower triangular matrices can have nonzero entries only on and below the main diagonal. Any entry on the main diagonal of a triangular matrix may be zero or not.

$$\begin{bmatrix} 2 & 0 & 0 \\ 8 & -1 & 0 \\ 7 & 6 & 8 \end{bmatrix}, \quad \begin{bmatrix} 3 & 0 & 0 & 0 \\ 9 & -3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 9 & 3 & 6 \end{bmatrix}$$

Scalar matrix

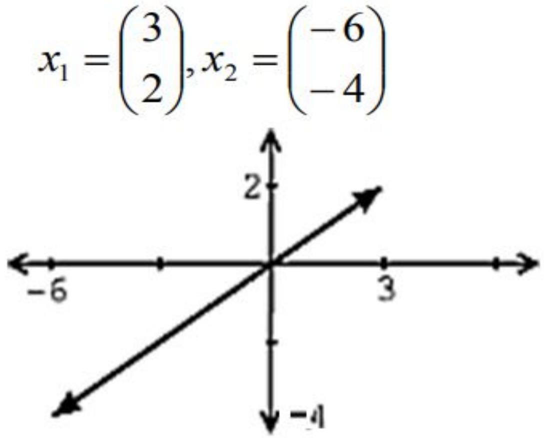
- A diagonal matrix whose main diagonal elements are equal to the same scalar
- A scalar is defined as a single number or constant

$$\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

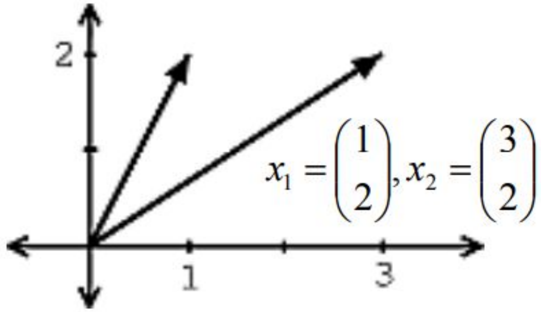
Linear Independence

- A set of d-dimensional vectors $x_i \in \mathbb{R}^d$, are said to be linearly independent if none of them can be written as a linear combination of the others.
- In other words,

$$c_1x_1 + c_2x_2 + \cdots + c_kx_k = 0,$$
$$iff\ c_1 = c_2 = c_3 \ldots = c_n = 0$$



Not linearly independent vectors



Linearly independent vectors

Span

- A span of a set of vectors $x_1, x_2, \dots, x_k$ is the set of vectors that can be written as a linear combination of $x_1, x_2, \dots, x_k$.

  $span(x_1, x_2, \dots, x_k)$
  $= \{c_1 x_1 + c_2 x_2 + \cdots + c_k x_k \mid c_1, c_2, \dots, c_k \in \mathbb{R}\}$

- Question:

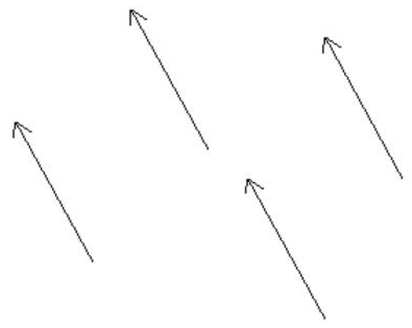
$$span\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right) = ?$$

Bases & Orthonormal Bases

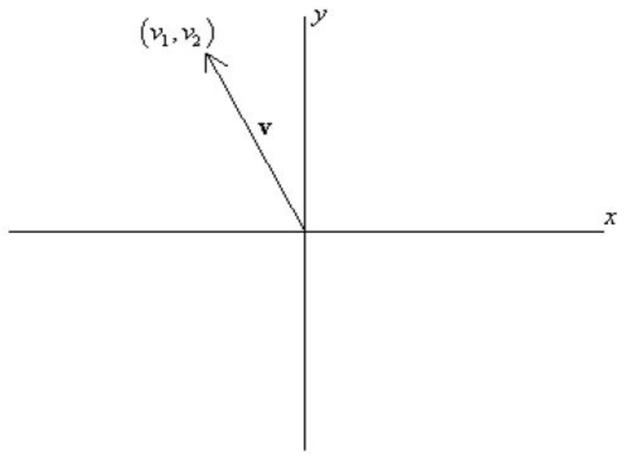
- A basis for $\mathbb{R}^d$ is a set of vectors which:
  - Spans $\mathbb{R}^d$, i.e. any vector in this d-dimensional space can be written as linear combination of these basis vectors.
  - Are linearly independent
- Clearly, any set of d-linearly independent vectors form basis vectors for $\mathbb{R}^d$
- Ortho-Normal: orthogonal + normal
  - Orthogonal: dot product is zero
  - Normal: magnitude is one

$$x = [1 \quad 0 \quad 0]^T$$
$$y = [0 \quad 1 \quad 0]^T$$
$$z = [0 \quad 0 \quad 1]^T$$

$$x \cdot y = 0$$
$$x \cdot z = 0$$
$$y \cdot z = 0$$



VS



Determinants

- **Given a square matrix A its determinant is a real number associated with the matrix.**
- **The determinant of A is written:**

det(A) or |A|

- **For a 2x2 matrix, the definition is**

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Source- 6502 : Mathematics for Engineers 2, James Burnett, Department of Mathematics, University College London

Determinants 2x2 examples

$$\det\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (1)(4) - (2)(3) = -2$$

$$\det\begin{pmatrix} -5 & 2 \\ -2 & 0 \end{pmatrix} = \begin{vmatrix} -5 & 2 \\ -2 & 0 \end{vmatrix} = (-5)(0) - (2)(-2) =$$

4

$$\det\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = (1)(4) - (2)(2) = 0$$

Determinants

- To define det(A) for larger matrices, we will need the definition of a minor M_{ij}
- The minor M_{ij} of a matrix A is the matrix formed by removing the *i*th row and the *j*th column of A

$$A = \begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & 3 \\ 2 & 7 & 0 \end{pmatrix}$$

M_{11}: remove row 1, col 1

$$M_{11} = \begin{pmatrix} 2 & 3 \\ 7 & 0 \end{pmatrix}$$

Determinants

- **To define det(A) for larger matrices, we will need the definition of a minor M_{ij}**
- **The minor M_{ij} of a matrix A is the matrix formed by removing the *i*th row and the *j*th column of A**

$$A = \begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & 3 \\ 2 & 7 & 0 \end{pmatrix}$$

M_{12} : remove row 1, col 2

$$M_{12} = \begin{pmatrix} -1 & 3 \\ 2 & 0 \end{pmatrix}$$

Determinants

- **To define det(A) for larger matrices, we will need the definition of a minor M_{ij}**
- **The minor M_{ij} of a matrix A is the matrix formed by removing the *i*th row and the *j*th column of A**

M_{13} : remove row 1, col 3

$$A = \begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & 3 \\ 2 & 7 & 0 \end{pmatrix}$$

$$M_{13} = \begin{pmatrix} -1 & 2 \\ 2 & 7 \end{pmatrix}$$

Determinants

- **To define det(A) for larger matrices, we will need the definition of a minor M_{ij}**
- **The minor M_{ij} of a matrix A is the matrix formed by removing the *i*th row and the *j*th column of A**

$$A = \begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & 3 \\ 2 & 7 & 0 \end{pmatrix}$$

M_{21} : remove row 2, col 1

$$M_{21} = \begin{pmatrix} 1 & -2 \\ 7 & 0 \end{pmatrix}$$

The formula for a 3x3 matrix

- **For a matrix**

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

- Its determinant is given by

$$|A| = a_{11}|M_{11}| - a_{12}|M_{12}| + a_{13}|M_{13}|$$

- From the formula for a 2x2 matrix:

$$|M_{11}| = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{23}a_{32}$$

The formula for a 3x3 matrix

- **For a matrix**

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

- Its determinant is given by

$$|A| = a_{11}|M_{11}| - a_{12}|M_{12}| + a_{13}|M_{13}|$$

- From the formula for a 2x2 matrix:

$$|M_{12}| = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{23}a_{31}$$

The formula for a 3x3 matrix

- **For a matrix**

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

- Its determinant is given by

$$|A| = a_{11}|M_{11}| - a_{12}|M_{12}| + a_{13}|M_{13}|$$

- From the formula for a 2x2 matrix:

$$|M_{13}| = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{31}a_{22}$$

3x3 Example

$$A = \begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & 3 \\ 2 & 7 & 0 \end{pmatrix}$$

$$|A| = 1x|M_{11}| - 1x|M_{12}| + (-2)x|M_{13}|$$

$$|A| = 1x\begin{vmatrix} 2 & 3 \\ 7 & 0 \end{vmatrix} - 1x\begin{vmatrix} -1 & 3 \\ 2 & 0 \end{vmatrix} + (-2)\begin{vmatrix} -1 & 2 \\ 2 & 7 \end{vmatrix}$$

$$= 1x(-21) -1x(-6) +(-2)x(-11) = 7$$

The formula for a 3x3 matrix

- **For the matrix**

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

- We used the top row to calculate the determinant:

$$|A| = a_{11}|M_{11}| - a_{12}|M_{12}| + a_{13}|M_{13}|$$

- However, we could equally have used **any row** of the matrix and performed a similar calculation

The formula for a 3x3 matrix

- **For the matrix**

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

- Using the **top** row:

$$|A| = a_{11}|M_{11}| - a_{12}|M_{12}| + a_{13}|M_{13}|$$

- Using the **second** row

$$|A| = -a_{21}|M_{21}| + a_{22}|M_{22}| - a_{23}|M_{23}|$$

- Using the **third** row

$$|A| = a_{31}|M_{31}| - a_{32}|M_{32}| + a_{33}|M_{33}|$$

The formula for a 3x3 matrix

$$|A| = a_{11}|M_{11}| - a_{12}|M_{12}| + a_{13}|M_{13}|$$

$$= -a_{21}|M_{21}| + a_{22}|M_{22}| - a_{23}|M_{23}|$$

$$= a_{31}|M_{31}| - a_{32}|M_{32}| + a_{33}|M_{33}|$$

- Notice the **changing signs** depending on what row we use:

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

The formula for a 3x3 matrix

- **Equally, we could have used any column as long as we follow the signs pattern**

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

- **E.g. using the first column:**

$$|A| = a_{11}|M_{11}| - a_{21}|M_{21}| + a_{31}|M_{31}|$$

The formula for a 3x3 matrix

- This choice sometimes makes it a bit easier to calculate determinants. e.g.

$$A = \begin{pmatrix} 1 & 1 & -2 \\ 0 & 2 & 3 \\ 0 & 1 & 1 \end{pmatrix}$$

- Using the first row:

$$|A| = 1x \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} - 1x \begin{vmatrix} 0 & 3 \\ 0 & 1 \end{vmatrix} + (-2)\,x \begin{vmatrix} 0 & 2 \\ 0 & 1 \end{vmatrix}$$

$$= 1x(-1) - 1x(0) + (-2)x(0) = -1$$

The formula for a 3x3 matrix

- **This choice sometimes makes it a bit easier to calculate determinants. e.g.**

$$A = \begin{pmatrix} 1 & 1 & -2 \\ 0 & 2 & 3 \\ 0 & 1 & 1 \end{pmatrix}$$

- **However, using the first column:**

$$|A| = 1x \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} - 0 + 0 = 1x(-1) = -1$$

A general formula for determinants

- For a 4x4 matrix we add up minors like the 3x3 case, and again use the same signs pattern

$$\begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix}$$

- Notice that if we think of the signs pattern as a matrix, then it can be written as $(-1)^{i+j}$

A general formula for determinants

- For a nxn matrix $A=(a_{ij})$ the **co-factors** of A are defined by

$$C_{ij} := (-1)^{i+j}|M_{ij}|$$

- The **determinant** of A is given by the formula

$$|A| = \sum_{i=1}^{n} a_{ij} C_{ij} \qquad \text{for any } j=1,2,...,n$$

- Or,

$$|A| = \sum_{j=1}^{n} a_{ij} C_{ij} \qquad \text{for any } i=1,2,...,n$$

A general formula for determinants

- **Consider the 2x2 matrix** $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

- **Add the second column to the first and calculate the determinant:**

$$\begin{vmatrix} a+b & b \\ c+d & d \end{vmatrix} = (a+b)d - b(c+d)$$
$$= (ad+bd) - (bc+bd)$$
$$= (ad-bc)$$
$$= |A|$$

A trick for calculating determinants

- In fact if you replace **any column** of a matrix by the **original column + a multiple of any other column** the determinant is unchanged.

- Similarly, if you replace **any row** of a matrix by the **original row** + **a multiple of any other row** the determinant is unchanged.

- WARNING: adding one row or column to itself will in general **change the determinant**

A trick for calculating determinants

- **Example:**

$$A = \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ b+c & a+c & a+b \end{pmatrix} \xrightarrow{C_2 \to C_2 - C_1} \begin{pmatrix} 1 & 0 & 1 \\ a & b-a & c \\ b+c & a-b & a+b \end{pmatrix}$$

$$\xrightarrow{C_3 \to C_3 - C_1} \begin{pmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ b+c & a-b & a-c \end{pmatrix}$$

- **So, using the top row:**

$$\begin{aligned} |A| &= (b-a)(a-c) - (c-a)(a-b) \\ &= ba - a^2 + ac - bc \\ &\quad -(ac - bc - a^2 + ab) \\ &= 0 \end{aligned}$$

More determinant properties

- If we take the transpose of a matrix, its determinant is unchanged: $|A| = |A^T|$
- For **diagonal** or **upper triangular** or **lower triangular** matrices, the determinant is the product of the **leading diagonal entries**:

$$\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33}$$

More determinant properties

- Multiplying a whole row (or column) by k multiplies the determinant by k.

- If a matrix is nxn then multiplying the matrix by k is the same as multiplying n rows by k. Hence, the determinant is multiplied by k^n.

$$\begin{vmatrix} ka & kb \\ c & d \end{vmatrix} = k\begin{vmatrix} a & b \\ c & d \end{vmatrix} \qquad \begin{vmatrix} ka & kb \\ kc & kd \end{vmatrix} = k^2\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

More determinant properties

- **If we swap two rows (or two columns), the determinant changes by a factor of (-1):**

$$\begin{vmatrix} c & d \\ a & b \end{vmatrix} = (-1)\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

- **If an entire row or column is zero, the determinant is zero**

$$\begin{vmatrix} 1 & 0 & -1 \\ 2 & 4 & 5 \\ 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} -1 & 0 & 7 \\ 3 & 0 & 5 \\ 1 & 0 & 4 \end{vmatrix} = 0$$

More determinant properties

- **The determinant of a product is the product of determinants:**

$$|A\,B| = |A|\;|B|$$

- **Example**

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} \quad |A| = -2,\ |B| = 2$$

$$A\,B = \begin{pmatrix} -1 & 2 \\ -1 & 6 \end{pmatrix} \quad \text{So, } |A\,B| = -4 = (-2)(2) = |A|\,|B|$$

Determinant of a Matrix: Applications

- Used for inversion
- If det(A) = 0, then A has no inverse

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \det(A) = ad - bc \qquad A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- Determinants can be used to see if a system of n linear equations in n variables has a unique solution.
- For more, please refer: “Several Applications of Determinants”, E. Ulrychov´a

Rank of a Matrix

- **Definition of row equivalence:**
  - **A matrix B is row equivalent to a matrix A if B result from A via elementary row operations.**
- **Let**

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, B_1 = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}, B_2 = \begin{bmatrix} 2 & 4 & 6 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, B_3 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 7 & 8 & 9 \end{bmatrix}$$

- **Since**

$$A = \begin{array}{c} (1) \\ (2) \\ (3) \end{array} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow{(1)\leftrightarrow(2)} B_1 = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$$

$$A = \begin{array}{c} (1) \\ (2) \\ (3) \end{array} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow{(1)=2*(1)} B_2 = \begin{bmatrix} 2 & 4 & 6 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$A = \begin{array}{c} (1) \\ (2) \\ (3) \end{array} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow{(2)=(2)-(1)} B_3 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 7 & 8 & 9 \end{bmatrix}$$

B_1, B_2, B_3 are row equivalent to A

Rank of a Matrix

- "row-rank of a matrix" counts the max. number of linearly independent rows.
- "column-rank of a matrix" counts the max. number of linearly independent columns.
- One application: Given a large system of linear equations, count the number of essentially different equations.
  - The number of essentially different equations is just the row-rank of the augmented matrix.

Evaluating the row-rank by definition

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$$

Linearly independent

$$\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$$

Linearly independent

$$\begin{bmatrix} 2 & 4 & 6 & 8 \end{bmatrix}$$

Linearly independent

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \end{bmatrix}$$

Linearly dependent

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

Linearly independent

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 6 & 8 \end{bmatrix}$$

Linearly independent

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \end{bmatrix}$$

Linearly dependent

$\Rightarrow$ Row-Rank = 2

Calculation of row-rank via RREF

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \end{bmatrix} \xrightarrow[\text{Row reductions}]{} \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Row-rank = 2

Row-rank = 2

Because row reductions do not affect the number of linearly independent rows

Calculation of column-rank by definition

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \end{bmatrix}$$

List all combinations of columns

$\Rightarrow$ Column-Rank = 2

Linearly independent??

$$\begin{array}{cccccccccc} \text{Y} & \text{Y} & \text{Y} & \text{Y} & \text{Y} & \text{Y} & \text{Y} & \text{Y} & \text{Y} & \text{Y} \\ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} & \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} & \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} & \begin{bmatrix} 1 \\ 4 \\ 8 \end{bmatrix} & \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 4 \end{bmatrix} & \begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 2 & 6 \end{bmatrix} & \begin{bmatrix} 1 & 1 \\ 1 & 4 \\ 2 & 8 \end{bmatrix} & \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 4 & 6 \end{bmatrix} & \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 4 & 8 \end{bmatrix} & \begin{bmatrix} 1 & 1 \\ 3 & 4 \\ 6 & 8 \end{bmatrix} \end{array}$$

$$\begin{array}{cccc} \text{N} & \text{N} & \text{N} & \text{N} \\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} & \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix} & \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 4 \\ 2 & 6 & 8 \end{bmatrix} & \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 4 & 6 & 8 \end{bmatrix} \end{array}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 6 & 8 \end{bmatrix} \quad \text{N}$$

Theorem

Given any matrix, its row-rank and column-rank are equal.

- In view of this property, we can just say the "rank of a matrix". It means either the row-rank of column-rank.

Why row-rank = column-rank?

- If some column vectors are linearly dependent, they remain linearly dependent after any elementary row operation

- For example, $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 4 & 6 & 8 \end{bmatrix}$ are linearly dependent

$$\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} - 2\begin{bmatrix} 3 \\ 1 \\ 6 \end{bmatrix} + \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 10 \\ 2 \\ 4 \end{bmatrix} - 2\begin{bmatrix} 10 \\ 3 \\ 6 \end{bmatrix} + \begin{bmatrix} 10 \\ 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix} + \begin{bmatrix} 1 \\ 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Why row-rank = column-rank?

- Any row operation does not change the column-rank.
- By the same argument, apply to the transpose of the matrix, we conclude that any column operation does not change the row-rank as well.

Why row-rank = column-rank?

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Apply row reductions. row-rank and column-rank do not change.

Apply column reductions. row-rank and column-rank do not change.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The top-left corner is an identity matrix.

The row-rank and column-rank of this "normal form" is certainly the size of this identity submatrix, and are therefore equal.

Pivot/Leading entry

- Let A be a non-zero matrix. Then, in each non-zero row of A, the left most non-zero entry is called a pivot/leading entry.
- The column containing the pivot is called a pivotal column.
- For example, the entries a_{12} and a_{23} are pivots in

$$A = \begin{bmatrix} 0 & 3 & 4 & 2 \\ \mathbf{0} & \mathbf{0} & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{2} & \mathbf{1} \end{bmatrix}$$

Row Echelon Form (REF) of a Matrix

- A matrix is in row echelon form if the following conditions are satisfied:
  1. if the zero rows are at the bottom
  2. if the pivot of the (i + 1)-th row, if it exists, comes to the right of the pivot of the i-th row, i.e., the column subscript of the leading non-zero entries increases as the row subscript increases.
  3. if the entries below the pivot in a pivotal column are 0
- Some texts add the condition that the leading coefficient must be 1.

$$\begin{bmatrix} 0 & \boxed{2} & 4 & 2 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \boxed{1} & 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & \boxed{3} & 4 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix}, \begin{bmatrix} \boxed{1} & 2 & 0 & 5 \\ 0 & \boxed{2} & 0 & 6 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix}.$$

In Row Echelon Form

$$\begin{bmatrix} 0 & \boxed{1} & 4 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 1 \end{bmatrix} \text{ and } \begin{bmatrix} \boxed{1} & 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & \boxed{1} & 4 \end{bmatrix}.$$

Not In Row Echelon Form

Source: Linear Algebra through Matrices, Arbind K Lal and Sukant Pati

Elementary Row Operations on a Matrix

- A combination of the following operations will transform a matrix to row echelon form:
  1. Interchange any two rows.
     - Notation: R_{ij} indicates interchanging the *i*-th and *j*-th rows
  2. Multiply all elements of a row by a nonzero real number.
     - Notation: kR_i indicates multiplying the *i*-th row by *k*
  3. Add a multiple of one row to any other row.
     - Notation: $kR_i + R_j$ indicates adding *k* times the *i*-th row to the *j*-th row

Example: Finding the REF

Input Matrix:

$$\begin{bmatrix} 1 & -1 & 2 & -3 \\ 2 & 1 & -1 & 0 \\ -1 & 2 & -3 & 7 \end{bmatrix} \xrightarrow{-2R_1 + R_2} \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 3 & -5 & 6 \\ -1 & 2 & -3 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 3 & -5 & 6 \\ -1 & 2 & -3 & 7 \end{bmatrix} \xrightarrow{R_1 + R_3} \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 3 & -5 & 6 \\ 0 & 1 & -1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 1 & -1 & 4 \\ 0 & 3 & -5 & 6 \end{bmatrix} \xleftarrow{R_{23}} \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 3 & -5 & 6 \\ 0 & 1 & -1 & 4 \end{bmatrix}$$

Source: http://www.northlandprep.org/wp-content/uploads/2012/03/PC-7.3b.ppt

Example: Finding the REF

$$\begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 1 & -1 & 4 \\ 0 & 3 & -5 & 6 \end{bmatrix} \xrightarrow{-3R_2 + R_3} \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & -2 & -6 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{2}R_3} \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Practice

- What elementary row operations applied to

$$\begin{bmatrix} -2 & 1 & -1 & 2 \\ 1 & -2 & 3 & 0 \\ 3 & 1 & -1 & 2 \end{bmatrix} \xrightarrow{\boldsymbol{R_1 = 2R_2 + R_1}} \begin{bmatrix} 0 & -3 & 5 & 2 \\ 1 & -2 & 3 & 0 \\ 3 & 1 & -1 & 2 \end{bmatrix}$$

Practice

- What elementary row operations applied to

$$\begin{bmatrix} -2 & 1 & -1 & 2 \\ 1 & -2 & 3 & 0 \\ 3 & 1 & -1 & 2 \end{bmatrix} \xrightarrow{R_3 = \frac{1}{4}R_3} \begin{bmatrix} -2 & 1 & -1 & 2 \\ 1 & -2 & 3 & 0 \\ 0.75 & 0.25 & -0.25 & 0.5 \end{bmatrix}$$

Practice

- **Find a row echelon form for the given matrix.**

$$\begin{bmatrix} 1 & 2 & -3 \\ -3 & -6 & 10 \\ -2 & -4 & 7 \end{bmatrix} \xrightarrow{R_2 = 3R_1 + R_2} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 1 \\ -2 & -4 & 7 \end{bmatrix}$$

$$\xrightarrow{R_3 = 2R_1 + R_3} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xleftarrow{R_3 = -1R_2 + R_3} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Reduced Row Echelon Form

- **It is possible to continue applying elementary row operations to a REF matrix until every column that has a leading non-zero has 0's elsewhere. This is the Reduced Row Echelon Form (RREF) of the matrix.**

Example: Finding the RREF

Input Matrix: $\begin{bmatrix} 3 & 7 & -11 \\ 1 & 2 & -3 \\ 4 & 9 & -13 \end{bmatrix} \xrightarrow{R_1 = -3R_2 + R_1} \begin{bmatrix} 0 & 1 & -2 \\ 1 & 2 & -3 \\ 4 & 9 & -13 \end{bmatrix}$

$$\xrightarrow{R_3 = -4R_2 + R_3} \begin{bmatrix} 0 & 1 & -2 \\ 1 & 2 & -3 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \xleftarrow{R_{12}} \begin{bmatrix} 0 & 1 & -2 \\ 1 & 2 & -3 \\ 0 & 0 & 1 \end{bmatrix} \xleftarrow{R_3 = -R_1 + R_3} \begin{bmatrix} 0 & 1 & -2 \\ 1 & 2 & -3 \\ 0 & 1 & -1 \end{bmatrix}$$

Example: Finding the RREF

$$
\begin{tikzcd}
\begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \arrow[r, "R_1 = -2R_2 + R_1"] & \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \arrow[d, "R_1 = -R_3 + R_1"] \\
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \arrow[l, "R_2 = 2R_3 + R_2"']
\end{tikzcd}
$$

Practice

$$\begin{bmatrix} 2 & -1 & 1 & -3 & 5 \\ 1 & 2 & 3 & -1 & 16 \\ 7 & -3 & -2 & 2 & -11 \\ -1 & 1 & -1 & 5 & -9 \end{bmatrix} \xrightarrow{\text{RREF: List all the elementary row operations}} \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 7 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix}$$

- Consider last column as an additional column. Your actual matrix is 4×4
- All the elementary operation will also be performed on last column

REF

- **Theorem.** *Every matrix is row equivalent to a matrix in row echelon form.*
- **Proof**
  - Select a non-zero entry furthest to the left in the matrix.
  - If this entry is not in the first column, this means that the matrix consists entirely of zeros to the left of this entry, and we can forget about them.
  - So suppose this non-zero entry is in the first column. After an interchange of rows, we can find an equivalent matrix such that the upper left-hand corner is not O.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

REF

- Say the matrix is

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

- and $a_{11} \neq 0$.
- We multiply the first row by a $\frac{a_{21}}{a_{11}}$ and subtract from the second row. Similarly, for other rows.
- Then we obtain a matrix which has zeros in the first column except for all.
- Thus the original matrix is row equivalent to a matrix of the form

REF

- Thus, the original matrix is row equivalent to a matrix of the form

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a'_{22} & \cdots & a'_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & a'_{m2} & \cdots & a'_{mn} \end{pmatrix}.$$

- We then repeat the procedure with the smaller matrix

$$\begin{pmatrix} a'_{22} & \cdots & a'_{2n} \\ \vdots & & \vdots \\ a'_{m2} & \cdots & a'_{mn} \end{pmatrix}.$$

- We can continue until the matrix is In row echelon form

Systems of equations

- A system of equations is a group of two or more equations.
- To solve a system of equations means to find values for the variables that satisfy all of the equations in the system.
- Systems of equations can involve any number of equations and variables.

Linear equation

- A linear equation is an equation that may be put in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n + b = 0$$

where x_1, x_2, …, $x_n$are the variables (or unknowns), and b, a_1, a_2, …, a_n are the coefficients, which are often real numbers.

System of linear equations

- **Examples of linear systems**
  - **Suppose $a, b \in R$. Consider the system $ax = b$ in the variable x. if**
    - $a \neq 0$ **then the system has a unique solution** $x = \frac{\boldsymbol{b}}{\boldsymbol{a}}$
    - $a = 0$ **and**
      i. $b \neq 0$ then the system has no solution.
      ii. $b = 0$ then the system has infinite number of solutions, namely all $x \in R$.

System of linear equations

- System of two linear equations in two variables may be written in the general form

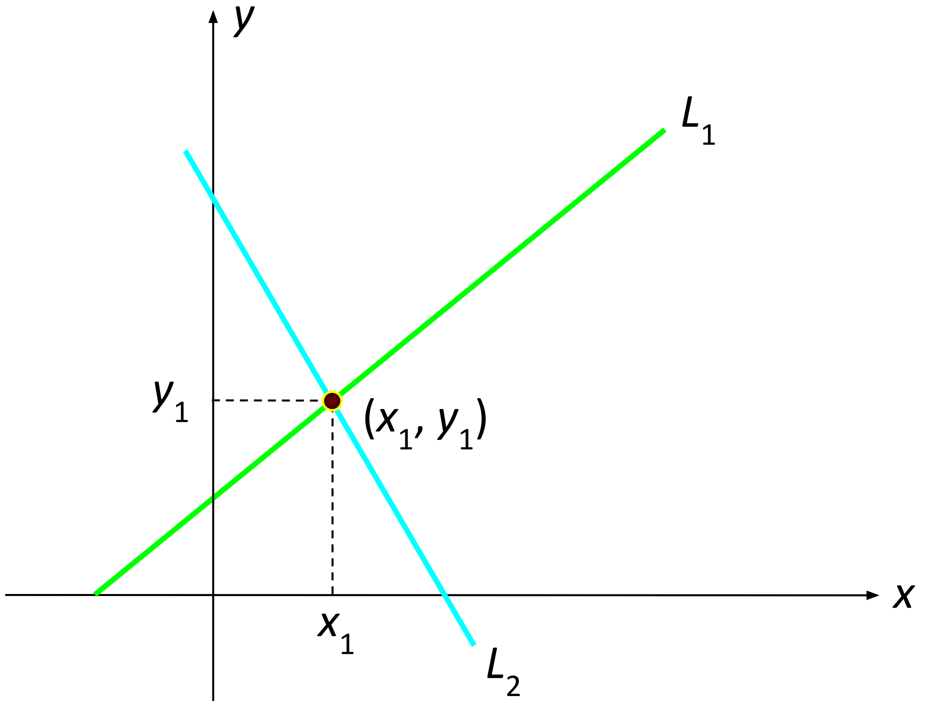
$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

where a_1, a_2, b_1, b_2, c_1, and c_2 are real numbers and $(a_1, b_1), (b_1, b_2) \neq 0$, i.e., neither a_1 and b_1 nor a_2 and b_2 are both zero.

- Graph of each equation in the system is a straight line in the plane, so that geometrically, the solution to the system is the point(s) of intersection of the two straight lines ℓ_1 and ℓ_2, represented by the first and second equations of the system.

System of linear equations

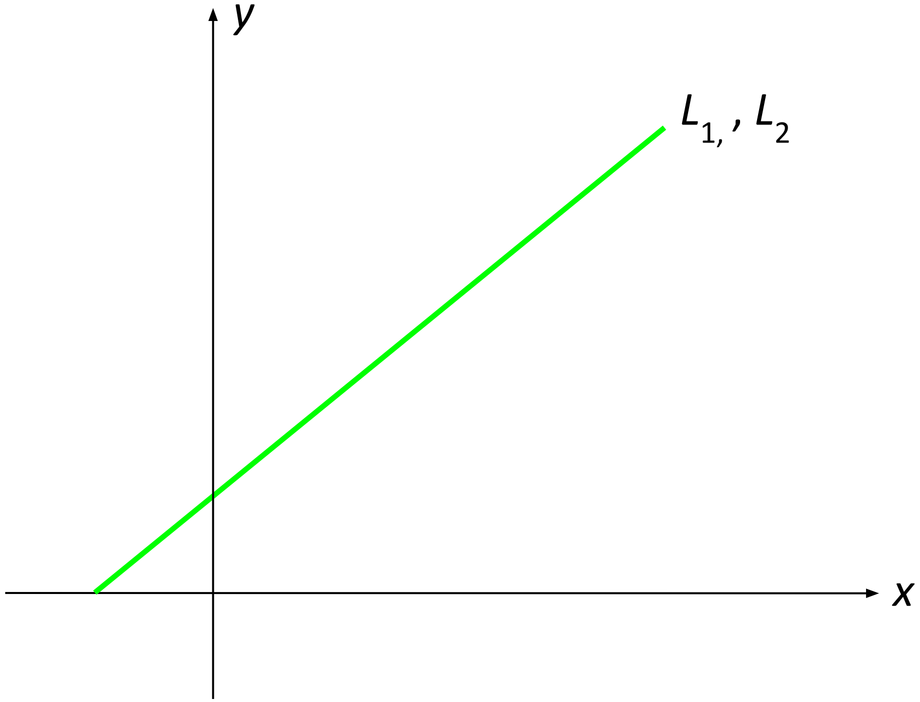
- **Given the two straight lines L_1 and L_2, one and only one of the following may occur:**
  1. **L_1 and L_2 intersect at exactly one point.**



Unique solution
(x_1, y_1)

System of linear equations

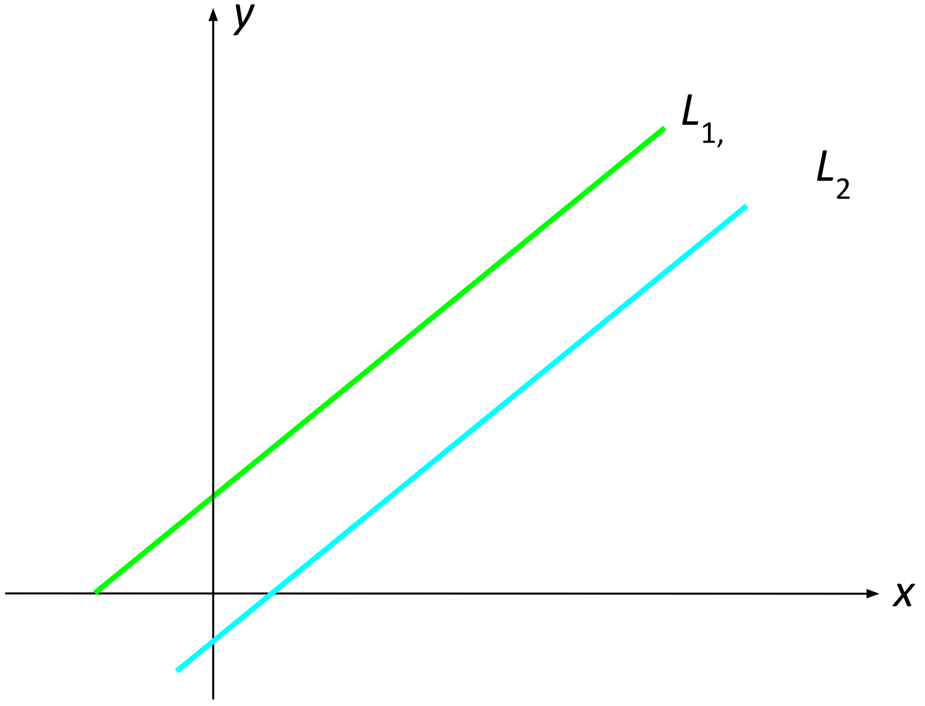
- Given the two straight lines L_1 and L_2, one and only one of the following may occur:
  2. L_1 and L_2 are coincident.



Infinitely many solutions

System of linear equations

- **Given the two straight lines L_1 and L_2, one and only one of the following may occur:**
    3. **L_1 and L_2 are parallel.**



No solutions

System of linear equations

- **Unique Solution** $(a_1b_2 - a_2b_1 \neq 0)$: **The linear system** $x - y = 3$ **and** $2x + 3y$ **has** $(x, y) = (4,1)$ **as the unique solution.**
- **No Solution** $(a_1b_2 - a_2b_1 = \mathbf{0}$ **but** $a_1c_2 - a_2c_1 \neq 0)$: **The linear system** $x + 2y = 1$ **and** $2x + 4y = 3$ **represent a pair of parallel lines which have no point of intersection.**
- **Infinite Number of Solutions** $(a_1b_2 - a_2b_1 = \mathbf{0}$ **and** $a_1c_2 - a_2c_1 = 0)$: **The linear system** $x + 2y = 1$ **and** $2x + 4y = \mathbf{2}$ **represent the same line.**
- **If the linear system** $ax + by = c$ **has**
  i. $(\mathrm{a}, \mathrm{b}) = (0, 0)$ **and** $\mathrm{c} \neq 0$ **then** $\mathrm{ax} + \mathrm{by} = \mathrm{c}$ **has no solution.**
  ii. $(\mathrm{a}, \mathrm{b}, \mathrm{c}) = (0, 0, 0)$ **then** $\mathrm{ax} + \mathrm{by} = \mathrm{c}$ **has infinite number of solutions, namely whole of** $\mathbb{R}^2$**.**

Different ways of looking at the linear system

- There are three ways of looking at the linear system Ax = b
  - Point of intersection of planes
  - Vector sum approach
  - Matrix multiplication approach.
- For the linear system $x - y = 3$ and $2x + 3y$
  - $(\mathrm{x}, \mathrm{y}) = (4{,}1)$ corresponds to the point of intersection of the corresponding two lines

Different ways of looking at the linear system

- There are three ways of looking at the linear system Ax = b
  - Point of intersection of planes
  - Vector sum approach
  - Matrix multiplication approach.
- For the linear system $x - y = 3$ and $2x + 3y$
  - Using matrix multiplication, the given system equals

Ax = b, where $\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{-1} \\ \mathbf{2} & \mathbf{3} \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 3 \\ 11 \end{bmatrix}$.

So the solution is $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b} = \frac{1}{5}\begin{bmatrix} \mathbf{3} & \mathbf{1} \\ \mathbf{-2} & \mathbf{1} \end{bmatrix}\begin{bmatrix} 3 \\ 11 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

Different ways of looking at the linear system

- **There are three ways of looking at the linear system Ax = b**
  - **Point of intersection of planes**
  - **Vector sum approach**
  - **Matrix multiplication approach.**
- **For the linear system** $x - y = 3$ **and** $2x + 3y$
  - **Re-writing** $Ax = \boldsymbol{b}$ **as,** $\begin{bmatrix} 1 \\ 2 \end{bmatrix} x + \begin{bmatrix} -1 \\ 3 \end{bmatrix} y = \begin{bmatrix} 3 \\ 11 \end{bmatrix}$ **gives us** $\mathbf{4} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \mathbf{1} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \end{bmatrix}$. **This corresponds to addition of vectors in the Euclidean plane.**

System of linear equations

- **A system of m linear equations in n variables $x_1, x_2, \dots, x_n$ is a set of equations of the form**

$$\begin{aligned}
\boldsymbol{a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n} &= \boldsymbol{b_1}\\
\boldsymbol{a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n} &= \boldsymbol{b_2}\\
\boldsymbol{a_{31}x_1 + a_{32}x_2 + \cdots + a_{3n}x_n} &= \boldsymbol{b_3}\\
\vdots \qquad\qquad\qquad & \vdots\\
\boldsymbol{a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n} &= \boldsymbol{b_m}
\end{aligned}$$

where for $1 \le i \le m$ **and** $1 \le j \le n$**;** $a_{ij}, b_i \in R$

- **The linear system given above is called** homogeneous **if** $b_1 = 0 = b_2 = \cdots = b_m$ **and** non-homogeneous**, otherwise.**

System of linear equations

- Let

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \text{ and } \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

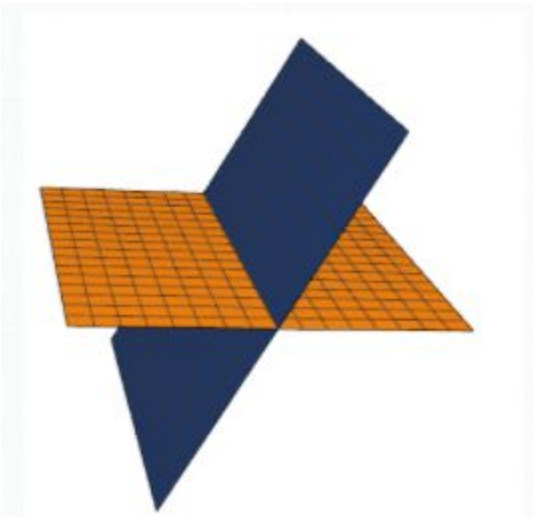
- Then, the system of m linear equations in n variables $x_1, x_2, \ldots, x_n$ can be re-written as $\boldsymbol{Ax} = \boldsymbol{b}$, where A is called coefficient matrix and the block matrix [A b] is called the augmented matrix

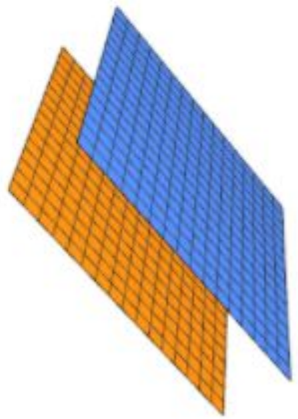
System of linear equations

- Consider a linear system $Ax = b$. Then
  1. a solution of Ax = b is a vector y such that the matrix product Ay indeed equals b.
  2. the set of all solutions is called the solution set of the system.
  3. this linear system is called consistent if it admits a solution and is called inconsistent if it admits no solution

System of linear equations

- **Underdetermined if n>m: “more unknowns than equations”**
  - cannot have a unique solution.
  - In this case, there are either infinitely many or no solutions.
  - For an example of this, refer to what can happen with only two planes in three dimensions:





- **Overdetermined if m>n: “more equations than unknowns”**
  - An overdetermined system may also have infinitely many or no solutions or may have a unique solution

Gauss-Jordan Elimination

- The Gauss-Jordan elimination method is a technique for solving systems of linear equations of any size.
- The operations of the Gauss-Jordan method are
  - Interchange any two equations.
  - Replace an equation by a nonzero constant multiple of itself.
  - Replace an equation by the sum of that equation and a constant multiple of any other equation.

Recall: Row-Reduced Form of a Matrix

- Each row consisting entirely of zeros lies below all rows having nonzero entries.
- The first nonzero entry in each nonzero row is 1 (called a leading/pivot 1).
- In any two successive (nonzero) rows, the leading 1 in the lower row lies to the right of the leading 1 in the upper row.
- If a column contains a leading 1, then the other entries in that column are zeros.

Row Operations

1. Interchange any two rows.
2. Replace any row by a nonzero constant multiple of itself.
3. Replace any row by the sum of that row and a constant multiple of any other row.

Terminology for the Gauss-Jordan Elimination Method

- **Unit Column**
  - A column in a coefficient matrix is in unit form if one of the entries in the column is a 1 and the other entries are zeros.
- **Pivoting**
  - The sequence of row operations that transforms a given column in an augmented matrix into a unit column.

The Gauss-Jordan Elimination Method

1. Write the augmented matrix corresponding to the linear system.
2. Interchange rows, if necessary, to obtain an augmented matrix in which the first entry in the first row is nonzero. Then pivot the matrix about this entry.
3. Interchange the second row with any row below it, if necessary, to obtain an augmented matrix in which the second entry in the second row is nonzero. Pivot the matrix about this entry.
4. Continue until the final matrix is in row-reduced form.

Gauss-Jordan Elimination

- **Solve the following system of equations:**

$$\begin{aligned} 2x + 4y + 6z &= 22 \\ 3x + 8y + 5z &= 27 \\ -x + \ \ y + 2z &= \ \ 2 \end{aligned}$$

- **Solution**
  - **First, we transform this system into an equivalent system in which the coefficient of x in the first equation is 1:**

$$\begin{aligned} 2x + 4y + 6z &= 22 \\ 3x + 8y + 5z &= 27 \\ -x + \ \ y + 2z &= \ \ 2 \end{aligned}$$

Multiply the equation by 1/2

$$\begin{aligned} x + 2y + 3z &= 11 \\ 3x + 8y + 5z &= 27 \\ -x + \ \ y + 2z &= \ \ 2 \end{aligned}$$

Example

- Solve the following system of equations:

$$\begin{aligned} 2x + 4y + 6z &= 22 \\ 3x + 8y + 5z &= 27 \\ -x + y + 2z &= 2 \end{aligned}$$

- Solution
  - Next, we eliminate the variable *x* from all equations except the first:

$$\begin{aligned} x + 2y + 3z &= 11 \\ 3x + 8y + 5z &= 27 \\ -x + y + 2z &= 2 \end{aligned}$$

Replace by the sum of -3 X the first equation + the second equation

$$\begin{aligned} x + 2y + 3z &= 11 \\ 2y - 4z &= -6 \\ -x + y + 2z &= 2 \end{aligned}$$

Example

- **Solve the following system of equations:**

$$\begin{aligned} 2x + 4y + 6z &= 22 \\ 3x + 8y + 5z &= 27 \\ -x + \ \ y + 2z &= \ \ 2 \end{aligned}$$

- **Solution**
  - **Next, we eliminate the variable *x* from all equations except the first:**

$$\begin{aligned} x + 2y + 3z &= 11 \\ 2y - 4z &= -6 \\ -x + \ \ y + 2z &= \ \ 2 \end{aligned}$$

Replace by the sum of the first equation + the third equation

$$\begin{aligned} x + 2y + 3z &= 11 \\ 2y - 4z &= -6 \\ 3y + 5z &= 13 \end{aligned}$$

Example

- Solve the following system of equations:

$$\begin{aligned} 2x + 4y + 6z &= 22 \\ 3x + 8y + 5z &= 27 \\ -x + y + 2z &= 2 \end{aligned}$$

- Solution
  - Then we transform so that the coefficient of y in the second equation is 1:

$$\begin{aligned} x + 2y + 3z &= 11 \\ 2y - 4z &= -6 \\ 3y + 5z &= 13 \end{aligned}$$

Multiply the second equation by 1/2

$$\begin{aligned} x + 2y + 3z &= 11 \\ y - 2z &= -3 \\ 3y + 5z &= 13 \end{aligned}$$

Example

- Solve the following system of equations:

$$\begin{aligned} 2x + 4y + 6z &= 22 \\ 3x + 8y + 5z &= 27 \\ -x + \ \ y + 2z &= \ \ 2 \end{aligned}$$

- Solution
  - We now eliminate y from all equations except the second:

$$\begin{aligned} x + 2y + 3z &= 11 \\ y - 2z &= -3 \\ 3y + 5z &= 13 \end{aligned}$$

Replace by the sum of the first equation + $(-2) \times$ the second equation

$$\begin{aligned} x \qquad + 7z &= 17 \\ y - 2z &= -3 \\ 3y + 5z &= 13 \end{aligned}$$

Example

- Solve the following system of equations:

$$\begin{aligned} 2x + 4y + 6z &= 22 \\ 3x + 8y + 5z &= 27 \\ -x + \ y + 2z &= \ 2 \end{aligned}$$

- Solution
  - We now eliminate y from all equations except the second:

$$\begin{aligned} x \qquad + 7z &= 17 \\ y - 2z &= -3 \\ 3y + 5z &= 13 \end{aligned}$$

← Replace by the sum of the third equation + (–3) × the second equation

$$\begin{aligned} x \qquad + 7z &= 17 \\ y - 2z &= -3 \\ 11z &= 22 \end{aligned}$$

Example

- **Solve the following system of equations:**

$$\begin{aligned} 2x + 4y + 6z &= 22 \\ 3x + 8y + 5z &= 27 \\ -x + y + 2z &= 2 \end{aligned}$$

- **Solution**
  - **Now we transform so that the coefficient of *z* in the third equation is 1:**

$$\begin{aligned} x \quad + 7z &= 17 \\ y - 2z &= -3 \\ 11z &= 22 \end{aligned}$$

Multiply the third equation by 1/11

$$\begin{aligned} x \quad + 7z &= 17 \\ y - 2z &= -3 \\ z &= 2 \end{aligned}$$

Example

- Solve the following system of equations:

$$\begin{aligned} 2x + 4y + 6z &= 22 \\ 3x + 8y + 5z &= 27 \\ -x + y + 2z &= 2 \end{aligned}$$

- Solution
  - We now eliminate z from all equations except the third:

$$\begin{aligned} x \quad + 7z &= 17 \\ y - 2z &= -3 \\ z &= 2 \end{aligned}$$

Replace by the sum of the first equation + (–7) $\times$ the third equation

$$\begin{aligned} x \quad &= 3 \\ y - 2z &= -3 \\ z &= 2 \end{aligned}$$

Example

- Solve the following system of equations:

$$\begin{aligned} 2x + 4y + 6z &= 22 \\ 3x + 8y + 5z &= 27 \\ -x + y + 2z &= 2 \end{aligned}$$

- Solution
- We now **eliminate** z from all equations except the third:

$$\begin{aligned} x \qquad\qquad &= 3 \\ y - 2z &= -3 \\ z &= 2 \end{aligned}$$

Replace by the sum of the second equation + 2 $\times$ the third equation

$$\begin{aligned} x \qquad &= 3 \\ y \quad &= 1 \\ z &= 2 \end{aligned}$$

Example

- **Solve the following system of equations:**

$$\begin{aligned} 2x + 4y + 6z &= 22 \\ 3x + 8y + 5z &= 27 \\ -x + y + 2z &= 2 \end{aligned}$$

- **Solution**
  - **Thus, the solution to the system is** $x = 3$**,** $y = 1$**, and** $z = 2$**.**

$$\begin{aligned} x \qquad\qquad &= 3 \\ y \qquad &= 1 \\ z &= 2 \end{aligned}$$

Augmented Matrices

- Matrices are rectangular arrays of numbers that can aid us by eliminating the need to write the variables at each step of the reduction.
- Solve the following system of equations:

$$\begin{aligned} 2x + 4y + 6z &= 22 \\ 3x + 8y + 5z &= 27 \\ -x + y + 2z &= 2 \end{aligned}$$

Coefficient Matrix

$$\left[\begin{array}{ccc|c} 2 & 4 & 6 & 22 \\ 3 & 8 & 5 & 27 \\ -1 & 1 & 2 & 2 \end{array}\right]$$

Example

$$\begin{aligned} 2x + 4y + 6z &= 22 \\ 3x + 8y + 5z &= 27 \\ -x + y + 2z &= 2 \end{aligned}$$

Multiply the equation by 1/2

$$\left[\begin{array}{ccc|c} 2 & 4 & 6 & 22 \\ 3 & 8 & 5 & 27 \\ -1 & 1 & 2 & 2 \end{array}\right]$$

$$\begin{aligned} x + 2y + 3z &= 11 \\ 3x + 8y + 5z &= 27 \\ -x + y + 2z &= 2 \end{aligned}$$

Replace by the sum of -3 X the first equation + the second equation

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 11 \\ 3 & 8 & 5 & 27 \\ -1 & 1 & 2 & 2 \end{array}\right]$$

$$\begin{aligned} x + 2y + 3z &= 11 \\ 2y - 4z &= -6 \\ -x + y + 2z &= 2 \end{aligned}$$

Replace by the sum of the first equation + the third equation

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 11 \\ 0 & 2 & -4 & -6 \\ -1 & 1 & 2 & 2 \end{array}\right]$$

$$\begin{aligned} x + 2y + 3z &= 11 \\ 2y - 4z &= -6 \\ 3y + 5z &= 13 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 11 \\ 0 & 2 & -4 & -6 \\ 0 & 3 & 5 & 13 \end{array}\right]$$

Example

$$\begin{aligned} x + 2y + 3z &= 11 \\ 2y - 4z &= -6 \\ 3y + 5z &= 13 \end{aligned}$$

Multiply the second equation by 1/2

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 11 \\ 0 & 2 & -4 & -6 \\ 0 & 3 & 5 & 13 \end{array}\right]$$

$$\begin{aligned} x + 2y + 3z &= 11 \\ y - 2z &= -3 \\ 3y + 5z &= 13 \end{aligned}$$

Replace by the sum of the first equation + (–2) $\times$ the second equation

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 11 \\ 0 & 1 & -2 & -3 \\ 0 & 3 & 5 & 13 \end{array}\right]$$

$$\begin{aligned} x \qquad + 7z &= 17 \\ y - 2z &= -3 \\ 3y + 5z &= 13 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -2 & -3 \\ 0 & 3 & 5 & 13 \end{array}\right]$$

Replace by the sum of the third equation + (–3) $\times$ the second equation

$$\begin{aligned} x \qquad + 7z &= 17 \\ y - 2z &= -3 \\ 11z &= 22 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 11 & 22 \end{array}\right]$$

Example

$$\begin{aligned} x \quad\quad + 7z &= 17 \\ y - 2z &= -3 \\ 11z &= 22 \end{aligned}$$

Multiply the third equation by 1/11

$$\left[\begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 11 & 22 \end{array}\right]$$

$$\begin{aligned} x \quad\quad + 7z &= 17 \\ y - 2z &= -3 \\ z &= 2 \end{aligned}$$

Replace by the sum of the first equation + (–7) $\times$ the third equation

$$\left[\begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\begin{aligned} x \quad\quad\quad &= 3 \\ y - 2z &= -3 \\ z &= 2 \end{aligned}$$

Replace by the sum of the second equation + 2 $\times$ the third equation

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\begin{aligned} x \quad\quad\quad &= 3 \\ y \quad\quad &= 1 \\ z &= 2 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Row Reduced Form of the Matrix

Example

- **Solve the following system of equations:**

$$\begin{aligned} 2x + 4y + 6z &= 22 \\ 3x + 8y + 5z &= 27 \\ -x + \ y + 2z &= \ 2 \end{aligned}$$

- **Solution**
  - **Thus, the solution to the system is** $x = 3$**,** $y = 1$**, and** $z = 2$**.**

$$\begin{aligned} x \qquad\qquad &= 3 \\ y \qquad &= 1 \\ z &= 2 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Row Reduced Form of the Matrix

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