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Credit Risk Modelling in Application

Catastrophe Swaps and Deep Learning Approaches

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Abstract

Keywords: CAT swaps, deep learning, credit risk, default swaps, Black-Cox model, Merton model, SOM

This work begins with examining various structural models with machine learning extensions to analysing credit risk. This is followed by the applications of credit risk modelling in the form of CAT swaps and CD swaps. The study of catastrophe (CAT) swaps is relatively new in literature and has received very little scholarly attention despite its extensive usage in the financial market. Both pre and ex-ante pricing model for CAT swaps are thoroughly exhumed with a final culmination in fuzzy network modelled deep learning pricing strategies for CAT swaps is provided. Credit Default swaps are extensively analysed and thoroughly explained.

Preface

Credit risk has been a fascinating concept and the subject of innumerable scholarly articles. It is only in the recent past that we have started using advanced methods - with the advent of technology like deep learning and machine learning - to get a more robust structure of pricing and analysing swaps. This work focuses, primarily on how the latter can be used to advantage and on a very prominent but less researched swap - catastrophe swaps - in addition to CDS and its pricing. The thesis is divided into two parts. The first part (up to Chapter 4) deals with the theory and the second part with the applications.
A conceptual introduction and subsequent exposition of credit risks is provided in **Chapter 1** beginning with a literature review which is followed by the essentials of credit risk modelling. **Chapter 2** is a collection of the mathematical tools used in the entirety of the document. No familiarity of concepts is assumed. This chapter lays the basis for **Chapter 3** where structural models of credit risk are discussed in detail. **Chapter 4** is an extension in the assessment of risk using artificial intelligence in portfolio management and credit risk assessment.
Chapter 5 introduces the reader to the Credit Default Swaps (CDS) through a brief introduction and a thorough literature review of all the concepts involved. The subsequent section of this chapter puts in place a formal structure to understand how the modelling and mathematics behind CDS works. Estimation of default probabilities and modelling of default correlations are also discussed in depth. The pricing of CDS using structural models discussed in Chapter 3 is also briefly explained.
Chapter 6 has been devoted to pricing catastrophe (CAT) swaps - an entity that has received close to no scholarly attention. After pricing the model using structural models, a double stochastic model for pricing CAT swaps is discussed in addition to its relation to the CDS model. A machine learning approach to estimating the loss due to natural catastrophe (eg. hurricanes) is explained with an example.
Chapter 7 is a brief discussion about the results and the future possibilities of this line of research. Conclusions regarding the nature of the novel line of approach using AI in credit risks are also discussed.
The author would like to thank his supervisor, **Dr (Prof.) Davide Radi**, for his guidance and recurrent feedback in this thesis from the inception. He recommended the baseline topic and his valuable suggestions regarding the layout have made this thesis possible.

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1. Credit Risk

1.1 Introduction

A swap is a special contract agreement between two parties which results in them agreeing to make regular payments to each other. Different types of swaps result in different agreements regarding currency, interest rate, and even payment timings. In this chapter we attempt to explain the notion of credit risk that surrounds the existence of swaps and the variants of this risk.

This will form the basis for the next chapters in this part which will be dedicated to structural models that estimate credit risks. While we understand that there also exist reduced form (intensity based) approaches to modelling credit risk, we abstain from mentioning these for the sake of brevity.

Robert Fullér[43] has thoroughly discussed the risks associated with swaps. The following segment assumes that one holds a portfolio of swaps and not just a single swap to list out the risks involved in swaps. In the case of a single swap portfolio, the only foreseeable risk is the opportunity risk, i.e., risk arising due to the the possibility of having chosen a different swap. While credit risks are not generally directly observable, many indicators and proxies are used to gauge the risk involved to a great degree of accuracy. The main risks involved with swaps are as follows:

- *Interest rate risk*

  The risk associated with interest rate can be seen a spread risk and a market risk simultaneously. In generic terms this risk emerges due to the inverse nature of the yield and the price of fixed-rate interest bearing debt. This is in turn affects the debt management instrument. This implies that a change in interest rate of the maturity involved will affect the instrument of debt management. A decrease in interest risk might be averse to a dealer whose strategy is based on a steady swap spread. This specific interest rate risk resulting from yield curve movements is referred to as a spread risk[25, 75].

  The market risk term of interest rate evolves during the lifetime of the swap. Consider the beginning of the swap. Its value is zero as no one has profited from it yet; neither has anyone been subjected to its risk against the counter-party. But as soon as it is signed over, the swap contract becomes sensitive to interest rate movements.

- *Currency Exchange Risk*

  This risk arises due to the differences in nominal currencies of the underlying assets of the swaps causing the interest rate risk to be accompanied by currency risk. International transactions often have multiple individuals from various currency regions involved making currency risks more tangible and relevant. The impact of this risk can be dampened or eliminated entirely by using interest rate swaps or hedging using suitable instruments which will lessen the gap between interest rate movements.

- *Credit Default Risk*

  Credit risk is defined as the probability of the counterparty in the swap agreement ceasing to exist, incorporating all the possible financial effects of the elimination.[14]. This insolvency can be due to anything ranging from bankruptcy to a change in the macroeconomic environment. While assessing the total credit risk one must take into account not just the current credit risk but also the dynamics of said credit risk over the time frame in question. This enables to understand the probability of default an arbitrary period of time.

- *Sovereign Risk*

  Sovereign risks arise in cross border interest rate swaps, swaps that have parties from two different countries. It is, so certain degree, a function of the countries' standing in the global market (which in turn is a function of political stability). Generally, sovereign risk is considered as a segment of credit risk except that credit risk is specific to counterparty and sovereign risk is specific to the country of the counterparty's operation. Factors such as polities, national taxes, restrictions affect the investment climate and increase the price of the swap.

- *Mismatch Risk*

  This type of risk evolves around and as a result of differences in notional maturity, swap coupon, floating index, principal, payment frequencies bet ween parallel agreements etc. With an increase in number of agreements comes an increase in complexity. This in turn increases the risk that the treasurer would face in not being able to hedge each possible position taken by way of an identical agreement with an opposite cash flow.

  Mismatch risks become more common in context with credit risks. If a mismatch has occurred as a result of difference in payment dates and if the treasurer thus pays a certain sum of money in quarterly installments as opposed to the counterparty paying on a yearly basis, and the counterparty defaults, the dealer will pay without receiving and will thus lose the interest rate cash flow.

- *Liquidity Risk*

  As the name suggests, liquidity risks arise from the ease of transferring the swap into liquid assets which depends on the secondary market structure and the structure of the swap itself with the independence of these two factors. There is a positive proportionality in the sense that the lesser developed the secondary market is, the harder it will be to find a new counterparty who needs the specific contract. Additionally, the more broader (less tailor-made) the contract is, the easier it will be to dispose of the contract (that is, find a new buyer). This risk is usually measured on a cost-per-time basis - the longer it takes to find a new counterparty, the costlier the agreement gets. In short, liquidity risk is concerned with the ease to exit a contract without incurring a price reduction.

- *Delivery Risk*

  Also called settlement risk, delivery risks usually occur when the two parties are of two different countries. This risk arises owing to the difference in the settlement hours of the two capital markets thereby causing the two agents to effect their payments to each other at different times of the day.

- *Systematic Risk*

  Systematic risks consider the probability of extensive disturbances that could occur even to the extent of the crash of financial institutions. In other words, this risk has its foundation on the nature of panic and loss of confidence that might arise in the system itself or on the status quo as a consequence of rapidly changing reality. The modelling of such a risk is only essential in the theoretical or philosophical framework simply because if in reality such incidents were to happen, they would render the pricing strategy irrelevant.

- *Basis Risk*

  The term basis refers to difference in between two prices. In the case of swaps it is the difference between two floating rate indices. This type of risk can arise due to two different ways. First, assume that a treasurer and a counterparty agree to a floating-floating interest swap to be paid with respect to two different indices. Second, it might occur when matching pair of swaps are paid for using two different indices like FTSE 100 and NASDAQ 100. The risk arises because

> of the different characteristics of the two indices, for instance how they fluctuate independent of each other.

Catastrophe risks are another form of risk which will be dealt with extensively in Chapter 4 through the introduction of CAT swaps. CAT swaps are prone to basis risk and interest rate risks due to the use of index triggers.

1.2 Literature Review

The initial interest in credit risk models was in order to set up a more robust quantitative estimate of capital - namely, economic capital - required to finance a bank's risk undertakings [14]. Some authors like Morris and Shin have devoted sufficient literature to the indicators of an institution's credit risk:

> *Insolvency risk* is the conditional probability of default due to deterioration of asset quality if there is no run by short term creditors. *Total credit risk* is the unconditional probability of default, either because of a (short term) creditor run or (long run) asset insolvency. *Illiquidity* risk" is the difference between the two, i.e., the probability of a default due to a run when the institution would otherwise have been solvent.[empahsis added][83]

Others like Eisenbach have contributed to gauging roll-over risk and a two-side inefficiency[42]. Bouvard *et al* discuss at length a plausible theory for optimal transparency of risk assessment of banks which allows for both investors and the banks to benefit mutually by side-stepping the information disclosure that could signal economic deterioration and subsequent withholding of information by the regulator[19]. At the same time, multiple case studies have been carried out on credit risk management and how they significantly affect the management of banks such as the research carried out by Alshatti[3]. Alshatti's work, although centered around Jordan banks, strongly suggests that credit risk management indicators have significant impacts on the financial performance of the commercial banks. The policy change-over recommendations that Alshatti suggests might border on standard practices, but allow some room for innovative research in the field of financial and corporate law.

Yet another case study, this one on India and its banks, by Parab *et al* approaches credit risk any analysing panel data for about forty banks in the country. By using various proxies for credit risk, such as, Credit Deposit Ratio, Capital Adequacy Ratio, Loan Loss Allowance to Non Performing Assets, Loan Loss Allowance to Assets etc, their model shows "positive and significant relationship between Loan Loss Allowance and all the three performance indicators used in the study"[87].

Various other case studies list out similar results for banks in coutnries such as Barbados, Pakistan, Malaysia and others[64, 99, 105, 106]. The non-scholarly literature on credit risk is extensive especially in the post banking crisis era of the years after 2009; needless to say, mostly negative. While the criticism makes one wonder, very little is usually spoken of the intricate mechanism of the risk itself. Multiple studies and texts have focused on studying credit risk and in managing it [37, 70]. There has also been recent advances, in negligible steps, to using machine learning to assess credit risk. Khandani[1], for example, uses ML to construct a non-linear parametric forecast of for consumer credit risk using consumer transactions and credit bureau data[65].

To be able to set the field to understand structural models of credit risk, it is imperative that groundwork of mathematical rigor be laid. In this interest, we take a mathematical digression to establish a few concepts that will be useful in both the following chapters of this part and in the second part of the thesis.

[1] See also Kruppa, Bao, and Wang [11, 67, 77].

2. Mathematical Pre-requisites

This chapter is a link between the theory of credit risk and its mathematical modelling in the subsequent chapters.

2.1 Stochastic Theory

Definition 2.1.1 A **stochastic process** is a sequence of events in which the outcome of any stage depends on a probability

A continuous time stochastic process $\{B(t) : t \geq 0\}$ that describes any macroscopic feature of a random walk has to satisfy the following properties:

1. $B(t_n) - B(t_{n-1}), B(t_{n-1}) - B(t_{n-2}), B(t_2) - B(t_1)$ are independent $\forall\ 0 \leq t_1 \leq t_2 \leq \ldots \leq t_n$. In other words, the process has independent increments.
2. The process has stationary increments, that is, the distribution of any arbitrary increment $B(t+h) - B(t)$ is independent of time
3. The process $\{B(t) : t \geq 0\}$ has almost surely continuous paths

Let us introduce Fatou's Lemma, which will be useful in some proofs for martingales.

Lemma 2.1.1 Fatou's Lemma If $X_n \geq 0$, then it is that $E[\liminf X_n] \leq \liminf E[X_n]$

2.1.1 Markov Process

Let us define the Markov process and then proceed to formalise the definition

Definition 2.1.2 A **Markov process** is a stochastic process that has the following properties

1. There are a finite number of possible outcomes (states).
2. The outcome of any stage depends only on the outcome of the previous stage.
3. The probabilities are constant over time.

Let x_0 be the vector that represents the initial state of a system that follows Markovian rules. Then the matrix M exists such that the state of the system after one time-step (or iteration) is given by Mx_0. In general the state of a system at $t = n$ is given by $M^n x_0$. This matrix is called the **transition matrix**.

2.2 Martingale Measure

Assume a probability space given by $(\Omega, \mathscr{F}, P)$. Unless otherwise stated, a random variable X means it is $\mathscr{F}$-measurable. At this stage we call a sequence of RVs $(S_n)_{n=0}^{\infty}$ as a stochastic process which is bounded if there is a $c \in \mathbb{R}$ such that $|S_n(\omega)| \leq c \;\; \forall n, \omega$.

2.2.1 Filtrations

Definition 2.2.1 A countable set of σ-fields $(\mathscr{F}_{n=0}^{\infty})$ is a filtration if $\mathscr{F}_0 \subset \mathscr{F}_1 \ldots \subset \mathscr{F}$.

Definition 2.2.2 A stochastic process $X = (X_n)$ is considered adapted to the filtration given by $(\mathscr{F}_n)$ if X_n is $\mathscr{F}_n$ measurable for all n.

Definition 2.2.3 A process $M = (M_n)_{n=0}^{\infty}$ is a Martingale if

1. (M_n) is adapted,
2. $M_n \in L^1 \forall n$,
3. $\mathbb{E}\left[M_{n+1} | \mathscr{F}_{n-1}\right] = M_n \forall n$ almost surely

If 3 does not hold with equality but $\mathbb{E}\left[M_{n+1} | \mathscr{F}_{n-1}\right] \geq M_{n-1} \forall n$, then M is a submartingale. It is a supermartingale if $\mathbb{E}\left[M_{n+1} | \mathscr{F}_{n-1}\right] \leq M_n \forall n$ almost surely.

We can better understand martingales from an example.

■ **Example 2.1** Define $\mathscr{F} = \sigma(X_1, X_2, \ldots X_n)$.If (X_n) is a sequence of i.i.d so that $\mathbb{E}[X_n] = 1 \forall n$ and we are able to find $c \in \mathbb{R}$ such that $|X_n| \le c \forall n$, then

$$S_n = \prod_{i=1}^{n} X_n$$

is a martingale. ■

Submartingales, on average, increase. Supermartingales on the other hand decrease. The proof of this is beyond the scope of this introductory mathematical digression.

2.2.2 Martingale Convergence

We prove an almost sure convergence of supermartingales[1], M_n, as $n \to \infty$. Assume (M_n) to be a stochastic process and with $a < b$ we can define $U_N[a,b]$ as the upcrossing numbers made in the interval $[a,b]$ by $M_1, \ldots, M_n$ and that means $U_N[a,b]$ is the largest κ such that there exists

$$0 \le s_1 < t_2 < \ldots < s_\kappa < t_\kappa \le N$$

such that $M_{s_i} \le a$, $M_{t_i} > b$ for all $i = 1, \ldots, \kappa$.

Lemma 2.2.1 **(Doob's Upcrossing Lemma)** Let M be a supermartingale. Then

$$(b-a)\mathbb{E}[U_N[a,b]] \le \mathbb{E}\left[(M_N - a)^-\right]$$

Lemma 2.2.2 If M is a supermatingale that is bounded in L^1 then $P[U_\infty[a,b] = \infty] = 1$

The proofs for these lemmas are given in the appendix of this document. Below we prove the first

[1]Martingales are a special case of this and can be handled by multiplying throughout by -1.

convergence theorem[2].

Theorem 2.2.1 First Martingale Convergence Theorem Assume M to be a supermartingale that is bounded in L^1. Then $M_\infty = \lim_{n\to\infty} M_n$ is an almost sure limit and exists. Further, $P[|M_\infty| < \infty] = 1$

Proof. Begin by defining

$$\Lambda_{a,b} = \{\omega : \liminf_n M_n(\omega) < a < b < \limsup_n M_n(\omega)\} \tag{2.1}$$

It is clear that $\Lambda_{a,b} \subset \{U_\infty[a,b] = \infty\}$ which has null probability from the last lemma. Since we have that

$$\{\omega : M_n(\omega) \text{ does not converge in } [-\infty,\infty]\} = \bigcup_{a,b\in\mathbb{Q}} \Lambda_{a,b} \tag{2.2}$$

we have almost surely $\liminf_n M_n = \limsup_n M_n$. Therefore,

$$P[M_n \text{ converges to some } M_\infty \in [-\infty,\infty]] = 1 \tag{2.3}$$

This and Fatou's lemma give us the result that

$$\mathbb{E}\left[|M_\infty|\right] = \mathbb{E}\left[\lim_{n\to\infty} \inf |M_n|\right] \leq \lim_{n\to\infty} \inf \mathbb{E}[|M_n|] \leq \sup_n \mathbb{E}[|M_n|] < \infty \tag{2.4}$$

which implies that $P[|M_\infty| < \infty] = 1$ and proves the theorem. ■

2.3 Geometric Brownian Motion

The interest in Brownian motion (BM) originated from the movement of coal particles suspended in a solution and later on from the mathematical formulation that led Einstein to informally conclude that

[2] It is interesting to note that $(C \cdot M)$ can be considered to be the stochastic integral $\int CdM$ and being a discrete analogue of the Riemann integral if M were a deterministic sequence.

atoms consisted of charged particles. Brownian motion is a macroscopic phenomenon emerging from the microscopic random movement of a particle in an n-dimensional space.

2.3.1 Lognormal Distributions

Assume $Y \sim N(\mu,\sigma^2)$, then $X = e^Y$ is a strictly positive random variable (RV) which is log-normally distributed - denoted as $lognorm(\mu,\sigma^2)$. X has the following distribution

$$f(x) = \begin{cases} \frac{1}{\sqrt{2\pi x^2\sigma^2}} e^{-\frac{(\ln(x)-\mu)^2}{2\sigma^2}} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \tag{2.5}$$

2.3.2 The GBM

A plain BM is not used for financial modelling (among other reasons) since the phase-space of the BM also includes negative values. Thus a transformation is employed - the Geometric Brownian Motion (GBM). A GBM is defined as $S(t)$ such that

$$S(t) = S_0 e^{X(t)} \tag{2.6}$$

with $S(0) = S_0 > 0$ and where $X(t)$ is the generic BM with drift (μ) given by

$$X(t) = \sigma B(t) + \mu t \tag{2.7}$$

Notice that the log-reduced form of the GBM takes us back to a BM: $X(t) = \ln(S(t)/S_0) \implies \ln(S(t)) = X(t) + \ln(S_0)$ which is normal with mean $\mu t + \ln(S_0)$ and variance $\sigma^2 t$. In other words, for each t, $S(t)$ is log-normally distributed.

Notice that for $t_0 < t_1 < \ldots < t_n = T$ the ratios of $S(t_{i+1})$ to $S(t_i)$ for $1 \leq i \leq n$ are independent RVs that are lognormal[3]. Define this ratio as $L_i \stackrel{def}{=} S(t_i)/S(t_{i-1})$.

■ **Example 2.2** Consider the two ratios

$$L_1 \stackrel{def}{=} \frac{S(t_1)}{S(t_0)} = e^{X(t_1)}$$

$$L_2 \stackrel{def}{=} \frac{S(t_2)}{S(t_1)} = e^{X(t_2)-X(t_1)}$$

They are independent and follow the log-normal distribution due to the face that the BM has independent increments according to the normal distribution; $X(t_1)$ and $X(t_2) - X(t_1)$ are normally distributed *i.i.d.*. However, the differences $S(t_1) - S(t_0)$ and $S(t_2) - S(t_1)$ are not independent. ■

Thus we can write the GBM as

$$S(t) = S_0 L_1 L_2 \dots L_n \tag{2.8}$$

as a product of the independent n log-normal RVs.

2.3.3 GBM as a Markov Process

Proposition 2.3.1 The future GBM state is independent of the present GBM state.

This means that $S(t+h)$ is independent of $\{S(k) : 0 \leq k < t\}$ given $S(t)$, or, the future state h steps from t is independent of the states in the past of t. This can be proven as follows

[3] In practical terms this implies that the percentage changes in the modelled entity, say stock prices, are independent. The changes themselves $(S(t_{i+1}) - S(t_i))$ are not.

Proof. Note that

$$
\begin{aligned}
S(t+h) &= S_0 e^{X(t+h)} \\
&= S_0 e^{X(t+h)+X(t)-X(t)} \\
&= S_0 e^{X(t)} e^{X(t+h)-X(t)} \\
&= S(t) e^{X(t+h)-X(t)}
\end{aligned}
$$

Since the BM has independent increments and as shown above $S(t+h)$ only depends on the future increment of the BM given by[4] $X(t+h)-X(t)$, we get the Markov property. ■

2.3.4 Moments of GBM

The moment generating function for a normally distributed RV $X \sim N(\mu, \sigma^2)$ is

$$M_X(s) = \mathbb{E}(e^{sX}) = e^{\mu s + \frac{\sigma^2 s^2}{2}} \qquad -\infty < s < \infty$$

This translates as following for a BM with drift as

$$M_{X(t)}(s) = \mathbb{E}(e^{sX(t)}) = e^{\mu ts + \frac{\sigma^2 ts^2}{2}}$$

since $X(t) \sim N(\mu t, \sigma^2 t)$.

We can now compute the moments for the GBM by using $s = 1, 2, \ldots$ to give

$$\mathbb{E}[S(t)] = S_0 e^{(\mu + \frac{\sigma^2}{2})t} \tag{2.9a}$$

$$\mathbb{E}[S^2(t)] = S_0^2 e^{2\mu t + 2\sigma^2 t} \tag{2.9b}$$

[4] It is interesting to note that $\{X(t+h)-X(t) : h > 0\}$ is also a BM with the same drift and variance. That is, given the present time GBM $S(t)$ the future process defines the same BM except with a new starting point given by $S(t)$

$$\mathrm{Var}[S(t)] = S_0^2 e^{2\mu t + \sigma^2 t}(e^{\sigma^2 t} - 1) \tag{2.9c}$$

The moments of any of the ratios $S(a)/S(b)$ can also be found in a similar manner. The generic form of the two is given by

$$\mathbb{E}[S(t)] = S_0 e^{\bar{r}t} \tag{2.10a}$$

$$\mathbb{E}\left[\frac{S(a)}{S(b)}\right] = e^{\bar{r}(a-b)} \tag{2.10b}$$

where $\bar{r} = \mu + \frac{\sigma^2}{2}$.

2.4 Itô Calculus

The BM is not a differentiable process, i.e., for a B< B_t, $\frac{dB_t}{dt}$ does not exist. However, when using BM in models, it is often the case that we attempt to find the difference of a function of type $f(B_t)$ over some infinitesimal time difference. Thus we are interested in $f(t, B_t)$ which has a implicit dependence on time since the BM has a dependence on time[5].

2.4.1 Calculus

Let us attempt to write the difference df in terms of the difference dB_t. Consider the Taylor expansion of f:

$$\delta f = f(x+\Delta x) - f(x) = (\Delta x)f'(x) + \frac{(\Delta x)^2}{2!}f''(x) + \frac{(\Delta x)^3}{3!}f'''(x) + \dots \tag{2.11}$$

For $x = B_t$ we see that

$$\Delta f = (\Delta B_t).f'(B_t) + \frac{(\Delta B_t)^2}{2!}f''(x) + \frac{(\Delta B_t)^3}{3!}f'''(x) + \dots \tag{2.12}$$

[5]If $\frac{dB_t}{dt}$ did exist, then it would be trivial to note that

$$df = \left(\frac{dB_t}{dt}f'(B_t)\right)dt$$

Consider the seconds term, that is $(\Delta B_t)^2$. B_t being a BM, it is that $E[(\Delta B_t)^2] = \Delta t$. The fundamental of Itô calculus tells us that we can make the substitution $(\Delta B_t)^2 = \Delta t$ and consider the other terms to be negligible. Thus the above expression reduces to

$$\Delta f = (\Delta B_t).f'(B_t) + \frac{\Delta t}{2} f''(x) \tag{2.13}$$

And that gives us

$$df(B_t) = f'(B_t)dB_t + \frac{1}{2} f''(B_t)dt \tag{2.14}$$

This final equation is called Itô Lemma and forms the core of Itô calculus.

Theorem 2.4.1 **(Itô's Lemma)** Let $f(t,x)$ be a smooth function of two variables and X_t be a stochastic process that satisfies $dX_t = \mu_t dt + \sigma_t dt$ for some BM B_t, then

$$df(t,X_t) = \left(\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial x} + \frac{1}{2}\sigma_t^2 \frac{\partial f}{\partial x^2} \right) dt + \frac{\partial f}{\partial x} dB_t \tag{2.15}$$

Proof. We have

$$\begin{aligned} df(t,X_t) &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX_t + \frac{\partial^2 f}{\partial x^2}(dX_t)^2 \\ &= \left(\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial x} + \frac{1}{2}\sigma_t^2 \frac{\partial^2 f}{\partial x^2} \right) dt + \sigma_t \frac{\partial f}{\partial x} dB_t + O(dtdB_t, dt^2) \end{aligned}$$

Ignoring the terms of order $dtdB_t$ and dt^2, we arrive at the lemma. ■

Definition 2.4.1 We define stochastic integration as the inverse of the differentiation as

$$F(t,B_t) = \int f(t,B_t)dB_t + \int g(t,B_t)dt \tag{2.16}$$

iff

$$dF = f(t,B_t)dB_t + g(t,B_t)dt$$

Let us consider a few examples.

■ **Example 2.3** Consider the BM with constant drift μ and constant variance σ given by $X_t = \mu t + \sigma B_t$. For this we have

$$dX_t = \mu dt + \sigma dB_t$$

■

■ **Example 2.4** Consider the function $f(x) = \frac{1}{2}x^2$. It is that

$$df(B_t) = B_t dB_t + \frac{1}{2}dt$$

On an equivalent note, we have

$$\frac{1}{2}B_T^2 = \int_0^T B_t dB_t + \int_0^T \frac{1}{2}dt = \int_0^T B_t dB_t + \frac{T}{2}$$

which implies that

$$\int_0^T B_t dB_t = \frac{1}{2}B_T^2 - \frac{T}{2}$$

■

■ **Example 2.5** Let $f(t,x) = \exp(\mu t + \sigma x)$. Then, we have,

$$df(t,B_t) = (\mu + \frac{1}{2}\sigma^2)f(t,B_t)dt + \sigma f(t,B_t)dB_t$$

Let us find the stochastic process, $X_t(t,B_t)$ such that

$$dX_t = \sigma X_t \cdot dB_t$$

This can be done by setting $\mu = -\frac{1}{2}\sigma^2$ in the function to give us $X_t(t,B_t) = \exp(-\frac{1}{2}\sigma^2 t + \sigma B_t)$ ■

■ **Example 2.6** Let $f(t,x) = t^2 + x^2$ and let $X_t = \mu t + \sigma B_t$. Then

$$df(t,X_t) = 2tdt + 2X_t dX_t + (dX_t)^2 = 2tdt + 2(\mu dt + \sigma dB_t) + \sigma^2 dt = (2t + 2\mu + \sigma^2)dt + 2\sigma dB_t$$

■

R There are multiple ways to implement and understand BM using Itô calculus, i.e., many versions of it. Another commonly used form is

$$df = f'(B_t)dB_t - \frac{1}{2}f''(B_t)dt \tag{2.17}$$

But the Itô integral form is the most intuitive one with the context of how the time variable fits into the application based on the fact that the future is unforeseeable is the basis of the whole theory.

2.4.2 Properties

Theorem 2.4.2 Let $B(t)$ be a BM, and let $\Delta(t)$ be a function of time that is nonrandom. Assume a stochastic process, $I(t)$, that satisfies

$$dI = \Delta(s)dB_s \quad \text{i.e.} \quad I(t) = \int \Delta(s)dB_S$$

where, I(0) = 0. Then, for each $t \geq 0$, $I(t)$ is distributed normally.

We next ask the question, what happens when we allow $\Delta(t)$ to be a random function of time?

Definition 2.4.2 Let X_t be a stochastic process. A process, Δ_t is called an **adapted process** with respect to X_t if, for all $t \geq 0$, the RV Δ_t depends only on X_s for $s \leq t$.

Consider the following examples to understand this.

■ **Example 2.7** Let X_t be a stochastic process.

1. $\Delta(t) = X_t$ is an adapted process.
2. The process $\Delta(t) = \min\{X_t, c\}$, where c is a constant, is an adapted process.
3. $\Delta(t) = \max_{\{0 \le t \le T\}} X_t$ is **not** an adapted process.
4. Given a stopping time τ, X_τ is an adapted process.

■

Next, we introduce another theorem that asserts that for a BM B_t, the Itô integral of an adapted process with respect to this BM is also a Martingale process.

Theorem 2.4.3 Let B_t be a BM. Then for all adapted processes given by $g(t, B_t)$, the integral

$$\int g(t, B_t) dB_s$$

is a Martingale if $g \in L^2$. That is

$$\int \int g^2(t, B_t) dt dB_t < \infty$$

■ **Example 2.8** The process B_t is itself and *adapted process*. To see this, note that

$$\int_0^t B_s dB_S = \frac{1}{2} B_t^2 - \frac{t}{2}$$

and $\mathbb{E}[B_t^2] = t$. Thus,

$$\mathbb{E}\left[\int_0^t B_s dB_s\right] = 0$$

Or, more generally,

$$\begin{aligned}\mathbb{E}\left[\int_{t_1}^{t_2} B_s dB_s|\mathscr{F}_{t_1}\right] &= \mathbb{E}\left[\left(\frac{1}{2}B_{t_2}^2 - \frac{t_2}{2}\right)|\mathscr{F}_{t_1}\right] - \left(\frac{1}{2}B_{t_1}^2 - \frac{t_1}{2}\right) \\ &= \frac{1}{2}(t_2 - t_1) + \frac{1}{2}B_{t_1}^2 - \frac{t_2}{2} - \frac{1}{2}B_{t_1}^2 + \frac{t_1}{2} = 0\end{aligned}$$

■

2.5 Options and Greeks

2.5.1 Options

An option gives the holder the right, but not an obligation, to sell (buy) the underlying asset of a call(put) at a specific price before the date of maturity. Options can provide added income or leverage and thus enhance a portfolio.

2.5.2 Greeks

The Greeks are a measure of the sensitivity of the trading option to various factors. Since the options are always convex with respect to to some of their parameters, the linear approximation of these always lie below the actual option prices. In mathematical wordings, for an option V, the Taylor expansion yields:

$$\Delta V = \frac{\partial V}{\partial S}\Delta S + \frac{\partial V}{\partial \sigma}\Delta\sigma + \frac{\partial V}{\partial t}\Delta t + \frac{\partial V}{\partial r}\Delta r + \frac{1}{2}\frac{\partial^2 V}{\partial^2 S}(\Delta S)^2 + O(\Delta^2) \tag{2.18}$$

Table 2.1 summarizes the main Greeks and Figure 2.1 gives the relationships between Greeks and long/short call/put options.

Greek	Symbol	Measures	Definition
Delta	$\Delta = \frac{\partial V}{\partial S}$	Equity exposure	Change in option pricing due to spot
Vega	$\nu = \frac{\partial V}{\partial \sigma}$	Volatility exposure	Change in option pricing due to volatility
Theta	$\Theta = \frac{\partial V}{\partial t}$	Time decay	Change in option pricing due to time
Gamma	$\Gamma = \frac{\partial^2 V}{\partial^2 S}$	Payout convexity	Change in delta due to spot
Rho	$\rho = \frac{\partial V}{\partial r}$	Interest rate exposure	Change in option pricing due to interest rates

Table 2.1: Summary table of Greeks

GREEKS /LONG|SHORT	Delta	Vega	Theta	Gamma	Rho
Long call	+	+	-	+	+
Short call	-	-	+	-	+
Long put	-	+	-	+	-
Short put	+	-	+	-	+

Figure 2.1: Greeks for Options

2.6 K-Means Clustering

The K-means is used in categorization by means of vector quantization[6]. It takes n observations and classifies them into k Voronoi cells, or 'clusters'. A numerical solution of the K-means clustering is considered computationally hard and is of the order of NP-Hard. In the unsupervised learning models, the K-means is similar to the expectation maximization algorithm. Below, we briefly discuss the naive K-means algorithm.

2.6.1 Naïve K-means Algorithm

The simplest algorithm used is the iterative refinement which is often, owing to its pervasive nature in literature, is simply called the K-means algorithm[7]. At starting time $t = 0$, given a set of k means - $\mu_1, \mu_2, \ldots, \mu_k$ it iterates two steps:

1. **Assignment:** Each observation is assigned to the cluster with the mean that is closest to its value

[6] The notion of the K-means comes from signal processing where it was first used.
[7] Note, however, that this is not the fastest algorithm.

by considering the least Euclidean distance.

$$S_i^{(t)} = \{x_p : \|x_p - \mu_i^{(t)}\|^2 \leq \|x_p - \mu_j^{(t)}\|^2 \forall j, 1 \leq j \leq k\} \tag{2.19}$$

where each each of the n entries, x_p are assigned only one $S^{(t)}$ even if there might occur overlaps.

2. **Update:** The means are recalculated for observations belonging to each cluster.

$$\mu_i^{(t)} = \frac{1}{|S_i^{(t)}|} \sum_{x_j \in S_i^{(t)}} x_j \tag{2.20}$$

Convergence is achieved at some t when there is no change between $\mu^{(t)}$ and $\mu^{(t-1)}$.

2.6.2 K-Means and Feature Learning

Feature learning is an integral part of unsupervised learning where the data is fed into the ML model and it finds patterns in the data. K-means clustering has been extensively used in this field (we will employ K-means clustering in estimating credit defaults among bank customers in Chapter 4). Unlabelled data can be categorized by the K-means clustering by projecting the input related datum into feature space via an encoder, indicator, or distance transform.

2.7 Replicating Portfolio and Black Scholes

In the original paper by Black-Scholes (BS), the BS partial differential equation (PDE) is derived using hedging arguments. A generalisation of this approach is the derivation of the PDE using a replicating portfolio, that is, one can express any derivative with a a unique self-financing strategy by using a risk-neutral measure to calculate its value. We assume a few instruments

- Riskless bond B evolving by $dB = rBdt$ where r is the risk-free interest rate.
- An underlying security which evolves by the Itô process $dS = \mu S dt + \sigma S dW$.

- An option V on the underlying security S which evolves based on Itô's lemma as

$$dV = \left(\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S}\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}dt\right)dt + \left(\sigma S\frac{\partial V}{\partial S}\right)dW \tag{2.21}$$

2.7.1 The Trading Strategy

Assume N assets which, at time t, have values $Z_1(t), Z_2(t), \dots, Z_N(t)$.

Definition 2.7.1 A **trading strategy** is an N-dimensional stochastic process, $a_1(t), \dots, a_N(t)$ which represents the allocation, at time t, into the assets.

For a portfolio, the t-time value is given by $\Pi(t) = \sum_{i=1}^{N} a_i(t)Z_i(t)$.

Definition 2.7.2 A trading strategy is called self-financing if the change of value of the portfolio is only due to a change in the value of assets and independent of inflows or outflows. That is, it is self-financing $\iff$:

$$d\Pi(t) = d\left(\sum_{i=i}^{N} a_i(t)Z_i(t)\right) = \sum_{i=1}^{N} a_i(t)dZ_i(t) \tag{2.22}$$

which can be rewritten, explicitly, in terms of the portfolio as

$$\Pi(t) = \Pi(0) + \sum_{i=1}^{N}\left(\int_0^t a_i(u)dZ_i(u)\right) \tag{2.23}$$

2.7.2 Arbitrage Opportunity

If an arbitrage opportunity on the portfolio has the following properties, it is a self-financing strategy:

$$\Pi(t) \leq 0$$

$$P\left[\Pi(T) > 0\right] = 1$$

In words, it means that the value of the portfolio is non positive for all times except at maturity when it is positive with complete certainty. In other terms, even if one starts with debt, at some future instant, one would have a non-funded wealth.

2.7.3 Derivation of BS PDE

The risky asset determines the payoff, $V(T)$, of a derivative at time T. We need to prevent self-arbitrage and so we set $\Pi(T) = V(T)$, that is, identify a self-financing trading strategy which will produce the same payoff as the derivative in question. Thus we have a replicating strategy and a replicating portfolio; a replicating strategy exists if the the economy is complete (all the derivative is attainable). Pricing by arbitrage is finding the value of a derivative using a replicative portfolio.

We begin by forming a self-financing portfolio which contains the right proportion of the stock S and bond B. So we use a replicating strategy $(\alpha(t), \beta(t))$ to give us the replicating portfolio given by $V(t) = \alpha(t)S(t) + \beta(t)B(t)$ to thereby find the value $(\alpha(t), \beta(t))$. Since we have a self-financing assumption, we have (dropping the time for ease of writing)

$$dV = \alpha dS + \beta dB \tag{2.24}$$

From EQ 2.21 for dV and dB and dS we get

$$\left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} dt\right) dt + \left(\sigma S \frac{\partial V}{\partial S}\right) dW = (\alpha \mu S + brB)\, dt + \alpha \sigma S dW \tag{2.25}$$

This implies that $\alpha = \frac{\partial V}{\partial S}$ by comparing coefficients. This in turn, gives us

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = brB \tag{2.26}$$

$$= br\left(\frac{V - aS}{b}\right) \tag{2.27}$$

$$= rV - rS\frac{\partial V}{\partial S} \tag{2.28}$$

On rearranging this, we get

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{2.29}$$

which is the Black Scholes PDE for a general option V.

2.8 Cox Processes

A Cox process is a generalisation of Compound Poisson process (and thus, sometimes also called a doubly stochastic Poisson process) and have widespread use in financial mathematics. It is a continuous time stochastic process. The intensity measure itself, is a random process governed by the Poisson distribution. The simulation would first require generating a Poisson process for the intensity λ_t and then generating a process based on this intensity.

Definition 2.8.1 Define a probability space with filtration $\mathscr{F}$ to be $(\Omega, \mathscr{F}, P)$. Let N_1 be a point process adopted to σ_t algebra. Assume ζ_t is a non-negative process which is σ-measurable with $t \geq 0$ and it is that $\int_0^t \zeta_m dm < \infty$ almost surely. N_t is a Cox (σ_t-doubly stochastic Poisson) process with intensity ζ_t if $\forall\ 0 \leq t_1 \leq t_2$ and $u \in \mathbb{R}$ we have

$$\mathbb{E}\left[e^{iu(N_{t_2}-N_{t_1})}|\sigma_{t_2}\right] = \exp\left[(e^{iu}-1)\int_{t_1}^{t_2} \zeta_m dm\right] \tag{2.30}$$

There are other ways to define a Cox process, namely using a Laplace transform of a Poisson process. Notice that Eq 2.30 is an implicit expression of the transform which we abstain from formally introducing for ease of readability.

3. Structural Models of Credit Risk

We can sum up the previous chapter and state that credit risk is a measure of the probability of monetary (or financial) loss owing to the change in the quality of credit of the market players. A major reason for the change of credit is the event called the default event which is usually predefined in the contract. For most large cap firms, this occurs when there is a failure to fulfil the debt obligations. In other words, the default is a singular no return event which results in the form ceasing to exist causing financial losses, often very large, to some of the security holders.

Structural models[1] consider defaulting as an event that occurs when the firm's assets cross its liability threshold negatively. It requires a good deal of assumptions regarding the firm asset dynamics, debt and capital structure to provide an intuitive and endogenous explanation for default. This section lists a few existing structural models and their advantages and disadvantages.

3.1 Merton Model

Developed in the early 1970 and established in print in 1974, the Merton model consists of an overtly simplified debt structure. It assumes a frictionless market and that the dynamics of the riskless asset is

[1] There also exist intensity-based approaches (reduced form approaches) to calculating the probability of default such as the Jarrow-Turnbull model(see [48] *et al*).

given by

$$dB_t = rB_t dt \tag{3.1}$$

where r is the short term interest rate. This simplification means that there are no transaction costs and there is an absence of taxes. Furthermore, it mandates that there are no restrictions on short-selling and borrowing while assets are continuously traded and perfectly divisible[2].

Then the asset dynamics of the firm follows the GBM with the physical measure given by:

$$dA_t = \mu A_t dt + \sigma A_t dW_t \tag{3.2}$$

where μ is average rate of return on the asset and σ is the volatility of the asset. W_t is a Brownian motion. In the Merton model, the debt is a single outstanding bond of face value K and maturity T. At T, if the asset's total value exceeds the debt (is higher than the debt), then the debt is paid in full and the rest of the amount is distributed among the shareholders. On the other hand, if $A_T < K$, default occurs and the bondholders exercise their right to liquidate the company and receive the liquidation value (which in this model is also the firm value).

3.1.1 Risk Structure and Yield Spread

In the case of a default, shareholders do not get anything but are also not required to pay anything extra in lieu of the principle of limited liability. Thus, shareholders have a cash-flow governed by, say E_T

$$E_T = (A_T - K)^+ = \max\{A_T - K, 0\} \tag{3.3}$$

[2]This is also the same as saying that there are no bankruptcy charges (liquidation value equals firm value) and that debt and equity are frictionless tradeable assets.

that is, we can consider equity as a European call option with strike K on the assets that the firm holds. The debtholders (bondholders), by a similar argument, receives, say D_T:

$$D_T = \min(A_T, K) = K - \max\{K - A_T, 0\} = L + \min\{A_T - K, 0\} \tag{3.4}$$

It is trivial that $E_T + D_T = A_T$ and thus one can derive one claim value simply by using the additivity of payoffs.

To price these, we can continue with the assumption of the Black-Scholes model for call and put option pricing and solving the replicating portfolio PDE[3]:

$$\begin{aligned} D_t &= Ke^{-r(T-t)} - P_t(A_t, K, T-t) \\ &= Ke^{-r(T-t)} - [Ke^{-r(T-t)}N(-d_2) - A_T N(-d_1)] \\ &= Ke^{-r(T-t)}N(d_2) + A_T N(-d_1) \end{aligned} \tag{3.5}$$

where

$$d_1 = \frac{\frac{\ln(A_t)}{K} + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \tag{3.6a}$$

$$d_2 = \frac{\frac{\ln(A_t)}{K} + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \tag{3.6b}$$

where P_t is the put price. We can assess the effects of various parameters by gauging the Greeks of the call option if we take into account the isomoprhic relationship between levered equity and the call option, which in this case will be $D_t = A_t - E_t$ and E_t is a call option. Thus we have the following relationships

$$\frac{\partial D_t}{\partial A_t} = 1 - \frac{\partial E_t}{\partial A_t} = 1 - N(d_1) \geq 0 \tag{3.7a}$$

$$\frac{\partial D_t}{\partial (T-t)} = -\frac{\partial E_t}{\partial (T-t)} < 0 \tag{3.7b}$$

[3]We solve it by considering it to be the heat equation and arriving at analytic solutions

$$\frac{\partial D_t}{\partial r} = -\frac{\partial E_t}{\partial r} < 0 \tag{3.7c}$$

$$\frac{\partial D_t}{\partial \sigma^2} = -\frac{\partial E_t}{\partial \sigma^2} < 0 \tag{3.7d}$$

$$\frac{\partial D_t}{\partial K} = -\frac{\partial E_t}{\partial K} > 0 \tag{3.7e}$$

Thus, we can summarise by writing debt value as a function $D_t \doteq D(A_t^+, K^+, \sigma^-, r^-)$ where the superscript corresponds to the sign of the derivative we arrived at in Eq(3.7). The yield to maturity, given by the solution of $D_t = Ke^{-y(T-t)}$ is given by

$$y_{t,T} = -\frac{\ln\left(\frac{D_t}{K}\right)}{T-t} \tag{3.8}$$

Thus, a zero coupon defaultable bond with unit face value and maturity at T will have Yield Spread

$$\begin{aligned} s_{t,T} &= y_{t,T} - r \\ &= -\frac{1}{T-t}\ln\left(\frac{Ke^{-r(T-t)}N(d_2) + A_t N(-d_1)}{K}\right) - r \\ &= -\frac{1}{T-t}\ln\left(N(d_2) + \frac{A_t}{Ke^{-r(T-t)}}N(d_2)\right) \\ &= f\left(d, \sigma, T-t\right) \end{aligned} \tag{3.9}$$

where the second term in side the logarithm is the inverse of the quasi-debt ratio[4]. The last step of Eq(3.9) comes from the fact that d_1 and d_2 can be written as functions of d.

3.1.2 Recovery and Probability of Default

Rewriting the debt price in a way so that it reflects the recovery rate and the default probability is a useful way to glean information from it. Considering debt value as the difference between a safe claim

[4] See Gibson *et al*[47].

$(Ke^{-r(T-t)})$ and a put option we get

$$\begin{aligned} D_t &= Ke^{-r(T-t)} - \left[Ke^{-r(T-t)}N(-d_2) - A_tN(-d_2)\right] \\ &= Ke^{-r(T-t)}\left[1 - N(-d_2)\left(1 - \frac{A_t}{Ke^{-r(T-t)}}\frac{N(-d_1)}{N(-d_2)}\right)\right] \end{aligned} \tag{3.10}$$

From Eq(3.5) we arrive at

$$\begin{aligned} D_t = Ke^{-r(T-t)} &\left[1 - Q(A_T < K)\left(1 - \frac{E_Q A_T \mathbf{1}_{\{A_T<K\}} e^{-r(T-t)}}{Ke^{-r(T-t)}Q(A_T < K)}\right)\right] \\ = Ke^{-r(T-t)} &\left[1 - p*(1 - E_Q(\delta_T | A_T < K))\right] \end{aligned} \tag{3.11}$$

where we denote the fractional recovery $\delta_T \equiv \frac{A_T}{K}$ and the probability of default as $p* \equiv Q(A_T < L)$ and use the conditional expectation property[5] to arrive at the last line.

3.1.3 Drawbacks

As mentioned earlier, the Merton model is an oversimplified model and is far from ideal. Some of the shortcomings are

- The stationarity of the debt structure does not allow for default to occur on a date other than the fixed date. Efforts have been made, however, to add this - the most notable is the Geske-Johnson model [24, 63]. Black-Cox also address this issue [16].
- The model incorrectly assumes that firm value is tradeable when in reality it is not directly observed.
- Interest rates must be considered to be stochastic. This is taken into consideration in the original paper by Merton [81] and he provides a generalisation.
- As opposed to observations, the calibrated Merton model contains the short end of the yield spread stuck close to zero for a long period of time.

[5] Namely, for a RV X, $E_p[X|B] = \frac{E_p[X1_{\{B\}}]}{P(B)}$

3.2 Black-Cox Model

Black-Cox model, instead of just admitting default at T, allows postulates that default occurs the first time when the value of firm assets drops below a time dependent barrier - $K(t)$. The default time is then given by

$$\tau = \inf\{t > 0 : A_t < K(t)\} \tag{3.12}$$

With the constraint that $K(t) \leq K \; \forall \; t$ and $K_t \leq K$ (for consistency of default definition). One way to enforce this is by taking this barrier to be an increasing function of time such as (not restricted to) $K(t) = K_0 e^{kt}, K_0 \leq Ke{-}kT$.

We are in now a position to write the risk neutral probability of defaulting before $t \leq T$, but first, we notice that

$$\{A_t < K(t)\} = \{W_t + \sigma^{-1}\left(r - \frac{\sigma^2}{2} - k\right)t < \sigma^{-1}\log\left(\frac{K_0}{A_0}\right)\}$$

From this it follows that

$$\begin{aligned} Q[0 \leq \tau < T] &= Q\left[\min_{s \leq t}\left(\frac{A_s}{K(s)}\right) \leq 1\right] \\ &= Q[\min_{s \leq t} X_s \leq \sigma^{-1}\log\left(\frac{K_0}{A_0}\right)] \end{aligned} \tag{3.13}$$

where $X_t = W_t + \xi t$, $\xi = \sigma^{-1}\left(r - \frac{\sigma^2}{2} - k\right)$. The solution to this[6] is given by

$$Q[\min_{s \leq t} X_t \leq d] = 1 - FP(-d; -m, t) \tag{3.14}$$

$$FP(d; m, t) := N\left(\frac{d - mt}{\sqrt{t}}\right) - e^{2md} N\left(\frac{-d - mt}{\sqrt{t}}\right) \quad d \geq 0 \tag{3.15}$$

And therefore

$$Q[0 \leq \tau < t] = 1 - FP(-d; -m, t) \tag{3.16}$$

[6]See Appendix A

where $d = \sigma^{-1} \log\left(\frac{K_0}{A_0}\right) < 0$ and $m = \sigma^{-1}\left(r - \frac{\sigma^2}{2} - k\right)$. The payoff for the equity holders at time of maturity is given by

$$(A_T - K)^+ \mathbf{1}_{\{\min_{s \leq T} X_s > d\}} = \left(e^{kt} A_0 e^{\sigma X_T} - K\right)^+ \mathbf{1}_{\{\min_{s \leq T} X_s > d\}} \tag{3.17}$$

This is equivalent to the barrier option that is down-and-out call option. This can be priced in a similar fashion to the Merton model. However, the value of this equity is smaller than the shared value that we calculated in the Merton model. Further, it is not monotone in volatility. On defining $d_t = \sigma^{-1} \log\left(\frac{K(t)}{A_t}\right)$ we have the value of the bond at any time t prior to default as

$$D_t = \int_t^T e^{r(t-s)} K(s) \left(-\partial_s FP\left(-d_t; -m, s-t\right)\right) ds + K e^{-r(T-t)} FP\left(-d_t; m, T-t\right) \tag{3.18}$$

In general, arriving at an analytic solution for F is not possible and numerical methods are used for approximations. The original work by Black and Cox arrives at a numerical approximation with a particular solution by using recursive techniques where the "terminal condition at each stage determined by the solution to the previous stage." Thus, computational methods are imperative to be able to solve equation 3.18.

4. Deep Learning in Credit Risk

4.1 Gauging Risk using Machine Learning

Machine learning (ML) has revolutionised the field of finance and financial economics with the possibility to form patterns and links which humans fail to gauge as important factors pertinent to the sector. This allows room for a wide range of applications starting from forecasting various indicators to actually providing distinct values to risk by analysing available data.

The advent of open-sourced data has ushered in the age of modernisation that the financial sector has always latched onto. In this section, we discuss a model that can be scaled widely to help assess risk. The key feature in any ML or DL model is to find suitable data for the model to 'learn' and discover underlying pattern linkages. This is a very challenging aspect for the purpose of gauging credit risk since this depends on more factors than we can possibly pool in. While it is a trivial matter for the computer to process, actually acquiring the data required for this is the herculean task. One way to navigate this issue is by setting proxies for risk.

4.1.1 The Model

Let us propose a model for the assessment of credit risk of a bank. The recommended proxies and indicators one can use, readily available in the form of panel data, include (not limited to):

- **Bank Specific:**
  - Advances to assets
  - Return on assets
  - Return on equity
  - Net interest margin
  - Credit-deposit ratio
  - Capital adequacy ratio
  - Gross non-performing asset ratio
  - Operating costs to assets
- **Macroeconomic Indicators:**
  - GDP
  - Inflation rate
  - Unemployment rate

The workflow of the proposed ML model is depicted in Figure 4.1 for ready reference. The thumb rule of ML models is *more data*. As many more proxies we add, the better we can assess credit risk. The next issue is the entity that we will use to measure risk itself. The output cannot be 'credit risk' because there is no tangible measure that exists for the same. In other words, the output has to be an indicator of risk that was not used in the input to the model (the ones mentioned above).

We can instead, have the model predict certain indicators of credit risk and then use those to make a safe assumption of credit risk. We propose using the number of outstanding loan payments as the

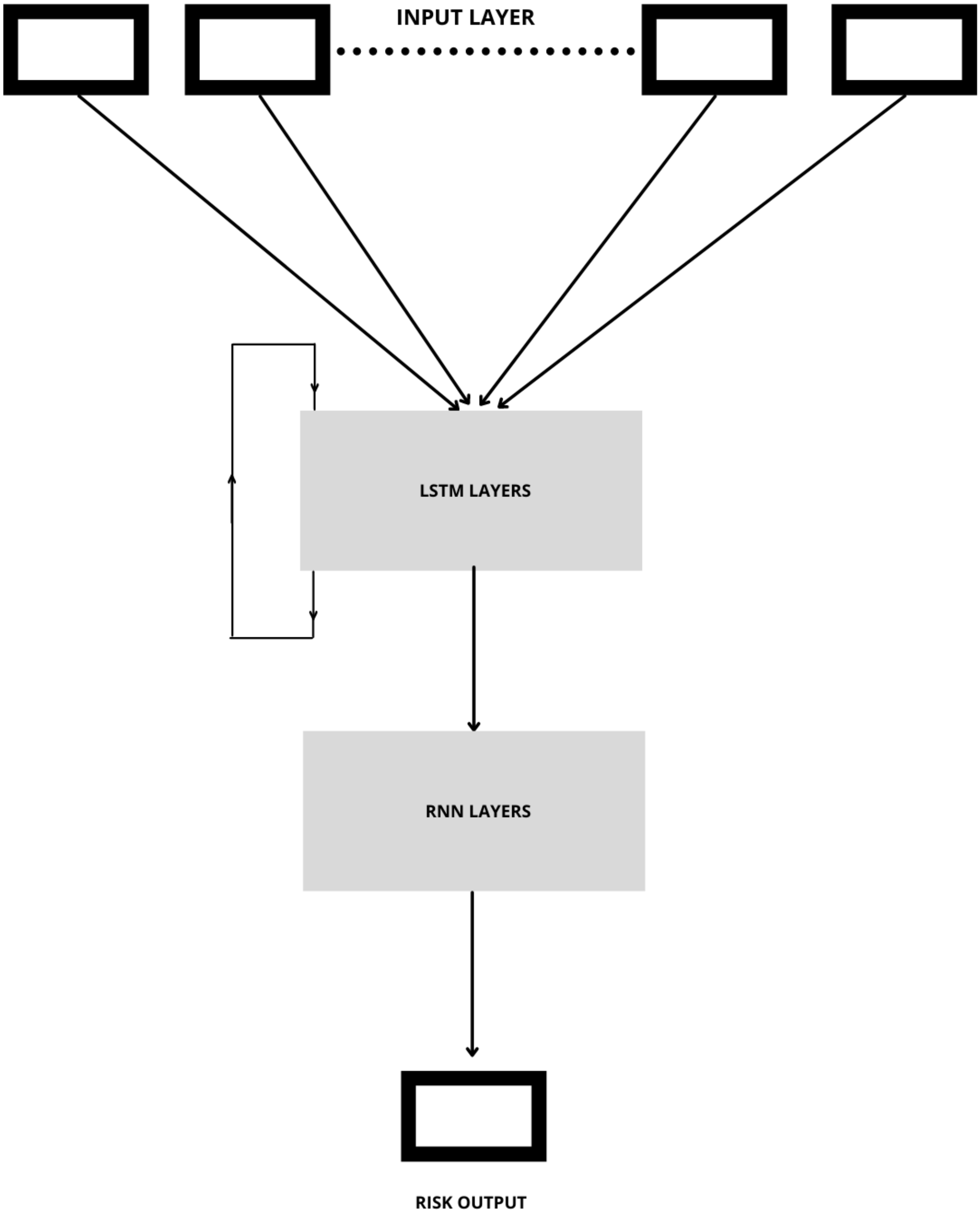

Figure 4.1: ML Architecture workflow for Credit Risk Assessment

```
_________________________________________________________________
Layer (type)                 Output Shape              Param #
=================================================================
lstm (LSTM)                  (None, 11, 64)            16896
_________________________________________________________________
dropout (Dropout)            (None, 11, 64)            0
_________________________________________________________________
lstm_1 (LSTM)                (None, 11, 64)            33024
_________________________________________________________________
dropout_1 (Dropout)          (None, 11, 64)            0
_________________________________________________________________
dense (Dense)                (None, 11, 6)             390
_________________________________________________________________
dropout_2 (Dropout)          (None, 11, 6)             0
_________________________________________________________________
dense_1 (Dense)              (None, 11, 1)             7
=================================================================
Total params: 50,317
Trainable params: 50,317
Non-trainable params: 0
_________________________________________________________________
```

Figure 4.2: Proposed architecture model output from Python

output of the model[1]. This will be the proxy to credit risk. In other words, the model will take up panel data of the entries in the list above and provides the output of outstanding loan payments. This is a prediction system.

A python implementation of the machine learning model is presented in Figure (4.2). The model consists of 2 LSTM layers with 2 dropout layers followed by a dense RNN layer of 390 parameters with another 20% dropout layer. The final dense layer is the output node of the neural network. The data itself will be split into the training set and the testing set. The training set itself can be further split into different portions which further increase the accuracy of the proposed model. The lack of data prevents us from actually implementing this model. However, this model can be used for banks who might be interested in analysing their credit risks. In the next section we solve such a problem with unsupervised and unlabelled data; this is different from the one proposed in this section.

4.2 Credit Card Fraud and Credit Risk

Unsupervised machine learning models that detects credit risk of customers of a bank using details about the can be formulated. The details themselves can be anything ranging from dummy variables

[1]There are other possible proxies that can be used such as the credit history, capacity to repay, capital, the loan's conditions, and associated collateral. In practice, the model can be used to provide the output for each of these proxies.

(such as political ideology being left or right annotated with binary variable, say, 1 and 0.) to the number of cars owned. Since credit risk cannot be explicitly measured as such, we detect the possibility of fraud in the application for loan by these customers.

Consider a number of customers who have applied for loans/credit cards or some other service that a bank offers. The probability that they might default on their loan (and thereby pose a credible default risk to the bank itself) is unknown and is a point of interest for the bank in assessing the credit risk the customer may cause and thereby posing a threat to the bank's credit stability in terms of swaps. We have, however, auxiliary information that they have submitted in interest of their application. For our purpose we generate randomly a high dimensional data set containing around 300 applicants and each applicant has 20 entry fields which translate into numbers. Such data sets can also be procured directly from banks. These could be dummy variables or entries denoting things ranging from their deposits in the bank to their pre-existing loan amounts. The data set consists of all of these details and more and will be used to train either a Self Organizing Map (SOM) or a supervised ANN model.

4.2.1 SOM

The self organizing map was an unsupervised ANN that introduced by Teuvo Kohonen in the 1980s and is a versatile unsupervised deep learning (DL) model. The benefit of an unsupervised learning model is that it works even if the 'correct; answer is not known and the neural network attempts to find patterns within the data.

The SOM has a sheet like architecture as can be seen in Figure 4.3 with the neurons being activated by differing patterns that it detects in the data. The primary 'output' of the SOM is categorization and it uses K-means clustering to perform this complex operation of classification based on hidden and visible patterns in complex data sets.

In this example, each customer and their corresponding entries will form one entry vector. The set of customer will form the input space. The SOM k-map, that is, the output space, and the input space are

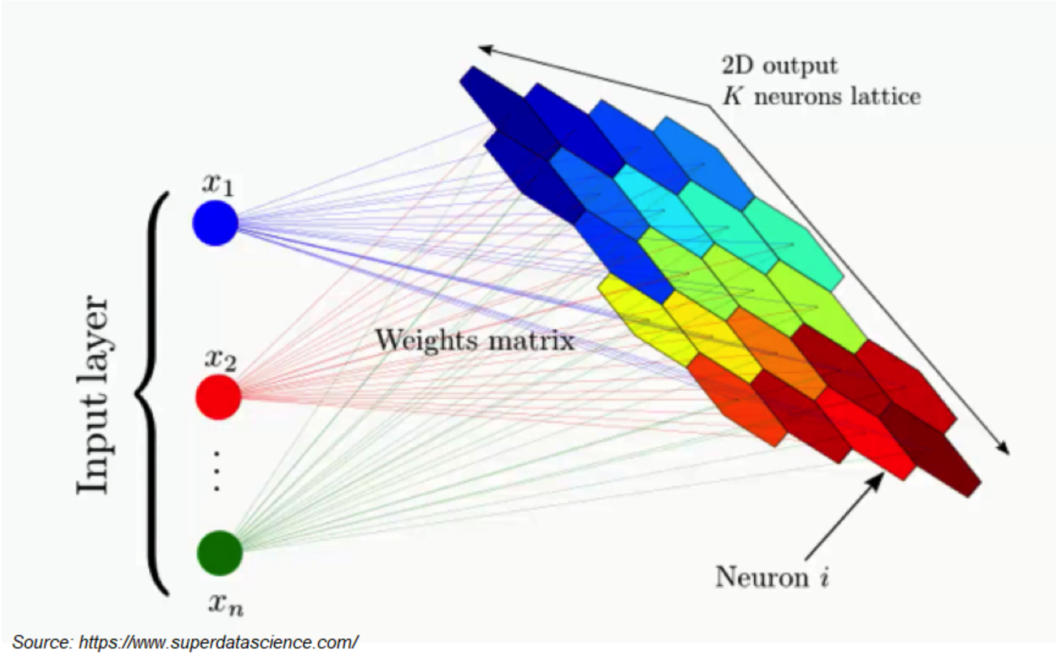

Figure 4.3: Self-Organizing Maps

connected by a neural network. The algorithm runs till the feature map stops changes and for each customer there is a winning node (or Best Matching Unit), that is the node which is the most similar to the customer. Once the SOM is created, the outliers to this map will be the fraudulent customers. Using an inverse mapping function on these outliers will give us the list of fraudulent customers from the original input space dimensions in terms of these customer IDs.

For each of the neurons in the network, we calculate the mean Euclidean distance between it and the neurons in its vicinity - this is called the Mean Inter-neuron Distance (MID). The outliers will be outside the neighbourhood, which has to be manually defined. The result from a python source code implementation is depicted in Figure 4.4. We spend some time in analysing the feature map.

The red circles are fraudsters who did not get approved for a loan/credit card, and the green squares are fraudsters who managed to got approved. The latter are more important to detection of fraud than the former. Thus the result has the MIDs and the winning node fraudsters who either got approved or rejected. The higher the value of the MID (see the colour-bar legend), the more of an outlier it represents - so in this case, the yellower it is, the more it is an outlier. Since we use normalised data, it

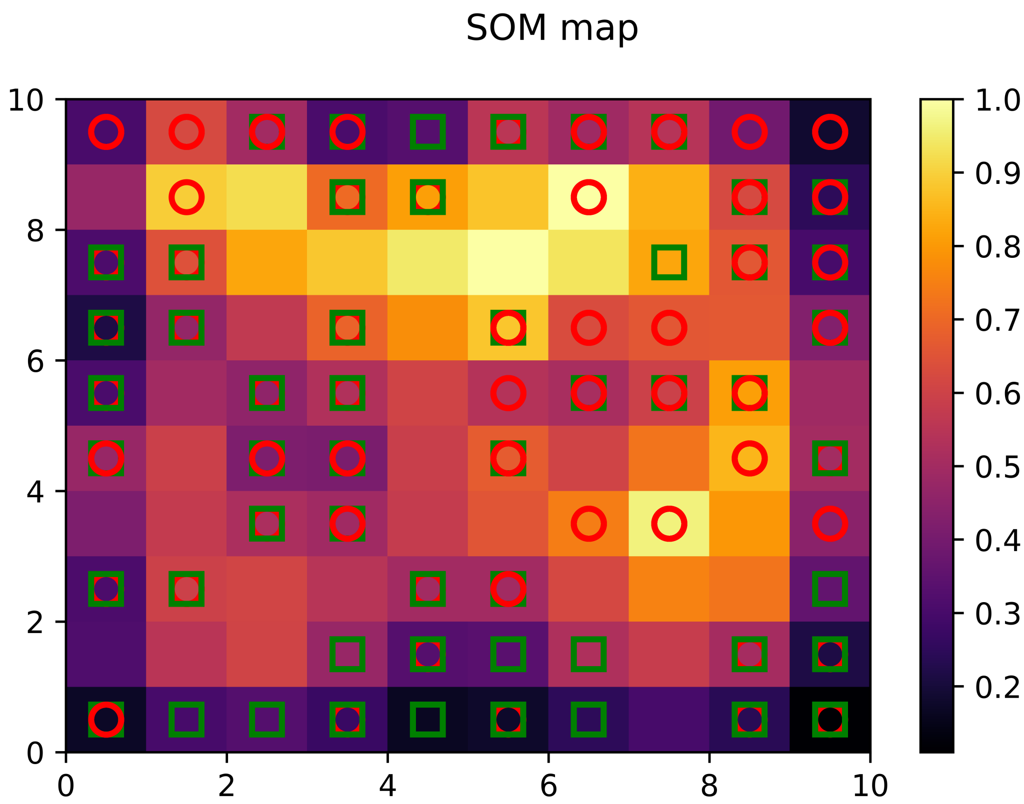

Figure 4.4: SOM feature map for Credit Fraud Detection

ranges from 0 to 1. Notice that at coordinates given by $(2,9)$, $(7,9)$, and $(8,4)$ are not areas of much concern since these groups of individuals were not approved for the loan/credit card. We have the following coordinates which have approval (green squares) inspite of high MID (outliers)

- (5,9)
- (8,8)
- (9,6)

We can use this information to find the user IDs of the fraudulent customers as reported in Table A.1. Thus we have implemented a DL model to detect fraud and this will be an asset to credit risk management for the banks.

II Applications

5. The Credit Default Swap

This chapter introduces Credit Default Swaps (CDS) and outlines various literature review on this controversial financial instrument. While various views are opted for, caution is taken to be able to provide sufficient arguments for either sides wherever possible. The sections that follow draw out the structural mechanism of the CDS and provide some empirical information on them and possible inferences from the same. A simple vanilla binary CDS is discussed in addition to the sensitivity of the assumptions to the claimant amount and a short section on how the analysis can be extended to the case where the payoff is contingent on multiple defaults and counterparty risk.

5.1 Introduction

CDS were first invented by Blythe Masters of JP Morgan in 1994[4] but didn't gain popularity until early 2000s. By 2007, the outstanding CDS value stood at over $62 trillion[32] (also see Fig5.1). Credit risk is the probability of default on a debt that may arise due to the borrower failing to make the required payments[92]. In essence it is the risk of the counter-party in the swap agreement ceasing to exist. The purpose of CDS is to let risks be traded like market risks. A CDS, therefore, provides protection against the risk of any form of credit event based on the agreement of the contract.

ISDA Market Survey

Notional amounts oustanding at year-end, all surveyed contracts, 1987-2009

Notional amounts in billions of US dollars

	Year-end outstandings for interest rate swaps	Year-end outstandings for currency swaps	Year-end outstandings for interest rate options	Total IR and currency outstandings	Total credit default swap outstandings	Total equity derivative outstandings
1987	$ 682.80	$ 182.80		$ 865.60		
1988	1,010.20	316.80	327.30	1,654.30		
1989	1,502.60	434.90	537.30	2,474.70		
1990	2,311.54	577.53	561.30	3,450.30		
1991	3,065.10	807.67	577.20	4,449.50		
1992	3,850.81	860.39	634.50	5,345.70		
1993	6,177.35	899.62	1,397.60	8,474.50		
1994	8,815.56	914.85	1,572.80	11,303.20		
1995	12,810.74	1,197.39	3,704.50	17,712.60		
1996	19,170.91	1,059.64	4,722.60	25,453.10		
1997	22,291.33	1,823.63	4,920.10	29,035.00		
1998				50,997.00		
1999				58,265.00		
2000				63,009.00		
2001				69,207.30	918.87	
2002				101,318.49	2,191.57	2,455.29
2003				142,306.92	3,779.40	3,444.08
2004				183,583.27	8,422.26	4,151.29
2005				213,194.58	17,096.14	5,553.97
2006				285,728.14	34,422.80	7,178.48
2007				382,302.71	62,173.20	9,995.71
2008				403,072.81	38,563.82	8,733.03
2009				426,749.60	30,428.11	6,771.58

Figure 5.1: ISDA Market Survey

5.2 Literature Review

Scholarly interest in CDS was initially to price these instruments[40, 88, 96]. In the recent years it has rapidly been extended to general analysis of financial economics[58]. While many scandals have had CDS involved - such as the scandal popularly called the London Whale incident which violated US anti-trust laws - some such as Napier Park and BlueMountain Risk[1] have made enormous profits piggybacking on CDS. Many hedging and trading activities involve CDS in some manner and are a driving component of financial regulation on a global scale including the Basel III guidelines[13]. An analysis by Stultz *et al* revealed that "in 2005 the gross notional amount of credit derivatives held by banks exceeds the amount of loans on their books."[82]. They also conclude that banks hedge low-risk loans more than high-risk loans and the use of credit derivatives by banks is limited due to the lack of robustness in banks to perform hedge accounting when using credit derivatives to hedge.

CDS have drawn widespread criticism, both in favour and against. It is interesting to note that in a friction-less world they would be redundant securities in a financial market but are nonetheless under scrutiny. One possible explanation is that there is sufficient evidence to suggest that CDS can alter the behaviour of the investors, regulators and firms in addition to being able to influence the underlying security's price[4].

Much literature has been devoted to expanding the vanilla binary CDS models to account for non-zero default recoveries such as Jarrow *et al*[59] and Hull *et al*[56] who assume that the claim is equal to the no default of the bond. Some other works assume that the claim is exactly equal to the value of the bond in the instant right before default. However, this is not in line with the bankruptcy laws in most countries as Jarrow and Turnbull write while discussing their reduced form approach where observable market data (credit spreads) are used to understand the market's gauging of the bankruptcy process and the subsequent pricing of credit risk derivatives[60].

[1] BlueMountain made profits in the London Whale incident.

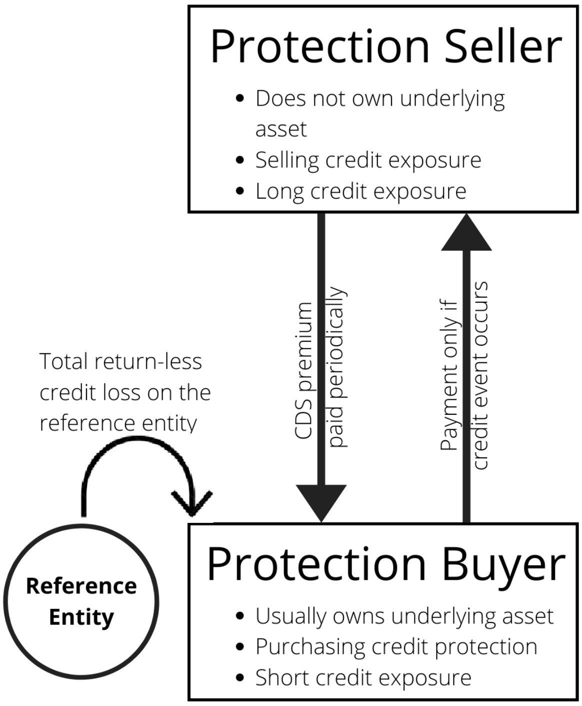

Figure 5.2: Credit Default Swap transaction

5.3 Formulation

When the contract is made, the default events are marked out. It could be anything ranging from market price of a commodity to a natural disaster. The agent selling the protection pays the buyer a periodic payment. The buyer, on the other hand, will only pay - the entire amount as agreed in the contract - if the defaulting event occurs. An example might help illustrate this concept with more clarity.

■ **Example 5.1** Consider that Robinson and Friday enter into a contract extending over a five-year period on the first day of the year. Friday, who only has access to water, sells the protection to Robinson, who only has access to pineapples. The contract, in this case, could be that if Friday's water endowment goes below one unit (the default event), Robinson has to pay Friday 6 units of water and 3 units of pineapples. Every six months, as long as the contract has not expired and the Friday's water endowment is above one unit, the contract requires Friday to pay Robinson one unit of water. If at the end of five years, the default event has not occurred, Robinson has received 10 units of water from Friday while Friday has received nothing. On the other hand, if the defaulting event did occur, then

Friday receives the payment of pineapples and water as mentioned above. ∎

In the real world, naturally, things are a bit more complex. CDS are usually used on mortgage payments as in the case of mortgage-backed securities, or to transfer the risk of municipal bonds' credit risks. It is a gamble. This gamble has to be modelled to maximise security of both the buyer and seller. The buyer has to ensure that he is investing in a maximum returns contract. The seller has to be certain that she will not lose money in premium payments (and accordingly hedge). This requires a mathematical foundation for pricing derivatives. For the sake of brevity, we consider a plain vanilla and binary CDS, valuing it without counterparty default risk. Let us begin with the assumption that default probabilities, interest rates, and recovery rates (δ) are independent.

The CDS contract provides the holder the protection from a credit event risk by a company - called the reference entity. The buyer of this insurance gets the right to sell the bonds - called reference obligation - issued by the company at sub-par values if the credit event occurs. The total par value that the buyer can sell the reference obligation is called the swap's notional principle. The buyer makes periodic payments to the insurance seller until the occurrence of the credit event or the maturity of the contract, whichever comes first. If the credit event occurs, the seller of the insurance compensates the buyer for the loss incurred as a result of the credit event. Usually the buyer is required to make a final accrual payment at the credit event and then the swap is covered. Sometimes the payment is done in cash settlement and this is $(100-Q)\%$ of the notional principle where Q is the mid-market price of the reference obligation which is determined by the calculation agent.

There are a variety of variations of the CDS. In a binary CDS, the payoff in the event of default is a fixed payment. In a basket CDS, there is a group of reference entities taken into account and the payoff occurs when the first of these entities defaults. Contingent CDS provides a payoff only when there is both a credit event and an additionally trigger such as some specific movement of the market or the default of another company.

We consider a plain vanilla binary CDS model without counterparty default risk assuming that interest rates, recovery rates, and default probabilities are independent. Relaxing these assumptions causes the complexity of the model which is beyond the scope of this text.

5.4 Estimating Default Probability

Valuing CDS first requires estimating the risk-free default probability of the default of the reference entity in future time instances. This estimation is primarily dependent on the price of the bonds that the reference entity issues. Assuming that a corporate bond sells less than a similar Treasury bond if and only if there is a default possibility, then we have

$$V_T - V_C = V_D \tag{5.1}$$

where V_T is the vale of the Treasury bond, V_C is the value of the corporate bond, and V_D is the present value of cost of defaults. Using V_D and by making assumptions on the recovery rate, one can estimate the probability of a company defaulting in future time instances. At times, the reference entity of interest might not have released many bonds. In such cases, we consider the bonds issues by another company which have the same risk of default as the reference entity[2].

■ **Example 5.2** Suppose a 3-year zero-coupon Treasury bond with 100 face value yields 5%. A corporate bond with similar parameters yields 5.5%. Assuming both rates are expressed with constant compounding, the value of the treasury bond and the value of the corporate bond are

$$V_T = 100e^{-0.05\times 3} = 86.07$$

$$V_C = 100e^{-0.055\times 3} = 84.79$$

[2] More often than not, there is a company in the same market and same field as the reference entity which has a similar if not same risk of default.

Thus, the present value of cost of default is given by

$$V_D = V_T - V_C = 86.07 - 84.79 = 1.28$$

We can define the risk-neutral probability of default during this three year lifetime of the bond as p and assuming that there are no recoveries in case of a default - the default causes a loss of 100 at the end of 3 years (maturity). Thus the expected loss is $100p$ and subsequently the present value of the expected loss is $100pe^{-0.05\times 3}$. Therefore

$$100pe^{-0.05\times 3} = 1.28$$

which gives us $p = 0.01487 = 1.487\%$. ■

Reality is often disappointing and complex than the example above This is because of two main reasons; firstly, the recovery rate of default is usually non-zero, and secondly, most of the corporate bonds out there are coupon bonds and not zero-coupons. As we have established, the payoff from the CDS in the event of a default is the difference between the reference obligation and its market value just after the instant of default (t). If we assume (and rightly to a certain extent) that the claim in the event of a default is the face value of the bond and the accumulated interest, then we have the payoff (P_{CDS}) from a typical CDS to be

$$P_{CDS} = L - RL[1 + A(t)] = L[1 - A(t)R] \tag{5.3}$$

where L is the notional principle, R is the recovery rate, and $A(t)$ is the interest accumulated up till time t.

5.5 General Analysis

In this section we extend our analysis by relaxing, ever so slightly, the assumptions we made on the nature of the the default and the recovery rate. This allows us to present a general analysis that can be extended with variant assumptions about the claim amount.

5.5.1 Discrete Time Defaults

We begin by choosing N bonds which are issued, for all practical purposes by the reference entity[3]. We make no other assumptions on the occurrence of defaults and they can occur on any of the bond maturity dates. Consider the maturity of the ith bond to be t_i and without loss of generality $t_1 < t_2 < \ldots t_N$. We define a few quantities:

- B_j: price of the jth bond today
- G_j: price of the jth bond today if there is no probability of default. This is also the same as the price of a Treasury bond that pays the same cash flow as the jth bond.
- $F_j(t)$: the jth bond's forward price due to a forward contract that matures at $t(< t_j)$ with the assumption that the bond is default free.
- $v(t)$: present value of USD 1 received at t with certainty.
- $C_j(t)$: claim of the jth bond holders in the event of default at $t(< t_j)$.
- $R_j(t)$: recovery rate for holders of jth bond in the event of default at $t(< t_j)$.
- α_{ij}: present value of the loss due to a default on the jth bond at t_i, relative to that value the bond might have had if there was no probability of default.
- p_i: risk-neutral probability of default at t_i.

Let us begin with a case where interest rates are deterministic and there is absolute certainty regarding claim amounts (C_j) and recovery rates (R_j). This allows us to state that price of the no-default value at

[3] Or by a corporate that has similar risk default structure as the reference entity - that is, the probability of default in a future time, as of the instantaneous date (today), is the same for both the reference entity *and* this corporate. This is similar to Hull and White's works[57].

t on bond j is given by $F_j(t)$. In the case of a default at t the bondholder makes a recovery at the rate $R_j(t)$ on bond j on the claim $C_j(t)$. Thus, it follows

$$\alpha_{ij} = v(t_i)\left[F_j(t_i) - R_j(t_i)C_j(t_i)\right] \tag{5.4}$$

There is a p_i probability of the loss of α_{ij} and so the total present value of the losses on bond j is given by

$$G_j - B_j = \sum_{i=1}^{j} p_i \alpha_{ij} \tag{5.5}$$

which in turn gives us

$$p_j = \frac{G_j - B_j - \sum_{i=1}^{j-1} p_i \alpha_{ij}}{\alpha_{jj}} \tag{5.6}$$

5.5.2 Relaxing Assumption on Recovery Rates

Next we relax the assumption on recovery rates and claim amounts. First we will assume that the claim amount is equal to the no-default value of the bond in the instant of default. Secondly we will assume that the claim amount is equal to the sum of the face value and the interest accrued at the time of default. Assuming that the recovery rate is equal to the expected value of the same in a risk-neutral world it is trivial to see that if default events, Treasury interest rates, and recovery rates are independent, then Eq(5.4) and Eq(5.5) hold for uncertain recovery rates, uncertain default probabilities, and stochastic interest rates.

We can reasonably rule out systematic risk in recovery rates so that the expected recovery rates in the world are the same as the ones in a risk-neutral world. Thus one can use historical data to estimate recovery rates. In line with this, since N bonds are issue, each of these bonds should have the same p_i. This is not entirely true and p_i varies according to t_j and the bond itself. For simplicity we will also assume that the recovery rate in time independent and that all bonds have same seniority in the event of default due to reference obligation. Thus the expected value of $R_j(t)$ is independent of time and we will denote this by $\hat{R}$.

5.5.3 Generic Time Defaults

We relax the assumption used to derive Eq(5.6) by allowing for defaults to occur at any time and not just on bond maturity dates as before. Let $q\delta t$ be the probability of default between time t and $t+\delta t$ as observed at time 0. Notice that is not the same as the hazard rate $h(t)$ which is so defined such that $h(t)\delta t$ the probability of default between t and $t+\delta t$ as seen at time t assuming that the time period between 0 and t has been default free. The variables $q(t)$ and $h(t)$ are related by

$$q(t) = h(t)\exp\left[\int_0^t h(\tau)d\tau\right] \tag{5.7}$$

Jarrow *et al* often define their default models using the hazard rate (*ibid.*), however, we will proceed with expressing it in terms of $q(t)$ such as Hull and White follow (*ibid.*). Henceforth, we refer to $q(t)$ as the default probability density. Thus having moved from discrete time to continuous time, it is a trivial exercise to see that we can assume $q(t)$ is constant and equal to q_i for $t_{i-1} < t < t_i$ and set

$$\beta_{ij} = \int_{t_{i-1}}^{t_i} v(t)\left[F_j(t) - \hat{R}C_j(t)\right] \tag{5.8}$$

using this, we are now in a position to use a similar approach as before to get

$$q_j = \frac{G_j - B_j - \sum_{i=1}^{j-1} q_i\beta_{ij}}{\beta_{jj}} \tag{5.9}$$

The integral in Eq(5.8) can be solved using standard procedures such as the Simpson's rule of numerical integration.

5.5.4 Expected Recovery Rates and Bond Yields

The default probability density we derived in Eq(5.9) has to be a non negative value, thus

$$B_j \leq G_j - \sum_{i}^{j-1} = 1 q_i \beta_{ij} \tag{5.10}$$

At the same time, the cumulative probability of default has to be less than 1 and so

$$\sum_{i=1}^{j} q_i \left(t_i - t_{i-1}\right) \leq 1 \tag{5.11}$$

which can be rewritten as

$$q_j \left(t_j - t_{j-1}\right) \leq 1 - \sum_{i=1}^{j-1} q_i \left(t_i - t_{i-1}\right) \tag{5.12}$$

Using Eq(5.12) and Eq(5.10) we get

$$B_j \geq G_j - \sum_{i=1}^{j-1} q_i \beta_{ij} - \frac{\beta_{jj}}{t_j - t_{j-1}} \left[1 - \sum_{i=1}^{j-1} q_i (t_i - t_{i-1})\right] \tag{5.13}$$

Thus we have an upper and lower bound on a bond maturing at t_j provided we specify the expected recovery rates and yields on bonds that mature at times $t < t_j$ imposed by the above equations. In general, Eq(5.10) and Eq(5.13) can be used to test if the recovery rate assumption allows consistency of the bond yields under the assumption.

5.6 Valuation of Plain Vanilla CDS

Let us consider a plain vanilla CDS with a 1 USD notional principle in addition to the usual assumptions where default rates, Treasury interest rates, and recovery rates are independent of each other. Additionally we also assume that the default event claim is the sum of the face value and assimilated interest. We begin by defining a few terms[4] as before

T	maturity of the CDS
q(t)	risk-neutral default probability density at time t
$\hat{R}$	expected recovery rate of the reference obligation in a risk-neutral world.
$u(t)$	present value of payments at the rate of 1 USD per year on payment dates that are between time zero and time t.
$e(t)$	present value of an accrual payment at time t equal to $t - t^*$ where t^* is the payment date immediately preceding time t.
$v(t)$	present value of 1 USD received at time t.
w	total payments per year made by credit default swap buyer.
s	value of w that causes the credit default swap to have the value 0.
π	the risk-neutral probability of no credit event during the life of the swap.
$A(t)$	Accrued interest on the reference obligation at time t as a percent of face value.

By definition, π can be calculated as

$$\pi = 1 - \int_0^T q(t)dt \tag{5.14}$$

The payments occur until time of maturity T or until the default event, whichever comes first. Considering default that occurs at $t < T$, the present value of payments is calculated as $w\,[u(t) + e(t)]$. If the bond matures before a default event, then the present values of payments is simply $wu(T)$. The expected present value of the payments can thus be ascertained as

$$w \int_0^T q(t)\,[u(t) + e(t)]\,dt + w\pi u(T)$$

[4] As we have shown previously, $\hat{R}$ is independent of the time of default and the same as recovery rate on the Treasury bonds which is used to calculate $q(t)$

With our assumption about the claim amounts, we can write the risk-neutral payoff from the CDS to be

$$1-[1+A(t)]\hat{R}=1-\hat{R}-A(t)\hat{R}$$

Thus we write the present value of the expected payoff from the CDS as

$$\int_0^T \left[1-\hat{R}-A(t)\hat{R}\right] q(t)v(t)dt$$

Since the value of the CDS to the buyer is the difference present value of the expected payoff and the present value of the payments made by the buyer, we can mathematically write

$$\int_0^T \left[1-\hat{R}-A(t)\hat{R}\right] q(t)v(t)dt - w\int_0^T q(t)\left[u(t)+e(t)\right]dt - w\pi u(T)$$

We are interested in the value of w, s, that makes this expression zero which is

$$\begin{aligned} s &= \frac{\int_0^T \left[1-\hat{R}-A(t)\hat{R}\right] q(t)v(t)dt}{\int_0^T q(t)\left[u(t)+e(t)\right]dt+\pi u(T)} \\ &\approx \frac{(1-\hat{R}-a\hat{R})\int_0^T q(t)v(t)dt}{\int_0^T q(t)\left[u(t)+e(t)\right]dt+\pi u(T)} \end{aligned} \tag{5.15}$$

Where a is the average value of $A(t)$ for $t\in[0,T]$.

The entity s is what is called the Credit Default Swap Spread. The CDS spread is the sum of the per year payments as a percentage of the notional principal for a CDS that has been newly issued.

5.6.1 Determining s

One can use approximate no-arbitrage arguments to better understand the determinants of s. Consider a portfolio that an investor forms of a T-year par yield bond that is issued by the reference entity and the CDS; the investor has eliminated most of the risks associated with the default on the bond. The approximate (at least) net return for the investor is given by $y-s$ where y is the yield to maturity on the bond. In a no-arbitrage set up, this is approximately equal to the T-year Treasury par yield denoted by, say, x. If it is that $y-s>x$ then it is profitable to buy a T-year par yield bond that was issued by the reference entity, then buy the CDS, and short a T-year par yield Treasury bond. On the other hand, if $y-s<x$ then it is profitable to buy the T-year par yield Treasury bond, short the T-year par yield bond from the reference entity, and sell the CDS. Thus, one can argue that it must hold that $y-s=x$. A brief analysis shows that the arbitrage is not perfect. We begin by defining[5]:

$s^\star$ y-x.

L notional principle of the CDS.

$A^\star(t)$ accrued interest as a percentage of the face value at time t on the underlying par yield corporate bond.

R Realised recovery rate when the default event occurs.

In the case of constant interest rates and a flat Treasury curve, the CDS spread is exactly $s^\star$ for a CDS with the credit event at time t being $L\left[1+A^\star(t)\right](1-R)$. This is made obvious when we consider the position of the investor who has bought both the CDS and a certain amount of the face valued L underlying corporate par yield bond when the spread is given by $s^\star(=y-x)$.Thus the investor receives the same cash flow as if he owned a Treasury par yield bond until T or until a credit event, whichever comes first. In case of a credit event at $(0<)t(<T)$ then the investor makes accrual payment at t (defining $t^\star$ to be the payment date right before t) such that the net pay-off of the CDS is

$$\begin{aligned}\text{Payoff} &= L\left[1+A^\star(t)\right](1-R) - L(y-x)(t-t^\star)\\ &= L[1+x(t-t^\star)] - LR(1+A^\star(t)]\end{aligned}$$

[5] $A^\star(t)$ is a bond issued by the reference entity with the same payment dates as the swap.

where we rewrote the last line due to the definition of $A^\star(t) = y(t-t^\star)$. Subsequently, the net value of the holding is given by

$$L[1+x(t-t^\star)]$$

This is the amount required to buy a par yield Treasury bond at face value L at the time t. In all circumstances, the portfolio of the investor replicates exactly the cash flows from the par yield Treasury bond implying that $s^\star$ is in fact the correct CDS spread. Spreads less than this value give rise to arbitrage. We can extend this idea to refer to a CDS that provides the payoff $(1-R)[1+A^\star(t)]$ as an idealized CDS[6].

With a flat Treasury curve and constant interest rates, we can correct for the difference between the payoffs of the idealized and real CDS. An analysis similar to Eq(5.15) yields

$$\begin{aligned} s^\star &= \frac{\left(1-\hat{R}\right)\int_0^T [1+A^\star(t)]\, q(t)v(t)dt}{\int_0^T q(t)[u(t)+e(t)]dt+\pi u(T)} \\ &\approx \frac{(1-\hat{R})(1+a^\star)\int_0^T q(t)v(t)dt}{\int_0^T q(t)[u(t)+e(t)]dt+\pi u(T)} \end{aligned} \tag{5.16}$$

where $a^\star$ is the average for $A^\star(t)$ for t.

Thus using Eq(5.15) and Eq(5.16) we can write the relation between the actual spread and the idealized spread to be

$$s = s^\star \frac{(1-\hat{R}-a\hat{R})}{(1-\hat{R})(1+a^\star)} \tag{5.17}$$

Various models and modifications have been proposed to relax the assumption on the constant nature of the interest rates[21] or the cases of a non-flat Treasury curves[39]. We leave these at the interest of the reader.

[6] In reality, the payoff from a CDS is not $(1-R)[1+A^\star]$ but is $1-R-A(t)R$ resulting in $s^\star$ over-estimating the true spread, s, of the CDS.

5.6.2 Binary CDS

For a binary CDS - which only provides a fixed payment at maturity or default event, whichever is first - can be analysed in a manner similar to what we have done prior. This gives the value of the spread providing a payoff of unit dollar in the event of a default as

$$s^{\text{binary}} = \frac{\int_0^T q(t)v(t)dt}{\int_0^T q(t)[u(t)+e(t)]dt + \pi u(T)} \tag{5.18}$$

5.6.3 Independence Assumptions

This section has assumed exclusively that the interest rates, recovery rates, and default probabilities are independent of each other. These are not always true in actual application. For example, one can hypothesize that the increase in interest rates can increase the chances of default. This in turn causes the high default probabilities to be associated with high discount rates of the payoffs and thereby reducing the CDS spreads.

Additionally, high default chances are usually associated with low market values for the bonds that the reference entity issues. This in turn increases the CDS spreads as is gives the buyer more right to sell the reference bond for its face value. Since these two effects act opposite to each other, there is an offset to some degree.

5.7 Pricing CDS

The credit default swap, as discussed, is a versatile financial instrument. It is therefore important that we devote a brief section on pricing the instrument. Much literature is readily available in this field and many models have been extended to price CDS. Works of Houweling *et al* and Hull and White focus on the empirical analysis of pricing CDS after calculating the present value[54]. More recent research, such as Radi *et al* and Hao *et al* use hybrid reduced for models - the hybrid framework CE model and fractional Vasicek interest model respectively[8, 50].

The valuation framework we adopt to price the CDS is in line with the previous chapters. We assume credit risk modelled by a Merton model - a frictionless market with the risky asset dynamics being governed by Eq 3.1 and the asset dynamics of a firm being governed by Eq 3.2.

In this model, we have the debt and equity of the firm at some time t as

$$D_t = V_t N(-d_1) + De^{-r(T-t)}N(d_2) \tag{5.19a}$$

$$E_t = V_t N(d_1) - De^{-r(T-t)}N(d_2) \tag{5.19b}$$

where the symbols have their usual meaning as discussed in Chapter 3.

Consider a CDS that is written on a unit dollar bond that has initial protection time of t_0. The final protection time is T such that the time to maturity is $T - t_0$. Assuming a random time of default (say, t_d) and a credit default spread given by s, then at time t_0; due to a constant interest rate r the premium leg is given by

$$\mathbf{PremLeg} = \mathbb{E}\left[s\int_{t_0}^{T} e^{-r(z-t_0)}\mathbf{1}_{\{t_d>z\}}dz\right] \tag{5.20}$$

The protection leg, in a similar manner, is given by

$$\mathbf{ProLeg} = \mathbb{E}\left[e^{-r(t_d-t_0)}(1-\delta)P\left(r(t_d),t_d;T\right)\mathbf{1}_{\{T\geq t_d>t_0\}}\right] \tag{5.21}$$

$\delta \in [0,1)$ is the recovery rate, which we assume is constant in this case in the lines of works by Cathcart *et al* [28] and Gündüz and Uhrig [49]. On equating these two we get the CDS spread, s, as

$$s\left(x(t_0),r(t_0),t_0;T\right) = \frac{\mathbb{E}\left[e^{-r(t_d-t_0)}(1-\delta)P\left(r(t_d),t_d;T\right)\mathbf{1}_{\{T\geq t_d>t_0\}}\right]}{\mathbb{E}\left[\int_{t_0}^{T} e^{-r(z-t_0)}\mathbf{1}_{\{t_d>z\}}dz\right]} \tag{5.22}$$

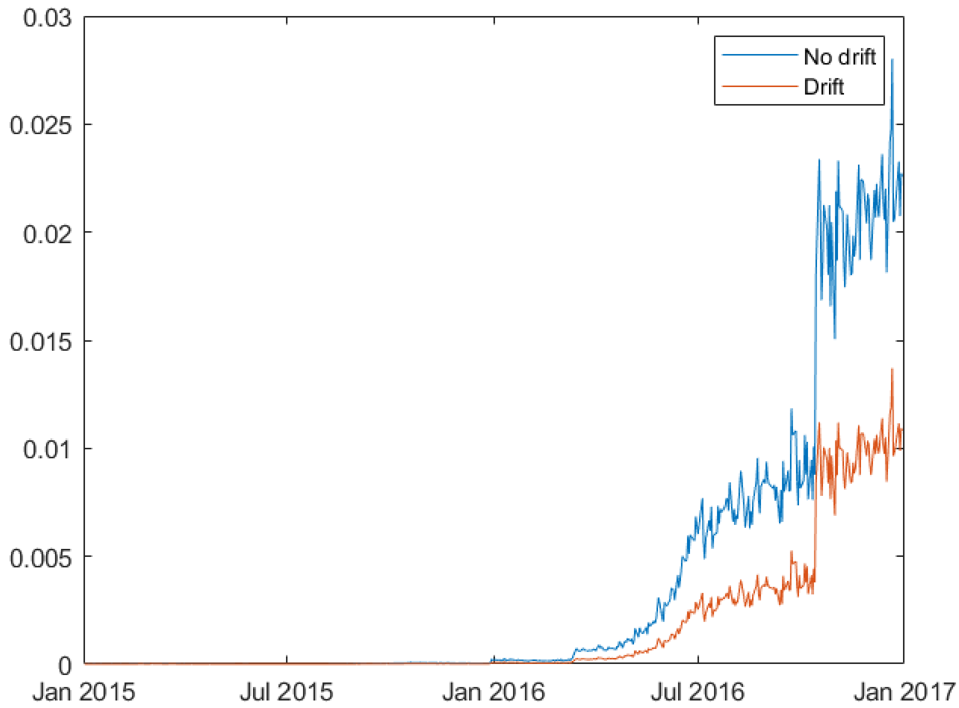

Figure 5.3: Numerical simulation of probability of default for Merton model

In our case, $P(r(t_d), t_d; T) = \exp(-r(T - t_0))$. From this, under the Merton model, as we have considered here, the closed form of the price of a single name (that is, the underlying is a reference obligation) CDS for a generic time t is simple to calculate.

The problem reduces to solving the two expectation values; the first one is the probability of survival while the second is the probability of default under the Merton model. The probability of default has been estimated in Chapter 3. The probability of default is simply $N(-d_2)$. Thus, we calculate it as follows

$$s(x(t_0), r(t_0), t_0; T) = \frac{(1-\delta)e^{-r(T-t_0)}e^{-r(t_d-t_0)}[1 - N(-d_2)]}{\int_{t_0}^{T} e^{-r(z-t_0)} N(-d_2) dz} \tag{5.23}$$

In general, this cannot be solved analytically. A numerical simulation can be performed on MATLAB and is depicted for two cases (with and without drift) in Figure 5.3 and can be readily replicated with the inbuilt MATLAB dataset[7].

[7] See appendix

A similar method can be followed to price a CDS under the Black-Cox structural model. It is interesting to note that since we use equity volatility (which is publicly available) in the place of the unobservable asset volatility, the CDS prices estimated in this method are neither very robust nor efficient in predicting the riskiness involved. There exist reduced form and hybrid models which have been used to overcome this. As we have previously discussed in chapter 3, machine learning has made sufficient strides in this field as well and shows scope for increase in conciseness.

6. Catastrophe Swaps

This chapter will deal with, sequentially, in application of risk modelling and provide offshoots to the same. The first section will focus on a meagerly researched financial instrument - catastrophe(CAT) swaps. Although CAT swaps have been extensively used in hedging against natural calamities in the billions of USDs, they have received very minimal scholarly treatment. An attempt is made at bridging the gap between the heavy use and little research on CAT swaps. The subsequent section introduces a fairly new concept in the field of risk modelling where deep learning is used to generate plausible models of risk while working with fuzzy sets. It will be shown that risk can be modelled by treating it to be information with fuzziness. The application of AI models in credit risk is also discussed in depth. Some models of AI in financial instrument analysis are proposed with their pseudo-codes wherever applicable.

Various insurance and reinsurance companies have used multiple instruments to lay-off the risk due to natural disasters since the early 1990s. CAT options were first offered by Chicago Board of Trade in 1995. The swap buyer paid a premium to the seller who paid the former a cash payment if the risk index exceeded a certain mark. If the index remained below a set mark for a period of time then the option was worthless and the seller retained the premium.

In this section the typical contract design is analysed and the current market for CAT swaps is studied. Further, following the two-step contingent claims pricing approach and model catastrophe as a doubly stochastic Poisson process - a Cox process - in line with the reasoning of Alex Braun *et al*[20]. It will be shown that in the continuous time limit, the Ornstein-Uhlenbeck process is well suited to represent the dynamics in a CAT swap pricing model.

6.1 CAT Swaps

In 1992, hurricane Andrew heavily hit the insurance agencies and inflicted insured losses amounting to $ 16 billion and over $25 billion (accounted for inflation, 2011) in overall losses [18, 89]. This was dwarfed by hurricane Katrina in 2005 which amounted to over $ 45 billion in overall losses[78]. Andrew caused the insolvency of eight insurance companies and pushed many to the brink of extinction[18]. While many companies attempted to boost their insurance capacity by buying reinsurance from those who had been less affected by hurricane Andrew, some in the insurance industry created a new instrument to bolster available capital. This was the birth of the catastrophe bond. The first CAT bonds were issued in 1997 and were defined (and still retain) as "a security that pays the issuer when a predefined disaster risk is realized, such as a hurricane causing $500 million in insured losses or an earthquake reaching a magnitude of 7.0"[90] on the Richter scale.

If one considers a no arbitrage condition then CAT swaps will behave similar to how CAT bonds do and should have high comparative yields and an immaterial return correlation with other asset classes[9, 27, 36].

6.1.1 Literature Review

CAT swaps are immune to most financial crises but are prone to other risks like interest rate risk and currency exchange risk which Lai, Parcollet and Lamond[68] have gauged succinctly. Poncet and Vaugirard[91] have studied a risk-neutralised CAT bond ignoring catastrophic risk but instead use simple diffusion process for the catastrophe index and a stochastic interest. While almost entirely unrealistic, it does give a closed form for the CAT bond price. Vaugirard[100, 102] has furthered this analysis by replacing the diffusion process with an arbitrage approach and subsequently a monte carlo approach. Lee and Yu[74] have put forth insurance model pricing by modelling loss dynamics using Poisson process. Jarrow[61] travels a different route and approaches CAT risk analogously to credit risk and provides a method to value CAT bonds based on the probability of a catastrophe occurring and the percentage of loss that it causes. Egami and Young[41] have considered a utility-based pricing of CAT bonds while Dieckmann[38] proposed a dynamical equilibrium model following the external habit process as discussed by Campbell and Cochrane[26].

CAT swaps, in contrast to CAT bonds, have received very little scholarly attention and almost none specifically to pricing CAT swaps apart from Braun (*ibid.*). Authors like Galeotti *et al* [44], Lane [72], and Lane, Morton and Mahul[71] have discussed an actuarial pricing method for CAT swaps and provide some empirical analysis of a few of these approaches. Aase modelled CAT derivative pricing by using an equilibrium model[1] and later amends it to be modelled by Markovian processes[2]. Apart from these, the majority of the literature on pricing CAT swaps prefer to use the no arbitrage frameworks. Noteworthy works include Cummins and Geman[34, 35] attempting to value CATs in Asian options with a constant jump amplitude diffusion process, Chang *et al*[31] developing a model accessing stochastic time change resulting in the rewriting of a compound Poisson process into a simple diffusion process, Baryshnikov[12] and Lee and Yu[74] adopting stochastic interest rates, and Bakshi and Madan [6, 7] providing a solution for CAT option price in its closed form.

Wu and Chung[107] successfully used a double stochastic Poisson process to price CAT bonds within the ambit Ornstein-Uhlenbeck intensity as an addition to the Cox *et al*[33] model for interest rate, and Jarrow and Yu[62] framework for default risk. The section on pricing CAT swaps will deal with this form of modelling.

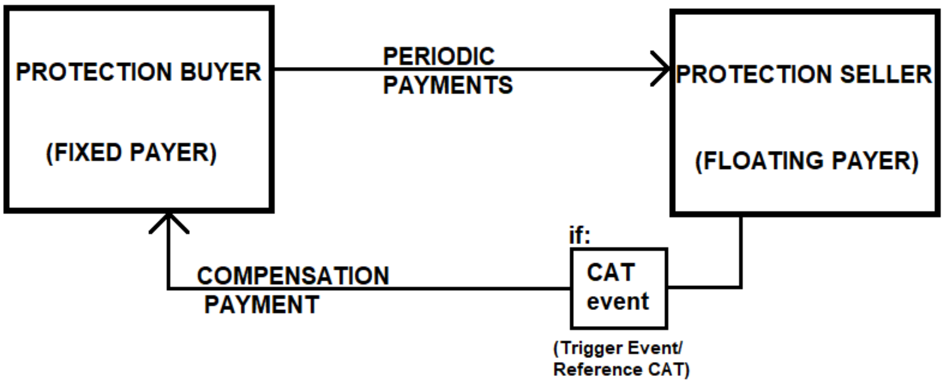

Figure 6.1: CAT Swap contract design

6.1.2 Contract Design

A perspicuous explanation of the design of a generic CAT swap is given in Fig6.1 and is explained explicitly in this section.

Catastrophe (CAT) swaps are financial instruments which transfer the risk of natural disasters among counter-parties. A generic contract consists of a protection buyer (fixed payer) making periodic payments to the protection seller (floating payer) who in turn pays the compensation payment to the fixed payer contingent on the binary event of the trigger event (reference CAT/covered event) occurring within the covered territory. The cover territory is the geographic extent of the coverage and the catastrophe has to strike within this region for the swap to be relevant. The reference peril (reference cat) is the trigger event and defines the type of disaster that the swap covers - this could include earthquakes, windfalls in hurricanes, and tsunamis. When a trigger event has occurred (or occurs), the assigned damage report provider assesses the initial estimate of the insurance losses resulting from the event and assigns a serial number to the case. This loss is refreshed continuously over a period of no more than 6 months and the final report is released within this time-frame. It is important to note that for CAT swaps, losses from different catastrophe's within the covered territory are not appended as an aggregate but are treated and filed separately. In other words, the trigger mechanism is related to individual catastrophe losses. CAT swaps exhibit an event threshold and an acceleration threshold (which is slightly larger than the former). The contract terminates if the estimate of the final loss crosses the event threshold (also called the attachment level) and this results in an immediate payoff to the fixed payer. This is also the case if the interim loss estimate exceeds the acceleration threshold. To make more explicit the concept of CAT swaps, we consider an example. The *Deutsche Bank Event Loss Swaps (ELS)* is one such CAT swap offered in the market used for wind storms (launched in 2006) with three thresholds - \$ 20 billion, \$ 30 billion, and \$50 billion. With a standard maturity of one calendar year, the notional amounts are staggered in lots of \$ 5 million. Similarly, swaps have been released by the Swiss Re as *Swiss Re Natural Catastrophe Swaps*. The Property Claim Services division of Insurance Services Offices works as the provider of the loss report. Risk swaps can also be executed via third-parties such as the Catastrophe Risk Exchange, which is web based, or directly as an over the counter market swap. Risk swaps are like standard CAT swaps and the present values of the two swap sides balance exactly. Additionally, there exist no up-front payments between the counter-parties.

6.1.3 Market Development

As we have established, CAT bonds have been around for almost three decades, the market for CAT swaps is taking its baby steps. Due to the OTC nature of the swaps, the information on transactions is at best derived from personal accounts[36]. However, there is rapid chatter of the acceleration of market size[35]. In 2009, the international Swaps and Derivatives Association (ISDA) set out a document template for CAT swaps referencing windstorm events in the US [23, 73]. All CAT counter-parties are insurance and reinsurance agencies which work within the ambit of the pre-defined terms to aide in transparency, reduce uncertainty, and increase liquidity which in turn ecourages market growth. According to Litzberger[76], Bantwal and Kutnreuther[10], and Cummin and Weiss [36] the reasoning of no arbitrage ensures that CAT swaps behave like CAT bonds and therefore has high material yields. In other words, for an investor, CAT swaps allow for synthetic exposure to catastrophe risk without actually having to invest in a CAT bond. Additionally, the protection seller (or buyer) do not need to be established as insurance regulator entities to be eligible as swap counter-parties. This ensures that CAT swaps can be used for investment[1] in addition to hedging. There is however a difference between ILW and CAT swaps that the latter is not an insurance contract. The regulations around the swaps are unambiguous and under the IFRS (or US GAAP) catastrophe swaps, are not eligible for reinsurance accounting due to them being pure index contracts. Irrespective, the swaps are economically equivalent to ILWs in spite of not having an indemnity trigger.

The following sections will first use structural models to price a CAT swap following which a double stochastic pricing model will be discussed in accordance with Braun's work[20]. Finally, we propose a deep learning dual model to estimate the CAT loss which forms an integral part of pricing a CAT swap.

6.2 Pricing CAT Swaps Using Structural Models

CAT swaps are considered special instruments and are rarely priced using structural models. But the approach mentioned here can be generalised for all models including reduced, and hybrid models. Caution must be taken, however, since these methods are far from robust.

The pricing of the swap requires us to find the spread of the instrument. We consider a **valuation framework** where time is discreet (but can easily be replaced by continuous time). Once we have derived the risk neutral price of the pure discounted bond maturing at T for some initial time t_0 we can proceed to calculate the spread of the instrument by equating the premium and protection legs [22, 49]. For simplicity, we assume a constant interest rate r since we are in the valuation framework of a toy

[1] An example for this is the negative bass trade between the CAT swaps and CAT bonds. This allows for a risk-arbitrage strategy exploits the price difference between the cash and derivative instrument. In case the spread on the bond is substantially larger than that on a similar sizeable swap, then this negative basis allows a positive carry can be employed in buying the bong and at the same time hedging for protection under the swap.

Merton Model. Thus, the risk-neutral price is given by

$$P\left(r(t_0), t_0; T\right) = e^{-r(T-t_0)} \tag{6.1}$$

where r is the default-free short-term interest rate which can be specified with a stochastic process. Assume s_{cat} to be the spread of the CAT swap that is paid in discrete time intervals, and the protection duration to be from t_0 to T giving a time to maturity as $T - t_0$. If t_d is the random time of default (in this case, the event of catastrophe) then the premium leg of the swap[2] is given by

$$\begin{aligned} \textbf{PremLeg} &= \mathbb{E}\left[s_{cat} \sum_{i=0}^{T-t_0} e^{-\int_{t_0}^{t_i} r du} \mathbf{1}_{\{t_d > t_i\}} \right] \\ &= \mathbb{E}\left[s_{cat} \sum_{i=0}^{T-t_0} e^{-r(t_i - t_0)} \mathbf{1}_{\{t_d > t_i\}} \right] \end{aligned} \tag{6.2}$$

where $\mathbf{1}_{\{.\}}$ is the indicator function. Consider a single protection payment, denoted by LGD which is operated at t_d. If we assume that at the instance of random default at t_d, δ pure discount default-free bonds which have the same maturity and payment as the initial security is given to the bondholder, then,

$$LGD = (1 - \delta) P\left(r(t_0), t_0; T\right) \tag{6.3}$$

This allows us to write the protection leg as

$$\begin{aligned} \textbf{ProLeg} &= \mathbb{E}\left[e^{-\int_{t_0}^{t_d} r du} (1 - \delta) P\left(r(t_d), t_d; T\right) \mathbf{1}_{\{T \geq t_d > t_0\}} \right] \\ &= \mathbb{E}\left[e^{-r(t_d - t_0)} (1 - \delta) P\left(r(t_d), t_d; T\right) \mathbf{1}_{\{T \geq t_d > t_0\}} \right] \end{aligned} \tag{6.4}$$

[2]We assume it analogous to the CDS (see Chapter 6)

Equating the premium and protection legs allows us to arrive at the value of CAT spread as

$$s_{cat} = \frac{\mathbb{E}\left[e^{-r(t_d-t_0)}(1-\delta)P(r(t_d),t_d;T)\mathbf{1}_{\{T\geq t_d>t_0\}}\right]}{\mathbb{E}\left[\sum_{i=0}^{T-t_0} e^{-r(t_i-t_0)}\mathbf{1}_{\{t_d>t_i\}}\right]} \tag{6.5}$$

The probability of default in case of a CAT swap is the same as the probability of a catastrophe occurring which in this case is modelled as based on the Merton model, and thus us given by $N(-d_2)$ as we have previously shown in Chapter 3. Thus, the expression for the price of the CAT swap can be readily reduced to

$$s_{cat} = \frac{(1-\delta)e^{-r(t_i-t_0}e^{-r(T-t_0)}[1-N(-d_2)]}{\sum_{i=0}^{T-t_0} e^{-r(t_i-t_0)}N(-d_2)} \tag{6.6}$$

The value of recovery is assumed to be a constant in the above analysis but is in fact not realistic to catastrophes. The recovery rate in case of a CAT swap depends on the financial loss cased by a catastrophe. While this can be estimated only ex-post, at the end of this chapter we propose a machine learning model to estimate this loss and thereby the recovery (δ) ex-ante.

6.3 Double Stochastic Pricing Model

Various pricing strategies for CAT swaps have been proposed in the past. Wang and Xu propose a reduced risk model with counterparty credit risk assuming a loss process that is generated by a doubly stochastic Poisson process[108]. A jump diffusion process, that is correlated with the loss process, models the share price process. They rely on the Vasicek model - which is known to be robust[5] - for the modelling of the default intensity and the interest rate.

Others like Lamond, Parcollet *et al* valuate CAT bonds exposed to currency risk[69]. The model considers an arbitrage approach to to price CAT bonds with the exchange rate, domestic interest rate, and foreign interest rate being modelled by a three-dimensional stochastic process (as an extension of an n-dimensional Brownian motion). They note that the price of the CAT bond is affected detrimentally by its correlation with domestic interest rate and by the volatility of the exchange rate, while the domestic interest rate has an increasing effect on it.

Braun, however, focuses on the two stage pricing model (*ibid.*). The essential ex-ante variables involved in the pricing of swaps would be the randomnesses of the catastrophes, their timing of occurrence, and the final loss estimate (made no more than 6 months after the catastrophe). In addition to being able to determine a fair CAT swap spread before the natural disaster, it is also important for market players to value the contracts when the catastrophe has already happened, right after the initial loss estimate report has been published.

The first phase - the re-estimation phase - would be under the circumstances of some knowledge of the timing of the catastrophe and an approximate idea of the damage it would cause. However, there exists uncertainty in the future development of the loss estimate, that is, there would be two cases. The first one is an ex-ante pricing of the CAT swaps, with the second being the pricing of the swap during the re-estimation phase. We will proceed to provide a continuous-time contingency model without market frictions such as taxes, bid-ask spreads, transaction costs, or government intervention - a friction-less model - as followed in Alexander Braun (2011) (*ibid.*).

6.3.1 Risk-neutral valuation

While the current industrial practice is focused on using actuarial approaches to price CAT swaps, we will employ a financial approach since the swaps are not insurance contracts. Merton[80] and Black-Scholes[17] have laid out modern option pricing theory with a no-arbitrage condition which has enabled a preference-free pricing of a risk-neutral pricing of derivatives - an equivalent martingale measure. It is foreseeable that such an equivalent martingale measure exists because of the absence of arbitrage (both weak and strong) in the market. But to prove that there exists a unique martingale measure, we must be able to prove that the market is complete. This will mean that all the contingent claims are can be replicated via securities that are available[51]. It is a well known fact that insurance markets are incomplete[52, 53, 97] and this prevents the uniqueness of a martingale measure. But as we have mentioned in the literature review for CAT swaps, the majority of the literature proceeds with contingency claims valuation frameworks and that natural disaster risk cannot be hedged with traditional forms of security. Authors like Cummins and Geman[34, 35], Chang *et al*[29–31], Geman and Yor[45] , Barshnikov *et al*[12], Muermann[84, 85], and Vaugirard[100–102] assume that the completeness of the market is preserved due to sufficiently liquid instruments and tradeables that are driven by the same source of randomness. This, in turn, suggests that replicating portfolios can be formed and by obtaining the market price of a CAT risk we can obtain a unique risk-neutral measure. Several financial exchanges have introduced contracts like CME hurricane index, and PCS index to establish possible liquid market for exchange-traded catastrophes, which despite being in this pre-formative stages, has a promising outlook in the long run.

One can still work with incomplete markets as shown by Merton[79] and Bakshi Madan[6] by treating natural risk (disaster) as unsystematic shocks which enables us to execute the risk neutrality of the market players. This is supported by the empirical analysis of Hoyt and Mcculough[55] and Cummins and Weiss[36].

The incomplete market can be overcome by employing a change of measure (such as, Esscher Transform in Gerber and Shiu[46]). Wang[104] introduced a distortion operator $g_\alpha(u) = \Phi\left[\Phi^{-1}(u) + \alpha\right]$ where Φ is the standard normal cumulative distribution and uses this to recover risk-neutral option pricing for the Black-Scholes model. In this the rest of this sub-section, however, we will assume that we have already managed to calculate the risk-neutral probability measure $\mathbb{Q}$ by changing the physical measure $\mathbb{P}$ using a suitable change of measure and will directly proceed to working in our model framework with risk-neutrality. $\mathbb{Q}$ will have the same characteristics of stochastic processes and distributions as $\mathbb{P}$.

6.3.2 Ex-ante Pricing

A more general sense of pricing the CAT swap deals with doing so before the catastrophe has struck in the covered territory. The prime uncertainty in this time-period is the stochastic number and timings of the disasters during the term of the contract and an uncertainty in the final loss they will cause. Consider a probability space $(\Omega, \mathscr{S}, \mathbb{Q})$. We further assume that the number of natural disasters $n_{t,t+\delta t}$ in an interval $(t, t+\delta t]$ is Poisson distributed with intensity $\lambda_{t,t+\delta t} = \int_t^{t+\delta t} \lambda(u) du$; that is

$$n_{t,t+\delta t} \sim \mathscr{P}(\lambda_{t,t+\delta t}), \quad \forall t \in (0, T-\delta t] \tag{6.7}$$

We could have also modelled using a Bernoulli trial implying a binomial distribution sum for the occurrence of a catastrophe at each time step. But this would be less relevant with respect to the total events till maturity because the binomial distribution converges to the Poisson distribution for large number of trials.

We will also allow for the cyclic nature of the natural disasters by employing the stochastic Poisson intensity governed by the mean-reverting Ornstein-Uhlenbeck process

$$d\lambda(t) = \kappa(\mu_\lambda - \lambda(t))\,dt + \sigma_\lambda dW_\lambda^{\mathbb{Q}}(t) \tag{6.8}$$

where κ is the mean reverting rate and the other symbols have their usual meanings (see Chapter 1). Thus we have introduced the arrival of the natural disaster using a doubly stochastic Poisson process, a Cox process. The twofold randomness is because Eq(6.8) generates the intensity of the Poisson process. The empirical evidence for using such a process has been well documented by Braun(*ibid.*; *et al*) and makes sense for climate change events like El Niño.

Associated with each catastrophe i is an i.i.d. Y_i with distribution function $F_y(X)$ and represents the stochastic final loss estimate of the i^{th} catastrophe. It is reasonable to assume that Y_i and $n_{t,t+\delta t}$ are stochastically independent. Furthermore, we assume that the release of loss estimate after a disaster is instantaneous and with no time delays. Therefore, we can write the aggregate final loss estimate within a period $(t, t+\delta t]$ as a compound Poisson process. This compound Poisson process has expectation $\lambda_{t,t+\delta t} E^{\mathbb{Q}}[Y_i]$ and thus

$$L_{t,t+\delta t} = \sum_{i=1}^{n_{t,t+\delta t}} \quad \forall t \in (0, T-\delta t] \tag{6.9}$$

Since the payoff of the cat swap is spontaneously triggered in the event of the final estimate crossing (or becoming equivalent) to the event threshold, we only aggregate each final loss separately and compared to the event threshold at the time of its occurrence. In other words, we do not aggregate losses from different events together as a whole in the entire duration of the contract. Due to this aforementioned premature termination of the instrument if the trigger event occurs, it is imperative that the pricing model captures path dependency. This can be done by sequential re-evaluation of Eq(6.9) for infinitesimal time step dt from the beginning of the contract to its maturity:

$$\lim_{\delta t \to 0} L_{t,t+\delta t} \equiv L_{t,t+dt} = dL_t \quad \forall t \in (0, T-\delta t] \tag{6.10}$$

Therefore, we now have a series of compound Poisson processes instead of one single process for the whole term. This ensures that in non-overlapping intervals, catastrophe arrivals are independent and the probability of one natural disaster in interval dt is approximately $\lambda_{t,t+dt}$ with the probability of more than one disaster being negligibly small.

As mentioned earlier, CAT swaps consist of a fixed payment which is the stream of regular premiums paid by the protection buyer (Fixed buyer) and a floating payment which is triggered by the CAT event. Usually, swap pricing is based on the separate valuation of these two with with the aim of balancing these through the fair spread s_{cat}:

$$PV_{float} = PV_{fix}(s_{cat}) \tag{6.11}$$

The first passage time (τ) for the series of compound process is defined as the earliest instant wherein the final loss estimate is equal to or crosses the event threshold (ET)

$$\tau \equiv \inf\{t \| L_{t,t+dt} \geq ET\} \tag{6.12}$$

The present value of the floating payment can be written as

$$PV_{float} = E_0^{\mathbb{Q}}\left[e^{-r\tau}\alpha N \mathbf{1}_{\tau \leq T}\right] \tag{6.13}$$

where we have considered a CAT swap with maturity T, notional N, and a payoff determined in percentage of the notional α. $\mathbf{1}$ is the indicator function and this specific case $\mathbf{1}_{\tau \leq T}$ is unity if $\tau \leq T$ and 0 otherwise, r is the instantaneous risk-free interest rate - the term structure is flat and is deterministic.

The fixed payment from the swap is constituted by the regular premiums and the accrual payment which is made in case τ does not coincide with the premium payment date. Thus

$$PV_{fix}(s_{cat}) = PV_{prem}(s_{cat}) + PV_{accr}(s_{cat}) \tag{6.14}$$

Assuming that ET has not been crossed, we can value the accrual bit of the above as

$$PV_{accr}(s_{cat}) = E_0^{\mathbb{Q}}\left[e^{-r\tau} s_{cat} N(\tau - t_{i-1}\mathbf{1}_{t_{i-1}} \leq \tau \leq t_i\right] \quad i = 1, 2, ..., n \tag{6.15}$$

Of we consider δt_i to be the length of the premium period $t_{i+1} - t_i$, then the buyer makes payment of

$s_{cat}N\delta t_i$ on premium dates t_i. Thus we can write

$$PV_{prem}(s_{cat}) = \sum_{i=1}^{n} e^{-rt_i} E_0^{\mathbb{Q}} \left[s_{cat} N \delta t_i \mathbf{1}_{\tau > t_i} \right] \tag{6.16}$$

Although we have to solve a time stoppage problem, we cannot analytically solve it due to the compound nature of the Poisson process as shown in Eq(6.9). Thus one resorts to Monte Carlo techniques to calculate s_{cat}, the fair spread.

6.3.3 Loss Re-estimation Phase Pricing

Once the swap has been initiated, the actual market-to-market price of the swap will fluctuate based on the occurrence of natural disasters and with the development of the loss estimates. When the interim loss estimates approach the threshold values, a spike in value must be observed. However, the ex-ante pricing model does not include this sensitivity which necessitates pricing CAT swaps during the re-estimation phase as well. This is the phase after the catastrophe has struck and the initial loss estimate has been published. The uncertainty now is about the value of the final loss estimate which will be captured (as in Braun[20]) by using a penurious[3] barrier option framework which will enable us to derive closed forms for the spread of the CAT swap.

We assume that the dynamics of the interim loss estimates in consideration of the CAT swap ar governed by a Geometric Brownian Motion (GBM)

$$\frac{dL^i(t)}{L^i(t)} = rdt + \sigma dW^{\mathbb{Q}}(t) \tag{6.17}$$

where σ is the volatility, and $dW^{\mathbb{Q}}(t)$ is the standard $\mathbb{Q}$-Wiener process. Using a diffusion process for the dynamics of loss estimates is in line with the methodology followed in usual CAT bond pricing and is well documented in Bakshi and Madan[6], Schmidli[98] and Biglani[15].

[3] Parsimonious or stingy.

The distance to the acceleration threshold AT($\geq$ET) at time t can be defined as

$$D_{AT}(t) = \frac{AT}{L^i(t)} \tag{6.18}$$

The distance to the event threshold can similarly be defined as

$$D_{ET}(t) = \frac{ET}{L^i(t)} \tag{6.19}$$

As in ex-ante pricing, we match the floating and fixed legs of payment using s_{cat} as in Eq(6.11) Assume a time $0 \leq s \leq T$ at which instant an interim loss for a specific cat L^i is available. Note that L^i is deterministic in the re-estimation phase while it was unknown in the ex-ante phase. Thus the interim loss process as given in Eq(6.17) starts at $L^i(s)$ and when this crosses the AT for the first time (say at t), then the protection buyer receives the payment. This can be otherwise expressed as the time when D_{AT} becomes unity for the first time. We can define the first passage time for the diffusion process as

$$\tau_d = \inf\{t \| L^i(t) \geq AT\} \tag{6.20}$$

The first passage time density can be written analytically as

$$h(\tau_d) = \frac{\ln\left(D_{AT(s)}\right)}{\sqrt{2\pi\sigma^2}\tau_d^{\frac{3}{2}}} \exp\left[-\frac{1}{2}\left(\frac{\ln(D_{AT}(s)) + \left(r - \frac{\sigma^2}{2}\tau_d\right)}{\sigma\sqrt{\tau_d}}\right)^2\right] \tag{6.21}$$

which is as established by Reiner[94, 95]. Since the compensation payment by the protection seller is made either in the case of maturity (at T) or if the interim loss estimate crosses event threshold, ET can be seen as the strike price of a binary European call option with a knock-out feature to avoid pricing some states twice. This feature will ensure that it lapses when $L^i(t) \geq AT$. To make this more concrete, consider the situation when $L^i(T) = AT \geq ET$ where it can be seen that both European call option and a up-and-in (UAI) one-touch option will pay off. Therefore, to calculate the floating leg, we need to combine UAI one touch and a binary up-and-out (UAO) call option which will pay if and only if we cross ET keeping that AT has not been crossed. This gives us

$$PV_{float} = UAI_{ot} + UAO_{call} \tag{6.22}$$

We employ results from Reiner[94, 95] to give the price of UAI one touch as

$$\begin{aligned} UAI_{ot} &= \int_s^T \alpha N e^{-r\tau_d} h(\tau_d) d\tau_d \\ &= \alpha N \int_s^T e^{-r\tau_d} h(\tau_d) d\tau_d \\ &= \alpha N Q(T) \end{aligned} \tag{6.23}$$

with $\Phi(x)$ is the standard normal cumulative distribution function (cdf), $s < u$ and

$$Q(u) = \int_s^u e^{-r\tau_d} h(\tau_d) d\tau_d = D_{AT}^{a+b}(s)\Phi(d_1) + D_{AT}^{a-b}\Phi(d_2)$$

$a = \frac{r}{\sigma^s} - \frac{1}{2}$

$b = \frac{\sqrt{(r-\frac{\sigma^2}{2})^2+2r\sigma^2}}{\sigma^2}$

$d_1 = \frac{-\ln(D_{AT}(s))-b\sigma^2 u}{\sigma\sqrt{u}}$

$d_2 = \frac{-\ln(D_{AT}(s))+b\sigma^2 u}{\sigma\sqrt{u}}$

Additionally, we can follow Rubenstein and Reiner [94, 95] to give the value of the binary UAO as

$$UAO_{call} = \alpha N \left(B_1(T) - B_2(T) + B_3(T) - B_4(T)\right) \tag{6.24}$$

where

$B_1(T) = e^{-rT}\Phi\left[x_1 - \sigma\sqrt{(T)}\right]$

$B_2(T) = e^{-rT}\Phi\left[x_2 - \sigma\sqrt(T)\right]$

$B_3(T) = e^{-rT}D_{AT}^{2a}(s)\Phi\left[-y_1 + \sigma\sqrt(T)\right]$

$B_4(T) = e^{-rT}D_{AT}^{2a}(s)\Phi\left[-y_2 + \sigma\sqrt(T)\right]$

with

$x_1 = \frac{-\ln(D_{ET}(s))+(a+1)\sigma^2 T}{\sigma\sqrt{T}}$

$x_2 = \frac{-\ln(D_{AT}(s))+(a+1)\sigma^2 T}{\sigma\sqrt{T}}$

$y_1 = \frac{-\ln(\frac{AT^2}{L^i(s)ET})+(a+1)\sigma^2 T}{\sigma\sqrt{T}}$

$y_2 = \frac{\ln(D_{AT}(s))+(a+1)\sigma^2 T}{\sigma\sqrt{T}}$

Once again, as before, the fixed payment segment consist of the premiums and the accrual in accordance with Eq6.14. The fixed payer pays $s_{cat}N\delta t_i$ on the remaining dates of the premium $s < t_i \leq T$. We can define the probability that the trigger event does not occur before u as the survival rate of the contract from s to u as $(1 - H(u))$ where $H(u) = \int_s^u h(\tau_d)d\tau_d$. Then the premium payments will be

$$\begin{aligned} PV_{prem}(s_{cat}) &= \sum_{i=1}^{n} s_{caT}N\delta t_i e^{-rt_i}(1 - H(t_i)) \\ &= Ns_{cat}\sum_{i=1}^{n}\delta t_i e^{-rt_i}(1 - H(t_i)) \\ &= Ns_{cat}\sum_{prem} \end{aligned} \tag{6.25}$$

where

$$\sum_{prem} = \sum_{i=1}^{n}\delta t_i e^{-rt_i}(1 - H(t_i))$$

with

$$H(u) = \int_s^u h(\tau_d)d\tau_d = D_{AT}^{2a}(s)\Phi(z_1) + \Phi(z_2)$$

and

$$z_1 = -\frac{\ln(D_{AT}(s) - (r - \frac{\sigma^2}{2})u}{\sigma\sqrt{u}}$$

$$z_2 = -\frac{\ln(D_{AT}(s) + (r - \frac{\sigma^2}{2})u}{\sigma\sqrt{u}}$$

The present value of the accrual at $t = s$ can be expressed as

$$\begin{aligned} PV_{accr} &= \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} s_{cat} N(\tau_d - t_{i-1}) e^{-r\tau_d} h(\tau_d) d\tau_d \\ &= s_{cat} N \left[\sum_{i=1}^{n} \int_{t_i-1}^{t_i} \tau_d e^{-r\tau_d} h(\tau_d) d\tau_d - \sum_{i=1}^{n} t_{i-1} \int_{t_i-1}^{t_i} e^{-r\tau_d} h(\tau_d) d\tau_d \right] \\ &= s_{cat} N \left[\sum_{i=1}^{n} (J(t_i) - J(t_{i-1})) - \sum_{i=1}^{n} t_{i-1}(Q(t_i) - Q(_{i-1})) \right] \\ &= s_{cat} N \Sigma_{accr} \end{aligned} \tag{6.26}$$

Where

$$\Sigma_{accr} = \left[\sum_{i=1}^{n} (J(t_i) - J(t_{i-1})) - \sum_{i=1}^{n} t_{i-1} (Q(t_i) - Q(_{i-1})) \right]$$

and

$$J(u) = \int_s^u \tau_d e^{-r\tau_d} h(\tau_d) d\tau_d$$

We can substitute equations 6.22-6.26 into equation 6.11 to get

$$\begin{aligned} PV_{float} &= PV_{fixed}(s_{cat}) \\ (UAI_{one-touch} + UAO_{call}) &= s_{cat} N (\Sigma_{prem} + \Sigma_{accr}) \\ \implies s_{cat} &= \frac{(UAI_{one-touch} + UAO_{call})}{N (\Sigma_{prem} + \Sigma_{accr})} \end{aligned} \tag{6.27}$$

This spread needs to be added to the results from the ex-ante pricing model when working in the re-evaluation paradigm.

6.3.4 Sensitivity to Interim Loss Estimates

We devote a small digression to the discussion of how this instrument is sensitive to the estimates of the interim loss. It is clear that a low $L^i(s)$ results in a worthless instrument under the barrier option because the re-estimation process will no longer be a major risk driver. This is due to the fact that $L^i(s)$ have large distance to the threshold from which we infer that there are low probabilities of the up-and-in-one-touch being triggered and subsequently the an in the money up-and-out call. The up-and-in one-touch mirrors a CAT swap without an ex-ante component in a parallel fashion. The value of $L^i(t)$ for a up-and-out call resembles an inverted dome. These can be visualised using numeric examples. Sensitivity of the instrument with respect to the time to maturity (T) of the instrument is also worth noting. The barrier option approach yields an additional value as high as that from an ex-ante only pricing strategy for all T.

6.3.5 Counterparty Default Risk and CDS

In addition to being affected and sensitive to the natural disaster risk, CAT swaps also have a great deal of sensitivity to counterparty default risks. This can occur in two ways, the first being when the protection seller defaults in the term of the contract, and second when the protection seller defaults after a trigger event has occurred. The first scenario results in the protection buying losing his hedge which now requires him to enter new contracts at the instant market conditions which could now be in opposition to him and unfavourable. The second scenario results in the protection seller being unable to pay the compensatory payment. The easiest way to incorporate this risk in the pricing model is to subtract the CDS spread from the CAT spread:

$$s_{cat}^{df} = s_{cat} - s_{CDS} \tag{6.28}$$

We use the CDS spread since the CDS is used a default risk measure in the market. Although FICO scores and agency credit ratings are also used, they are not as widespread and robust as the CDS[66]. Taking into account such a structure, the protection buyer will have successfully eliminated the risk of default from the contract for CAT swap. Thus Eq6.28 tells us that the default risk of the protection seller and the spread that protection buyer is willing to pay are inverse additive related - the higher the former, the lower the latter. In some cases, it so happens, that the investor is able to procure a market quote although CDS contracts are widely traded for various CAT swap counter-parties. The CDS pricing model is used in this case and has been used by Houweling and Vorst[54] *et al.* They begin with the general swapping identity

$$PV_{float}^{CDS} = PV_{fixed}^{CDS}(s_{cat}) \tag{6.29}$$

Call the stochastic time of default to be τ^{CDS} which is depicted by the first jump of Poisson process that has a constant hazard rate of λ^{CDS} under a $\mathbb{Q}$. Since there is a lack of statistically significant correlation between CAT risks and default (due to lack of publicly available empirical evidence) it is safe to assume that the hazard rate is independent of the Poisson intensity of the catastrophe which allows corporate defaults to occur any time - before, during, or after the catastrophe. Considering a recovery rate of δ the present value of the floating leg for the CDS is given by

$$PV_{float}^{CDS} = \mathbb{E}_0^{\mathbb{Q}}\left[e^{-r\tau^{CDS}}(1-\delta)N\mathbf{1}_{\tau^{CDS}\leq t_i}\right] \tag{6.30}$$

The fixed leg is the sum of the premiums ($s_{CDS}N\Delta t_i$) fulfilled on t_i - the date of the CAT swap itself - and the accrual payment ($s_{CDS}N(\tau^{CDS}-t_{i-1})$) as long as it is not the same as the premium payment

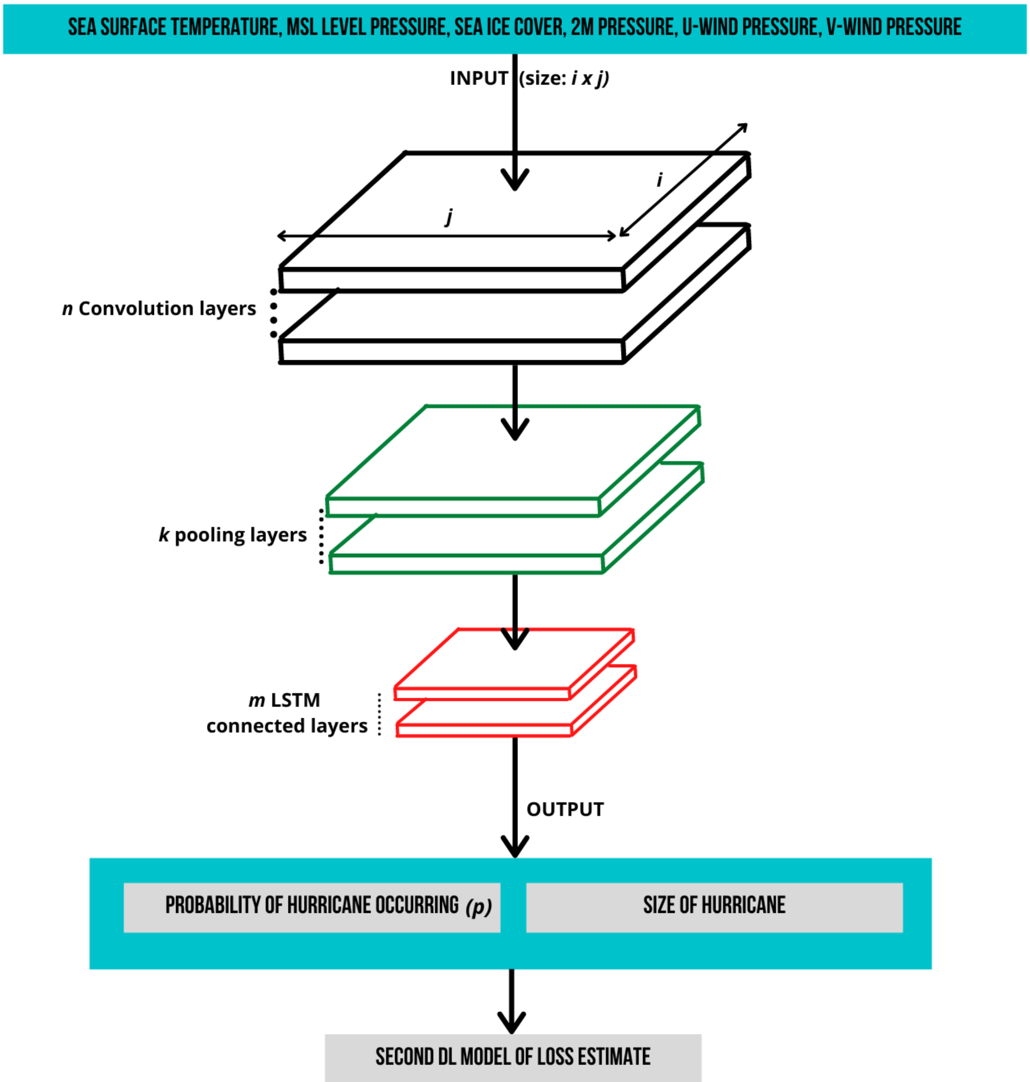

Figure 6.2: Proposed workflow to estimate CAT loss to expedite pricing process

date, and so, is given by

$$
\begin{aligned}
PV^{CDS}_{fixed} = \sum_{i=1}^{n} e^{-rt_i} E_0^{\mathbb{Q}} \left[s_{CDS} N \Delta t_i \mathbf{1}_{\tau^{CDS} > t_i} \right] \\
+ \mathbb{E}_0^{\mathbb{Q}} \left[e^{-r\tau^{CDS}} s_{CDS} N \left(\tau^{CDS} - t_{i-1} \right) \mathbf{1}_{t_{i-1} \leq \tau^{CDS} \leq t_i} \right] \quad (6.31)
\end{aligned}
$$

6.4 ML Estimation of CAT Loss

The estimation of CAT loss is fundamental to the pricing of the CAT swap as we have shown in the previous sections of this chapter. This purports us to consider how machine learning can augment this task in providing ex ante values to the loss which is accounted for within the recovery rate δ. Figure 6.2 depicts a proposed workflow for the ML model to estimate losses for hurricanes[4]. The inputs to this model are images in multiple channels (RGB, IR, UV) which provide data about sea surface temperature, pressure, wind pressure, and other factors. This model can be later replaced by a deep UNET model to provide more robust predictions since the UNET architecture has been shown to be more effective on CNN based analysis and have been previously implemented by the author on satellite image segmentation of 8 channel images to highlight geographical features[93, 103]. This toy model produces a probability of occurrence of hurricane and the size of the hurricane by learning on data of previous hurricane sizes and weather conditions. This is in turn placed into a second ANN or RNN model that estimates the interim loss ex ante.

[4]This can easily be extended for other natural disasters

7. Conclusion

7.1 Future Possibilities

The research presented here can be considered a small step in the direction of involving augmented work of man and machine in the field of credit swaps and financial instrument analysis. CAT swaps present a tangibly solvable problem that can be assessed and carried forward.

There is a major scope of scholarly work and research in modelling the pricing model for CAT swaps to contribute to the scanty literature in this field. The introduction of estimating catastrophe losses using machine learning in this work ushers in the new possibilities of research and marketable intervention which can provide better insight into pricing this financial instrument.

Since this is a new field, there is a high chance of development of novel methods in pricing these instruments. Moreover, the field of machine learning has found a strong foothold in the field of finance primarily in stock market analysis but not in pricing financial instruments.

7.2 Conclusion

This work has introduced and expounded on the concept of credit risk and provided a mathematical background to understand the same. A brief and gentle introduction to machine learning in the field of credit risk analysis specifically pertaining to customer default in banks.

Various structural models with their risk structure and recovery rates were discussed with special focus on their application in pricing catastrophe swaps. CAT swap pricing strategies were also discussed and presented extensively. A final ML possibility in assessing the recovery leg of the instrument. Finally, the credit default swap is thoroughly examined with a valuation of a plain vanilla CDS and on how to price a CDS.

Bibliography

[1] Knut Aase. “An Equilibrium Model of Catastrophe Insurance Futures and Spreads”. In: *The Geneva Papers on Risk and Insurance Theory* 24.1 (1999), pages 69–96. DOI: 10.1023/a:1008785300001 (cited on page 62).

[2] Knut K. Aase. “A Markov Model for the Pricing of Catastrophe Insurance Futures and Spreads”. In: *The Journal of Risk and Insurance* 68.1 (2001), page 25. DOI: 10.2307/2678131 (cited on page 62).

[3] Ali Sulieman Alshatti. “The effect of credit risk management on financial performance of the Jordanian commercial banks”. In: *Investment management and financial innovations* 12,№ 1 (contin. 2) (2015), pages 338–345 (cited on page 5).

[4] Patrick Augustin et al. “Credit Default Swaps: Past, Present, and Future”. In: *Annual Review of Financial Economics* 8.1 (2016), pages 175–196. DOI: 10.1146/annurev-financial-121415-032806 (cited on pages 41, 43).

[5] Simon Babbs. “Generalised Vasicek models of the term structure”. In: *Appl. Stochast. Models Data Anal.* 1 (Jan. 1993), pages 49–62 (cited on page 67).

[6] Gurdip Bakshi and Dilip Madan. "Average Rate Claims with Emphasis on Catastrophe Loss Options". In: *The Journal of Financial and Quantitative Analysis* 37.1 (2002), page 93. DOI: 10.2307/3594996 (cited on pages 62, 69, 73).

[7] Gurdip S. Bakshi and Dilip B. Madan. "Average Rate Contingent Claims with Emphasis on Catastrophe Loss Options". In: *SSRN Electronic Journal* (2001). DOI: 10.2139/ssrn.271610 (cited on page 62).

[8] Luca Vincenzo Ballestra, Graziella Pacelli, and Davide Radi. "Modeling CDS spreads: A comparison of some hybrid approaches". In: *Journal of Empirical Finance* 57 (2020), pages 107–124 (cited on page 57).

[9] Vivek Bantwal and Howard Kunreuther. "A Cat Bond Premium Puzzle?" In: *Journal of Psychology and Financial Markets* 1 (June 1999). DOI: 10.1207/S15327760JPFM010107 (cited on page 61).

[10] Vivek Bantwal and Howard Kunreuther. "A Cat Bond Premium Puzzle?" In: *Journal of Psychology and Financial Markets* 1 (June 1999). DOI: 10.1207/S15327760JPFM010107 (cited on page 64).

[11] Wang Bao, Ning Lianju, and Kong Yue. "Integration of unsupervised and supervised machine learning algorithms for credit risk assessment". In: *Expert Systems with Applications* 128 (2019), pages 301–315 (cited on page 6).

[12] Y. Baryshnikov, A. Mayo, and D.R. Taylor. "Pricing of CAT bonds". In: *Preprint* (Aug. 2001) (cited on pages 62, 69).

[13] "basel III: A Global Regulatory Framework for More Resilient Banks and Banking Systems - Revised Version". In: (June 2011). URL: https://www.bis.org/publ/bcbs189.htm (cited on page 43).

[14] Basle Committee on Banking Supervision Basle. "CREDIT RISK MODELLING: CURRENT PRACTICES AND APPLICATIONS". In: *BIS Org* (1999). URL: https://www.bis.org/publ/bcbs49.htm (cited on pages 2, 5).

[15] Francesca Biagini, Yuliya Bregman, and Thilo Meyer-Brandis. "Pricing of catastrophe insurance options written on a loss index with reestimation". In: *Insurance: Mathematics and Economics* 43 (Oct. 2008), pages 214–222. DOI: 10.1016/j.insmatheco.2008.05.016 (cited on page 73).

[16] Fischer Black and John C Cox. "Valuing corporate securities: Some effects of bond indenture provisions". In: *The Journal of Finance* 31.2 (1976), pages 351–367 (cited on page 29).

[17] Fischer Black and Myron Scholes. "The Pricing of Options and Corporate Liabilities". In: Mar. 2019, pages 3–21. DOI: 10.1142/97898147595880001 (cited on page 68).

[18] Eric Blake et al. "The Deadliest, Costliest, and Most Intense United States Tropical Cyclones From 1851 to 2006 (and Other Frequently Requested Hurricane Facts)". In: (Jan. 2007) (cited on page 61).

[19] Matthieu Bouvard, Pierre Chaigneau, and Adolfo De Motta. "Transparency in the financial system: Rollover risk and crises". In: *The Journal of Finance* 70.4 (2015), pages 1805–1837 (cited on page 5).

[20] Alexander Braun. "Pricing catastrophe swaps: A contingent claims approach". In: *Insurance: Mathematics and Economics* 49.3 (2011), pages 520–536. ISSN: 0167-6687. DOI: https://doi.org/10.1016/j.insmatheco.2011.08.003. URL: http://www.sciencedirect.com/science/article/pii/S0167668711000849 (cited on pages 61, 65, 73).

[21] Damiano Brigo and Aurélien Alfonsi. “Credit default swap calibration and derivatives pricing with the SSRD stochastic intensity model”. In: *Finance and stochastics* 9.1 (2005), pages 29–42 (cited on page 55).

[22] Damiano Brigo and Fabio Mercurio. *Interest rate models-theory and practice: with smile, inflation and credit*. Springer Science & Business Media, 2007 (cited on page 65).

[23] William Brodsky. “ISDA® International Swaps and Derivatives Association, Inc”. In: (Jan. 2010) (cited on page 64).

[24] David S Bunch and Herb Johnson. “A simple and numerically efficient valuation method for American puts using a modified Geske-Johnson approach”. In: *The Journal of Finance* 47.2 (1992), pages 809–816 (cited on page 29).

[25] James Butkiewicz and Kim Long. “"Predicting Interwar Business Cycles with the Interest Rate Yield Spread."” In: (Feb. 2003) (cited on page 2).

[26] John Y. Campbell and John H. Cochrane. “By Force of Habit: A Consumption-Based Explanation of Aggregate Stock Market Behavior”. In: *Journal of Political Economy* 107.2 (1999), pages 205–251. DOI: 10.1086/250059 (cited on page 62).

[27] Guy Carpenter. 2008 (cited on page 61).

[28] Lara Cathcart and Lina El-Jahel. “Pricing defaultable bonds: a middle-way approach between structural and reduced-form models”. In: *Quantitative Finance* 6.3 (2006), pages 243–253 (cited on page 57).

[29] Carolyn Chang, Jack Chang, and Weili Lu. “Pricing Catastrophe Options in Discrete Operational Time”. In: *Insurance: Mathematics and Economics* 43 (Dec. 2008), pages 422–430. DOI: 10.1016/j.insmatheco.2008.08.003 (cited on page 69).

[30] Carolyn Chang, Jack Chang, and WeLi Lu. "Pricing Catastrophe Options with Stochastic Claim Arrival Intensity in Claim Time". In: *Journal of Banking Finance* 34 (Jan. 2010), pages 24–32. DOI: 10.1016/j.jbankfin.2009.06.019 (cited on page 69).

[31] Carolyn W. Chang, Jack S. K. Chang, and Min-Teh Yu. "Pricing Catastrophe Insurance Futures Call Spreads: A Randomized Operational Time Approach". In: *The Journal of Risk and Insurance* 63.4 (1996), page 599. DOI: 10.2307/253473 (cited on pages 62, 69).

[32] *Chart; ISDA Market Survey; Notional amounts outstanding at year-end, all surveyed contracts, 1987–present*. URL STATUS: dead March 7, 2012. URL: http://www.isda.org/statistics/pdf/ISDA-Market-Survey-annual-data.pdf (cited on page 41).

[33] J.C. Cox, J.E. Ingersoll, and S.A. Ross. "A theory of the term structure of interest rates". In: *Econometrica* 53.2 (1985), pages 385–407 (cited on page 62).

[34] J. Cummins and Hélyette Geman. "Pricing Catastrophe Insurance Futures and Call Spreads: An Arbitrage Approach". In: *The Journal of Fixed Income* 4 (Jan. 1995), pages 46–57. DOI: 10.3905/jfi.1995.408128 (cited on pages 62, 69).

[35] J. Cummins, Hlyette Geman, and Anthony Santomero. "An Asian Option Approach to the Valuation of Insurance Futures Contracts". In: *Review of Futures Markets* 13 (Apr. 1997) (cited on pages 62, 64, 69).

[36] J. Cummins and Mary Weiss. "Convergence of Insurance and Financial Markets: Hybrid and Securitized Risk-Transfer Solutions". In: *Journal of Risk Insurance* 76 (Sept. 2009), pages 493–545. DOI: 10.2139/ssrn.1260399 (cited on pages 61, 64, 69).

[37] Arnaud De Servigny and Olivier Renault. *Measuring and managing credit risk*. McGraw-Hill, 2004 (cited on page 6).

[38] Stephan Dieckmann. "By Force of Nature: Explaining the Yield Spread on Catastrophe Bonds". In: *SSRN Electronic Journal* (2010). DOI: 10.2139/ssrn.1082879 (cited on page 62).

[39] Darrell Duffie. "Credit swap valuation". In: *Financial Analysts Journal* 55.1 (1999), pages 73–87 (cited on page 55).

[40] Darrell Duffie and Kenneth Singleton. *Credit Risk: Pricing Measurement and Management*. Jan. 2003 (cited on page 43).

[41] Masahiko Egami and Virginia R. Young. "Indifference prices of structured catastrophe (CAT) bonds". In: *Insurance: Mathematics and Economics* 42.2 (2008), pages 771–778. DOI: 10.1016/j.insmatheco.2007.08.004 (cited on page 62).

[42] TM Eisenbach. "Rollover Risk as Market Discipline: A Two-Sided Inefficiency, forthcoming". In: *Journal of Financial Economics* (2016) (cited on page 5).

[43] Robert Fuller. "An Introduction to Financial Management". In: *Lecture Notes Supported by the TEMPUS project No. S JEP-11097-96* (1997) (cited on page 1).

[44] Marcello Galeotti, M. Gürtler, and Christine Rehan. "Accuracy of Pricing Models for CAT Bonds – An Empirical Analysis". In: 2009 (cited on page 62).

[45] Hélyette Geman and Marc Yor. "Stochastic time changes in catastrophe option pricing". In: *Insurance: Mathematics and Economics* 21 (Dec. 1997), pages 185–193. DOI: 10.1016/S0167-6687(97)00017-6 (cited on page 69).

[46] Hans Gerber and Elias Shiu. "Actuarial Bridges to Dynamic Hedging and Option Pricing". In: *Insurance: Mathematics and Economics* 18 (Feb. 1996), pages 183–218. DOI: 10.1016/0167-6687(96)85007-4 (cited on page 69).

[47] Brian Gibson, Phillip McClellend, and Gregory Phillip. “Quasi-Debt and Quasi-Equity in the Financial Statements of Small Firms”. In: *Small Enterprise Research: The Journal of SEAANZ* 12 (Dec. 2004), page 72. DOI: 10.5172/ser.12.2.72 (cited on page 28).

[48] Jia-Wen Gu et al. “On modeling economic default time: a reduced-form model approach”. In: *Computational Economics* 47.2 (2016), pages 157–177 (cited on page 25).

[49] Yalin Gündüz and Marliese Uhrig-Homburg. “Does modeling framework matter? A comparative study of structural and reduced-form models”. In: *Review of Derivatives Research* 17.1 (2014), pages 39–78 (cited on pages 57, 65).

[50] Ruili Hao, Yonghui Liu, and Shoubai Wang. “Pricing credit default swap under fractional Vasicek interest rate model”. In: *Journal of Mathematical Finance* 2014 (2013) (cited on page 57).

[51] M. Harrison and David Kreps. “Martingale and multiperiod securities markets”. In: *J. Econ. Theory* 20 (Jan. 1979) (cited on page 69).

[52] John Heaton and Deborah Lucas. “The Effects of Incomplete Insurance Markets and Trading Costs in a Consumption-Based Asset Pricing Model”. In: *Journal of Economic Dynamics and Control* 16 (July 1992), pages 601–620. DOI: 10.1016/0165-1889(92)90051-F (cited on page 69).

[53] John Heaton and Deborah Lucas. “The Effects of Incomplete Insurance Markets and Trading Costs in a Consumption-Based Asset Pricing Model”. In: *Journal of Economic Dynamics and Control* 16 (July 1992), pages 601–620. DOI: 10.1016/0165-1889(92)90051-F (cited on page 69).

[54] Patrick Houweling and Ton Vorst. "Pricing default swaps: Empirical evidence". In: *Journal of International Money and Finance* 24 (Dec. 2005), pages 1200–1225. DOI: 10.1016/j.jimonfin.2005.08.009 (cited on pages 56, 79).

[55] Robert Hoyt and Kathleen Mccullough. "Catastrophe Insurance Options: Are They Zero-Beta Assets?" In: *Journal of Insurance Issues* 22 (Oct. 2011), pages 147–163 (cited on page 69).

[56] John Hull and Alan White. "The impact of default risk on the prices of options and other derivative securities". In: *Journal of Banking & Finance* 19.2 (1995), pages 299–322 (cited on page 43).

[57] John C Hull and Alan D White. "Valuing credit default swaps I: No counterparty default risk". In: *The Journal of Derivatives* 8.1 (2000), pages 29–40 (cited on page 48).

[58] Robert Jarrow. "The Economics of Credit Default Swaps". In: *Annual Review of Financial Economics* 3 (Oct. 2010). DOI: 10.2139/ssrn.1646373 (cited on page 43).

[59] Robert A Jarrow and Stuart M Turnbull. "Pricing derivatives on financial securities subject to credit risk". In: *The journal of finance* 50.1 (1995), pages 53–85 (cited on page 43).

[60] Robert A Jarrow and Stuart M Turnbull. "The intersection of market and credit risk". In: *Journal of Banking & Finance* 24.1-2 (2000), pages 271–299 (cited on page 43).

[61] Robert A. Jarrow. "A simple robust model for Cat bond valuation". In: *Finance Research Letters* 7.2 (2010), pages 72–79. ISSN: 1544-6123. DOI: https://doi.org/10.1016/j.frl.2010.02.005. URL: http://www.sciencedirect.com/science/article/pii/S154461231000019X (cited on page 62).

[62] Robert A. Jarrow and Fan Yu. "Counterparty Risk and the Pricing of Defaultable Securities". In: *The Journal of Finance* 56.5 (2001), pages 1765–1799. DOI: 10.1111/0022-1082.00389 (cited on page 62).

[63] Hann-Shing Ju et al. “Evaluation of conducting capital structure arbitrage using the multi-period extended Geske–Johnson model”. In: *Review of Quantitative Finance and Accounting* 44.1 (2015), pages 89–111 (cited on page 29).

[64] Bakri Abdul Karim, Norlina Kadri, and Kelvin Lee Yong Min. “Credit Risk and Banks’ Profitability in Malaysia”. In: *UNIMAS Review of Accounting and Finance* 1.1 (2018) (cited on page 6).

[65] Amir E Khandani, Adlar J Kim, and Andrew W Lo. “Consumer credit-risk models via machine-learning algorithms”. In: *Journal of Banking & Finance* 34.11 (2010), pages 2767–2787 (cited on page 6).

[66] Vinod Kothari. “Credit default swaps”. In: Mar. 2012, pages 79–98. ISBN: 9780470822920. DOI: `10.1002/9781118390412.ch3` (cited on page 79).

[67] Jochen Kruppa et al. “Consumer credit risk: Individual probability estimates using machine learning”. In: *Expert Systems with Applications* 40.13 (2013), pages 5125–5131 (cited on page 6).

[68] Van SON Lai, Mathieu Parcollet, and Bernard Lamond. “The Valuation of Catastrophe Bonds With Exposure to Currency Exchange Risk”. In: *International Review of Financial Analysis* 33 (2014), pages 243–252. ISSN: 1057-5219. DOI: `https://doi.org/10.1016/j.irfa.2014.02.014`. URL: `http://www.sciencedirect.com/science/article/pii/S1057521914000374` (cited on page 61).

[69] Van SON Lai, Mathieu Parcollet, and Bernard Lamond. “The Valuation of Catastrophe Bonds with Exposure to Currency Exchange Risk”. In: *SSRN Electronic Journal* (Jan. 2014). DOI: `10.2139/ssrn.2399966` (cited on page 67).

[70] David Lando. *Credit risk modeling: theory and applications*. Princeton University Press, 2009 (cited on page 6).

[71] Morton Lane and Olivier Mahul. “Catastrophe Risk Pricing: An Empirical Analysis”. In: *The World Bank, Policy Research Working Paper Series* (Jan. 2008) (cited on page 62).

[72] Morton N. Lane. “Pricing Risk Transfer Transactions”. In: *ASTIN Bulletin* 30.2 (2000), pages 259–293. DOI: 10.2143/ast.30.2.504635 (cited on page 62).

[73] B. Lazzerini et al. “Preface ISDA 2009”. In: *ISDA 2009 - 9th International Conference on Intelligent Systems Design and Applications* (Jan. 2009). DOI: 10.1109/ISDA.2009.4 (cited on page 64).

[74] Jin-Ping Lee and Min-Teh Yu. “Pricing Default-Risky CAT Bonds With Moral Hazard and Basis Risk”. In: *Journal of Risk Insurance* 69.1 (2002), pages 25–44. DOI: 10.1111/1539-6975.00003. URL: https://onlinelibrary.wiley.com/doi/abs/10.1111/1539-6975.00003 (cited on page 62).

[75] Aldo Letizia. “Credit Spread Widening Risk and its Effects on Banks’ Economic Capital”. In: *SSRN Electronic Journal* (Dec. 2010). DOI: 10.2139/ssrn.1800584 (cited on page 2).

[76] Robert Litzenberger, David Beaglehole, and Craig Reynolds. “Assessing Catastrophe Reinsurance-Linked Securities as a New Asset Class”. In: *The Journal of Portfolio Management* 23 (Dec. 1996), pages 76–86. DOI: 10.3905/jpm.1996.076 (cited on page 64).

[77] Xiaomeng Ma and Shuliang Lv. “Financial credit risk prediction in internet finance driven by machine learning”. In: *Neural Computing and Applications* 31.12 (2019), pages 8359–8367 (cited on page 6).

[78] Lynne McChristian. “HURRICANE ANDREW AND INSURANCE: THE ENDURING IMPACT OF AN HISTORIC STORM”. In: *Insurance Information Institute, Florida Of-*

fice (Aug. 2012), pages 4–19. URL: `https://www.iii.org/sites/default/files/paperHurricaneAndrewfinal.pdf` (cited on page 61).

[79] Robert Merton. "Option Prices When Underlying Stock Returns Are Discontinuous". In: *Journal of Financial Economics* 3 (Jan. 1976), pages 125–144. DOI: `10.1016/0304-405X(76)90022-2` (cited on page 69).

[80] Robert Merton. "The Theory of Rational Option Pricing". In: *Bell J Econ Manage Sci* 4 (Mar. 1973), pages 141–183. DOI: `10.1142/9789812701022_0008` (cited on page 68).

[81] Robert C Merton. "On the pricing of corporate debt: The risk structure of interest rates". In: *The Journal of finance* 29.2 (1974), pages 449–470 (cited on page 29).

[82] Bernadette Minton, Rene Stulz, and Rohan Williamson. "How Much Do Banks Use Credit Derivatives to Hedge Loans?" In: *Journal of Financial Services Research* 35 (Feb. 2009), pages 1–31. DOI: `10.2139/ssrn.1084058` (cited on page 43).

[83] Stephen Morris and Hyun Song Shin. *Illiquidity component of credit risk*. Technical report. working paper, Princeton University, 2009 (cited on page 5).

[84] Alexander Muermann. "Actuarially consistent valuation of catastrophe derivatives". In: (Aug. 2003) (cited on page 69).

[85] Alexander Muermann. "Market Price of Insurance Risk Implied by Catastrophe Derivatives". In: *North American Actuarial Journal* 12 (Apr. 2006). DOI: `10.1080/10920277.2008.10597518` (cited on page 69).

[86] Futures Options. *OTHER DERIVATIVES. John C. Hull*. 2008 (cited on page 108).

[87] Champa Ramkrishna Parab and MR Patil. "Credit Risk and Public and Private Banks' Performance in India: A Panel Approach". In: *Researchers World* 9.2 (2018), pages 34–43 (cited on page 6).

[88] Yiannos Pierides. “The pricing of credit risk derivatives”. In: *Journal of Economic Dynamics and Control* 21 (Aug. 1997), pages 1579–1611. DOI: 10.1016/S0165-1889(97)00049-3 (cited on page 43).

[89] Gonzalo Pita et al. “Analysis of Hurricane Andrew Insurance Claim Data for Residential Buildings”. In: *Advances in Hurricane Engineering* (2012). DOI: 10.1061/9780784412626.091 (cited on page 61).

[90] Andy Polacek. “Catastrophe bonds: A primer and retrospective”. In: *Chicago Fed Letter* (Jan. 2018). DOI: 10.21033/cfl-2018-405 (cited on page 61).

[91] Patrice Poncet and Victor Vaugirard. “The valuation of nature-linked bonds with exchange rate risk”. In: *Journal of Economics and Finance* 25 (Sept. 2001), pages 293–307. DOI: 10.1007/BF02745890 (cited on page 61).

[92] “Principles for the Management of Credit Risk – final document”. In: *BIS; Basel Committee on Banking Supervision* (Dec. 2000). URL: http://www.bis.org/publ/bcbs75.htm (cited on page 41).

[93] Olaf Ronneberger, Philipp Fischer, and Thomas Brox. “U-net: Convolutional networks for biomedical image segmentation”. In: *International Conference on Medical image computing and computer-assisted intervention*. Springer. 2015, pages 234–241 (cited on page 81).

[94] M. Rubinstein and E. Reiner. “Breaking Down the Barriers”. In: *Risk* 4(8) (1991), pages 28–35 (cited on pages 74, 75).

[95] M. Rubinstein and E. Reiner. “Unscrambling the Binary Code”. In: *Risk* 4(9) (1991), pages 75–83 (cited on pages 74, 75).

[96] Philipp Sch. “Pricing Credit Risk Derivatives”. In: (Apr. 1999) (cited on page 43).

[97] Harris Schlesinger and Neil A. Doherty. “Incomplete Markets for Insurance: An Overview”. In: *Foundations of Insurance Economics: Readings in Economics and Finance*. Edited by Georges Dionne and Scott E. Harrington. Dordrecht: Springer Netherlands, 1992, pages 134–155. ISBN: 978-94-015-7957-5. DOI: 10.1007/978-94-015-7957-56. URL: https://doi.org/10.1007/978-94-015-7957-56 (cited on page 69).

[98] Hanspeter Schmidli. “Modelling PCS options via individual in- dices”. In: (Oct. 2020) (cited on page 73).

[99] Muhammad Sadiq Shahid, Faid Gul, and Khawar Naheed. “Credit Risk and Financial Performance of Banks: Evidence from Pakistan”. In: *NUML International Journal of Business & Management* 14.1 (2019), pages 144–155 (cited on page 6).

[100] V.E. Vaugirard. “Valuing catastrophe bonds by Monte Carlo simulations”. In: *Applied Mathematical Finance* 10.1 (2003). cited By 18, pages 75–90. ISSN: 1350486X. DOI: 10.1080/1350486032000079741 (cited on pages 62, 69).

[101] Victor Vaugirard. “A canonical first passage time model to pricing nature-linked bonds”. In: *Economics Bulletin* 7 (Jan. 2004), pages 1–7 (cited on page 69).

[102] Victor E Vaugirard. “Pricing catastrophe bonds by an arbitrage approach”. In: *The Quarterly Review of Economics and Finance* 43.1 (2003), pages 119–132. ISSN: 1062-9769. DOI: https://doi.org/10.1016/S1062-9769(02)00158-8. URL: http://www.sciencedirect.com/science/article/pii/S1062976902001588 (cited on pages 62, 69).

[103] Vivin Vinod. “Image segmentation using Deep UNET on orthophotos”. 2021 (cited on page 81).

[104] Shaun Wang. “A Class of Distortion Operators for Pricing Financial and Insurance Risks”. In: *The Journal of Risk and Insurance* 67 (Mar. 2000). DOI: 10.2307/253675 (cited on page 69).

[105] Stephen Muthii Wanjohi, Joel Githinji Wanjohi, and James Muchiri Ndambiri. “The effect of financial risk management on the financial performance of commercial banks in Kenya”. In: *International Journal of Finance and Banking Research* 3.5 (2017), pages 70–81 (cited on page 6).

[106] Anthony Wood and Shanise McConney. “The impact of risk factors on the financial performance of the commercial banking sector in Barbados”. In: *Journal of governance & regulation* 7, Iss. 1 (2018), pages 76–93 (cited on page 6).

[107] Yang-Che Wu and San-Lin Chung. “Catastrophe risk management with counterparty risk using alternative instruments”. In: *Insurance: Mathematics and Economics* 47.2 (2010), pages 234–245. DOI: 10.1016/j.insmatheco.2010.04.002 (cited on page 62).

[108] Yajuan XU and Guojing WANG. “Pricing catastrophe options with counterparty credit risk in a reduced form model”. In: *Acta Mathematica Scientia* 38 (Jan. 2018), pages 347–360. DOI: 10.1016/S0252-9602(17)30136-4 (cited on page 67).

III

Appendix

A. Supplementary Proofs and Derivations

A.1 First Passage Time for BM with Drift

Let $\bar{M}_t^{\xi} = \max_{s \leq t} X_t$ and $\underline{M}_t^{\xi} = \min_{s \leq t} X_t$ denote the maximum and minimum processes for a BM given by $X_t = W_t + \xi t$ respectively where $\xi \in \mathbb{R}$ is a constant drift. The joint probability distribution for $\left(\bar{M}_t^{\xi}, X_t\right)$ and $\left(\underline{M}_t^{\xi}, X_t\right)$ can be written using the following function

$$FP(a,b;\xi,t) := N\left[\frac{a-\xi t}{\sqrt{t}}\right] - e^{2\xi b} N\left[\frac{a-2b-\xi t}{\sqrt{t}}\right], \quad -\infty < a \leq b < \infty, t \geq 0 \tag{A.1}$$

Proposition A.1.1 The following hold $\forall\, a \in \mathbb{R}, b > geq0:$

1. $P\left[\bar{M}_t^{\xi} \leq b, X_t \leq a\right] = FP(a \wedge b, b; \xi, t)$
2. $P\left[\bar{M}_t^{\xi} \leq b\right] = FP(b; \xi, t) := FP(b, b; \xi, t)$
3. $P\left[\underline{M}_t^{\xi} \geq -|b|, X_t \geq a\right] = FP(-a \wedge b, b; -\xi, t)$
4. $P\left[\bar{M}_t^{\xi} \geq -|b|\right] = FP(b; \xi, t) := FP(b; -\xi, t)$

Skeletal Proof:

Proof. Proving the first formula is sufficient since the other three can be derived from the first simply by changing the direction of the BM and using the definition of the minimum and maximum of a BM. The first formula os trivial to prove for $\xi = 0$:

$$\begin{aligned} P\left[\bar{M}_t^{\xi} \le b, W_t \le a\right] &= P\left[W_t \le a\right] - P\left[\bar{M}_t^{0} \le b, W_t \le a\right] \\ &= P\left[W_t \le a\right] - P\left[\bar{M}_t^{0} \le b, W_t \ge 2b - a\right] \\ &= P\left[W_t \le a\right] - P\left[W_t \ge 2b - a\right] \\ &= N\left[\frac{a}{\sqrt{t}}\right] - N\left[\frac{a-2b}{\sqrt{t}}\right] := FP(a \wedge b, b; 0, t) \end{aligned} \tag{A.2}$$

We extend this to the case of $\xi \neq 0$ with the use of the Girsanov Theorem for the change of measure taken to be with Radon-Nikodyn derivative

$$Z := \left(\frac{dP^{(\xi)}}{dP}\right)|_{\mathscr{F}_t} = e^{-\xi W_t - \frac{1}{2}\xi^2 t}$$

According to the theorem, $\underline{M}_t^{\xi}$, X_t, and P^{ξ} are distributed identical to $\bar{M}_t^0$, W_t under P. This gives us

$$\begin{aligned} E\left[\mathbf{1}_{\{\bar{M}_t^{\xi} \le b\}} \mathbf{1}_{\{X_t \le a\}}\right] &= E^{\xi}\left[Z^{-1} \mathbf{1}_{\{\bar{M}_t^{\xi} \le b\}} \mathbf{1}_{\{X_t \le a\}}\right] \\ &= E^{\xi}\left[e^{\xi(X_t + \xi t) + \frac{1}{2}\xi t^2} \mathbf{1}_{\bar{M}_t^{\xi} \le b} \mathbf{1}_{X_t \le a}\right] \\ &= \int_{-\infty}^{a} \left[e^{\xi(y+\xi t) - \frac{1}{2}\xi t^2}\right] \left[\int_{y \vee 0}^{b} \left[\partial_x \partial_y F(y, x; , 0, t)\right] dx\right] dy \\ &= \int_{-\infty}^{a} e^{\xi(y+\xi t) - \frac{1}{2}\xi t^2} \frac{1}{\sqrt{t}} \left[\phi\left(-\frac{|y|}{\sqrt{t}}\right) - \phi\left(y - \frac{2b}{\sqrt{t}}\right)\right] dy \end{aligned} \tag{A.3}$$

Where $\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$. On completing the square in the integral and solving the result gives us the desired formula ■

A.2 Martingale Proofs

A.2.1 Doob's Upcrossing Lemma

This is a proof of the Doob's Upcrossing Lemma which is used to prove that supermartingales are almost surely convergent.

Proof. Define $C_1 = \mathbb{1}\{M_0 < a\}$. Then we recursively define

$$C_n = \mathbb{1}\{C_{n-1} = 1, M_{n-1}\} + \mathbb{1}\{C_{n-1} = 0, M_{n-2} < a\} \tag{A.4}$$

Thus whenever $X < a$, $C_n = 1$. This continues till $M > b$ and then $C_n = 0$ and so on. Thus we can write

$$(C \cdot M)_N = \sum_{k=1}^{N} C_k (X_{k+1} - X_k) \geq (b-a)U_N[a,b] - (M_N - a)^- \tag{A.5}$$

Thus, each time X does an upcrossing of [a,b], it picks up at least $(b-a)$. And so, since M is a supermartingale and C is previsible and bounded, then $(C \cdot M)$ is also a supermartingale, we have $\mathbb{E}[(C \cdot M)_N] \leq 0$ thereby proving the lemma. ■

A.2.2 Supermartingale Boundedness Lemma

Proof. It is that

$$(b-a)\mathbb{E}[U_N[a,b]] \leq |a| + \sup_{n \in \mathbb{N}} \mathbb{E}[|M_n|] < \infty \tag{A.6}$$

and on using hte bounded convergence theorem we can write it as

$$(b-a)\mathbb{E}[U_\infty[a,b]] \leq |a| + \sup_{n \in \mathbb{N}} \mathbb{E}[|M_n|] < \infty \tag{A.7}$$

This gives us $\mathbb{P}[U_\infty[a,b] < \infty] = 1$. ■

A.3 Learning Algorithm for a SOM

The motivation behind a SOM is from the human brain handling different sensory information in different parts of the cerebral cortex. The algorithm for the SOM proceeds as in Algorithm 1 depicted. The feature map Φ depicts useful statistical traits of the input space. On being given some input vector x, the feature map will provide the weight vector $w_{I(\mathbf{x})}$ and its corresponding winning neuron given by $I(\mathbf{x})$. The former provides the image space coordinates of the winning neuron in the input space.

Algorithm 1: SOM Algorithm

Result: SOM Mapped Image

transform input vectors;

initialisation `// assign random weights for neurons`

while *feature map changing == True* **do**

  sampling `// training vectors from input space`

  matching `// find winning neuron`

  updating `// apply weight update equation`

end

`// visualisation`

for *i, x in enumerate(input vectors)* **do**

  w = $I(\mathbf{x})$ `// winning neuron for x`

  plot(w[0], w[1]) show()

end

Here, the winning neuron, $I(\mathbf{x})$ is that neuron which has the weight which is closest to the input vector and given by $\min\{d_j(\mathbf{x})\}$ where

$$d_j(\mathbf{x}) = \sum_{i=1}^{D} \left(x_i - w_{ji}\right)^2 \tag{A.8}$$

The weight updating equation is given by

$$\Delta w_{ji} = \eta(t) T_{j,I(\mathbf{x})}(t) \left(x_i - w_{ji}\right) \tag{A.9}$$

where $\eta(t)$ is the learning rate of the model and $T_{j,I(\mathbf{x})}(t)$ is some Gaussian neighbourhood. If considered in its entirety, the analytical weight vector ($\mathbf{W}_I(i)$) update algorithm for a neuron I at the $i+1$th iteration is given by

$$W_I(i+1) = W_I(i) + T_{j,I(\mathbf{x})}(i) \cdot \eta(i) \cdot (\mathbf{X(i)} - W_I(i)) \tag{A.10}$$

A.4 Binomial Lattice Model Approximation to GBM

The binomial lattice model (BLM) is a good approximation for the GBM. For a BLM, $S_n = S_0 Y_1 Y_2 \dots Y_n$, $n \geq 0$ where Y_i are $i.i.d.$ RVs which are distributed as $P(Y = u) = p$, and $P(Y = d) = 1 - p$. Thus, the parameters $0 < p < 1$ and $0 < d < 1 + r < u$ completely define the BLM. We provide a brief description of what these parameters must be for the BLM to approximate to the GBM. This can be trivially justified when, for large n, by the central limit theorem

$$\ln(Y_1 Y_2 \dots Y_n) = \sum_{i=1}^{n} \ln(Y_i) \approx X(t) \sim N(\mu t, \sigma^2 t) \tag{A.11}$$

This in turn gives us

$$S_n = S_0(Y_1 Y_2 \dots Y_n) \approx S_0 e^{X(t)} = S(t) \tag{A.12}$$

which follows from the previous equation by raising both sides to the exponent and multiplying with S_0.

Parameter Specification

For any fixed t we can rewrite the GBM in the same way as the BLM provided we set $t_i = \frac{it}{n}$ and divide the interval $(0,t]$ into n equal slices. Using pre-defined terms, we see that $\ln(L_i) \sim N\left(\frac{\mu t}{n}, \frac{\sigma^2 t}{n}\right)$. Thus we need to equate the mean and variance of this (for a lognormal distribution) an the definition of the BLM to find suitable values of u, p, d. That is, we find u, p, d such that $E(Y) = E(L) \wedge \operatorname{Var}(Y) = \operatorname{Var}(L)$, or equivalently, $E(Y) = E(L) \wedge E(Y^2) = E(L^2)$. This reduces to solving:

$$pu + (1-p)d = e^{\frac{\mu t}{n} + \frac{\sigma^2 t}{2n}} \tag{A.13a}$$

$$pu^2 + (1-p)d^2 = e^{2\frac{\mu t}{n} + 2\frac{\sigma^2 t}{n}} \tag{A.13b}$$

This does not have a unique solution. However, by forcing[1] $ud = 1$. And let $\nu = \mu + \frac{\sigma^2}{2}$ we re-write the above equations as

$$ud = 1 \tag{A.14a}$$

$$pu + (1-p)d = e^{\nu \frac{t}{n}} \tag{A.14b}$$

$$pu^2 + (1-p)d^2 = e^{(2\nu + \sigma^2)\frac{t}{n}} \tag{A.14c}$$

This enables us to write p in terms of u and d and subsequently solve for u and d as

$$p = \frac{e^{\nu \frac{t}{n}} - d}{u - d} \tag{A.15}$$

$$d = e^{\sigma \sqrt{\frac{t}{n}}} \tag{A.16}$$

$$e^{-\sigma \sqrt{\frac{t}{n}}} \tag{A.17}$$

[1] This has the effect on the BLM stock price that an up followed by a down (or vice versa) leaves the price alone.

	1 ID	2 Equity	3 EquityVol	4 Liability	5 Rate	6 Drift
1	'Firm 1'	26406000	0.7103	40000000	0.0500	0.0306
2	'Firm 2'	2.6817e+07	0.3929	35000000	0.0500	0.0300
3	'Firm 3'	39770000	0.3121	35000000	0.0500	0.0310
4	'Firm 4'	29470000	0.4595	32000000	0.0500	0.0302
5	'Firm 5'	25280000	0.6181	40000000	0.0500	0.0305

Figure A.1: Data set for point method

A.5 Numerical Estimation of Probability of Default

There are two main numerical paths to choose to solve the Merton model for the probability of default. The first one solves a two-by-two system of nonlinear equations while the second one uses time series method. The first method solves for (V_t, σ) by solving the option pricing formula and considering an unobservable asset volatility by replacing it with the equity volatility given by $\sigma_E = \frac{V}{E}N(d_1)\sigma$. The second method only works if there is a time series available for all the parameters of the model. Given n observation time series, this MATLAB model solves n equations to calibrate the time series asset value:

$$\begin{aligned} E_1 &= V_1 N(d_1) - De^{-r_1 T_1} N(d_2) \\ &\vdots \\ E_n &= V_n N(d_1) - De^{-r_n T_n} N(d_2) \end{aligned} \tag{A.18}$$

This is used to calculate the asset volatility by taking standard deviation of the log returns. The first method is a more complete-picture version and returns not just the probability of default but also the distance to default, firm asset's current value, and the annualised firm asset volatility. If we proceed with the time series approach, for the implementation, we use the inbuilt dataset from MATLAB for demonstrating the probability of default, both with and without drift. The code and output (Figure A.2) for these methods is given below.

```
load MertonData.mat
E = MertonData.Equity;
EVol = MertonData.EquityVol
```

```
probdef =                ast =

    0.0638                   1.0e+07 *
    0.0008
    0.0000                    6.4210
    0.0026                    6.0109
    0.0344                    7.3063
                              5.9906
                              6.3231
dist2def =

    1.5237               astvol =
    3.1679
    4.4298                    0.3010
    2.7916                    0.1753
    1.8196                    0.1699
                              0.2263
                              0.2511
```

Figure A.2: Command window output MATLAB

	1 Dates	2 Equity	3 Liability	4 Rate
1	2015-01-02	3.6762e...	21000000	0.0305
2	2015-01-05	3.7362e...	21000000	0.0307
3	2015-01-06	3.5343e...	21000000	0.0305
4	2015-01-07	3.6829e...	21000000	0.0310
5	2015-01-08	3.6535e...	21000000	0.0302
6	2015-01-09	3.5716e...	21000000	0.0301
7	2015-01-12	3.6110e...	21000000	0.0301
8	2015-01-13	3.6456e...	21000000	0.0301
9	2015-01-14	3.7990e...	21000000	0.0304
10	2015-01-15	3.7568e...	21000000	0.0304

Figure A.3: First 10 entries of the time series

```
D = MertonDataTS.Liability;
Rate = MertonDataTS.Rate;
Drift = MertonData.Drift;
[probdef,dist2def,ast,astvol] = mertonmodel(E,EVol,D,Rate,'Drift',Drift)
```

```
load MertonData.mat
Dates = MertonDataTS.Dates;
E = MertonDataTS.Equity;
D = MertonDataTS.Liability;
Rate = MertonDataTS.Rate;
PD0 = mertonByTimeSeries(E,D,Rate); %Merton model without drift
PD1 = mertonByTimeSeries(E,D,Rate,'Drift',0.20); %Merton model with 20%
    drift
plot(Dates, PD0, Dates, PD1)
legend('NoDrift', 'Drift')
```

The website linked here provides the comparison of the results of these two methods on MATLAB.

A.6 Market Price of CAT risk

In case the underlying process is a GBM, in accord with Hull [86], one can recover the market price of the risk. Following the approach of Hull, we assume that there are two derivatives $G_1(\xi)$ and $G_2(\xi)$ on the variable ξ which is itself not a traded asset but has dynamics with drift a and volatility b of the form (dW_t is the standard Weiner process):

$$d\xi_t = a\xi_t dt + b\xi_t dW_t \tag{A.19}$$

The derivatives themselves follow the process:

$$dG_1(\xi) = \mu_1 G_1(\xi_t)dt + \sigma_1 G_1(\xi_t)dW_t \tag{A.20a}$$

$$dG_2(\xi) = \mu_2 G_2(\xi_t)dt + \sigma_2 G_2(\xi_t)dW_t \tag{A.20b}$$

Thus, only the Weiner process of the underlying variable dynamics causes the only uncertainty which affects the prices of these derivatives.

The trivial way to produce a risk-free portfolio (say, Ω) is, then, to buy $\sigma_2 G_2(\xi)$ of the first derivative and to sell $\sigma_1 G_1(\xi)$ of the second one, eliminating dW_t:

$$\Omega(G_1(\xi_t), G_2(\xi_t)) = \sigma_2 G_2(\xi) G_1(\xi_t) - \sigma_1 G_1(\xi) G_2(\xi_t) \tag{A.21}$$

The marginal change of this portfolio is calculated as

$$\begin{aligned} d\Omega &= \sigma_2 G_2(\xi_t) dG_1 - \sigma_1 G_1(\xi_t) dG_2 \\ &= [\mu_1 \sigma_2 G_2(\xi_t) G_1(\xi_t) - \mu_2 \sigma_1 G_1(\xi_t) G_2(\xi_t)]\, dt \end{aligned} \tag{A.22}$$

The risk-free interest rate that this portfolio yields in the next marginal time period is given by

$$d\Omega = r\Omega dt \tag{A.23}$$

Some algebraic manipulation yields

$$\frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2} = \kappa_\xi \tag{A.24}$$

where κ_ξ is the risk's market price corresponding to the underlying ξ. Thus the risk-premium that investors require to hold this asset over and above the risk-free rate is given by $\mu - r = \kappa_\xi \sigma$.

A.7 Simpson's Rule of Numerical Integration

Here we derive the Simpson's rule of numerical integration of the definite integral $\int_a^b f(x)dx$. We begin by using Taylor polynomials. Let $x_0 = a$, $x_1 = \frac{a+b}{2}$, $x_2 = b$. Since $x_1 - x_0 = h = x_2 - x_1$, we can rewrite $x_1 = a + \frac{b-a}{2} = a + h$. Thus

$$\begin{aligned}\int_{a=x_0}^{b=x_2} f(x)dx = \int_{x_0}^{x_2} \Big[f(x_1) + f'(x_1)(x-x_1) + \frac{f''(x_1)}{2!}(x-x_1)^2 \\ + \frac{f^{(3)}(x_1)}{3!}(x-x_1)^3 + \frac{f^{(4)}(\xi)}{4!}(x-x_1)^4\Big]dx\end{aligned} \tag{A.25}$$

Solving the linear integrals and re-arranging terms with the substitutions we get

$$\begin{aligned}\int_{a=x_0}^{b=x_2} f(x)dx &= 2hf(x_1) + \frac{h^3}{3}\left[\frac{f(x_0) - 2f(x_1) + f(x_2)}{h^2} - \frac{h^2}{12}f^{(4)}(\xi)\right] + \frac{h^5}{60}f^{(4)}(\xi) \\ &= \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)] - \frac{h^5}{90}f^{(4)}(\xi)\end{aligned} \tag{A.26}$$

Thus the Simpson's rule is

$$\int_a^b f(x)dx \approx \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right] \tag{A.27}$$

A.8 SOM Code

The following is the python implementation of detecting credit card frauds using the SOM.

```
# -*- coding: utf-8 -*-
"""
Created on Fri Jun 18 11:48:03 2021
@author: vivin
"""
import numpy as np
import matplotlib.pyplot as plt
import pandas as pd
from minisom import MiniSom
import matplotlib.pyplot as plt

#Read Data from applications.csv file
data = pd.read_csv('applications.csv')
X = data.iloc[:, :-1].values
#notice we do not need to use data
#related to whether the application was passed or not

#normalise data
from sklearn.preprocessing import MinMaxScaler
sc = MinMaxScaler()
X = sc.fit_transform(X)

#initialise SOM
som = MiniSom(x=10, y=10, input_len=len(X[1,:]), sigma=1.0, learning_rate
    =0.25)
som.random_weights_init(X) #random initialise weights

#train the SOM
som.train_random(data=X, num_iteration=1000)

#winning nodes
mappings = som.win_map(X)

#Visualising the data
fig = plt.figure()
plt.inferno()
plt.pcolor(som.distance_map().T)
plt.colorbar()
#outliers
#red circle = customer did not get approval
#green square = customer got approval
markers=['o','s']
colors=['r','g']
```

```
y = data.iloc[:, -1].values #dependent variable
#used only for plotting markers

fig.suptitle("SOM map")
fig.dpi=1000
for i,x in enumerate(X):
    w = som.winner(x)
    plt.plot(w[0]+0.5, w[1]+0.5,markers[y[i]],
            markeredgecolor=colors[y[i]],
            markerfacecolor='None',markersize=10,
            markeredgewidth=2)
plt.show()


#find the fraudulent customers
frauds = np.concatenate((mappings[(5,9)],
                        mappings[(8,8)],
                        mappings[(9,6)]),axis=0)
frauds = sc.inverse_transform(frauds) #Final list of fraudsters
```

A.9 Fraudster Table

Table A.1: List of fraudulent customers

Customer ID	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	X11	X12	X13	X14
15,766,183	0	24.500	0.500	2	11	8	1.500	1	0	0	0	2	280	825
15,808,662	0	46	4	2	5	3	0	1	0	0	0	2	100	961
15,684,722	0	27.670	1.500	2	7	4	2	1	0	0	0	1	368	1
15,772,329	0	28.080	15	1	10	9	0	1	0	0	0	2	0	13,213
15,734,649	0	40.830	10	2	11	8	1.750	1	0	0	0	2	29	838
15,646,082	0	18.830	4.415	1	8	8	3	1	0	0	0	2	240	1
15,717,629	0	25.170	2.875	2	14	8	0.875	1	0	0	0	2	360	1
15,757,188	0	20.830	3	2	6	4	0.040	1	0	0	0	2	100	1
15,604,536	0	20.500	11.835	2	8	8	6	1	0	0	0	2	340	1
15,635,598	0	24.580	0.670	2	6	8	1.750	1	0	0	0	2	400	1
15,647,191	0	32.330	0.540	2	13	4	0.040	1	0	0	0	2	440	11,178
15,776,545	0	25	11	1	6	4	4.500	1	0	0	0	2	120	1
15,792,107	0	33.920	1.585	1	1	1	0	1	0	0	0	2	320	1
15,623,369	0	44.330	0	2	8	4	2.500	1	0	0	0	2	0	1
15,793,896	0	28.420	3.500	2	9	4	0.835	1	0	0	0	1	280	1
15,650,591	0	26.080	8.665	2	6	4	1.415	1	0	0	0	2	160	151
15,813,192	0	24.920	1.250	2	1	1	0	1	0	0	0	2	80	1
15,633,608	0	22.920	11.585	2	13	4	0.040	1	0	0	0	2	80	1,350
15,675,450	0	18.830	9.540	2	6	4	0.085	1	0	0	0	2	100	1
15,766,906	1	29.830	3.500	2	8	4	0.165	0	0	0	0	2	216	1
15,763,108	1	28.670	14.500	2	2	4	0.125	0	0	0	0	2	0	287
15,723,989	1	22.250	9	2	6	4	0.085	0	0	0	0	2	0	1
15,699,340	1	23.170	0	2	8	4	0	0	0	0	0	3	184	1
15,732,884	1	17.920	0.205	2	6	4	0.040	0	0	0	0	2	280	751
15,644,400	1	34.580	0	2	8	4	0	0	0	0	0	3	184	1
15,684,440	1	33.670	2.165	2	8	4	1.500	0	0	0	0	3	120	1
15,736,510	1	41.500	1.540	2	3	5	3.500	0	0	0	0	2	216	1
15,736,420	1	22.080	2.335	2	4	4	0.750	0	0	0	0	2	180	1

15,726,167	1	36.170	18.125	2	9	4	0.085	0	0	0	0	2	320	3,553
15,765,093	1	31.570	0.625	2	4	4	0.250	0	0	0	0	2	380	2,011
15,742,297	1	25.250	1	2	6	4	0.500	0	0	0	0	2	200	1
15,729,377	1	18.170	10.250	2	8	8	1.085	0	0	0	0	2	320	14
15,737,542	1	33.750	2.750	2	3	5	0	0	0	0	0	2	180	1
15,748,986	1	48.080	3.750	2	3	5	1	0	0	0	0	2	100	3
15,769,980	1	20.670	0.415	2	8	4	0.125	0	0	0	0	2	0	45
15,596,797	1	28.170	0.125	1	4	4	0.085	0	0	0	0	2	216	2,101
15,571,415	1	37.580	0	2	8	4	0	0	0	0	0	3	184	1
15,565,714	1	42.750	4.085	2	6	4	0.040	0	0	0	0	2	108	101
15,608,688	1	15.170	7	2	10	4	1	0	0	0	0	2	600	1
15,609,070	1	18	0.165	2	11	7	0.210	0	0	0	0	2	200	41
15,650,313	1	37.500	0.835	2	10	4	0.040	0	0	0	0	2	120	6
15,575,438	1	41.170	1.250	1	9	4	0.250	0	0	0	0	2	0	196
15,565,996	1	26.250	1.540	2	9	4	0.125	0	0	0	0	2	100	1
15,592,914	1	20.750	5.085	1	5	4	0.290	0	0	0	0	2	140	185
15,581,871	1	16.500	0.125	2	8	4	0.165	0	0	0	0	2	132	1
15,611,973	1	18.330	1.210	1	10	2	0	0	0	0	0	2	100	1
15,593,178	1	31.080	1.500	1	9	4	0.040	0	0	0	0	1	160	1
15,573,077	1	44.830	7	1	8	4	1.625	0	0	0	0	2	160	3
15,636,626	1	62.750	7	2	10	9	0	0	0	0	0	2	0	13
15,645,571	1	20.420	1.085	2	11	4	1.500	0	0	0	0	2	108	8
15,638,272	1	33.580	0.335	1	13	4	0.085	0	0	0	0	2	180	1
15,599,152	1	16.920	0.335	1	4	4	0.290	0	0	0	0	1	200	1
15,642,001	1	34.830	2.500	1	9	4	3	0	0	0	0	1	200	1
15,600,027	1	29.830	2.040	1	14	8	0.040	0	0	0	0	2	128	2
15,588,019	1	47.170	5.835	2	9	4	5.500	0	0	0	0	2	465	151
15,615,670	1	36.420	0.750	1	2	4	0.585	0	0	0	0	2	240	4
15,599,535	1	39.420	1.710	1	7	4	0.165	0	0	0	0	1	400	1
15,575,146	1	39.500	1.625	2	8	4	1.500	0	0	0	0	2	0	317