cs476

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1 Course Information

grade: 4 assignments

1.1 genral definition of financial deriustive contract

A financial option/derivative is a financial contract at time t=0, The value of the contract at future expiry T is determined by the market price of underlying asset at T
S_t is underling price at time t, it is a stochoastic process.
Knowing the future value of contract in relation to the underlying price allows it to be used as an insurance
Holder bought insurance, how much should holder pay today
Writer bought risk (uncertainty) how much should the writer recieve
premium V_0

1.2 payoff function

$V_T = payoff(S_T)$

Definition 1.2.1 (European option)
call option: right to buy at a preset price K at time T put option: right to sell at a preset price L at time T

Definition 1.2.2 (American option)
The right can be exerciesed any time from now to exprity T

Definition 1.2.3 (holder)
buyer of the option, enters a long position, +

Definition 1.2.4 (Writer)
seller of the option, enters a short position, -
eg. -100 share means sell 100 share of balabala

let V(S(t),t) or V_t denote the option value at time t. recall S(t) is price of item at time t
question: is the first argument a function or a value
when know at T, $payoff(S_T)$
question: what is fair value V_0 of the option today
What are payoff functions

$$V_T = payoff(S_T) = max(S_T - K, 0)$$

there can be many payoff functions

1.3 one perioad binomial, fair value of option, arbitrage, put-call parity

consider an 1-perioad binomial case Auume: T=1 and up probablity is p=0.1 20-¿22/18
consider a call with K=21

then option value today is

$$.1 * 1 + .9 * 0 = .1$$

side
cash account continously compounds at risk free rate
borrowing moeny from a bank is selling a bond
depositing money to bank == buying a bond
let a bond has value $\beta(t)$ at time t,

$$\frac{d\beta(t)}{\beta(t)} = rdt$$

solving this ODE by integrating both sides,

$$log(\beta(T)) - log(\beta(t)) = r(T - t)$$

Discounting: 1 year ago,

$$\beta(T) = 1 \rightarrow \beta(t) = e^{-r(T-t)}$$

Discounting: 1 year ago,

$$\beta(t) = 1 \rightarrow \beta(T) = e^{r(T-t)}$$

An aaritage is trading opportunity to make a no-risk profit greater than that of a bank deposit which earns interest $r \geq 0$
Example: buy one share of stock and borrow 100 (sell bonds)

$$H_0 = 1 * S_0 - 100 \text{ or } H_0 = \{S_0, -100\}$$

the value at time t:

$$H_t = S_t - 100e^{rt}$$

Mathematical characterization of an Arbitrage Strategy
A profolio with initail $H_0 = 0$ but $H^T > 0$ is arbitrage
question: interest rate?

1.3.1 Put and Call Parity

Assume stock S_t does not pay divident, interest rate $r \geq 0$, no arbitrage. Then at any time $t \in (0, T)$ European call C_t and put P_t, with same stike K and expiry T, on the same underlying, satisfies

$$C_t = P_t + S_t - Ke^{-r(T-t)}$$

$$C_t - P_t = S_T - K$$

and put them all back in time $T \rightarrow t$

1.4 1-period in binomial model

- option replication and heging
- computing option fair value
- risk neutrual valuation

hedge the uncertainty

1.4.1 Pricing by Replication

qustion what is V_t Assume:

- $S_t > 0$
- no arbitrage
- length of time interval $\delta t > 0$

stock: S_t $S^u_{t+1} = uS_t$ $S^d_{t+1} = dS_t$
bond: $e^{-rt} \to 1$
option: V_t $V^u_{t+1} = uV_t$ $V^d_{t+1} = dV_t$
At t, construct portfolio$\{\delta_t S_t, n_t \beta_t\}$ so that:
Buy η bond and δ_t stock

$$\eta_t + uS_t\delta_t = V^n_{t+1}$$

$$\eta_t + dS_t\delta_t = V^d_{t+1}$$

n is bond, δ_t is amount of stock
Note solution of η, δ is unique
No arbitrage, then

$$V_t = \delta_t S_t + \eta_t e^{-rt}$$

in other words

$$\{V_t, -\delta_t S_t\}$$

is risk free
I can construct combination of stock and bond st the value of the stock+value of bond equals value of option at time t+1
recall $S^u_T = 22 = 1.1S_0$ $S^d_T = 18 = 0.9S_0$
r=0 K=21
we get

$$22\delta + 1\eta = 1$$

$$18\delta + 1\eta = 0$$

$\delta = \frac{C^u_T - C^d_T}{(u-d)S_0}$ $\quad \eta = -4.5$ (sell bond borrow cash)

1.4.2 risk neutral valuation

note: no arbitrage assumge implies $d \leq e^{rt} \leq u$
consider

$$\psi^u + \psi^d = e^{-rt}$$

$$uS_t\psi^u + dS_t\psi^d = S_t$$

the unique solution is $\psi^u = e^{-rt}q^*$, $\psi^d = e^{-rt}(1 - q^*)$

$$q* = \frac{e^{rt} - d}{u - d} \quad q^* \in (0, 1)$$

we can then treat q as a probability

we get

$$S_t = e^{-rt}(q^* u S_t + (1-q^*) d S_t) = e^{-rt} E^Q(S_{t+1})$$

Where E^Q uses q^* as probablity
Let $\{\delta_t S_t, \eta_t \beta_t\}$ be replicating portfolio
by writing β_t as $(q^* \cdot 1 + (1-q^*) \cdot 1)$

$$V_t = E^Q(V_{t+1})$$

risk neutral valuation:

$$\beta_t = e^{-rt}\beta_{t+1}$$
$$S_t = e^{-rt}E^Q(S_{t+1})$$
$$V_t = e^{-rt}E^Q(V_{t+1})$$

where V_t is derivative on S.
for the example
risk neutrual probability is

$$q^* = \frac{e^{rt}-d}{u-d} = \frac{1-0.9}{1.1-0.9}$$

2 logarithmic return

$$\Delta X_n = log(\frac{S_{n+1}}{S_n}) \quad X_n = log(S_n)\Delta X_n = X_{n+1} - X_n$$

simple return $\frac{S_{n+1}}{S_n} \approx \Delta X_n$
so ΔX_n is Δh with p and $-\Delta h$ with 1-p

3 Discrete random walk and its limit

price $S_0 \to uS_0$
log price $X_0 \to x_0 + h$, both with probablity p
Properities of descrete random walk:

- $\Delta X)n$ has identical independent distribution for any n
- moments:
  $E(\Delta X_n) = p\Delta h - (1-p)\Delta h = (2p-1)\Delta h = \alpha_0$
  $E(\Delta X_n^2) = p(\Delta h)^2 + (1-p)(\Delta h)^2 = (\Delta h)^2$
  $var(\Delta X_n) = \Delta h^2 - (2p-1)^2(\Delta h)^2$

$prob\left(\frac{\sum_{N-1}^{n=0} \Delta X_n - N\alpha_0}{\sigma_0\sqrt{N}} \leq x\right) \to F_{normal}(x)$
in finance, standard deviation is volatility σ, or variance σ^2
the expected log return in an interval of Δt is $\alpha\Delta$, variance $\sigma^2\Delta t$
so when $\sigma\alpha$ is given, we chan choose $p, \Delta h$ to satisfy $E(\Delta X_n)$ and $var(\Delta X_n)$
one frequent choice:

$$u = e^{\sigma\sqrt{\Delta t}}$$

$$d = e^{-\sigma\sqrt{\Delta t}}$$
$$p = 1/2 + \frac{1\alpha}{2\sigma}\sqrt{\Delta t}$$

note:

- option value does not depend on α, it depends only on volatility
- CRR leads to a nondrifting tree
- there are many differnet choices for (u,d,p) since there are 2 equation and 3 parameters
- one can also set up no-arbitrage lattice(set p) for which the expected rate of return is r

an algorithm

- choose u d p st. $log(u) = \Delta h = \sigma\sqrt{\Delta t}$ Δt small enough st
$$u = e^{-\sigma\sqrt{\Delta t}} \leq e^{r\Delta t} \leq e^{\sigma\sqrt{\Delta t}}$$
- construct lattice of prices e.g. $u = e^{\sigma\Delta t}$
$$S_{j+1}^{n+1} = ...S_j^{n+1} = ...$$
- at exprity T, value of option is $V_j^N = payoff(S_j^N)$

4 Divident

at t=0, the underlying pays divident D $D = \rho S(t_d^-)$ at $t = t_d$, $0 \leq t_d \leq T$

$$S(t_d^+) = S(t_d^-) - D$$

option is divident protected
$$V(S(t), t) \equiv V(t)\text{is continous funciton of t}$$
$$V(S(T_d^d), t_d^-) = V(S(T_d^+), t_d^+) = V(S(T_d^-) - D, t_d^+)$$

4.1 brownian motion

$$dX = \alpha dt + \sigma dZ$$

αdt is drift term, σ is volatility dZ is random term

$$dZ = \phi\sqrt{dt} \quad \phi \sim N(0,1)$$

$$E[dx] = \alpha dt$$
$$Var[dx] = E[dX - E[dx]]^2 = \sigma^2 dt$$

discrete model: x has p possibility to go upward, q to go downward

$$E[\Delta x] = (p - q)\Delta h$$
$$E[\Delta X]^2 = \Delta h^2$$

$$VAR[\Delta X] = 4pq(\Delta h)^2$$

$$E[X_n - X_0] = n(p-q)\Delta h = \frac{t}{\Delta t}(p-q)\Delta h$$

$$VAR[X_n - X_0] = n4pq(\Delta h)^2$$

if we take $\Delta h = \sigma\sqrt{\Delta t}$ and $p - q = \frac{\alpha}{\sigma}\sqrt{\Delta t}$

$$p, q = 0.5[1 \pm \frac{\alpha}{\sigma}\sqrt{\Delta t}]$$

$$E[X_n - X_0] = \alpha t$$

$$Var[X_n - X_0] = t\sigma^2(1 - \frac{\alpha^2}{\sigma^2}\Delta t)$$

4.2 Standard Brownian motion

p=0.5 $\Delta X = \sqrt{\Delta T}$ A standard Brownian motion Z_t has

- Z(0)=0
- $Z(t + \Delta t) - Z(t) \sim N(0, \Delta t)$
- $Z(t_2) - Z(t_1)$ is independent

Ito's Process
$dX_t = a \cdot dt + b \cdot dZ_t$ a deterministic trend + random flucturation of standard Brownian
Black Scholes Model:

$$\frac{dS_t}{S_t} = \mu \cdot dt + \sigma \cdot dZ_t$$

important propperties of Z_t, let $\psi_t \sim N(0, 1)$

$$dZ_t = \psi_t\sqrt{dt} \quad (dZ_t)^2 = dt$$

note variance of ΔZ_t^2 goes to zero quadratically

4.3 Geometric Brownian motion

4.4 Ito's lemma

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5.1 risk neutral pricing

$$E^Q(\frac{S(t_{n+1})}{S(t_n)}) = e^{\Delta t} = 1 + r\Delta t + O((\Delta t)^2)$$

this implies: under risk neutral probability,

$$\frac{dS_t}{S_t} = rdt + \sigma dZ_t^Q$$

Z_t^Q is a standard Brwonain

Under continous model, risk neutral pricing becomes

$$V(S,t) = e^{-r(T-t)} E^Q(payoff(S_T))$$

where $E^Q()$ assumes underlying asset price follows

$$\frac{dS_t}{S_t} = rdt + \sigma dZ_t^Q$$

5.2 MC for pricing European option

$$V(S_0, 0) \approx \frac{1}{M} \sum_{M}^{j=1} (e^{-rt} payoff((S_T)^j))$$

recall we can generate $S_t = S_0 e^{(r-0.5\sigma^2)t + \sigma Z_t}$
we can generate M S_T accordingly

5.3 MC for pricing European path dependent option

Definition 5.3.1 (Barrier Option)
a barrier option come/ceases to exist when a barrier has been crossed e.g.(up/down, in/out)

Definition 5.3.2 (Up-Out)
if S_t crosses a up-barrier S_u, option pays nothing
otherwise, standard payoff

Definition 5.3.3
Asian call payoff

$$max(0, \frac{1}{T} \int_0^T S_t dt - K)$$

we need to make this discrete

$$A_n = \frac{1}{n}((n-1)A_{n-1} + S(t_n)) = \frac{n-1}{n} A_{n-1} + \frac{1}{n} S(t_n)$$

If we use M simulations, sampling error is $O(\frac{1}{\sqrt{M}})$ becasue CLT
CI:

$$\hat{\sigma} = \left[\frac{\sum_M^{j=1} (Y^j - V^M)^2}{M-1} \right]^{\frac{1}{2}}$$

approcimating,

$$V^M - V^0 \approx N(0, \frac{\hat{\sigma}}{\sqrt{M}})$$

where $V_0 = E(Y)$

$$v_0 \in \left[V^M - 1.96\hat{\sigma}/\sqrt{M}, V^M + 1.96\hat{\sigma}/\sqrt{M} \right]$$

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6.1 Euler Method and Time Stepping

$$\frac{dS_t}{S_t} = rdt + \sigma(S_t, t)dZ_t$$

integrate, we get (approximately)

$$S_{n+1} = S_n + S_n(r\Delta t + \sigma(S_n, t_n)\sqrt{\Delta t}\phi_n) \quad \phi_n \sim N(0, 1)$$

6.2 Error in Eular Method

the above equation comes from approximating integral from aboveabove equation

$$\left|E(S(T)) - E(S_T^{\Delta t})\right| \leq C\Delta t$$

S(T) is a price realization to

$$\frac{dS_t}{S_t} = rdt + \sigma(S_t, t)dZ_t$$

$S_T^{\Delta t}$ is approximation with time step Δt

4.8.2 course note

Balance Time Stepping Error and Sampling Error
Assume a approximation has time stepping accuracy Δt Sample error is (M simulation) $O(\frac{1}{\sqrt{M}})$
if we want total error $= O(\Delta t)$, we need

$$M \approx \frac{C}{(\Delta t)^2}$$

complexity = O(timestep * samples)
$= O((\frac{T}{\Delta t})M)$
$= O(\frac{M}{\Delta t})$
$= O(\frac{1}{(\Delta t)^3})$

$$error = O(\frac{1}{complexity^{\frac{1}{3}}})$$

thus to reduce error by factor of 10, we increase computation by 10^3

6.3 Dynamic Trading Performance Analysis via MC simulation

in addition to back testing, we can perform hedging analysis based on a stochastic model
Goal:

- How good is hedging strategy
- how risky
- how we measure the risk

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7.1 dynamic hedging analysis

number of units in underlying is

$$\delta_j^n = \frac{V_{j+1}^{n+1} - V_j^{n+1}}{(u-d)S_j} \approx \frac{dV}{dS}(S_j^n, t_n)$$

initially, we have

$$B_0 = V_0 - \delta_0 S_0$$

$$\text{protfolio } \pi = -V + \delta S + B \text{ has value } \pi_0 = 0$$

$$\pi_{t_{n+1}^+} = \pi_{t_{n+1}^-}$$

$$B_{n+1} = B_n e^{r\Delta t} + (\delta_n - \delta_{n+1}) S_{n+1}$$

interpretation: if $\delta_{n+1} > \delta_n$, buy $\delta_{n+1} - \delta_n$ units
if $\delta_{n+1} < \delta_n$, sell $\delta_{n+1} - \delta_n$ units
At T, liquid the portfolio which has value

$$\pi_N = -V(S_N, t_N) + \delta_{N-1} S_N + B_{N-1} e^{r\Delta t}$$

- π_N is random if it's 0, perfect hedge
- $\pi_N > 0$, profit for writer
- $\pi_N > 0$, a loss scenario
- π_n is hedging error, we consider discounted relative $P\&L$

$$P\&L = \frac{e^{-rT}\pi_N}{V(S_0, 0)}$$

7.2 computing delta under binomial lattice and MC pricing

7.3 Delta neutral, gamma neutral, bega neutral hedging

7.3.1 Delta Neutral

$\pi = \{-V, \delta S, B\}$

$$\frac{d\pi}{dS} = -\frac{dV}{dS}(S,t) - \delta \equiv 0$$

7.3.2 gamma neutral

$$\frac{d\pi}{dS} = \frac{d^2\pi}{dS^2} = 0$$

let

$$\pi = \{-V, \delta_s S, \delta_i I, B\}$$

$$-\frac{dV}{dS} + \delta_S + \delta_I \frac{dI}{dS} = 0$$
$$-\frac{d^2V}{dS^2} + 0 + \delta_I \frac{d^2I}{dS^2} = 0$$

hence

$$di = \frac{\frac{d^2V}{dS^2}}{\frac{d^2I}{dS^2}}$$

8 Models for correlated processes, generate correlated standard normal, simulate correlated process, BS PDE

Model for correlated process:

$$\frac{dS^1}{S^1} = \mu^1 dt + \sigma^1 dZ^1$$
$$\frac{dS^2}{S^2} = \mu^1 dt + \sigma^2 dZ^2$$
$$E(dZ^1 dZ^2) = \rho dt$$

to generate this model, we observe $dZ^i = \phi^i \sqrt{dt}$ Thus we genrate 2 correlated standard normal with

$$E(\phi^1 \phi^2) = \rho$$

for any any $\sum \phi^i x_i$
its mean is 0, its variance is $x^t Q x$
for positive semidefinte maxtrix Q, we can $Q = G^T G, G = chol(Q)$
we generate $\phi = G^T \epsilon, \epsilon\ _{iid} N(0,1)$
we can show $\phi\phi^T = Q$

8.1 BD PDE

assume now volatility is a $\sigma(S_t, t)$
if we construct profolio $\{V_t, -\delta_t S_t\}$ and choose $\delta_t = \frac{\delta V}{\delta S}$
then we have $\delta \pi_t = f(S_t, t)$ does not have ϕ term (deterministic)
and according to no arbitrage, $d\pi_t = r\pi_t dt$
thus we have PDE, lec 13