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1 introduction and background

The non-intuitive behavior of photon-mirror-beamsplitter system results from features of quantum mechanics called *superposition and interference*

the second beam aplitter has caused two paths (in superposition) to interfere, resulting in cancelation of the 0 path

2 linear albegra

Definition 2.0.1 (Hilbert space)
finite-dimensional complex vector space H

2.1 Dual Vectors

- linearity in second argument
$$\langle v, \sum_i \lambda_i w_i \rangle = \sum_i \lambda_i \langle v, w_i \rangle$$
- conjugate-commutativity
$$\langle v, w \rangle = \langle w, v \rangle^*$$
- nonnegativity
$$\langle v, v \rangle \geq 0$$
with equality iff v=0

c* means complex conjugate of number c
$v \cdot w = \sum_i v_i^* w_i$

Definition 2.1.1 (H^*)
let H be a hilbert space, H^* is the set of linear maps $H \to C$
We denote elements of H^* by $\langle x|$, where $\langle X|$ has action

$$\langle X| : |\psi\rangle \to \langle X|\psi\rangle \in C$$

The set of maps H^* is a complex vector space itself, is called the the *dual vector space* associated with H.
The vector $\langle x|$ is called the dual of $|x\rangle$, it's derived from transposing $|x\rangle$ to row matrix and taking complex conjugate
Note dot product of complex vectors are
$$\mathbf{A} \cdot \mathbf{B} = \sum_i a_i^* b_i$$

Definition 2.1.2 (Euclidean norm of $|\psi\rangle$)
norm of a vector $|\psi\rangle$ is $\||\psi\rangle\| \equiv \sqrt{\langle\psi|\psi\rangle}$

Definition 2.1.3 (Kronecker delta function$\delta_{i,j}$)
it is 1 whenever i=j, 0 othervise

Theorem 2.1.1 (dual basis)
the set $\{\langle b_n|\}$ is an orthonormal basis for H^* called the dual basis

2.2 operators

outer product

$$(|\psi\rangle\langle\varphi|)|\alpha\rangle = |\psi\rangle(\langle\varphi|\alpha\rangle) = (\langle\varphi|\alpha\rangle)|\psi\rangle$$

outer project of a vector with itself defines a lienar operators $|\psi\rangle\langle\psi|$

Definition 2.2.1 (orthogonal projector)
it projects a vector in H to 1 dimensional subspace of H spanned by $|\psi\rangle$. sunch an operator is called an orthogonal projector

Theorem 2.2.1
let $B = \{|b_n\rangle\}$ be an orthonormal basis for vector space $\mathbb{H}$, then every line operator T on H can be written as

$$T = \sum_{b_n, b_m \in B} T_{n,m}|b_n\rangle\langle b_m|$$

where $T_{n,m} = \langle b_n|T|b_m\rangle$

the set of all linear operators on vector space H forms a new complex vector space $L(H)$
vectors in L(H) are linear operators on H the basis vectors for L(H) are all possible outer products of pairs of basis vectors from B $\{|b_n\rangle\langle b_m|\}$
the action of T is then

$$T : |\psi\rangle \to \sum_{b_n, b_m \in B} T_{n,m}\langle b_m|\psi\rangle|b_n\rangle$$

$T_{n,m}$ is matrix entry of T

Definition 2.2.2 (identity operator/resolution of the identity)
for any orthonormal basis $B = \{|b_n\rangle\}$ identity operator is

$$1 = \sum_{b_n \in B} |b_n\rangle\langle b_n|$$

- $H \to C \in H^*$ corresponds to some vector $\langle\varphi'|$
- adjoint of T, $T^\dagger$ sends $|\varphi\rangle \to |\varphi'\rangle$

Definition 2.2.3
adjoint of T, $T^\dagger$ is linear operator on H^* that satisfies

$$(\langle\psi|T^\dagger|\varphi\rangle)^* = (\langle\varphi|T|\psi\rangle) \quad \forall|\psi\rangle, |\varphi\rangle \in H$$

$T^\dagger$ is just complex conjugate transpose of T, also called Hermitean conjugate, adjoint of T

Definition 2.2.4 (unitary)
U is unitary if $U^\dagger = U^{-1}$

Definition 2.2.5 (Hermitean)
an operator T in Hilbert space H is called Hermitean (self adjoint) if

$$T^\dagger = T$$

Definition 2.2.6 (projector, orthogonal projecter)
A projector on vector space H is a linear operator P that satisfies $P^2 = P$, an orthogonal projector also satisfies $P^\dagger = P$

Theorem 2.2.2
if $T == T^\dagger \quad T|\psi\rangle == \lambda|\psi\rangle$, then $\lambda \in R$

Definition 2.2.7 (trace)
$Tr(A) = \sum_{b_n} \langle b_n|A|b_n\rangle$ where $\{|b_n\rangle\}$ is any orthonormal basis

2.3 spectral Theorem

Definition 2.3.1 (normal)
A is normal operator if

$$AA^\dagger = A^\dagger A$$

both unitary and hermitean operators are normal

Theorem 2.3.1 (spectral Theorem)
for every normal operator T acting on a finite-dimentional Hilbert space H, there is an orthonormal basis of H consisting of eigenvectors $|T_i\rangle$ of T
We refer to T written in its own eigenbasis as the spectral decomposition of T.
The set of eigenvalues of T is called the sepctrum of T

$T = \sum_i T_i|T_i\rangle\langle T_i|$ where T_i are eigen values, $|T_i\rangle$ are eigenvectors
another way of saying the spectral theorem is

Theorem 2.3.2
For every finite dimensional normal matrix T there is a unitary matrix P such that $T = PDP^\dagger$ where D is a diagonal matrix
diagonal entries of D are eigenvalues of T. columns of P encode eigenvectors of T

2.4 Functions of Operators

with spectral theorem we can write any normal operator T to

$$T = \sum_i T_i |T_i\rangle\langle T_i|$$

Note each $|T_i\rangle\langle T_i|$ is a projector
Talor series for any function f acting on an operator T will have

$$f(T) = \sum_m a_m T^m$$

If T is written in diagonal form, then

$$f(T) = \sum_i f(T_i)|T_i\rangle\langle T_i|$$

2.5 Tensor Products

- For any $c \in \mathbf{C}, |\psi_1\rangle \in H_1, |\psi_2\rangle \in H_2$
$$c(|\psi_1\rangle \otimes |\psi_2\rangle) = (c|\psi_1\rangle) \otimes |\psi_2\rangle = |\psi_1\rangle \otimes (c|\psi_2\rangle)$$
- for any $|\psi_1\rangle$, $|\varphi_1\rangle \in H_1$, $|\psi_2\rangle \in H_2$
$$(|\psi_1\rangle + |\varphi_1\rangle) \otimes |\psi_2\rangle = |\psi_1\rangle \otimes |\psi_2\rangle + |\varphi_1\rangle \otimes |\psi_2\rangle$$
- for any $|\psi_1\rangle$, $|\varphi_1\rangle \in H_1$, $|\psi_2\rangle \in H_2$
$$|\psi_1\rangle \otimes (|\psi_2\rangle + |\varphi_2\rangle) = |\psi_1\rangle \otimes |\psi_2\rangle + |\psi_1\rangle \otimes |\varphi_2\rangle$$

$$(A \otimes B)(|\psi_1 \otimes |\psi_2\rangle) \equiv A|\psi_1\rangle \otimes B|\psi_2\rangle$$

3 Qubits and the framework of quantum mechanics

Quantum information is the result of reformulating information theory in this quantum framework

3.1 State of a Quantum System

the light-splitter example is an example of a 2-state quantum system:
a photon that follow one of two paths, which we identify by $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$
and noted a path state of the photo can be described as $\begin{pmatrix}\alpha_0\\\alpha_1\end{pmatrix}$ with $|\alpha_0|^2 + |\alpha_1|^2 = 1$

Definition 3.1.1 (State Space Postulate)
the state of a system is described by a unit vector in a Hilbert Space $\mathbb{H}$

$\mathbb{H}$ can be infinite dimensional.

In practice, we cannot distinguish a continous state from a discrete state having small spacing
we label one basis vector with $|0>$, one with $|1>$, they are orthogonal to each other
we represent general state by $\alpha_0|0> + \alpha_1|1>$ with $|\alpha_0|^2 + |\alpha_1|^2 = 1$
α_0 and α_1 are complex coefficient, often called *amplitude* of basis state $|0>$ and $|1>$
amplitude α can be decomposed uniquely as a product $e^{i\theta}|\alpha|$ where $|\alpha|$ is non-negative corresponding to magnitude of α, $e^{i\theta}$ has norm 1.
value θ is *phase*, $e^{i\theta}$ is *phase vector*
the state is described by unit vector means $|\alpha_0|^2 + |\alpha_1|^2 = 1$
this is called *normalization constriant*
vector $e^{i\theta}|\phi>$ is equivalent to the state described by $|\phi>$
example of ϕ would be $|0> + |1>$
question: is this example not considering it has 100% appearing in 0 and 1 state
On the other hand, relative phase factors between 2 orthogonal states in superposition are physically significant, and the state described by the vector
$|0> + |1>$ is physically different from $|0> + e^{i\theta}|1>$

Theorem 3.1.1 (State SpacePostulate)
we can describe the most general state $|\phi>$ of a single qubit by a vector of form

$$|\phi>= cos(\theta/2)|0> + e^{i*k} sin(\theta/2)|1>$$

on a classical computer, a *classical bit* could be represented by a 0/1,
there is also probablitistic classical bit $\begin{pmatrix} p_0 \\ p_1 \end{pmatrix}$
we can represent two probabilities by 2-dimensional unit vector
now we go back to quantum bit, which is described by a complex unit vector $|\phi>$ in 2 dimentional Hilbert space. Up to a (physically insignificant) flobal phase factor, such a vector can be written in the form

$$|\phi>= cos(\theta/2)|0> + e^{i*k} sin(\theta/2)|1>$$

this state vector is often depicted as a point on a 3-dimentional sphere, the *Bloch Sphere*
points on the sphere can be expressed in Cartesian corrdinates as

$$(x, y, z) = (sin\theta cos\phi, sin\theta sin\phi, cos\theta)$$

3.2 Time Evolution of a Closed System

Theorem 3.2.1 (Evolution Postulate)
The time-Evolution of state of a *closed* quantum system is described by a unitary operator. that is for any evolution of the closed system there exsist a unitary operator U s.t. if initial state is $|\psi_i>$ then after evolution the state will be

$$|\psi_2>= U|\psi_1>$$

3.3 Composite Systems

How to described a closed system of n qubits. how it envolves and what happens when we measure it.

we treat it as a composition of subsystems

Theorem 3.3.1 (Composition of Systems Postulate)
When two physical systems are treated as one combined system. the state space of combined physical system is product space $H_1 \otimes H_2$ of the state space H_1, H_2 of the component subsystems.
If first system is in state $|\psi_1 >$, second system in state $|\psi_2 >$, the state of combined system is

$$|\psi_1 > \otimes |\psi_2 >$$

We write the joint state like $|\psi_1\rangle| \otimes \psi_2\rangle$ or $|\psi_1\rangle|\psi_2\rangle$

If 2 qubits are allowed to intreact, the closed system includes both qubits together, it might not be possible to write the state in product form. Then we say the qubits are *entangled*

The state of composite system is a vector in 4-dimensional tensor product space of 2 constituent qubits. The 4-dimensional state vectors formed by tensor product of the 2 2-dimensional state vectors form a sparce subset of all 4-dimensional state vectors.

Most 2-qubits states are entangled.

A state is tangled if the equation has no solution of α and β

$$|\psi\rangle = (\alpha_0|0\rangle + \alpha_1|1\rangle)(\beta_0|0\rangle + \beta_1|1\rangle)$$

for example

$$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle|0\rangle + \frac{1}{\sqrt{2}}|1\rangle|1\rangle$$

This is EPR pair. If we use X to first qubit, Y to second qubit,
the system $|\psi_1\rangle \otimes |\psi_2\rangle$ is mapped to

$$X|\psi_1\rangle \otimes I|\psi_2\rangle = (X \otimes I)(|\psi_1\rangle \otimes |\psi_2\rangle)$$

3.4 Measurement

state of a single-qubit system is represented as a vector in Hilbert space

The evolution of state of a system during a Measurement is not unitary

if the state $\sum_i \alpha_i|i >$ is provided as input. it will output i with probability $|\alpha_i|^2$ and leave the system in state $|i >$

Theorem 3.4.1
For a given orthonormal basis $B = \{|\varphi_i\}$ of a state space H_A for a system A, it's possible to perform a Von Neumann measurement on system H_A with respect to basis B that, given a state

$$|\psi\rangle = \sum_i \alpha|\varphi_i\rangle$$

outputs label i with probablity $|\alpha_i|^2$ and leaves the system in state $|\varphi_i\rangle$
For state $|\psi\rangle = \sum_i \alpha_i|\varphi_i\rangle$, note that $\alpha_i = \langle\varphi_i|\psi\rangle$ and thus

$$|\alpha_i|^2 = \alpha_i * \alpha_i = \langle\psi|\varphi_i\rangle\langle\varphi_i|\psi\rangle$$

One slight generalization of Von Neumann measurements:

A Von Neumann Measurement is special kind of projective Measurement

Recall orthogonal projection is operator P with
Note: $A^\dagger$ stands for congugate transpose of A

$$P^2 = P, P^\dagger = P$$

For any decomposition of identy operator $I = \sum_i P_i$ into orthogonal projectors P_i, there exisits projective Measurement that outputs i with probablity $p(i) = \langle\psi|P_i|\psi\rangle$ and leaves the system in renormalized state $\frac{P_i|\psi\rangle}{\sqrt{p(i)}}$.

In other words, this measurement projects the input state $|\psi\rangle$ into one of the orthogonal subspaces corresponding to the projection operators P_i, with probablity of sqaure of amplitude of component of $|\psi|\rangle$ in that subspace

Von Neumann measurement is special case of a projective measurement where all projectors P_i has rank one (in other words, are of $|\psi_i\rangle\langle\psi_i|$)

The simplest example of a Von Neumann measurement is a complete measurement in the computational basis. This can be viewed as following decompositon

$$I = \sum_{i\in\{0,1\}^n} P_i$$

where $P_i = |i\rangle\langle i|$

projective measurements are often described as *observable* An observable is a Hermitean operator M acting on state space of the system it has decomposition

$$M = \sum_i m_i P_i$$

where P_i is orthogonal projector on eigenspace of M with real eigenvalue m_i

Measuring the observable corresponds to performing a projective measurement with respect to the decomposition $I = \sum_i P_i$ where the measurement outcome i corresponds to eigenvalue m_i

$$\||\psi\rangle\| \equiv ...$$

4 circuit model of computation

uniform families of reversible circuits: model of computation

Circuits are networks composed of wires that carry bit values to gates that perform elementary operations on the bits

acyclic, bits move through circuit in a linear fashion, wires never feed back to a prior location in the circuit.

A circuit has n wires. input -¿ enter at most one gate at one time -¿ output

A family of circuits is a set of circuits $\{C_n | n \in Z^+\}$

Definition 4.0.1
a set of gates is universal if for any f, n,m $f : \{0,1\}^n \rightarrow \{0,1\}^m$
a circuit can be constructed for f using only gates from that set

for circuits model, measuring complexity,

we can measure the number of gates/depth of the circuit.

the time slices (different from gate as concurrency)

bits, number of lines

$\triangle$