Chernoff bound/inequality:-

Statement:- Let $x_1, \dots, x_n$ be arbituary r.v. with $P_i = Pr(X_i = 1)$ Then $N = \sum_{i=1}^{n} P_i$ is the expected value & it holds that,

$$Pr\left(\sum_{i=1}^{n} X_i > (1+\beta) N\right) \leq \left(\frac{e^{\beta}}{(1+\beta)^{1+\beta}}\right)^{N}.$$

Lets relatively consider the expression,

$$Pr\left(\sum_{i=1}^{n} X_i \geq t\right) = P\left(e^{a\sum_{i=1}^{n} \cancel{x}_i} > e^{at}\right)$$

by using Markov's inequality of $P(X \geq t) \leq \frac{E[X]}{t}$,

$$\leq \frac{E\left[e^{a\sum_{i=1}^{n} X_i}\right]}{e^{at}}$$

$$\leq e^{-at} E\left[e^{a\sum_{i=1}^{n} X_i}\right]$$

$$\leq e^{-at} \prod_{i=1}^{n} E\left[e^{aX_i}\right]$$

let $t = (1+\beta)N$. and

$a = \ln(1+\beta)$.

$$P\left(\sum_{i=1}^{n} X_i \leq (1+\beta)N\right) \leq e^{-\ln(1+\beta)\cdot(1+\beta)N} \cdot \prod_{i=1}^{n} E\left[e^{\ln(1+\beta)X_i}\right]$$

(by logarithmic rules)

$$\leq (1+\beta)^{-N(1+\beta)} \prod_{i=1}^{n} E\left[(1+\beta)^{X_i}\right]$$

$E\left[(1+\beta)^{X_i}\right]$ with $X_i \in [0,1]$, given $P_i = P_r(X_i = 1)$.

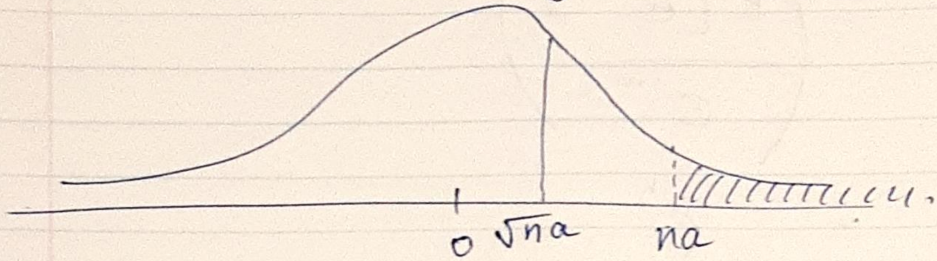
with Prob of P_i, $(X_i = 1)$, thus.

$$E\left[(1+\beta)^{X_i}\right] = P_i(1+\beta) + (1-P_i)(1).$$
$$= P_i(1+\beta) + (1-P_i).$$
$$= P_i + \beta P_i + 1 - P_i$$
$$= 1 + \beta P_i \qquad (1+ax = e^{ax})$$
$$\leq e^{\beta P_i}$$

Thus replacing it in the main expression

$$P\left(\sum_{i=1}^{n} X_i \leq (1+\beta)N\right) \leq (1+\beta)^{-N(1+\beta)} \prod_{i=1}^{n} e^{\beta P_i}$$
$$\leq (1+\beta)^{-N(1+\beta)} e^{\beta \sum_{i=1}^{n} P_i}$$
$$\leq (1+\beta)^{-N(1+\beta)} e^{\beta N}.$$
$$= \frac{e^{\beta N}}{(1+\beta)^{N(1+\beta)}}$$
$$= \left[\frac{e^{\beta}}{(1+\beta)^{(1+\beta)}}\right]^{N}.$$

Hoeffding's inequality :- Proof (General).



Hoeffding's inequality :- If x_i is equally likely to be -1 or 1 and $a>0$ then

$$P(x_1 + \cdots + x_n \geq na) \leq e^{-na^2/2}.$$

Consider,

$$x_1 + \cdots + x_n \geq na,$$

we relatively write it as, $e^{s(x_1+\cdots+x_n)} \geq e^{sna}$,
where $s>0$ (assumption).

$$P\left(e^{s(x_1+\cdots+x_n)} > e^{sna}\right) \leq \frac{E\left[e^{s(x_1+\cdots+x_n)}\right]}{e^{sna}}$$

given by Markov's inequality of,

$$P(x \geq a) \leq \frac{E[x]}{a}$$

Thus

$$= \frac{E\left[e^{sx_1+\cdots+sx_n}\right]}{e^{sna}}$$

$$= E\left[e^{sx_1} \cdots e^{sx_n}\right] / e^{sna}$$

Since all x_i' are i.i.d,

$$= \left(E\left[e^{sx_1}\right]\right)^n / e^{sna}$$

$$= \left(\frac{E[e^{sX_1}]}{e^{sa}} \right)^n$$

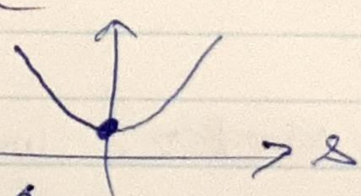
$$= \ell^n.$$

Since we have our $X_i \in [\pm 1]$,

$$\left(\frac{E[e^{sX_1}]}{e^{sa}} \right)^n = \left(\frac{E\left[\frac{1}{2}(e^{s} + e^{-s})\right]}{e^{sa}} \right)^n$$

$$= \left[\frac{\frac{1}{2}(e^{s} + e^{-s})}{e^{sa}} \right]^n$$

On seeing numerator



The denominator

On conclusion, for small 's', $\boxed{\ell < 1}$

If $s = a$,

$$\leq e^{-na^2/2}$$

$$\therefore \; P[X_1 + \cdots + X_n \geq na] \leq e^{-na^2/2}$$

Hence proved.