

Quantile functions

6 marks

Quantile functions. Suppose a continuous random variate X has a strictly increasing cumulative distribution function $F_X(x)$, a continuous density $f_X(x)$, and a quantile function $Q_X(p)$.

- a. **(2 marks)** Suppose $U \sim U(0, 1)$. Define a random variate $Y = Q_X(U)$. Prove that $Pr(Y \leq a) = F_X(a)$ for any value of a , and hence that Y has the same distribution as does X .

let x be a rv of X

$$\begin{aligned} P(Y \leq a) &= P(F_X(Y) \leq F_X(a)) \\ &= P(U \leq F_X(a)) \\ &= P(F_U(F_X(a))) \\ &= F_X(a) \end{aligned}$$

we are done

- b. **(4 marks)** Let $Y = aX + b$ for some constants $a > 0$ and b . Prove that the quantile function $Q_Y(p)$ for Y is related to that of X as

$$Q_Y(p) = aQ_X(p) + b.$$

$$\begin{aligned} P(aQ_X(p) + b < Y) \\ &= P(Q_X(p) < (Y - b)/a) \\ &= P(Q_X(p) < X) \\ &= P(Q_X(p) < Q_X(U)) \\ &= P(p < U) \end{aligned}$$

$$\begin{aligned} P(Q_Y(p) < Y) \\ &= P(Q_Y(p) < Q_Y(U)) \\ &= P(p < U) \end{aligned}$$

we see these two are equal